Journal of the Operations Research Society of Japan
Vol. 64, No. 1, January 2021, pp. 1–11

NASH EQUILIBRIA FOR INFORMATION DIFFUSION GAMES ON WEIGHTED CYCLES AND PATHS

Tianyang Li Maiko Shigeno
University of Tsukuba

(Received July 8, 2019; Revised July 8, 2020)

Abstract The information diffusion game, which is a type of non-cooperative game, models the diffusion process of information in networks for several competitive firms that want to spread their information. Recently, the game on weighted graphs was introduced and pure Nash equilibria for the game were discussed. This paper gives a full characterization of the existence of pure Nash equilibria for the game on weighted cycles and paths according to the number of vertices, the number of players and weight classes.

Keywords: Game theory, information diffusion game, pure Nash equilibrium, weighted cycle, weighted path

1. Introduction

Modeling and analyzing information flows and spread of rumors on social networks have been attracting many researchers in social sciences and information sciences (e.g. [3]). One such model is the information diffusion game by Alon et al. [1]. This game, a non-cooperative game on graphs, models the diffusion process of competitive information, i.e., products or brands in e-marketing, where each firm makes a decision to obtain as many customers as possible. Each firm, called a player in this game model, selects influencers, to which the player can give its information initially. Since the goal of this model is to select locations where a player sends his/her competitive information, the information diffusion game is regarded as a variation of the competitive facility location games on graphs, which are derived from Hotelling's model known as the ice-cream vendor problem [6].

Whereas some studies have investigated variations of the information diffusion game based on real-world applications [8, 9, 14], some researches have characterized this simple game theoretically and mainly investigated the existence of pure Nash equilibria. A pure Nash equilibrium (PNE) is an optimal strategy for each player with respect to his/her opponents' strategies. The existence of PNEs on the information diffusion game has been characterized for some classes of underlying graphs. Alon et al. [1] and Takehara et al. [13] discussed for graphs of small diameters. Small and Mason [11] dealt with the information diffusion game on trees. They proved that there is always a PNE for two players on trees, but not for more than two players. Roshanbin [10] showed the existence of PNEs for two players on paths, cycles, trees, unicycles and grids. Etesami and Basar [5] investigated on hypercubes and grids. Some of these results were extended to multi-players by Bulteau et al. [2]. PNEs exist for any number of players playing on paths and cycles except for three players on paths of length six or more. For the information diffusion game on grids, there exists a PNE for two players but not for three players. For the information diffusion game on hypercubes, there exist PNEs for any number of players except for three. Sukenari et al. [12] characterized PNEs on toroidal grids for two players. Following this line of research, it is important to clarify classes where a PNE always exists. Investigating the existence of PNEs for a special class of graphs would contribute to analyzing the information diffusion game, even though the graphs are fundamental e.g. paths and cycles.

Recently, Ito et al. [7] discussed the information diffusion game on weighted graphs, where the utility of each player is calculated by the sum of weights of infected vertices instead of the number of infected vertices in the unweighted version. The weight of a vertex represents the favorability level of a customer. If the weight of a vertex is negative, it signifies an unfavorable customer. Ito et al. established the complexity for the problem of asking whether there exists a PNE on certain graph classes. Yamaguchi and Ono [15] discussed the information diffusion game on weighted cycles with two players, where the weight is restricted to nonnegative values. We extend their results to multi-players and allow negative weight. We discuss the full characterization of the existence of PNEs on weighted cycles and paths. The purpose of our study is to clarify which subclass of the games always has a strategy profile that is PNE. The subclasses of the games are classified by weight of vertices and the number of vertices. We say a game admits a PNE if it has a strategy profile that is PNE. In this paper we give

- a proof that PNEs always exist for two players on arbitrary-weight cycles of 3, 4, and 5 vertices,
- a positive-weight such that PNE does not exist for $k(\geq 3)$ players on cycles of $n(\geq k+1)$ vertices,
- a proof that PNEs always exist for $k(\geq 3)$ players on cycles of k and $k-1$ vertices,
- a proof that PNEs always exist for two players on nonnegative-weight paths,
- a weight allowed negative value such that PNE does not exists for two players on paths of $n(\geq 5)$ vertices, and a positive-weight such that PNE does not exist for $k(\geq 3)$ players on paths of $n(\geq k+1)$ vertices, and
- a proof that PNEs always exist for $k(\geq 3)$ players on paths of k vertices.

The rest of the paper is organized as follows. In Section 2, we give the formal definition of the information diffusion game. In Section 3, we characterize the existence of PNEs on cycles and paths according to the relation of the numbers of vertices and players and to the relation to the restriction of weights that are allowed for various values, negative or not.

