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A NOTE ON A NEARLY UNIFORM PARTITION INTO COMMON INDEPENDENT SETS OF TWO MATROIDS

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Abstract The present note is a strengthening of a recent paper by K. Takazawa and Y. Yokoi (A generalized-polymatroid approach to disjoint common independent sets in two matroids, *Discrete Mathematics* (2019)). For given two matroids on E, under the same assumption in their paper to guarantee the existence of a partition of E into k common independent sets of the two matroids, we show that there exists a nearly uniform partition $\mathcal{P}$ of E into k common independent sets, where the difference of the cardinalities of any two sets in $\mathcal{P}$ is at most one.

Keywords: Combinatorial optimization, matroid, common independent sets, nearly uniform partition

1. Introduction

K. Takazawa and Y. Yokoi [8] have very recently shown a new approach to the problem of partitioning the common ground set of two matroids into common independent sets by means of generalized polymatroids. They successfully give a unifying view on some results of J. Davies and C. McDiarmid [1] and D. Kotlar and R. Ziv [5] and extend them by the generalized-polymatroid approach.

A partition $\mathcal{P}$ of a finite nonempty set E is said to be *nearly uniform* if the cardinality difference of every pair of sets in $\mathcal{P}$ is at most one. Researchers' attention has been drawn to the existence of a nearly uniform partition of the ground set of a combinatorial system into disjoint objects of the system such as branchings ([7, Sec. 53.6]) and matchings ([1, 4]). In the present note we show that the generalized-polymatroid approach in [8] reveals the existence of a nearly uniform partition $\mathcal{P}$ of E into common independent sets of two matroids under the same assumption in [8].

In Section 2 we describe the result of Takazawa and Yokoi [8] in a general form, which is basically a dynamic programming formulation. Then, in Section 3, under the same assumption in the paper [8] to guarantee the existence of a partition of E into k common independent sets of the two matroids, we show that there exists a nearly uniform partition $\mathcal{P}$ of E into k common independent sets. Section 4 gives some concluding remarks.

2. The Generalized-Polymatroid Approach of Takazawa and Yokoi

We follow the definitions and notation given in [8] (and in our Appendix). A brief survey about fundamental facts about matroids, polymatroids, generalized polymatroids, and submodular/supermodular functions is given in the appendix for readers' convenience. Also see [2, 3, 6, 7, 9].

Let E be a nonempty finite set. For each $i = 1, 2$ let $\mathbf{M}_i = (E, \mathcal{I}_i)$ be a matroid on E with $\mathcal{I}_i \subseteq 2^E$ being a family of independent sets. For a given positive integer $k \geq 2$ let

$\mathbf{M}_i^k = (E, \mathcal{I}_i^k)$ be the union matroid of k copies of $\mathbf{M}_i$ for each $i = 1, 2$, and we assume that $E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k$.

Now, consider the problem of partitioning the ground set E of the two matroids $\mathbf{M}_i$ $(i = 1, 2)$ into k common independent sets as follows:

(P): Find a partition $\mathcal{P} = \{X_1, \cdots, X_k\}$ of E into k disjoint subsets $X_j \subseteq E$ $(j = 1, \cdots, k)$ such that $X_j \in \mathcal{I}_1 \cap \mathcal{I}_2$ for all $j = 1, \cdots, k$.

Here we allow an empty component $X_j = \emptyset \in \mathcal{I}_1 \cap \mathcal{I}_2$, just by technical reason for the arguments in the sequel. (It should be noted that if we can partition E into k possibly empty common independent sets, then we can partition E into k nonempty common independent sets when $k \leq |E|$.)

