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# ASSET ALLOCATION WITH ASSET-CLASS-BASED AND FACTOR-BASED RISK PARITY APPROACHES

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*Abstract* The asset allocation strategy is important to manage assets effectively. In recent years, the risk parity strategy has become attractive to academics and practitioners. The risk parity strategy determines the allocation for asset classes in order to equalize their contributions to overall portfolio risk. Roncalli and Weisang (2016) propose the use of "risk factors" instead of asset classes. This approach achieves the portfolio diversification based on the decomposition of portfolio risk into risk factor contribution. The factor-based risk parity approach can diversify across the true sources of risk whereas the asset-class-based approach may lead to solutions with hidden risk concentration. However, it has some shortcomings. In our paper, we propose a methodology of constructing the well-balanced portfolio by the mixture of asset-class-based and factor-based risk parity approaches. We also propose the method of determining the weight of two approaches using the diversification index. We can construct the portfolio dynamically controlled with the weight which is adjusted in response to market environment. We examine the characteristics of the model through the numerical tests with seven global financial indices and three factors. We find it gives the well-balanced portfolio between asset and factor diversifications. We also implement the backtest from 2005 to 2018, and the performances are measured on a USD basis. We find our method decreases standard deviation of return and downside risk, and it has a higher Sharpe ratio than other portfolio strategies. These results show our new method has practical advantages.

**Keywords**: finance, portfolio optimization, risk parity, factor investing

## 1. Introduction

The asset allocation strategy is very important to manage assets effectively. The mean-variance model developed by Markowitz is employed to solve the asset allocation problem. The investor can determine the portfolio weight uniquely for his risk tolerance using the asset's expected rate of return and the covariance matrix. However, the portfolio is highly sensitive to the small change in parameters, especially expected rate of return. Qian[8] proposes risk parity strategy which equalizes each asset's contribution to overall portfolio risk without considering the expected rate of return but the covariance matrix. The portfolio is less affected by a specific asset and it contributes to the asset risk management. Many researches show the performance is superior to other portfolio strategies(Chaves *et al.*[2], Lohre *et al.*[6]).

While explicitly pursuing asset diversification, this methodology may lead to the strategy with hidden risk concentration. The optimal asset-class-based risk parity portfolio depends on how many assets are included in the portfolio. Roncalli and Weisang[12] propose the use of "risk factors" instead of asset classes for the risk parity portfolio. It is based on the idea that asset returns are driven by common risk factors, and achieves the portfolio diversification based on the decomposition of portfolio risk into risk factor contributions.

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Following this research, there are some previous studies. Shimizu and Shiohama[13] construct the factor risk parity portfolio using six factor indices for Japanese stock market, and illustrate it outperforms the market-capitalization-weighted portfolio through the empirical analysis. Shimizu and Shiohama[14] also construct the factor risk parity portfolios for Japanese, the US, UK, and Euro stock markets, and verify the effectiveness. They find that it is a prospective investment strategy that captures factor risk premiums for global equity investments. Komata[5] constructs the factor risk parity portfolio using ten global bond fund indices, and compares its performance with those of the asset risk parity portfolio and the equally weighted portfolio. The result shows that the cumulative return of factor risk parity portfolio is the best, but the Sharpe ratio is the worst because of high risk. The empirical results show the different by asset universe. But this strategy can control the factor risk, and it is an effective method for capturing factor risk premiums.

However, this method has the following two shortcomings. The first is that the factors may lead to the overinvestment in a specific asset-class by ignoring the asset diversification. The second is that it is difficult to obtain a unique solution for the optimization problem, which is shown by Braga[1].

In this paper, we propose a methodology of constructing the well-balanced portfolio by the mixture of the asset-class-based and factor-based risk parity approaches. It can explicitly consider both asset-class and factor diversifications. We also propose the method of determining the weight of two approaches using the diversification index. We can control the portfolio with the weight adjusted dynamically based on time-varying market environment. We examine our model through the numerical tests with seven global financial indices and three factors. We find it gives the well-balanced portfolio between asset and factor diversifications.

Our paper is organized as follows. We explain the basic methodologies of asset-class-based and factor-based risk parity strategies in Section 2. We propose a new method of risk parity strategy in Section 3. We call it "mixed risk parity approach". We calculate the risk parity portfolio using real financial data, and examine the characteristics of the model through numerical analysis in Section 4. We implement eighteen-year backtest and evaluate the performance for the mixed strategy in Section 5. We find our method has many practical advantages. We provide the concluding remarks and future tasks in Section 6.

## 2. Risk Parity Approach

### 2.1. Asset-class-based risk parity approach

#### 2.1.1. Asset risk contribution

We consider a set of $n$ assets $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ with the covariance matrix $\mathbf{\Sigma}$ of asset return. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$ be the vector of portfolio's asset weight. The most commonly used definition of each asset's risk contribution is based on the Euler's homogeneous function theorem. The risk contribution of asset $i$ is calculated as

$$\mathcal{RC}(\mathcal{A}_i) = x_i \cdot \frac{\partial \mathcal{R}(\boldsymbol{x})}{\partial x_i} = \frac{\partial \mathcal{R}(\boldsymbol{x})}{\partial x_i / x_i}, \tag{2.1}$$

where $\mathcal{R}(\boldsymbol{x})$ is the portfolio risk. It can be interpreted as the sensibility of portfolio risk to the relative change of portfolio weight. When the standard deviation is employed as a risk measure or $\mathcal{R}(\boldsymbol{x}) = \sigma_P = \sqrt{\boldsymbol{x}^\top \mathbf{\Sigma} \boldsymbol{x}}$, Equation (2.1) can be replaced as

$$\mathcal{RC}(\mathcal{A}_i) = x_i \cdot \frac{(\mathbf{\Sigma}\boldsymbol{x})_i}{\sigma_P}, \tag{2.2}$$

where $(\boldsymbol{\Sigma x})_i$ shows $i$-th component of the vector $\boldsymbol{\Sigma x}$. The risk contribution satisfies Equation (2.3), which shows that the total portfolio risk equals the sum of risk contribution of each asset by the Euler's homogeneous function theorem.

$$\mathcal{R}(\boldsymbol{x}) = \sum_{i=1}^{n} \mathcal{RC}(\mathcal{A}_i). \tag{2.3}$$

### 2.1.2. Formulation

The asset-class-based risk parity strategy equalizes its risk contribution across all assets. We solve the optimization problem (2.4) so that the portfolio risk can be equally diversified to each asset-class. Specifically, the objective function is defined to minimize the difference between the proportion of each risk contribution to the portfolio risk and the reciprocal of the number of assets. The first constraint shows the sum of portfolio weights is one, and the second constraint shows the long-only investment. Short selling is not allowed for many institutional investors to avoid the extreme leverage position.

$$\begin{aligned} &\textbf{Minimize} \quad \sum_{i=1}^{n} \left( \frac{\mathcal{RC}(\mathcal{A}_i)}{\sigma_P} - \frac{1}{n} \right)^2 \\ &\textbf{subject to} \quad \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0 \, (i = 1, \ldots, n). \end{aligned} \tag{2.4}$$

Roncalli[10] describes the following three advantages for the portfolio with equally weighted risk contributions (ERC hereafter) such as asset-class-based risk parity strategy; (1) well-diversification in terms of risk and weights, (2) no expected returns hypothesis, and (3) less sensitivity to the small change in the covariance matrix than the mean-variance portfolio.

## 2.2. Factor-based risk parity approach

A factor is any characteristic which can explain the risk and return of financial assets. There are two main categories, style factors and macroeconomic factors. Style factors such as value, growth, size and momentum can drive return within a specific asset class. Macroeconomic factors such as interest rate, inflation, credit, and economic growth can describe risk and return across asset classes.

Roncalli and Weisang[12] describe their motivation of proposing the 'risk factor parity' approach as follows. The first is to develop an approach of diversifying across the true sources of risk because even the methodologies which explicitly pursue asset-class diversification may lead to solutions with hidden risk concentration. For example, such a portfolio which consists of four bonds and one stock can be diversified in terms of asset-class, but it may lead to a risk concentrated portfolio in terms of interest rate. The second is to resolve the duplication invariance problem called by Choueifaty *et al.*[3], affecting in particular the ERC approach. The risk factor parity approach can be expected to prevent us from the inherent instability of the ERC solution created when introducing a duplicated asset.