2. Information Diffusion Game

Let $G = (V, E)$ be an undirected connected graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set E, where n is the number of vertices on G. Suppose that there are k players and assume that $k \leq n$. In the information diffusion game, each player chooses one vertex to send his/her message so as to spread it to as many vertices as possible. During the process of the game, each vertex is in one of three states: uninformed, informed, or deadlocked (these states are originally explained by using "color" for players [1]). Initially, every vertex is uninformed. At the first round of the process, each player sends his/her message to the chosen vertex. A vertex that receives one kind of messages sourced by a player enters the informed state, while a vertex that receives different kinds of messages sourced by more than one players enters the deadlocked state. At each round of the process, every vertex that becomes informed at the previous round sends the received message to all of its adjacent uninformed vertices. Then, an uninformed vertex that receives only one kind of messages enters the informed state, while an uninformed vertex that receives more than one kinds of messages enters the deadlocked state. This process is iterated until there is no vertex that newly becomes informed. The choice of each player at the beginning of the game is represented by the strategy profile $\boldsymbol{s} = (s(1), s(2), \ldots, s(k))$, where $s(p)$ is the index of vertex chosen by player p. Without loss of generality, we assume that $s(1) \le s(2) \le \cdots \le s(k)$. For strategy profile $\boldsymbol{s}$, the utility of player p, denoted by $U_p(\boldsymbol{s})$, is given by the number of informed vertices that receive the message of player p at the end of the diffusion process. Let $V_p(\boldsymbol{s})$ be the set of informed vertices that receive the message of player p when we start with strategy profile $\boldsymbol{s}$. Then, the utility of player p is equal to $|V_p(\boldsymbol{s})|$, i.e., $U_p(\boldsymbol{s}) = |V_p(\boldsymbol{s})|$. Throughout this paper, we use the expression "player p locates on $s(p)$ and obtains vertices in $V_p(\boldsymbol{s})$." When each vertex v is given a numerical weight $w(v)$, the weighted version of the information diffusion game defines the utility of player p as $U_p(\boldsymbol{s}) = \sum_{v \in V_p(\boldsymbol{s})} w(v)$. The original game without giving the vertex weight is regarded as a special case of the weighted version where all vertex weights are 1.

Given a strategy profile $\boldsymbol{s} = (s(1), \ldots, s(k))$, we denote a strategy profile where player p relocates to v_i from $v_{s(p)}$ as $\boldsymbol{s}_{-p,i}$, i.e.,

$$s_{-p,i}(j) = \begin{cases} s(j) & j \neq p \\ i & j = p. \end{cases}$$

A strategy profile $\boldsymbol{s}$ is called a pure Nash equilibrium (PNE) if $U_j(\boldsymbol{s}) \ge U_j(\boldsymbol{s}_{-j,i})$ for any player j and any $v_i \in V$. A PNE implies that no player can increase his/her utility by changing his/her strategy while the other players keep their strategies. The PNE is one of the foundational solution concepts in non-cooperative games.

3. The Existence of Pure Nash Equilibria

To specify a subclass of games where a PNE always exists, the game is investigated by classes of the underlying graphs, size of graphs, number of players and so on. In this section, we consider three weight classes; uniform, nonnegative, and arbitrary, each of which is given by $\{(\overbrace{a, \ldots, a}^{n}) \mid a > 0\}$, $\mathbb{R}^n_{\ge 0}$ and $\mathbb{R}^n$. Note that the game under the uniform weight class is equivalent to that on an unweighted graph. In the following subsections, we characterize the existence of PNEs by weight classes on cycles and paths.

3.1. On cycles

We assume that the edge set is given by $E = \{(v_i, v_{i+1}) \mid i = 1, \ldots, n-1\} \cup \{(v_n, v_1)\}$. For convenience, notation v_{n+t} stands for v_t for any integer t, e.g., $v_{n+1} = v_1$ and $v_{-1} = v_{n-1}$. We know that the information diffusion games on unweighted cycles always admit PNEs for any number of players (e.g. [2, 10]). Yamaguchi and Ono [15] studied the existence and non-existence of PNEs for the information diffusion game on cycles with nonnegative weights for two players ($k = 2$). They proved that the games on cycles with 3, 4 and 5 vertices admit PNEs under nonnegative weight and that there are positive weights for which the game does not admit PNEs on cycles with more than 5 vertices. For $n = 3, 4$, and 5, we can extend their result to the arbitrary weight class, which can be shown by a similar proof of [15] by considering which vertices have negative weights.

- When $n = 3$, without loss of generality, we assume that $v_1 \in \arg\max_{v \in V} w(v)$ and $w(v_2) \ge w(v_3)$. If $w(v_2) < 0$, then $\boldsymbol{s} = (v_1, v_1)$ is a pure Nash equilibrium. Otherwise, $\boldsymbol{s} = (v_1, v_2)$ is a pure Nash equilibrium.
- When $n = 4$, we also assume that, without loss of generality, $v_1 \in \arg\max_{v \in V} w(v)$ and $w(v_2) \ge w(v_4)$ (if $w(v_2) < w(v_4)$, we turn over the cycle, i.e., we regard the cycle v_1-v_2-v_3-v_4 as v_1-v_4-v_3-v_2). We first consider the case of $w(v_2) < 0$. If $w(v_3) \ge 0$, then $\boldsymbol{s} = (v_1, v_3)$ is a pure Nash equilibrium. Otherwise, $\boldsymbol{s} = (v_1, v_1)$ is. We next consider the case of