Let ρ be the rank function of $\mathbf{M} = (E, \mathcal{I})$, $\mathcal{I}^k$ the union matroid $\mathbf{M}^k$ of k copies of $\mathbf{M} = (E, \mathcal{I})$, ρ^k the rank function of the union matroid $\mathbf{M}^k = (E, \mathcal{I}^k)$, $\rho^{\#}$ the dual supermodular function of ρ, $\mathrm{P}(\rho)$ the submodular polyhedron associated with submodular ρ, and $\mathrm{P}(\rho^{\#})$ the supermodular polyhedron associated with supermodular $\rho^{\#}$ (see Appendix). Also for any family $\mathcal{F}$ of subsets of E denote by $\mathrm{Conv}(\mathcal{F})$ the convex hull of characteristic vectors $\chi_X \in \mathbb{R}^E$ for all $X \in \mathcal{F}$.

Theorem 2.1 ([8])**.** *Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid with $E \in \mathcal{I}^k$. Define*

$$\mathcal{F} = \{X \mid X \in \mathcal{I},\ E \setminus X \text{ can be partitioned into } k-1 \text{ sets in } \mathcal{I}\}. \tag{2.1}$$

Then we have

$$\mathrm{Conv}(\mathcal{F}) = \mathrm{P}(\rho) \cap \mathrm{P}((\rho^{k-1})^{\#}) \subseteq [0,1]^E. \tag{2.2}$$

Remark 1. Note that $E \setminus X$ can be partitioned into $k-1$ sets in $\mathcal{I}$ if and only if X is a co-spanning set of the union matroid $\mathbf{M}^{k-1} = (E, \mathcal{I}^{k-1})$ (see Appendix). In Theorem 2.1 the right-hand side of (2.2) is the intersection of the submodular polyhedron $\mathrm{P}(\rho)$ and the supermodular polyhedron $\mathrm{P}((\rho^{k-1})^{\#})$, which is nonempty by the assumption that $E \in \mathcal{I}^k$ (implying $(\frac{1}{k}, \cdots, \frac{1}{k}) \in \mathrm{P}(\rho) \cap \mathrm{P}((\rho^{k-1})^{\#})$) and is integral.

Hence a set $X \in \mathcal{F}$ can be found efficiently and we can further apply this process for $k \leftarrow k-1$, $E \leftarrow E \setminus X$ and $\mathbf{M} \leftarrow \mathbf{M}^E$ (the restriction of $\mathbf{M}$ on the updated E). We can repeat this process to obtain a partition $\{X_1, \cdots, X_k\}$ of E into k independent sets $X_j \in \mathcal{I}$ $(j = 1, \cdots, k)$. Although we have more direct and efficient algorithms to find a partition $\{X_1, \cdots, X_k\}$ of E into independent sets $X_j \in \mathcal{I}$ $(j = 1, \cdots, k)$, Theorem 2.1 gives a basis for the generalized-polymatroid approach to Problem (**P**) of Takazawa and Yokoi [8].

Now we have the following theorem, based on Theorem 2.1.

Theorem 2.2 ([8])**.** *Consider two matroids $\mathbf{M}_i$ $(i = 1, 2)$ such that $E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k$. Let $\ell \in \{0, 1, \cdots, k-1\}$ and let $\{X_1, \cdots, X_\ell\}$ be a set of disjoint ℓ common independent sets of $\mathbf{M}_i$ $(i = 1, 2)$.* Putting $F = E \setminus \bigcup_{j=1}^{\ell} X_j$, define for each $i = 1, 2$*

$$\mathcal{F}_i^{\ell}(F) = \{X \subseteq F \mid X \in \mathcal{I}_i,\ F \setminus X \text{ can be partitioned into } k-\ell-1 \text{ sets in } \mathcal{I}_i\}. \tag{2.3}$$

Then we have

$$\mathrm{Conv}(\mathcal{F}_1^{\ell}(F) \cap \mathcal{F}_2^{\ell}(F)) \subseteq \mathrm{P}(\rho_1^F) \cap \mathrm{P}(((\rho_1^F)^{k-\ell-1})^{\#}) \cap \mathrm{P}(\rho_2^F) \cap \mathrm{P}(((\rho_2^F)^{k-\ell-1})^{\#}), \tag{2.4}$$

where ρ_i^F is the rank function of the restriction of $\mathbf{M}_i$ on F.