### 2.2.1. Factor risk contribution

We explain the risk contribution of factors by reference to Roncalli and Weisang[12]. We denote a set of $m$ factors $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ and assume the following linear factor model,[1]

$$\boldsymbol{R} = \boldsymbol{AF} + \boldsymbol{\epsilon}, \tag{2.5}$$

[1]We assume the number of assets $n$ is larger than the number of factors $m$.

where $\boldsymbol{R}$ is a vector of asset returns, $\boldsymbol{F}$ is a vector of factor returns, and $\boldsymbol{\epsilon}$ is an error term. $\boldsymbol{A}$ is the "loadings" matrix which shows the relationship between assets and factors. A vector of portfolio's factor exposure $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^\top$ satisfies

$$\boldsymbol{y} = \boldsymbol{A}^\top \boldsymbol{x}. \tag{2.6}$$

We denote $\tilde{\boldsymbol{F}} = (\tilde{\mathcal{F}}_1, \ldots, \tilde{\mathcal{F}}_{n-m})^\top$ is an arbitrary set of $(n-m)$ factors spanning the space of the assets' idiosyncratic risks $\boldsymbol{\epsilon}$, and $\tilde{\boldsymbol{y}}$ represents the portfolio's exposures to the additional factors $\tilde{\boldsymbol{F}}$. The following decomposition can be written as

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{B}^+ & \tilde{\boldsymbol{B}}^+ \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \tilde{\boldsymbol{y}} \end{pmatrix}, \tag{2.7}$$

where $\boldsymbol{B} = \boldsymbol{A}^\top$, $\boldsymbol{B}^+$ is the Moore-Penrose inverse of $\boldsymbol{B}$, $\tilde{\boldsymbol{B}}^+$ is any $n \times (n-m)$ matrix that spans the left nullspace of $\boldsymbol{B}^+$, and $\boldsymbol{A}^+ = (\boldsymbol{B}^+)^\top$. The risk contributions of factors and additional factors can be calculated as the following equations, respectively.

$$\mathcal{RC}(\mathcal{F}_j) = y_j \cdot \frac{\partial \mathcal{R}(\boldsymbol{x})}{\partial y_j} = (\boldsymbol{A}^\top \boldsymbol{x})_j \cdot \left( \boldsymbol{A}^+ \frac{\partial \mathcal{R}(\boldsymbol{x})}{\partial \boldsymbol{x}^\top} \right)_j, \tag{2.8}$$

$$\mathcal{RC}(\tilde{\mathcal{F}}_j) = \tilde{y}_j \cdot \frac{\partial \mathcal{R}(\boldsymbol{x})}{\partial \tilde{y}_j} = (\tilde{\boldsymbol{B}} \boldsymbol{x})_j \cdot \left( \tilde{\boldsymbol{B}} \frac{\partial \mathcal{R}(\boldsymbol{x})}{\partial \boldsymbol{x}^\top} \right)_j. \tag{2.9}$$

These risk contributions satisfy

$$\mathcal{R}(\boldsymbol{x}) = \sum_{j=1}^{m} \mathcal{RC}(\mathcal{F}_j) + \sum_{j=1}^{n-m} \mathcal{RC}(\tilde{\mathcal{F}}_j). \tag{2.10}$$

### 2.2.2. Formulation

In this paper, we consider the standard deviation as a risk measure. Equations (2.8) and (2.9) can be replaced as follows.

$$\mathcal{RC}(\mathcal{F}_j) = (\boldsymbol{A}^\top \boldsymbol{x})_j \cdot \frac{(\boldsymbol{A}^+ \boldsymbol{\Sigma} \boldsymbol{x})_j}{\sigma_P}, \tag{2.11}$$

$$\mathcal{RC}(\tilde{\mathcal{F}}_j) = (\tilde{\boldsymbol{B}} \boldsymbol{x})_j \cdot \frac{(\tilde{\boldsymbol{B}} \boldsymbol{\Sigma} \boldsymbol{x})_j}{\sigma_P}. \tag{2.12}$$

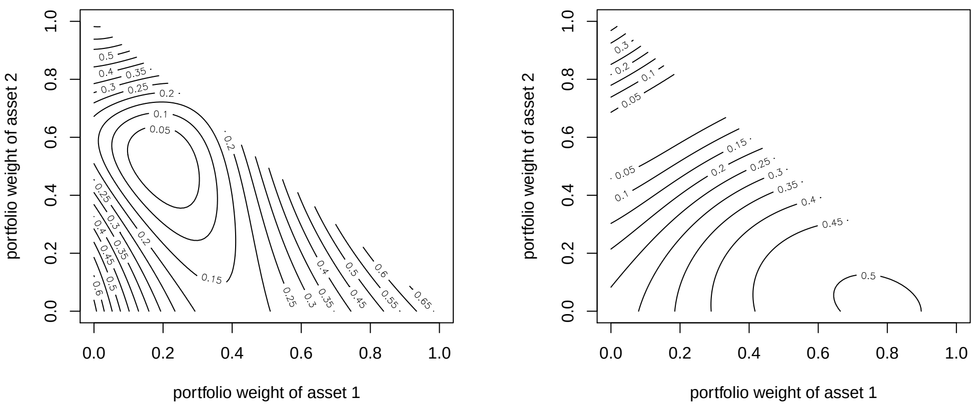
The factor-based risk parity strategy equalizes its risk contribution across all factors, and the portfolio risk can be equally diversified to each factor. We solve the optimization problem (2.13) as well as the asset-class-based risk parity formulation (2.4).[2]

$$\begin{aligned} &\textbf{Minimize} \quad \sum_{j=1}^{m} \left( \frac{\mathcal{RC}(\mathcal{F}_j)}{\sigma_P} - \frac{1}{m} \right)^2 \\ &\textbf{subject to} \quad \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0 \, (i = 1, \ldots, n). \end{aligned} \tag{2.13}$$

It is difficult to obtain a unique solution because the number of factors is usually smaller than the number of assets, and it is degenerated at the minimum optimal value of zero for

[2]We assume that the factor returns can explain the dynamics of asset returns well, and the ideal risk contribution to each factor should be $\frac{1}{m}$ even when portfolio risk $\sigma_P$ includes idiosyncratic risk.

the optimization problem. Figure 1 shows the contour lines of objective function values of optimization problems (2.4) and (2.13). We consider three assets and two factors.[3] The left graph shows the contour lines for Problem (2.4) which equalizes risk contributions of each asset. We find the function is convex. On the other hand, the contour lines in the right graph have different form from the left graph. It is derived by solving Problem (2.13) for the factor-based risk parity approach. We find there exists feasible region of portfolio weights where risk contributions are equally allocated to each factor, and it has a zero objective function value.




(a) Problem (2.4) for asset-class-based risk parity

(b) Problem (2.13) for factor-based risk parity

Figure 1: Contour lines of objective function values

### 2.2.3. Roncalli-Weisang(RW) method

Roncalli and Weisang[12] transformed Problem (2.13) to (2.14). The portfolio standard deviation is minimized, subject to the factor-based risk parity constraint as

$$\begin{aligned} &\textbf{Minimize} && \sigma_P \\ &\textbf{subject to} && \frac{\mathcal{RC}(\mathcal{F}_j)}{\sigma_P} = \frac{1}{m}\,(j = 1, \ldots, m), \\ & && \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0\,(i = 1, \ldots, n). \end{aligned} \tag{2.14}$$

We call it the RW method hereafter. However, we find the RW method tends to concentrate the weights in specific assets through our numerical experiments.

## 3. Mixed Approach of Considering Both Asset-Class-Based and Factor-Based Risk Parity

### 3.1. Proposed method: Mixed risk parity approach

We summarize the characteristics of two risk parity approaches. The asset-class-based risk parity approach has asset-class diversification which represents the differences among market

[3]Three assets are developed country stock, government bond and high-yield bond, and two factors are equity and interest rate.

participants and differences among countries, and it is robust to the change in parameters. However, the portfolio is dependent on the asset universe, and it may lead to a solution with hidden risk concentration on a specific factor.

On the other hand, the factor-based risk parity approach achieves factor diversification across assets, and it can diversify across the true sources of risk. It is a robust approach relatively to the change of the portfolio universe, and it has also the duplication invariance property of producing the same portfolio, regardless of whether the asset is duplicated. However, it is difficult to obtain a unique solution for the optimization problem whereas the asset-class-based approach gives a unique solution. It might lead to the overinvestment in a specific asset-class by ignoring the asset diversification.

As above-described, both asset-class-based and factor-based approaches have pros and cons. Actually, asset-class and factor risk diversifications of each approach are in a trade-off relationship; the asset-class-based risk parity approach gives the equal risk contribution among assets. On the other hand, the factor-based risk parity approach also gives the biased asset risk contributions. We need to consider the method of mixing both approaches in order to introduce those advantages and achieve the well-balanced diversification in perspective of both asset and factor risk contributions. Also, Idzorek and Kowara[4] demonstrate that "*either (risk-factor-based or asset-class-based) approach may be superior over a given time period in a series of real-world optimizations based on historical data.*" This shows that it is difficult to adopt only one of the approaches over time because we cannot find the advantage beforehand. Therefore, we attempt to achieve the better performance by receiving the benefits from both asset-class and factor through the well-balanced diversification on a long-term periods.