$w(v_2) \geq 0$. If $w(v_2) + w(v_3) < 0$, then $\boldsymbol{s} = (v_1, v_1)$ is a pure Nash equilibrium, because $w(v_3) < 0$. Otherwise, $\boldsymbol{s} = (v_1, v_2)$ is a pure Nash equilibrium if $w(v_1) + w(v_4) \geq 0$. For the remaining case, that is, $w(v_2) + w(v_3) \geq 0$ and $w(v_1) + w(v_4) < 0$, $\boldsymbol{s} = (v_2, v_2)$ is a pure Nash equilibrium.

- When $n = 5$, we define $Z_i = w(v_i) + w(v_{i+1})$. Note that the relationships of relative values of Z_i are divided into two cases; Z_1, Z_2 and Z_3 are at least as large as Z_4 and Z_5; and Z_1, Z_2 and Z_4 are at least as large as Z_3 and Z_5. For the former case, without loss of generality, we can assume $Z_1 \geq Z_3$. If $Z_3 \geq 0$, then $\boldsymbol{s} = (v_2, v_3)$ is a pure Nash equilibrium, because $U_1(\boldsymbol{s}) = Z_1$ and $U_2(\boldsymbol{s}) = Z_3$. Otherwise, $\boldsymbol{s} = (v_2, v_2)$ is a pure Nash equilibrium, because Z_3, Z_4 and Z_5 are negative. For the latter case, if $Z_4 < 0$, then $\boldsymbol{s} = (v_2, v_2)$ is a pure Nash equilibrium, because Z_3, Z_4 and Z_5 are negative. Thus, we assume that $Z_4 \geq 0$. If the values of Z_i except for Z_4 are negative, $\boldsymbol{s} = (v_4, v_4)$ is a pure Nash equilibrium. If $Z_1 \geq Z_2$ and $Z_1 \geq 0$, then $\boldsymbol{s} = (v_2, v_4)$ is a pure Nash equilibrium. Otherwise, that is, $Z_2 > Z_1$ and $Z_2 \geq 0$, then $\boldsymbol{s} = (v_2, v_5)$ is.

Combining the result of [15] with the above discussion, we obtain the following.

Theorem 3.1. *Suppose that $k = 2$. If $n = 3, 4$ and 5, then the information diffusion game on the cycle admits PNEs for arbitrary vertex weights. If $n \geq 6$, we have positive weights for which the information diffusion game on the cycle does not admit any PNE.*

In the following, we prove that there exist positive weights for which the game with three or more players ($k \geq 3$) does not admit any PNE. First, we introduce properties for the game with positive weights.

Lemma 3.1. *Assume that $n \geq k$ and a weight of each vertex is positive. Strategy profile $\boldsymbol{s}$ is not a PNE if there exist players p_1 and p_2 with $s(p_1) = s(p_2)$.*

Proof. When players p_1 and p_2 choose the same vertex, their utilities are zeros. Each of them can increase his/her utility by relocating to any unoccupied vertex. □

For convenience, players 0 and $k+1$ stand for players k and 1, respectively.

Lemma 3.2. *Assume that a given weight is positive. On cycles, strategy profile $\boldsymbol{s}$ is not a PNE if there exists player p such that both $s(p) - s(p-1)$ and $s(p+1) - s(p)$ are even.*

Proof. The set of vertices obtained by player p is

$$V_p(\boldsymbol{s}) = \{v_i \mid i = \lceil \frac{s(p) + s(p-1) + 1}{2} \rceil, \ldots, \lfloor \frac{s(p+1) + s(p) - 1}{2} \rfloor\},$$

which is equal to $\{v_i \mid i = \frac{s(p)+s(p-1)+2}{2}, \ldots, \frac{s(p+1)+s(p)-2}{2}\}$ since $s(p) - s(p-1)$ and $s(p+1) - s(p)$ are even. For the strategy profile $\boldsymbol{s}' = \boldsymbol{s}_{-p,s(p)+1}$, that is, when player p moves to $v_{s(p)+1}$ from $v_{s(p)}$, $V_p(\boldsymbol{s}') = \{v_i \mid i = \lceil \frac{s(p)+s(p-1)+2}{2} \rceil, \ldots, \lfloor \frac{s(p+1)+s(p)}{2} \rfloor\} \supset V_p(\boldsymbol{s})$, which implies that the utility is increased because the weight of the additional vertex is positive. □

We call vertex v_i *internal* in a strategy profile if vertices v_{i-1}, v_i, and v_{i+1} are chosen by players. Note that the utility of a player who chooses an internal vertex v_i is equal to $w(v_i)$.