If the intersection of the four polyhedra on the right-hand side of (2.4) *contains an integral point, i.e., a characteristic vector χ_{X^*} of some $X^* \subseteq F$, then we have $X^* \in \mathcal{F}_1^{\ell}(F) \cap \mathcal{F}_2^{\ell}(F)$.*

*When $\ell = 0$, regard $\{X_1, \cdots, X_\ell\}$ as an empty family and $\bigcup_{j=1}^{\ell} X_j = \emptyset$.

In particular, if the intersection of the four polyhedra on the right-hand side of (2.4) *is integral, then the inclusion relation* (2.4) *holds with equality and there exists a set* $X \in \mathcal{F}_1^\ell(F) \cap \mathcal{F}_2^\ell(F)$.

Remark 2. Note that $\mathrm{P}(\rho_1^F) \cap \mathrm{P}(((\rho_1^F)^{k-\ell-1})^\#)$ and $\mathrm{P}(\rho_2^F) \cap \mathrm{P}(((\rho_2^F)^{k-\ell-1})^\#)$ are integral for any matroids $\mathbf{M}_i$ $(i = 1, 2)$, due to Theorem 2.1, but their intersection does not necessarily contain an integral point. Since by the assumption that $E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k$ the vector $(\frac{1}{k}, \cdots, \frac{1}{k})$ belongs to the intersection of the four polyhedra in (2.4) for $\ell = 0$, if the intersection of the four polyhedra is integral (or more generally contains an integral point), there exists a set $X_1 \in \mathcal{F}_1^0(F) \cap \mathcal{F}_2^0(F)$. Then for $F = E \setminus X_1$ we can apply the same arguments to find $X_2 \in \mathcal{F}_1^1(F) \cap \mathcal{F}_2^1(F)$, and repeatedly carry out this process to find a desired partition $\{X_1, \cdots, X_k\}$ into common independent sets. Also see the proof of Theorem 3.1.

Remark 3. Takazawa and Yokoi [8] considered the case when the intersection of the first two polyhedra in (2.4) is a generalized polymatroid and so is that of the last two. They showed that such matroids are given by laminar matroids (special gammoids) and the matroids without $(k + 1)$-spanned elements considered by Kotlar and Ziv [5]. Since the nonempty intersection of two integral generalized polymatroids is integral, they thus showed that every pair of matroids from among laminar matroids and the matroids without $(k + 1)$-spanned elements considered by Kotlar and Ziv [5] satisfies the assumptions of Theorem 2.2, for which our problem (**P**) is efficiently solvable.

3. Nearly Uniform Partitions

Let us further examine the generalized-polymatroid approach of Takazawa and Yokoi given by Theorems 2.1 and 2.2 for the problem of partitioning two matroids into common independent sets.

Theorem 3.1. *Let* $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2)$ *be matroids and* k *be a positive integer such that* $E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k$. *If every* $\mathrm{P}(\rho_i^F) \cap \mathrm{P}(((\rho_i^F)^{k-\ell-1})^\#)$ *for* $i = 1, 2$ *appearing in* (2.4) *in Theorem* 2.2 *is an integral generalized polymatroid, then there exists a nearly uniform partition of* E *into* k *common independent sets of* $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2)$.

Proof. Put $\lambda = |E|/k$ and define $\lambda^+ = \lceil \lambda \rceil$ and $\lambda^- = \lfloor \lambda \rfloor$. It follows from Theorem 2.2 and the assumptions of the present theorem that if we find X_j for $j = 1, \cdots, \ell$ by the procedure described in Remark 1, then for each $i = 1, 2$ the polyhedron given by

$$\mathrm{P}(\rho_i^F) \cap \mathrm{P}(((\rho_i^F)^{k-\ell-1})^\#) \cap \{x \in \mathbb{R}^F \mid \lambda^- \le x(F) \le \lambda^+\} \tag{3.1}$$