Based on these characteristics, we propose a new risk parity method which balances asset and factor diversifications by solving the following problem (3.1) so that the differences from the equal risk contributions can be reduced.

$$\begin{aligned} &\textbf{Minimize} \quad (1-\lambda)\sum_{i=1}^{n}\left(\frac{\mathcal{RC}(\mathcal{A}_i)}{\sigma_P}-\frac{1}{n}\right)^2+\lambda\sum_{j=1}^{m}\left(\frac{\mathcal{RC}(\mathcal{F}_j)}{\sigma_P}-\frac{1}{m}\right)^2 \\ &\textbf{subject to} \quad \sum_{i=1}^{n}x_i=1, \quad x_i\geq 0\,(i=1,\ldots,n), \end{aligned} \tag{3.1}$$

where $\lambda$ is a weight to the factor-based risk parity, and we call it "factor parity weight", or "weight" for simplicity. $1-\lambda$ is a weight to the asset-class-based risk parity. We construct the portfolio to achieve not only factor diversification but also asset-class diversification. We adopt the asset-class-based risk parity strategy for $\lambda = 0$, and the factor-based risk parity strategy for $\lambda = 1$. We can control the balance between asset-class and factor diversifications by adjusting the weight $\lambda$.

### 3.2. Determining the weight based on diversification index

The solution is obtained, given a weight $\lambda$. We show the relationship between the different weights and risk contributions in Figure 2.

The graphs are obtained by using seven assets and three factors introduced in Section 4. The left graph shows the asset risk contributions. We find the asset-class-based risk parity portfolio($\lambda = 0$) has the equal risk contributions(14.3%), and each risk contribution becomes different as the weight $\lambda$ becomes large. The asset risk contributions drastically change around $\lambda = 1$ to achieve the factor-based risk parity. The right graph shows the factor risk contributions. We find the factor risk parity portfolio($\lambda = 1$) has the equal risk

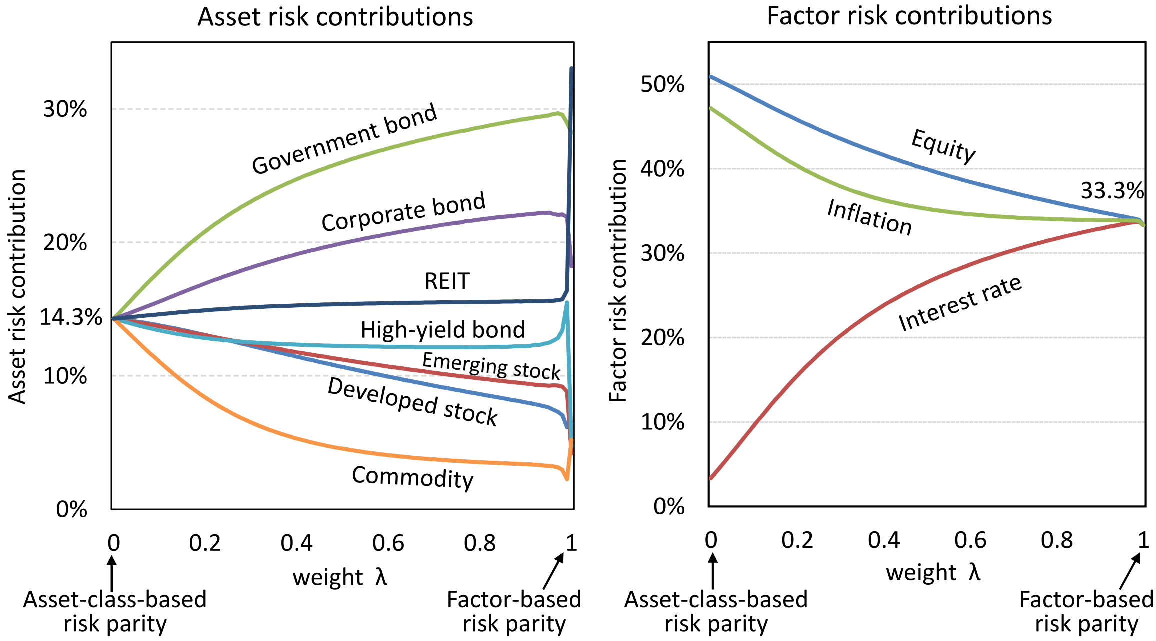



Figure 2: Relationship between the weight $\lambda$ and risk contributions

contributions(33.3%), and each risk contribution becomes different as the weight $\lambda$ becomes small.

The selection of the weight $\lambda$ strongly affects the portfolio weights. We need to find how to decide the weight which gives the well-balanced portfolio between asset-class and factor diversifications. We propose the method of determining the weight $\lambda$ using the diversification index(DI) to consider the difference of diversification between asset and factor. We define the DI so that the value can be small when assets or factors are diversified.

We give the larger weight to either asset or factor with larger DI in order to achieve the well-balanced diversification.[4] We determine the weight $\lambda$ for factor-based risk parity, and $1 - \lambda$ for asset-class-based risk parity as

$$\lambda = \frac{DI(\mathcal{F})}{DI(\mathcal{A}) + DI(\mathcal{F})}, \tag{3.2}$$

where $DI(\mathcal{A})$ and $DI(\mathcal{F})$ denote DIs of assets and factors, respectively. For example, when $DI(\mathcal{A}) = 0.8$, and $DI(\mathcal{F}) = 0.4$, $\lambda$ is $0.333 (= \frac{0.4}{0.8+0.4})$. When asset risk contributions are more concentrated than factor risk contributions, we weight the asset-class risk parity rather than factor to diversify the asset-class. Thus, we can obtain the well-balanced strategy between both of the diversifications.

We adopt the commonly used three diversification indices to measure portfolio diversification; portfolio diversification index (PDI), Herfindahl index (HI), and Gini index (GI). The PDI can be calculated by asset or factor returns, but the HI and GI are driven by risk contributions of assets or factors which are calculated using portfolio. We normalize these diversification indices so that the maximum can be one and the minimum can be zero. When the portfolio is perfectly concentrated in a specific asset or factor, the normalized DI

[4]The effects of asset and factor diversification indices may be different from each other, but we treat them equally without considering the difference among both effects, because the goal is to achieve the diversification well regardless of either assets or factors.

can be one. On the other hand, the normalized DI can be zero for the perfectly diversified portfolio. We calculate the weight $\lambda$ for the diversification indices as follows.

**(1) PDI(Portfolio Diversification Index) method**

Rudin and Morgan[11] construct a PDI which measures the number of unique investments in a portfolio. It is useful to assess the diversification benefits across the universe. The PDI is defined as

$$PDI = 2\sum_{i=1}^{n} i \cdot RS_i - 1, \quad RS_i = \frac{l_i}{\sum_{j=1}^{n} l_j} \; (i = 1, \dots, n), \tag{3.3}$$

where $n$ is the number of variables, $l_i$ is the eigenvalue of $i$-th principal component, and $RS_i$ is the $i$-th proportion of the total variance. The minimum value is 1 and the maximum value is $n$. The normalized PDI is denoted by $PDI^\star$, and defined as

$$PDI^\star = \frac{n - PDI}{n - 1}. \tag{3.4}$$

$PDI^\star = 1(PDI = 1)$ when all of assets are perfectly correlated. It indicates that the portfolio has a low diversification potential. Otherwise, $PDI^\star = 0(PDI = n)$ when all of assets are mutually uncorrelated. It indicates that the portfolio has a high diversification potential.

We set the weight $\lambda$ using the PDI as

$$\lambda = \frac{PDI^\star(\mathcal{F})}{PDI^\star(\mathcal{F}) + PDI^\star(\mathcal{A})}, \tag{3.5}$$

where $PDI^\star(\mathcal{A})$ and $PDI^\star(\mathcal{F})$ are the normalized PDI to asset and factor returns, respectively. The PDI is independent of the portfolio weight because it is calculated only using returns. The weight is calculated exogenously, apart from the optimization problem.