For a positive value α, let $w^{2\alpha}$ be a weight, where the values of five consecutive vertices are $2\alpha, 1, 2\alpha + 1, 1$ and 2α and the values of the remaining vertices are 2, namely,

$$w^{2\alpha}(v_i) = \begin{cases} 2\alpha & (i = 1, 5) \\ 1 & (i = 2, 4) \\ 2\alpha + 1 & (i = 3) \\ 2 & (\text{otherwise}) \end{cases}$$

(See Figure 1).

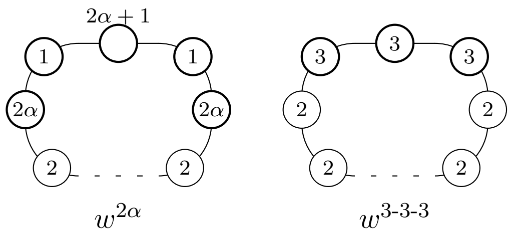


Figure 1: Specified vertex weights for cycle graphs

Lemma 3.3. *For k (≥ 4) players and $k+2 \leq n \leq \alpha(k-3)+4$, the information diffusion game on the n-cycle does not admit any PNE for weight $w^{2\alpha}$.*

Proof. Let $W = \{v_1, v_2, v_3, v_4, v_5\}$ and S be the set of vertices chosen by players. When $|W \cap S| \geq 4$, a player who locates on v_2 or v_4 has a utility equivalent to its weight 1. The player can increase his/her utility by relocating to any unoccupied vertex of weight 2.

When $|W \cap S| = 3$, we consider four cases classified by whether or not each of v_4 and v_5 is included in S. (i) If $v_4 \in S$ but $v_5 \notin S$, we have $s(3) = 4$ and $s(4) > 5$. In this case, player 4 obtains vertices with weight 2 only, i.e., $V_4(\boldsymbol{s}) \subseteq V \setminus W$. By relocating to v_5, player 4 can increase his/her utility. (ii) If $v_4, v_5 \notin S$, then $v_1, v_2, v_3 \in S$ which implies that v_2 is internal. Thus, the player located on v_2 can increase his/her utility by relocating to an unoccupied vertex. (iii) If $v_4, v_5 \in S$, but neither (i) nor (ii) applies by considering the symmetry of weight between v_1 and v_5 (i.e., neither $v_2 \in S, v_1 \notin S$ nor $v_1, v_2 \notin S$), then $v_1 \in S$ and $v_2, v_3 \notin S$. In this case, the player on v_1 can increase his/her utility by relocating to v_3. (iv) the remaining case is $v_1, v_3, v_5 \in S$ but $v_2, v_4 \notin S$. Lemma 3.2 implies that this case is not a PNE.

When $|W \cap S| = 2$, player 3 and player k, each of which respectively locates on the vertices in $V \setminus W$ with the smallest and largest indices, cannot obtain unoccupied vertices in W simultaneously. In this case, $k-3$ players obtain a share of the utility from $V \setminus W$. When $|W \cap S| \leq 1$, the number of players satisfying $V_p(\boldsymbol{s}) \subseteq V \setminus W$ is not less than $k-3$. These players obtain a share of the utility from $V \setminus W$. The average of their utilities is no more than

$$\frac{\sum_{v \in V \setminus W} w(v)}{k-3} = \frac{2(n-5)}{k-3} < 2\alpha,$$

which implies that there exists a player whose utility is less than 2α. Then the player can increase his/her utility by relocating to an unoccupied vertex $\hat{v}$ with $w(\hat{v}) \geq 2\alpha$, where such vertex $\hat{v}$ always exists when $|W \cap S| \leq 2$. □

Even when the weight is bounded by a small number such as 3, we have a weight such that no PNE is admitted on cycles with $k \geq 3$. We define weight $w^{3\text{-}3\text{-}3}$ for a cycle so that weights of three consecutive vertices are 3 and weights of remaining vertices are 2, namely,

$$w^{3\text{-}3\text{-}3}(v_i) = \begin{cases} 3 & (i = 1, 2, 3) \\ 2 & (\text{otherwise}). \end{cases}$$

Lemma 3.4. *For k (≥ 3) players and $n \geq 3k-3$, the information diffusion game on the cycle does not admit any PNE for the weight $w^{3\text{-}3\text{-}3}$.*

Proof. Let $W = \{v_1, v_2, v_3\}$ and S be the set of vertices chosen by players. If $|W \cap S| = 3$, then there are three consecutive unoccupied vertices in $V \setminus W$ because there are at least $2k-3$ unoccupied vertices while $k-3$ occupied vertices in $V \setminus W$. Thus, a player who locates on v_2 can increase his/her utility by relocating to an appropriate vertex in such three consecutive unoccupied vertices.

When $|W \cap S| = 0$, player 1 (resp. player k) can increase his/her utility by relocating to an appropriate vertex in W if $|V_1(\boldsymbol{s}) \cap W| \leq 1$ (resp. $|V_k(\boldsymbol{s}) \cap W| \leq 1$).