is an integral generalized polymatroid (due to Fact 3 in Appendix and the fact that the intersection of two integral generalized polyhedra is intregral) and contains the uniform vector $(\frac{1}{k-\ell}, \cdots, \frac{1}{k-\ell})$ in $\mathbb{R}^F$ and hence there exists a set $X \in \mathcal{F}_1^\ell(F) \cap \mathcal{F}_2^\ell(F)$ with $\lambda^- \le |X| \le \lambda^+$.† Hence there exists a partition $\{X_1, \cdots, X_k\}$ of E into common independent sets of $\mathbf{M}_i$ $(i = 1, 2)$ such that $\lambda^- \le |X_j| \le \lambda^+$ for all $j = 1, \cdots, k$. □

Remark 4. Since laminar matroids and the matroids considered in [5] satisfy the assumptions required in Theorem 3.1 as shown by Takazawa and Yokoi [8], for every pair of such matroids $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2)$ with $E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k$ there exists a nearly uniform partition of E into k common independent sets.

†Note that we have initially $\lambda^- \le |E|/k \le \lambda^+$ and hence $\lambda^- \le |X_1| \le \lambda^+$, and then we have $\lambda^- \le (|E| - |X_1|)/(k-1) \le \lambda^+$. So we can show by induction that for F in (2.4) we have $\lambda^- \le |F|/(k-\ell) \le \lambda^+$ for $\ell = 0, 1, \cdots, k-1$.

Remark 5. Theorem 3.1 can be given in a more general form as described in Theorem 2.2. That is, it suffices to impose that the intersection of the four polyhedra in (2.4) and $\{x \in \mathbb{R}^F \mid \lambda^- \le x(F) \le \lambda^+\}$ with $\lambda^- = \lfloor |E|/k \rfloor$ and $\lambda^+ = \lceil |E|/k \rceil$ contains an integral point.

For *general* matroids $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2)$ we also have the following. Define for each $i = 1, 2$

$$\mu_i^* = \min\{\mu \in \mathbb{Z}_{>0} \mid E \in \mathcal{I}_i^{\mu}\}, \tag{3.2}$$

which is the covering index for matroid $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2)$. A *subpartition* of E is a set of disjoint subsets of E.

Theorem 3.2. *Let $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2)$ be arbitrary matroids and k be a positive integer such that $E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k$. Suppose that $\mu_1^* \le \mu_2^* < k$. Then there exists a nearly uniform subpartition $\{X_1, \cdots, X_{k-\mu_2^*-1}\}$ of E such that*

- $X_\ell \in \mathcal{I}_1 \cap \mathcal{I}_2$ *for* $\ell = 1, \cdots, k - \mu_2^* - 1$,
- $E \setminus (X_1 \cup \cdots \cup X_{k-\mu_2^*-1}) \in \mathcal{I}_1^{\mu_2^*+1} \cap \mathcal{I}_2^{\mu_2^*+1}$.

Proof. For $\ell = 1, \cdots, k - \mu_2^* - 1$, under the assumption of the present theorem, for each $i = 1, 2$ we have $\emptyset \in \mathcal{F}_i^{\ell}(F)$ in (2.3), so that $\mathcal{F}_i^{\ell}(F)$ is actually $\mathcal{I}_i$. Hence the argument in the proof of Theorem 2.2 can be adapted for obtaining a nearly uniform subpartition $\{X_1, \cdots, X_{k-\mu_2^*-1}\}$ of E satisfying the conditions of the present theorem. □

Similarly we can show the following, a corollary of Theorem 2.1, which may be folklore.

Corollary 3.1. *For an arbitrary matroid $\mathbf{M} = (E, \mathcal{I})$ with $E \in \mathcal{I}^k$ there exists a nearly uniform partition of E into k independent sets of $\mathbf{M}$.*

It should be noted that Corollary 3.1 holds for any general matroid $\mathbf{M} = (E, \mathcal{I})$ with $E \in \mathcal{I}^k$, but for two matroids $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2)$ with $E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k$ we need additional conditions to guarantee the existence of a nearly uniform partition of E into common independent sets, in general, such as those given in Theorem 3.1.