**(2) HI(Herfindahl Index) method**

The HI is usually employed to measure market concentration. In our context, the vector $\boldsymbol{p} = (p_1, ..., p_n)^\top$ represents the risk contributions of assets or factors, and the sum of $p_i$ equals 1. The HI and normalized HI can be calculated as [5]

$$H(\boldsymbol{p}) = \sum_{i=1}^{n} p_i^2, \quad H^\star(\boldsymbol{p}) = \frac{nH(\boldsymbol{p}) - 1}{n - 1}. \tag{3.6}$$

The minimum value of $H(\boldsymbol{p})$ is $1/n$ and the maximum value of $H(\boldsymbol{p})$ is 1. $H^\star(\boldsymbol{p}) = 0(H(\boldsymbol{p}) = 1/n)$ when each $p_i$ is equal or $p_i = 1/n$, and it represents the maximum diversification. Otherwise, $H^\star(\boldsymbol{p}) = 1(H(\boldsymbol{p}) = 1)$ when any $p_i$ is equal to one, and it shows a portfolio is completely concentrated.

We set $\lambda(\boldsymbol{x})$ using the HI as

$$\lambda(\boldsymbol{x}) = \frac{H^\star(\mathcal{RC}(\mathcal{F})/\sigma_P)}{H^\star(\mathcal{RC}(\mathcal{A})/\sigma_P) + H^\star(\mathcal{RC}(\mathcal{F})/\sigma_P)}, \tag{3.7}$$

where $\mathcal{RC}(\mathcal{A})$ is a vector of risk contribution of assets, and $\mathcal{RC}(\mathcal{F})$ is that of factors. The weight $\lambda$ is obtained endogenously in the optimization problem because the risk contributions of asset and factor are calculated using the portfolio weights which affect the calculation of HI.

[5] The normalized index is defined in Roncalli and Weisang[12].

**(3) GI (Gini Index) method**

The GI and normalized GI can be calculated as

$$G(\boldsymbol{p}) = \frac{2\sum_{i=1}^{n} i p_{i:n}}{n\sum_{i=1}^{n} p_{i:n}} - \frac{n+1}{n}, \quad G^{\star}(\boldsymbol{p}) = \frac{nG(\boldsymbol{p})}{n-1}, \tag{3.8}$$

where $p_{i:n}\,(i = 1, \ldots, n)$ is the ordered statistics of $(p_1, ..., p_n)$. The minimum value of GI is zero and the maximum value is $1 - 1/n$; $G^{\star}(\boldsymbol{p}) = 0(G(\boldsymbol{p}) = 0)$ for maximum diversification, and $G^{\star}(\boldsymbol{p}) = 1(G(\boldsymbol{p}) = 1 - 1/n)$ for minimum diversification.

We set $\lambda(\boldsymbol{x})$ as Equation(3.9) using the GI. It is also defined endogenously as well as the HI because of depending on the portfolio weights $\boldsymbol{x}$.

$$\lambda(\boldsymbol{x}) = \frac{G^{\star}(\mathcal{RC}(\mathcal{F})/\sigma_P)}{G^{\star}(\mathcal{RC}(\mathcal{A})/\sigma_P) + G^{\star}(\mathcal{RC}(\mathcal{F})/\sigma_P)}. \tag{3.9}$$

## 4. Empirical Analysis

### 4.1. Setting

We conduct the empirical analysis to examine the characteristics of proposed method. We use the following index returns of seven asset-classes and three factors.[6] Qian[9] stated that *"Risk parity portfolio should have a balanced risk contribution from three sources: (1) equity risk; (2) interest rate risk; (3) inflation risk"*. In reference to Qian[9], we use the same three factors to construct the factor-based risk parity portfolio.

**Seven asset-class returns:**

1. Developed country stock: MSCI World Index
2. Emerging country stock: MSCI Emerging Markets Index
3. Government bond: Bloomberg Barclays Global Treasury Total Return Index Value Unhedged
4. Corporate bond: Bloomberg Barclays Global-Aggregate Total Return Index Value Hedged USD
5. High-yield bond: Bloomberg Barclays Global High Yield Total Return Index Value Unhedge
6. Commodity: Bloomberg Commodity Index
7. REIT: S&P Global REIT Index

**Three factor returns:**

1. Equity: MSCI ACWI (All Country World Index)
2. Interest rate: Bloomberg Barclays Global-Aggregate Total Return Index Value Unhedged USD minus the below-described inflation factor return [7]
3. Inflation: Bloomberg Barclays Global Inflation-Linked Total Return Index Value Unhedged USD minus the above-described government bond return [8]

**Data period:** October 2000 $\sim$ October 2018 (Monthly)

The parameters used to construct the portfolio are estimated using whole data period.

**Currency:** USD

The average returns and standard deviations of each asset are shown in Table 1, and those of each factor are shown in Table 2. We find the developed and emerging stocks have

[6]The data is obtained from Bloomberg Professional.

[7]The real interest rate is estimated, using the Fisher equation.

[8]We estimate the inflation return based on BEI(BreakEven Inflation rate) which is the difference between the government bond yield and inflation-linked bond yield, and we assume the sufficiently small risk premium for the inflation-linked bond.

as similar standard deviations as commodity and REIT, and these have larger risk than bond in Table 1. Table 2 shows the equity factor also has the largest risk.

Table 1: Assets' mean returns and standard deviations(S.D.)

| | Developed stock | Emerging stock | Government bond | Corporate bond | High-yield bond | Commodity | REIT |
|---|---|---|---|---|---|---|---|
| Mean | 3.56% | 7.44% | 4.46% | 5.20% | 8.19% | −0.11% | 5.87% |
| S.D. | 14.93% | 21.76% | 6.72% | 6.32% | 9.79% | 15.89% | 17.97% |

Table 2: Factors' mean returns and standard deviations

| | Equity | Interest rate | Inflation |
|---|---|---|---|
| Mean | 3.64% | 3.29% | 1.25% |
| S.D. | 15.31% | 7.25% | 4.52% |

Figure 3 shows the cumulative returns with the charts of depicting the consecutive decline of return for each factor. The bars are depicted in the above chart at the month when each factor return declines for three consecutive months. We find each factor declines consecutively at different time points. Equation (4.1) shows the correlation matrix of factor returns. The maximum correlation is 0.41 between equity and inflation factor returns. The factor correlations are almost smaller than asset correlations.

$$\begin{array}{l} \\ \text{equity} \\ \text{interest rate} \\ \text{inflation} \end{array} \begin{array}{c} \begin{array}{ccc} \text{equity} & \text{interest rate} & \text{inflation} \end{array} \\ \begin{pmatrix} 1.00 & -0.05 & 0.41 \\ -0.05 & 1.00 & -0.63 \\ 0.41 & -0.63 & 1.00 \end{pmatrix} \end{array}. \tag{4.1}$$

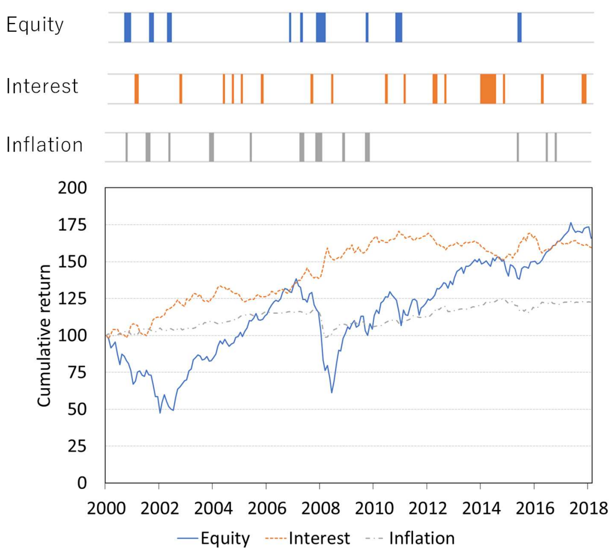



Figure 3: Cumulative returns of each factor

The problems of obtaining the mixed strategy are solved using by the Nelder-Mead method which is provided by "optim" function of R software(https://cran.r-project.org/).

The problems of RW strategy are solved using Numerical optimizer (Ver.20) mathematical programming software package developed by NTT DATA Mathematical System, Inc. on Windows 10 personal computer which has Intel Core i7-6700K 4.2 GHz CPU and 32 GB memory.

## 4.2. Result

The loadings matrix $\boldsymbol{A}$ is calculated as

$$\boldsymbol{A} = \begin{array}{l} \\ \text{Developed stock} \\ \text{Emerging stock} \\ \text{Government bond} \\ \text{Corporate bond} \\ \text{High-yield bond} \\ \text{Commodity} \\ \text{REIT} \end{array} \begin{array}{c} \begin{array}{ccc} \text{equity} & \text{interest rate} & \text{inflation} \end{array} \\ \begin{pmatrix} 0.981 & -0.046 & -0.075 \\ 1.187 & 0.387 & 0.678 \\ -0.022 & 1.181 & 1.008 \\ 0.074 & 0.950 & 1.266 \\ 0.368 & 0.498 & 1.116 \\ 0.323 & 0.753 & 1.522 \\ 0.707 & 0.833 & 1.292 \end{pmatrix} \end{array}. \tag{4.2}$$

The optimal portfolio weights, asset risk contributions and factor risk contributions are calculated using the loadings matrix $\boldsymbol{A}$.