Finally, we consider the case of $1 \leq |W \cap S| \leq 2$. When $v_3 \notin S$, suppose that player p locates on the vertex with the smallest index in $V \setminus W$ among occupied vertices. That is, $s(p-1) < 3$ and $s(p) > 3$ hold. In this case, player p can increase his/her utility by relocating to $v_{s(p-1)+1}$. By the symmetry of weight between v_1 and v_3, the remaining case is $v_1, v_3 \in S$ and $v_2 \notin S$, that is, $s(1) = 1$ and $s(2) = 3$. If $s(3) > 4$, then player 2 can increase his/her utility by relocating to v_2. If $s(3) = 4$ and $s(k) = n$, then there are three consecutive unoccupied vertices in $V \setminus W$ in which player 1 or 3 can increase his/her utility by relocating to the one vertex. □

The results of Lemmas 3.3 and 3.4 are summarized as follows.

Theorem 3.2. *For $k \geq 3$ and $n \geq k+2$, we have positive weights for which the information diffusion game on the cycle with k players does not admit any PNE.*

Proof. The case not covered by Lemmas 3.3 and 3.4 is $k = 3$ and $n = k + 2$. For this case, weight $(2, 1, 3, 1, 2)$ provides an example where no PNE exists. □

We next discuss the existence of PNEs of the information diffusion game on arbitrary weighted cycles with a small number of vertices.

Theorem 3.3. *Suppose that $k \geq 3$. If $n = k$ or $n = k + 1$, then the information diffusion games on cycles admit a PNE for arbitrary vertex weights.*

Proof. We denote the number of players who locate on vertex v_i in a strategy profile $\boldsymbol{s}$ by $P(v_i)$, i.e., $P(v_i) = |\{p \mid s(p) = i\}|$.

We first consider the case of $n = k$. If a given weight is nonnegative, then a strategy profile $\boldsymbol{s}^0$ given by $s^0(i) = i$ for $i = 1, \dots, k$ is a PNE. If a given weight is negative, then a strategy profile such that every player locates on v_1 is a PNE. Otherwise, by applying the following two processes, called steps, to the strategy profile $\boldsymbol{s}^0$, we create a strategy profile so that no player obtains negative weight vertices. A maximal interval of consecutive negative vertices $v_h, v_{h+1}, \dots, v_{h+l}$ means the interval of vertices such that $w(v_i) < 0$ for all $i = h, \dots, h + l$, but $w(v_{h-1}) \geq 0$ and $w(v_{h+l+1}) \geq 0$.

Step 1 For a maximal interval of consecutive negative vertices, $v_h, v_{h+1}, \dots, v_{h+l}$, if $l \geq 1$, then players who locate on this interval relocate to the next vertex outside the interval, i.e.,

$$s(j) = \begin{cases} h-1 & j = h, \dots, h + \lfloor \frac{l}{2} \rfloor \\ h+l+1 & j = h + \lfloor \frac{l}{2} \rfloor + 1, \dots, h + l. \end{cases}$$

Step 2 For a vertex v_h with negative weight and $P(v_h) = 1$, if $P(v_{h-1}) > 1$ (resp. $P(v_{h+1}) > 1$), then the player located on v_h relocates to vertex v_{h+1} (resp. v_{h-1}). Otherwise, the player located on v_h relocates to vertex v with $P(v) > 1$ or with $P(v) \geq 1$ and $w(v) < 0$ if exists.

Step 2 is applied after Step 1 is applied for every maximal interval of consecutive negative vertices with length of 2 or more ($l \geq 1$). Except for the case in which there exists only one negative weight vertex, the strategy profile obtained from these steps satisfies the following conditions.

P1 Every unoccupied vertex has negative weight. Conversely, every vertex v with negative weight satisfies $P(v) \neq 1$.

P2 The utility of each player is equal to the weight of the vertex located on or zero.

Condition P2 is derived from the structure of cycles. Thus, no player can increase his/her utility by relocating. The remaining case is in which vertex v_h has negative weight but weights of the others are nonnegative. If $n \geq 4$, then a player located on v_h relocates to a vertex except for v_{h-1} and v_{h+1}. The obtained strategy profile is a PNE. If $n = 3$, then the player relocates on v_{h+1}. Then the utility of player on v_{h-1} is $w(v_{h-1}) + w(v_h)$. If the utility is nonnegative, this profile strategy is a PNE. Otherwise we obtain a PNE strategy by relocating the player on v_{h-1} to v_{h+1}.