Table 3 shows the normalized diversification indices of asset-class and factor for each PDI, HI and GI. The weights $\lambda$ calculated using three diversification indices are almost equal to each other.

Table 3: Weights $\lambda$ in each mixed strategy

| | $PDI^{\star}$ | $H^{\star}$ | $G^{\star}$ |
|---|---|---|---|
| $DI(\mathcal{A})$ | 0.723 | 0.027 | 0.252 |
| $DI(\mathcal{F})$ | 0.504 | 0.025 | 0.181 |
| $\lambda$ | 0.411 | 0.477 | 0.418 |

We compare the mixed strategies of using five factor parity weights, the RW strategy and the mixed strategies of employing the weight $\lambda$ obtained by diversification indices in Table 4.[9]

The asset-class-based risk parity portfolio is constructed for $\lambda = 0$. All asset risk contributions are perfectly equal to 14.29%, but factor risk contributions are concentrated into two factors: equity risk(50.90%) and inflation risk(47.16%). On the other hand, the factor-based portfolio is constructed for $\lambda = 1$. All factor risk contributions are perfectly equal to 33.33%, but about 80% of the portfolio weight is allocated by three assets: government bond(42.55%), corporate bond(24.37%), and REIT(16.82%). Additionally, we find it is difficult to obtain a unique solution from the optimization problem for $\lambda = 1$.

The loadings matrix $\boldsymbol{A}$ shows that the exposure of commodity to the inflation risk is 1.522 which is larger than other exposures. As the weight $\lambda$ for the factor-based risk parity increases, the weight of commodity decreases from 10.80% to 4.44% due to the decrease in the inflation risk contribution. The exposure of the government bond to the interest rate risk is 1.181, which is relatively high. The interest rate factor risk contribution is 3.34% for

[9]Note that the sum of the factor risk contributions ($\sum_{j=1}^{m} \mathcal{RC}(\mathcal{F}_j)/\mathcal{R}(\boldsymbol{x})$) is not always 100% due to the existence of additional factor risk contribution. We set the optimal solution for asset-class-based risk parity as an initial value to solve the optimization problem (2.13), and obtain the optimal solution for factor-based risk parity($\lambda = 1$). The values for $\lambda = 1$ in Table 4 are dependent on the initial value.

the asset-class-based risk parity strategy. In contrast, it is 33.33% for the factor-based risk parity strategy, which is much larger than that for the asset-class-based strategy. Therefore, the weight of government bond increases from 28.55% to 42.55% when we change the weight $\lambda$ from 0 to 1. The portfolio weights of mixed strategies are located between the asset-class-based ($\lambda = 0$) and the factor-based ($\lambda = 1$) risk parity strategies.

Table 4: Portfolio weights, and risk contributions

| | asset | mixed strategy | | | factor | RW | mixed strategy | | |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda =$ | 0 | 0.25 | 0.5 | 0.75 | 1 | strategy | PDI | HI | GI |
| Portfolio weight($x_i$) | | | | | | | | | |
| Developed stock | 9.84% | 8.51% | 7.21% | 6.12% | 4.80% | 12.64% | 7.65% | 7.72% | 7.87% |
| Emerging stock | 6.55% | 5.68% | 5.02% | 4.53% | 2.14% | 0.00% | 5.23% | 5.27% | 5.31% |
| Government bond | 28.55% | 36.31% | 39.21% | 40.54% | 42.55% | 37.18% | 38.46% | 38.34% | 38.33% |
| Corporate bond | 21.82% | 23.76% | 25.51% | 26.62% | 24.37% | 33.23% | 24.97% | 24.87% | 24.75% |
| High-yield bond | 14.22% | 12.01% | 11.44% | 11.30% | 4.88% | 14.94% | 11.57% | 11.59% | 11.53% |
| Commodity | 10.80% | 5.69% | 3.58% | 2.88% | 4.44% | 2.01% | 4.09% | 4.18% | 4.23% |
| REIT | 8.22% | 8.05% | 8.03% | 8.02% | 16.82% | 0.00% | 8.03% | 8.03% | 7.99% |
| Percent asset risk contribution($\mathcal{RC}(\mathcal{A}_i)/\sigma_P$) | | | | | | | | | |
| Developed stock | 14.29% | 12.66% | 10.66% | 8.94% | 6.42% | 20.07% | 11.35% | 11.46% | 11.70% |
| Emerging stock | 14.29% | 12.71% | 11.21% | 10.01% | 4.19% | 0.00% | 11.71% | 11.80% | 11.88% |
| Government bond | 14.29% | 22.04% | 26.02% | 28.28% | 28.22% | 29.20% | 24.89% | 24.71% | 24.63% |
| Corporate bond | 14.29% | 17.56% | 19.94% | 21.47% | 18.23% | 30.23% | 19.22% | 19.09% | 18.96% |
| High-yield bond | 14.29% | 12.65% | 12.24% | 12.14% | 4.70% | 17.75% | 12.32% | 12.34% | 12.26% |
| Commodity | 14.29% | 7.35% | 4.54% | 3.62% | 5.20% | 2.74% | 5.20% | 5.33% | 5.38% |
| REIT | 14.29% | 15.04% | 15.38% | 15.52% | 33.04% | 0.00% | 15.30% | 15.28% | 15.19% |
| Percent factor risk contribution($\mathcal{RC}(\mathcal{F}_j)/\sigma_P$)) | | | | | | | | | |
| Equity | 50.90% | 44.63% | 39.93% | 36.52% | 33.33% | 33.33% | 41.42% | 41.66% | 41.92% |
| Interest rate | 3.34% | 18.03% | 26.55% | 31.09% | 33.33% | 33.33% | 24.15% | 23.74% | 23.52% |
| Inflation | 47.16% | 39.01% | 35.25% | 34.12% | 33.33% | 33.33% | 36.15% | 36.31% | 36.26% |
| Overall portfolio risk($\sigma_P$) | | | | | | | | | |
| | 8.21% | 7.62% | 7.35% | 7.21% | 7.60% | 6.57% | 7.42% | 7.44% | 7.44% |

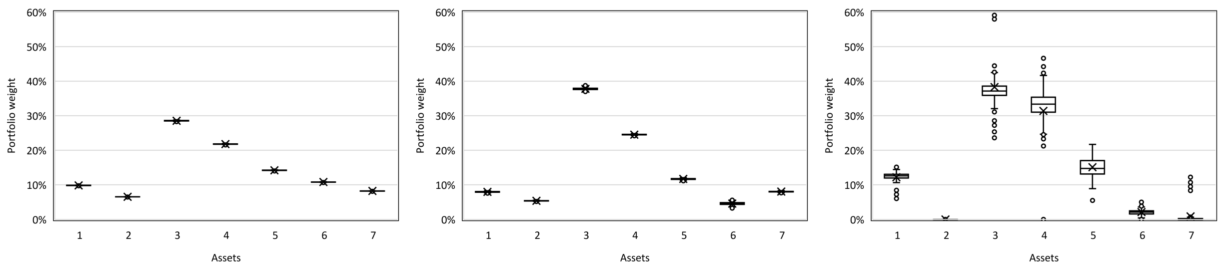
The different solutions are obtained for both the factor-based ($\lambda = 1$) and RW strategies even with equal factor risk contributions. It is difficult to obtain a unique solution from the optimization problem (2.13) or (3.1) for the mixed strategy of $\lambda = 1$ as previously mentioned whereas a unique solution can be derived by the problem (2.14) for the RW strategy. We find the zero portfolio weights for the emerging stock and REIT in the RW strategy because these assets have the largest volatilities as in Table 1. The RW strategy controls these portfolio weights to reduce portfolio risk, and it has an advantage of obtaining a unique solution, but it might be an extreme solution. The mixed strategy based on diversification index is located between $\lambda = 0.25$ and 0.5 as shown in Table 3.

### 4.3. Robustness check

The asset-class-based method is robust to the small change of parameters, but the factor-based method is sensitive to it. We give a small noise to each component of the covariance matrix[10] to examine the robustness of the proposed method, and we calculate the portfolio weights.

$$\sigma_{ij}^{noise(k)} = \sigma_{ij} \cdot (1 + \varepsilon_{ij}^{(k)}),\ (i, j = 1, \ldots, n\,(i \neq j);\, k = 1, \ldots, M), \tag{4.3}$$

where $n$ is the number of assets, $M$ is the number of samples. The number of $\varepsilon$ is $n(n+1)/2$. Each noise $\varepsilon$ is normally distributed with zero mean and 0.01 standard deviation, and these are mutually uncorrelated. Figure 4 shows the box plots of the portfolio weights to see the robustness among methods, using 10,000 samples of seven assets for the three methods. We confirm the robustness of the asset-class-based method to the small change of the covariance matrix in the left graph. However, we find the RW method is highly sensitive to noises as shown in the right graph though it gives a unique solution even for the factor-based strategy. We show the result of the proposed method using the normalized HI in the middle. We find it is a little bit more sensitive to parameters than the asset-class-based method, but much more robust than the RW method (factor-based method).




(a) asset-based method($\lambda = 0$) (b) proposed method(HI) (c) RW method

Figure 4: Box plots of the portfolio weights for robustness check among three methods

## 4.4. Examining portfolio duplication invariance property

Choueifaty *et al.*[3] propose the three invariance properties, and one of them is duplication invariance property. It is the property that portfolio construction process can produce the same portfolio, regardless of whether the asset is duplicated.

We suppose a universe where the government bond is duplicated. The asset-class-based strategy($\lambda = 0$), the mixed strategy based on the HI, and RW strategy are shown in Table 5. Each left column shows the original portfolio weight, and each right column shows the portfolio weight with duplicated asset. The weight of government bond in the right column shows the sum of the original and duplicated government bond which is indicated in italics.

We find the RW method almost retains the duplication invariance property. Even when the asset is duplicated, other asset weights are almost invariant. In contrast, the portfolio weights of asset-class-based and mixed strategies are affected when the asset is duplicated. However, we find the mixed strategy has more duplicate invariant property than the asset-class-based method.

## 4.5. Examining the uniqueness of the solution of mixed strategies

We examine the objective function forms for the three mixed strategies. We depict the forms in the three-asset and two-factor case to check the uniqueness numerically as well as

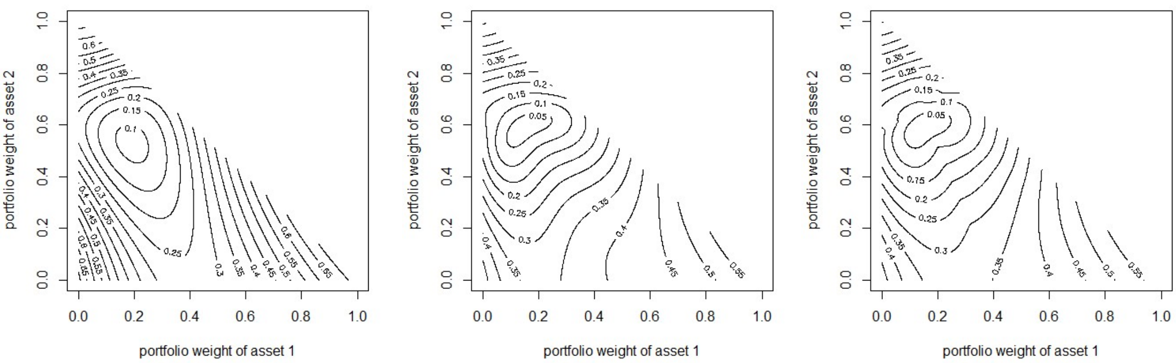
[10]The covariance matrix with noise is not guaranteed to be positive definite in theory, but it becomes positive definite in this example due to small noises.

those used to depict Figure 1 because it is difficult to prove the uniqueness of the solutions theoretically.

Figure 5 shows the contour lines of objective function values for the three mixed strategies, and we find each function is convex. It is difficult to obtain a unique solution for the factor-based approach, but it is expected that we can obtain a unique and an optimal solution in the mixed approach.[11]

Table 5: Portfolio weights of original and duplicated government bond index universes among three methods

| | Asset-class-based method($\lambda = 0$) | | Proposed method (HI) | | RW method | |
|---|---|---|---|---|---|---|
| | Original | Duplicated | Original | Duplicated | Original | Duplicated |
| Developed stock | 9.84% | 8.52% | 7.72% | 7.58% | 12.64% | 11.95% |
| Emerging stock | 6.55% | 5.60% | 5.27% | 5.01% | 0.00% | 0.00% |
| Government bond | 28.55% | *40.80%* | 38.34% | *47.54%* | 37.18% | *36.74%* |
| Corporate bond | 21.82% | 17.09% | 24.87% | 17.58% | 33.23% | 32.61% |
| High-yield bond | 14.22% | 12.08% | 11.59% | 10.25% | 14.94% | 17.12% |
| Commodity | 10.80% | 9.03% | 4.18% | 5.61% | 2.01% | 1.58% |
| REIT | 8.22% | 6.89% | 8.03% | 6.43% | 0.00% | 0.00% |




PDI strategy HI strategy GI strategy

Figure 5: Contour lines of objective function values for three mixed strategies

## 5. Backtest

It is well-known that the risk of financial asset return, their correlations and their exposure to each risk factor are time-varying. It makes the optimal risk parity portfolio different over time. We then employ the dynamic method of determining the weight $\lambda$ using the diversification index so that the weight can be adapted dynamically to market environment in order to construct the well-balanced portfolio between asset-class and factor diversifications. We implement the backtest under the following setting. Suppose we invest in seven assets as well as the empirical analysis. The length of periods of estimating the parameters

[11]We can obtain the optimal solutions stably in the seven-asset and three-factor case.

to construct the portfolio is sixty months. The performances are measured on a USD basis from October 2005 to October 2018. The portfolio is rebalanced monthly on the first day of each month, and the loadings matrix and risk contributions are estimated using each rolling window of sixty months.

### 5.1. Diversification index

Figure 6 reports the transition of weight $\lambda$ in the mixed strategy based on each diversification index. The factor parity weights based on the three diversification indices are similar to each other over time. However the weights based on the PDI changes dynamically, compared with the HI and GI. When the financial crisis occurred in 2008, the factor returns became highly correlated. It makes $PDI(\mathcal{F})$ rise, and therefore the weight $\lambda$ also becomes large. Moreover, the weights based on the HI and GI almost continue to be flat, but they have decreased since 2015 because the correlations between factors are gradually decreasing and the factors are easy to be diversified.

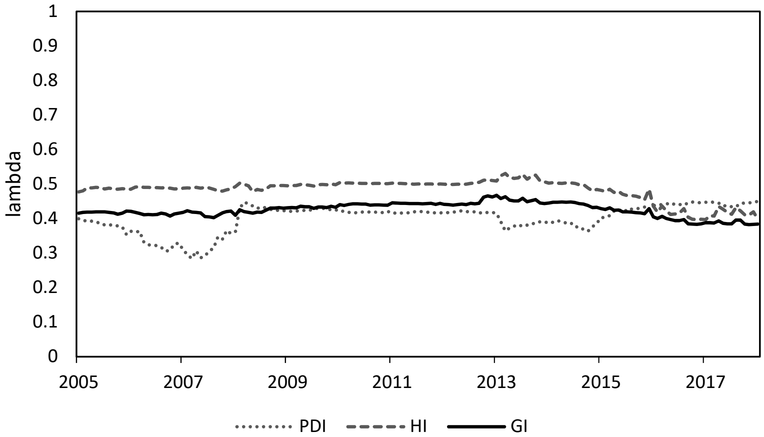



Figure 6: Weights $\lambda$ of using each diversification index in mixed strategy

### 5.2. Performance Measures

We evaluate the results of backtest employing five kinds of performance measures as follows.

**(1) Sharpe ratio**

We evaluate investment efficiency between return and risk using the Sharpe ratio, or

$$SR = \frac{\overline{r}_P - r_f}{\sigma_P}, \tag{5.1}$$

where $\overline{r}_P$ is a portfolio mean return and $r_f$ is a risk-free rate. We use one month USD LIBOR as the risk-free rate.

**(2) VaR (Value at Risk)**

The VaR represents the potential maximum loss on a given confidence level $\alpha$. We adopt $\alpha = 0.95$ in this paper.

$$VaR(\alpha) = \min\{V : P[-r_P > V] \leq 1 - \alpha\}. \tag{5.2}$$

**(3) CVaR (Conditional VaR)**

The CVaR is defined as the average loss beyond the VaR.

$$CVaR(\alpha) = \mathrm{E}[-r_P | - r_P \geq VaR(\alpha)]. \tag{5.3}$$

**(4) Maximum Drawdown**

We employ Maximum DrawDown(MDD) which is the maximum loss from a peak to a trough of a portfolio, before a new peak is attained.

$$MDD = \frac{\min_{t\in(0,T)}(\min_{t\in(0,\tau)} P(\tau) - P(t))}{\max_{t\in(0,T)} P(t)} - 1, \tag{5.4}$$

where $T$ is the total number of testing periods and $P(t)$ is the portfolio value at time $t$.