When $n = k + 1$, we assume that, without loss of generality, $w(v_n) = \min_{v \in V} w(v)$. If $\min_{v \in V \setminus \{v_n\}} w(v) \geq 0$, then the strategy profile $s^0(i) = i$ for $i = 1, \dots, k$ is a PNE. Thus, we consider the case of $\min_{v \in V \setminus \{v_n\}} w(v) < 0$. Similarly to the case of $n = k$, Steps 1 and 2 are applied iteratively. The obtained strategy profile with these steps is a PNE, except in the case in which there exists only one vertex v_h such that $w(v_h) < 0$ in $v_1, \dots, v_{n-1}$. For this exception, if v_h is not adjacent to v_n, we obtain a PNE in the same manner as in the case of $n = k$ with only one negative weight vertex. Therefore, we assume that $h = n - 1$. Since Step 1 makes $P(v_{n-2}) \geq 2$, the utility of a player on v_1 becomes $w(v_1) + w(v_n) + w(v_{n-1})$. If this utility is nonnegative, then this strategy profile is a PNE. Otherwise, the player on v_1 relocates to v_2, then the obtained strategy profile is a PNE. $\square$

3.2. On paths

Assume that vertices are indexed sequentially along a path, i.e., $E = \{(v_i, v_{i+1}) \mid i = 1, \dots, n-1\}$. We know that the information diffusion game on the unweighted paths possesses PNEs when the number of players is not three [2, 4].

Different from the case on cycles, a PNE always exists on nonnegative weighted paths with two players.

Theorem 3.4. *The information diffusion games on weighted paths with two players possess PNEs if the weights are nonnegative.*

Proof. Let $Y_h = \sum_{i=h+1}^{n} w(v_i) - \sum_{i=1}^{h-1} w(v_i)$. Because $Y_1 \geq 0$, $Y_n \leq 0$ and $Y_h \geq Y_{h+1}$ for any $h = 1, \dots, n-1$, $\hat{h} := \max\{h \mid Y_h \geq 0\}$ is well defined. We consider the strategy profile $s(1) = \hat{h}$ and $s(2) = \hat{h} + 1$. It is obvious that the utility does not increase by relocating player 1 to v_i for $i = 1, \dots, \hat{h} - 1$ and by relocating player 2 to v_i for $i = \hat{h} + 2, \dots, n$. By relocating player 2 to v_i for $i = 1, \dots, \hat{h} - 1$, his/her utility becomes less than or equal to $\sum_{i=1}^{\hat{h}-1} w(v_i)$ which is no more than the previous utility $\sum_{i=1}^{\hat{h}+1} w(v_i)$, since $Y_{\hat{h}} \geq 0$. Similarly, the utility of player 1, $\sum_{i=1}^{\hat{h}} w(v_i)$, decreases by the player relocating to v_i for $i = \hat{h}+2, \dots, n$ since $Y_{\hat{h}+1} < 0$. $\square$

With respect to arbitrary weights, we can prove that a PNE exists for two players on paths with $n = 3$ and 4 in a similar manner on cycles by checking every case. By the way, we have a weight in which no PNE is admitted for two players when $n \geq 5$. Let $\hat{w}^A$ be given by

$$\hat{w}^A(v_i) = \begin{cases} 1 & (i = 1) \\ n - 2 & (i = 2) \\ \lceil \frac{n}{2} \rceil & (i = n) \\ -1 & (\text{otherwise}) \end{cases}$$

(see Figure 2). Let $ol(v_i)$ be a vertex where player 2 obtains the maximum utility when player 1 locates on v_i. We can observe that $ol(v_1) = v_2, ol(v_2) = v_n, ol(v_3) = ol(v_4) = v_2$, $ol(v_l) = v_1$ for $l = 5, \dots, n$, which implies that there exists no PNE.

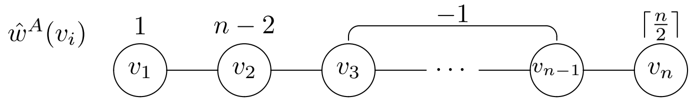


Figure 2: Weight $\hat{w}^A$ on n-path

Theorem 3.5. *For two players ($k = 2$) and $n \geq 5$, the information diffusion game on the n-path does not admit any PNE for the weight $\hat{w}^A$.*

We next consider the case of $k \geq 3$. From a similar observation for unweighted cases [4], we can see that a strategy profile is not a PNE for a nonnegative weight if condition

$$s(1) + 1 = s(2) \text{ and } s(k-1) + 1 = s(k) \tag{3.1}$$

is not satisfied. Let $\hat{w}^{2\alpha}$ be a weight for paths, where the values of both ends of the path and their neighbors are 2α and 1, respectively, and the values of the remaining vertices are 2, namely,

$$\hat{w}^{2\alpha}(v_i) = \begin{cases} 2\alpha & (i = 1, n) \\ 1 & (i = 2, n-1) \\ 2 & (\text{otherwise}) \end{cases}$$

(see Figure 3).

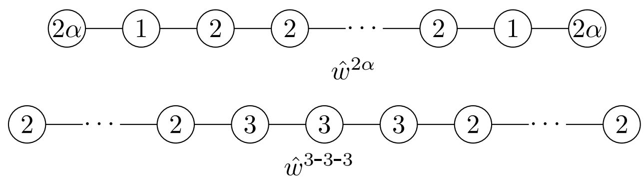


Figure 3: Specified vertex weights for paths

Lemma 3.5. *For k (≥ 3) players and $k + 1 \leq n \leq (k-2)\alpha + 3$, where α is a positive integer more than 2, the information diffusion game on the n-path does not admit any PNE for the weight $\hat{w}^{2\alpha}$.*

Proof. If $s(1) \geq 2$ for strategy profile $\boldsymbol{s}$, then players except for player 1 split the utility of weights between v_3 and v_n, i.e.,

$$\sum_{j=2}^{k} U_j(\boldsymbol{s}) \leq \sum_{i=3}^{n} w(v_i) = 2(n-4) + 2\alpha + 1 \leq 2\alpha(k-1) - 1.$$

Hence, there exists player j whose utility is less than 2α. Since player j increases his/her utility by relocating to v_1, $\boldsymbol{s}$ is not a PNE.