**(5) Turnover**

Turnover shows the measure of rebalancing portfolio weights, and it is proportion to a fee. Therefore, it is one of the important measures when we evaluate a practical performance. We define the portfolio turnover as the average sum of absolute values of each trade across $n$ assets.

$$Turnover = \frac{1}{T-1}\sum_{t=1}^{T-1}\sum_{i=1}^{n}\left| x_i^{t+1} - \frac{r_i^{t+1}x_i^t}{\sum_{j=1}^{n} r_j^{t+1}x_j^t}\right|, \tag{5.5}$$

where $x_i^t$ is a portfolio weight to asset $i$ at time $t$, and $r_i^{t+1}$ is a rate of return of asset $i$ from $t$ to $t+1$.

## 5.3. Performance

We examine the mixed strategy for five constant weights and the RW strategy. Table 6 shows the summary statistics and performance measures. The increase in the factor parity weight gives the decrease in the standard deviation, and the downside risk measures of VaR, CVaR and maximum drawdown. It implies that it can control tail risk of the portfolio. We find that the increase in $\lambda$ occurs a high turnover and therefore the factor-based risk parity portfolio($\lambda = 1$) has the largest turnover. We find the standard deviation of RW strategy is smaller than that of the factor-based portfolio for $\lambda = 1$ because the RW method minimizes the standard deviation in the formulation.

Table 6: Summary statistics and performance measures for mixed strategy with constant factor parity weights $\lambda$

| | asset | mixed strategy | | | factor | RW |
|---|---|---|---|---|---|---|
| Factor parity weight ($\lambda$) | 0 | 0.25 | 0.5 | 0.75 | 1 | strategy |
| Annual average return | 3.21% | 3.37% | 3.48% | 3.40% | 2.94% | 2.74% |
| Annual standard deviation | 8.81% | 8.28% | 7.97% | 7.79% | 7.94% | 7.56% |
| Skewness | −1.16 | −1.16 | −1.16 | −1.19 | −1.38 | −1.17 |
| Excess kurtosis | 3.43 | 3.25 | 3.29 | 3.27 | 4.39 | 2.42 |
| 95%-VaR | 3.81% | 3.58% | 3.42% | 3.22% | 3.18% | 3.24% |
| 95%-CVaR | 6.17% | 5.73% | 5.49% | 5.39% | 5.64% | 5.48% |
| Maximum drawdown | −30.5% | −28.0% | −26.3% | −25.7% | −26.14% | −25.3% |
| Sharpe ratio | 0.1994 | 0.2316 | 0.2541 | 0.2502 | 0.1870 | 0.1707 |
| Turnover | 2.72% | 3.39% | 4.03% | 4.96% | 23.34% | 19.57% |

Figure 7 shows the box plots of portfolio weights over time, asset risk contributions and factor risk contributions to each weight $\lambda$, and the RW strategy in the backtest period. The risk contribution of the equity factor(No.1) is relatively large and stable to other two factors for $\lambda = 0$ in the bottom of Figure 7. The interest rate and inflation factors are widely distributed, and these values change over time. All asset risk contributions also change over time, and the risk contribution of government bond(No.3), corporate bond(No.4) and REIT(No.7) are relatively large for $\lambda = 1$. These empirical weights of factor-based strategy are more widely distributed than those of RW strategy, and the turnover of RW strategy is also smaller than that of the factor-based portfolio for $\lambda = 1$.

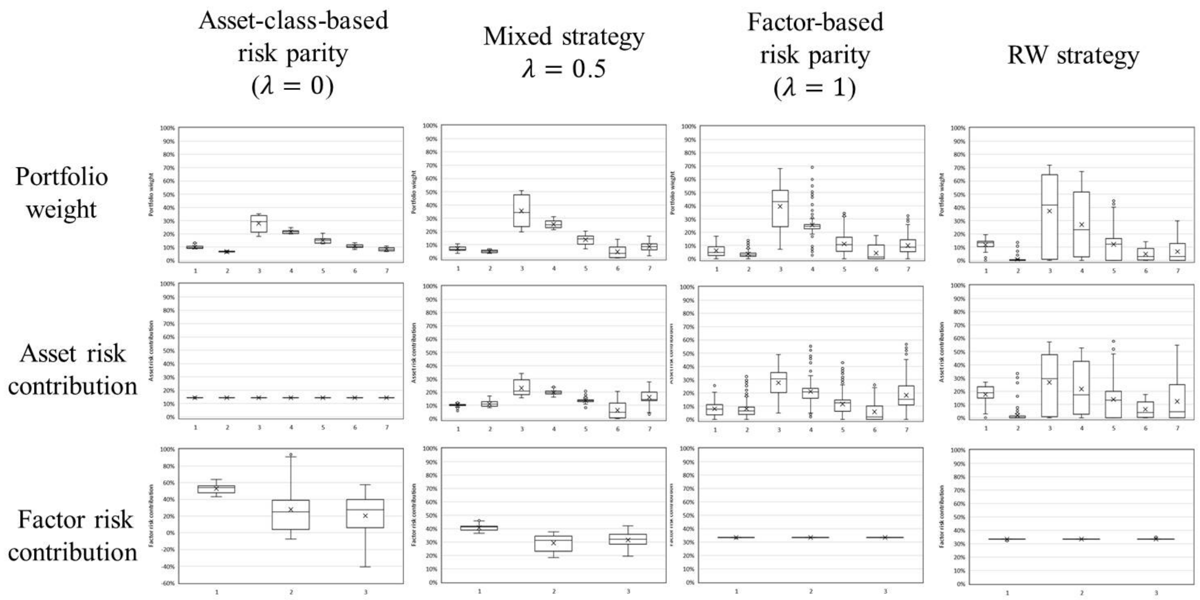



Figure 7: Box plots of portfolio weights over time, and asset and factor risk contributions for mixed strategy with constant factor parity weights and RW strategy

We examine the RW strategy and the mixed strategy with time-dependent factor parity weights based on the diversification index. Table 7 shows the summary statistics and performance measures, and Figure 8 shows the box plots of portfolio weights over time, and asset and factor risk contributions.

Table 7: Summary statistics and performance measures for mixed strategy with time-dependent factor parity weights $\lambda$

| | mixed strategy | | |
|---|---|---|---|
| | PDI | HI | GI |
| Annual average return | 3.44% | 3.51% | 3.49% |
| Annual standard deviation | 8.08% | 8.04% | 8.06% |
| Skewness | −1.17 | −1.15 | −1.16 |
| Excess kurtosis | 3.43 | 3.29 | 3.32 |
| 95%-VaR | 3.45% | 3.43% | 3.44% |
| 95%-CVaR | 5.57% | 5.54% | 5.56% |
| Maximum drawdown | −26.7% | −26.6% | −26.7% |
| Sharpe ratio | 0.2459 | 0.2559 | 0.2532 |
| Turnover | 3.79% | 3.91% | 3.88% |

According to Table 7, the three mixed strategies have the large Sharpe ratios, which show high investment efficiency. The HI strategy has the highest efficiency among them. Figure 9 shows the cumulative returns of each mixed strategy relative to the asset-class-based risk parity strategy. The positive values indicate the mixed strategy outperforms the

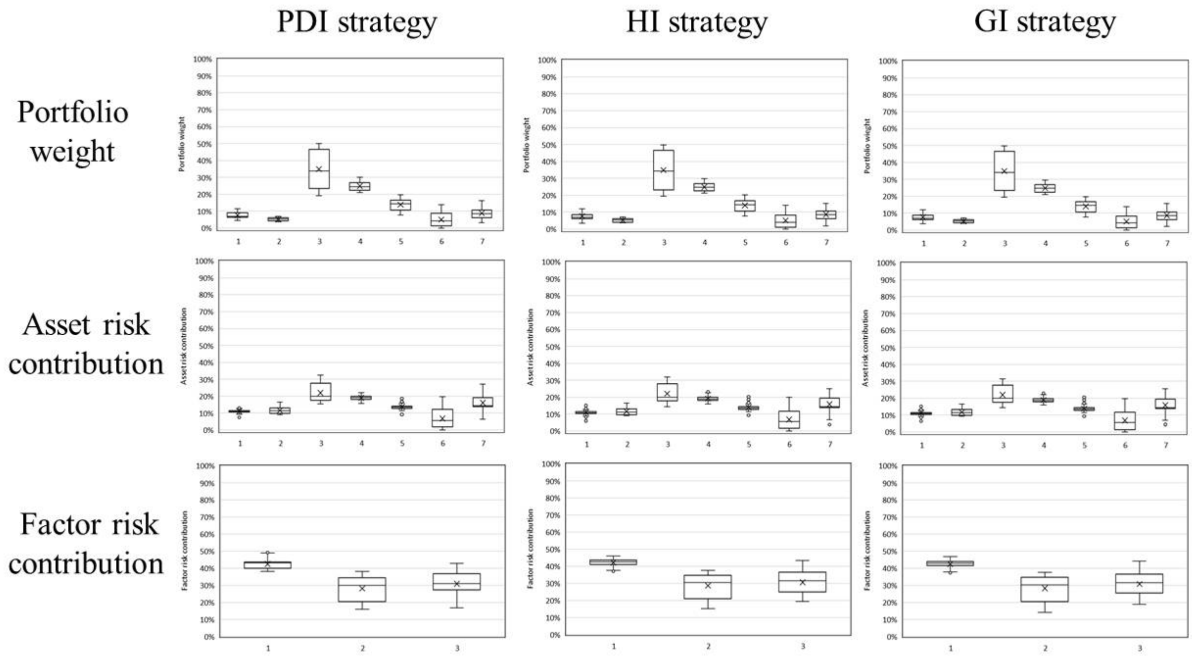



Figure 8: Box plots of portfolio weights over time, asset risk contributions and factor risk contributions of mixed strategy with time-dependent factor parity weights

asset-class-based strategy. The mixed strategies, having a large allocation to the government bond(No.3), outperform at times of the market downturn around 2009. After that period, these strategies underperform due to the overweight to the government bond from 2009 to 2011. In the future task, we need to develop the estimation method that is not necessarily based on historical data to improve performance. Additionally, our strategies have outperformed the traditional asset-class-based strategy since 2014. The portfolio weight of the commodity(No.6) decreases, but those of other assets increase, because the exposure of the commodity to the inflation factor decreases in this period and it allows the other assets to make the risk contributions to the inflation factor. Compared with Figure 7, the mixed strategies with time-dependent factor parity weights based on the diversification index are well-balanced between the asset-class-based and the factor-based risk parity strategies, and it leads to a noteworthy performance.

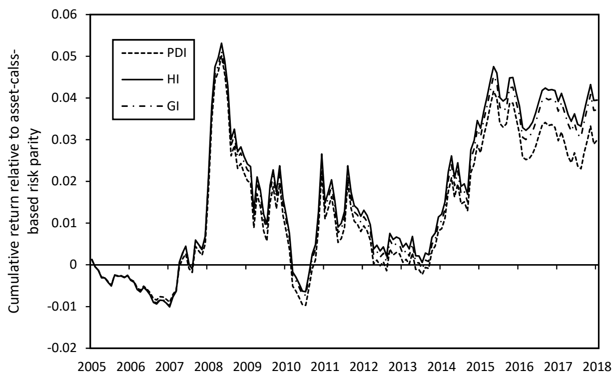



Figure 9: Relative cumulative return of mixed strategies with time-dependent factor parity weights vs. asset-class-based risk parity strategy

### 5.4. Do the proposed strategies outperform other portfolio strategies?

We compare the proposed mixed strategy with minimum variance strategy and equally weighted strategy in the same universe as the backtest.[12]

The minimum variance strategy minimizes the portfolio variance with no short selling constraint.

$$\boldsymbol{x}^* = \arg\min \left\{ \boldsymbol{x}^\top \Sigma \boldsymbol{x} \,|\, \mathbf{1}^\top \boldsymbol{x} = 1,\ \boldsymbol{x} \geq \mathbf{0} \right\}, \tag{5.6}$$

where $\mathbf{1}$ is a vector of ones. The equally weighted strategy has an equal portfolio weight for each asset.

$$x_i = \frac{1}{n}\ (i = 1, \ldots, n) \tag{5.7}$$

Table 8 shows the result for comparison. Maillard *et al.*[7] shows the volatility of risk parity portfolio is located between those of the minimum variance and the equally weighted portfolios. We find the same result as shown in Maillard *et al.*[7], and the proposed strategy is superior to the other portfolio strategies.

Figure 10 also reports the cumulative return of the minimum variance, the asset-class-based risk parity and the mixed strategy (HI) with time-dependent factor parity weights relative to the equally weighted strategy. We find the asset-class-based risk parity strategy is located between equally weighted and minimum variance strategies. The mixed strategy reduces the drawdown in 2009, and it obtains almost the same returns as or greater returns than the equally weighted strategy since 2010.

Table 8: Comparison of the proposed strategy and other portfolio strategies

| | HI strategy (*1) | Minimum variance | Risk parity(*2) | Equally weighted |
|---|---|---|---|---|
| Annual average return | 3.51% | 2.47% | 3.21% | 3.35% |
| Annual standard deviation | 8.04% | 6.76% | 8.81% | 11.42% |
| Skewness | −1.15 | −1.21 | −1.16 | −1.14 |
| Excess kurtosis | 3.29 | 3.64 | 3.25 | 3.29 |
| 95%-VaR | 3.43% | 3.16% | 3.81% | 5.29% |
| 95%-CVaR | 5.54% | 5.10% | 6.17% | 8.20% |
| Maximum drawdown | −26.60% | −20.50% | −30.52% | −41.37% |
| Sharpe ratio | 0.2559 | 0.1502 | 0.1994 | 0.1662 |
| Turnover | 3.91% | 8.31% | 2.72% | 2.87% |

(*1) Mixed strategy based on the HI in Table 7
(*2) $\lambda = 0$ (asset-class-based strategy) in Table 6

## 6. Conclusion

In this paper, we discuss the asset-class-based and the factor-based risk parity approaches. We point out some shortcomings of both approaches and we propose a new risk parity

[12] Chaves *et al.*[2] compare the risk parity strategy against other asset allocation strategies: equally weighted portfolio, minimum variance portfolio, a naive mean-variance optimization, and the 60/40 portfolio. Lohre *et al.*[6] consider four alternative risk-based strategies for benchmarking the diversified risk parity strategy: equally weighted portfolio, minimum variance portfolio, risk parity, and most-diversified portfolio. In our paper, we adopt the minimum variance portfolio, the equally weighted portfolio, and the asset-class risk parity portfolio for comparison with the mixed strategy as well as these references, and examine the performances of the mixed strategy.

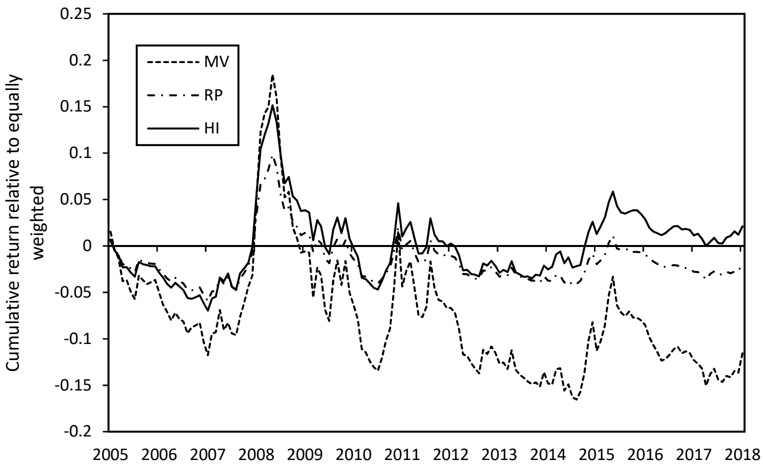



Figure 10: Cumulative return of minimum variance, asset-class-based risk parity and mixed strategy (HI) with time-dependent factor parity weights relative to equally weighted strategy

method which obtains the well-balanced portfolio between asset-class and factor diversifications to address these problems. We also propose the method of determining the weight of two approaches based on the diversification index.

In the empirical analysis, we find that the asset with larger exposure to risk factors tends to change its portfolio weight. Our method gives the well-balanced portfolio in consideration of asset and factor diversifications. We also implement the backtest under the assumption of the investment on global financial assets. The mixed strategy with time-dependent factor parity weight based on the diversification index leads to the reduction of the portfolio risk, and it has a higher Sharpe ratio than other portfolio strategies. We find that the proposed method improves the efficiency of asset management and can obtain the benefit of both the asset-class-based and the factor-based risk parity strategies. In the future research, we need to compare the proposed method with the different risk parity strategies using a downside risk measure.

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