If $s(1) = 1$ and $s(k) = n$ for a PNE $\boldsymbol{s}$, we have $s(2) = 2$ and $s(k-1) = n-1$ from (3.1). Thus this case occurs only when $k \geq 4$. If $s(3) = 3$, then the utility of player 2 is one and can be increased by this player relocating to an unoccupied vertex. If $s(3) = l > 3$, then the utility of player 2 is $1 + 2\lfloor \frac{l-3}{2} \rfloor$, which is less than the utility when this player relocates to $l-1$, $2 + 2\lfloor \frac{l-3}{2} \rfloor$. Either of both cases contradicts that $\boldsymbol{s}$ is a PNE. $\square$

Even when the weight is bounded by a small number such as 3, we have a weight such that no PNE is admitted on paths with $k \geq 3$. We define weight $\hat{w}^{3\text{-}3\text{-}3}$ as

$$\hat{w}^{3\text{-}3\text{-}3}(v_i) = \begin{cases} 3 & (i = \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1) \\ 2 & (\text{otherwise}) \end{cases}$$

(see Figure 3). The following property is also shown in a similar manner as the proof of Lemma 3.4, where we have to note that a player whose utility is less than four can increase his/her utility if v_1 and v_2 are unoccupied.

Lemma 3.6. *For k (≥ 3) players and $n \geq 3k - 3$, the information diffusion game on the path does not admit any PNE for weight $\hat{w}^{3\text{-}3\text{-}3}$.*

The above two lemmas imply the following.

Theorem 3.6. *For $k \geq 3$ and $n \geq k+1$, we have positive weights for which the information diffusion game on the path with k players does not admit any PNE.*

Finally, we consider the remaining case of $n = k$. In a similar manner as the proof of Theorem 3.3, we can prove the following.

Theorem 3.7. *Suppose that $k \geq 2$. If $n = k$, then the information diffusion game on the path admits a PNE for arbitrary vertex weights.*

Proof. When applying Step 1 in the proof of Theorem 3.3, we modify the manner of relocation as $s(j) = l + 1$ for $j = 1, \ldots, l$ if a maximal interval of consecutive negative vertices is located at the end of the path, e.g., $v_1, v_2, \ldots, v_l$. $\square$

4. Discussion

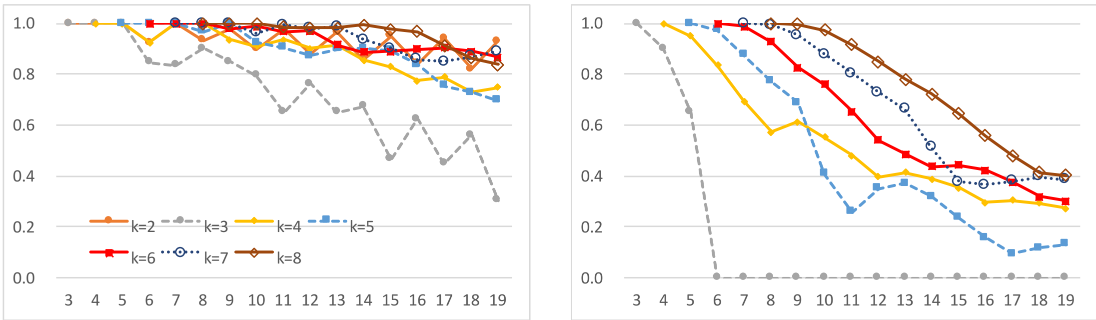
To specify the demarcation where information diffusion games always have PNEs for weighted graphs, we investigated cycles and paths with weight classes, i.e., uniform, nonnegative and arbitrary. We gave full characterization for the existence of PNEs, the summary of which is given in Table 1, where "A" means that PNEs always exist for arbitrary weight, "P" means that PNEs always exists for nonnegative weight, but we have weight, including negative value, for which PNEs are not admitted, "U" means that PNEs always exists for uniform weight, but we have a nonnegative weight for which PNEs are not admitted. Moreover, "–" means that no PNE exists even for uniform weight.

Table 1: Full characterization for existence of PNEs on paths and cycles

cycle					
# of players	n				
	$=k$	$=k+1$	$=k+2$	$=k+3$	$\geq k+4$
$k=2$		A	A	A	U
$k \geq 3$	A	A	U	U	U

path				
# of players	n			
	$=k$	$=k+1$	$=k+2$	$\geq k+3$
$k=2$	A	A	A	P
$k=3$	A	U	U	–
$k \geq 4$	A	U	U	U

Our results indicate which cases always admit PNEs and which do not. It is also interesting how PNEs are found [7] and how many instances have PNEs even though several instances do not admit any PNE in a subclass of the game. We finally display the computational results that evaluate the ratio of the existence of PNEs for all patterns of vertex weight which is given by $w(v) \in \{2, 3\}$. Two weight patterns are regarded to as equivalent if they coincide with each other by applying reflection and rotation of its weight. Our computational evaluation checked whether each weight pattern has a PNE or not. The ratio of instances having a PNE for the set of instances consisting of all possible weight patterns was evaluated. Figure 4 illustrates the existence ratios of PNEs for every k of no more than 9 and $k \leq n \leq 19$, where the horizontal axis represents the number of vertices n. Since PNEs always exist when $k = 2$ for paths, we exclude this case in the plot. Generally speaking, the existence ratio of PNEs on paths tends to be less than on cycles. In paths, the larger the number of players k is, the higher the existence ratio tends to be. However, such a tendency does not appear in cycles. While our computational results were restricted for instances where its weights were 2 and 3, investigating properties of weight patterns for which PNEs exist is interesting future work.



(a) on cycles (b) on paths

Figure 4: Existence ratios of PNEs corresponding to number of vertices, represented on the horizon axis, and number of players

Acknowledgement

This research is supported in part by the JSPS Grant-in-Aid for Scientific Research (B) under Grant No. 16H03118 and 20H02382.

References

[1] N. Alon, M. Feldman, A.D. Procaccia, and M. Tennenholtz: A note on competitive diffusion through social networks. *Information Processing Letters*, **110** (2010), 221–225.

[2] L. Bulteau, V. Froese, and N. Talmon: Multi-player diffusion games on graph classes. *Internet Mathematics*, **12** (2016), 363 – 380.

[3] D. Easley and J. Kleinberg: *Networks, Crowds, and Markets: Reasoning about a Highly Connected World* (Cambridge University Press, 2010).

[4] H. Enomoto, M. Hachimori, S. Nakamura, M. Shigeno, Y. Tanaka, and M. Tsugami: Pure-strategy Nash equilibria on competitive diffusion games. *Discrete Applied Mathematics*, **244** (2018), 1–19.

[5] S. R. Etesami and T. Basar: Complexity of equilibrium in competitive diffusion games on social networks. *Automatica*, **68** (2016), 100–110.

[6] H. Hotelling: Stability in competition. *The Economic Journal*, **39** (1929), 41–57.

[7] T. Ito, Y. Otachi, T. Saitoh, H. Satoh, A. Suzuki. K. Uchizawa, R. Uehara, K. Yamanaka, and X. Zhou: Competitive diffusion on weighted graphs. In F. Dehne, JR. Sack, and U. Stege (eds.): *Algorithms and Data Structures* (WADS 2015. Lect. Notes Comput. Sci. 9214, 2015) 422–433.

[8] S. Jafari and H. Navidi: A game-theoretic approach for modeling competitive diffusion over social networks. *Games* **9** (2018), doi:10.3390/g9010008.

[9] M.A.M. Ali, S.F.F. Ardestani, A. Aliahmadi, and F. Barzinpour: A novel game theoretic approach for modeling competitive information diffusion in social networks with heterogeneous nodes. *Physica A: Statistical Mechanics and its Applications*, **466** (2017), 570–582.

[10] E. Roshanbin: The competitive diffusion game in classes of graphs. *Proceedings of the 10th International Conference on Algorithmic Aspects in Information and Management (AAIM '14)*, (Lect. Notes Comput. Sci., 8546, 2014) 275–287.

[11] L. Small and O. Mason: Nash equilibria for competitive information diffusion on trees. *Information Processing Letters*, **113** (2013) , 217–219.

[12] Y. Sukenari, K. Hoki, S. Takahashi, and M. Muramatsu: Pure Nash equilibria of competitive diffusion process on toroidal grid graphs. *Discrete Applied Mathematics*, **215** (2016), 31–40.

[13] R. Takehara, M. Hachimori, and M. Shigeno: A comment on pure-strategy Nash equilibria in competitive diffusion games. *Information Processing Letters*, **112** (2012), 59–60.

[14] V. Tzoumas, C. Amanatidis, and E. Markakis: A game-theoretic analysis of a competitive diffusion process over social networks. *Proceedings of the Eighth Workshop on Internet and Network Economics (WINE '12)*, (Lect. Notes Comput. Sci., 7695, 2012) 1–14.

[15] R. Yamaguchi and H. Ono: Nash equilibria for competitive diffusion games on weighted cycles. *IPSJ SIG Technical Reports* **3B-1** (2017), 1–5 (in Japanese).

Maiko Shigeno
Faculty of Engineering, Information and Systems
University of Tsukuba
Tsukuba, 305-8573, Japan
E-mail: `maiko@sk.tsukuba.ac.jp`