4. Concluding Remarks

We have shown that under the same assumption in [8] that makes the generalized-polymatroid approach of Takazawa and Yokoi work, there also exists a nearly uniform partition into common independent sets.

It is interesting to identify the class of pairs of matroids for which every intersection of the four polyhedra in (2.4) is integral and computationally tractable, which is left open. Besides the way of using generalized polymatroids in [8] there may be the case when the intersection of the first and the fourth polyhedra in (2.4) is a generalized polymatroid and so is the intersection of the second and the third.

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A. Fundamental Facts about Matroids and Submodular Functions

We briefly give some definitions and fundamental facts about matroids, polymatroids, generalized polymatroids, and submodular/supermodular functions from a polyhedral point of views, which are used in the present paper. For general information relevant to the subject of this paper see [2, 3, 6, 7, 9] (the notations used here mostly follow [3]).

Let E be a nonempty finite set and $\mathbf{M} = (E, \mathcal{I})$ be a *matroid* on E with a family of *independent sets* (we omit the axioms for independent sets). A maximal independent set is called a *base*. A set $X \subseteq E$ is called a *spanning set* of $\mathbf{M}$ if there exists a base B of $\mathbf{M}$ such that $B \subseteq X$. A set function $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ defined by

$$\rho(X) = \max\{|Y| \mid Y \subseteq X,\ Y \in \mathcal{I}\} \tag{A.1}$$

is called the *rank function* of $\mathbf{M}$. The rank function ρ satisfies the *submodularity* inequalities

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y) \qquad (\forall X, Y \subseteq E). \tag{A.2}$$

Matroid $\mathbf{M}$ is uniquely determined by each of the family of independent sets, the family of bases, the family of spanning sets, and the rank function, associated with $\mathbf{M}$. The family of complements $E \setminus B$ of all bases B of $\mathbf{M}$ is the family of bases of a matroid on E, which is called the *dual* matroid of $\mathbf{M}$ is denoted by $\mathbf{M}^*$. For the rank function ρ of $\mathbf{M}$ we denote the rank function of the dual matroid $\mathbf{M}^*$ by ρ^*. The dual rank function ρ^* is given by

$$\rho^*(X) = |X| - \rho(E) + \rho(E \setminus X) \qquad (\forall X \subseteq E). \tag{A.3}$$

Any set function $f : 2^E \to \mathbb{R}$ is called a *submodular function* if it satisfies the submodularity inequalities (A.2) with ρ being replaced by f. The negative of a submodular function is called a *supermodular function*. Given a submodular function $f : 2^E \to \mathbb{R}$ with $f(\emptyset) = 0$, the *submodular polyhedron* associated with f is defined by

$$\mathrm{P}(f) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : x(X) \leq f(X)\}, \tag{A.4}$$

where $x(X) = \sum_{e \in X} x(e)$. (When $\mathrm{P}(f) \cap \mathbb{R}^E_{\geq 0} \neq \emptyset$, it is called a *polymatroid* and there uniquely exists a monotone nondecreasing submodular function f' such that $\mathrm{P}(f) \cap \mathbb{R}^E_{\geq 0} = \mathrm{P}(f') \cap \mathbb{R}^E_{\geq 0}$.) Also the *base polyhedron* associated with f is defined by

$$\mathrm{B}(f) = \{x \in \mathrm{P}(f) \mid x(E) = f(E)\}. \tag{A.5}$$

In a dual manner, given a supermodular function $g : 2^E \to \mathbb{R}$ with $g(\emptyset) = 0$, the *supermodular polyhedron* associated with g is defined by

$$\mathrm{P}(g) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : x(X) \geq g(X)\} \tag{A.6}$$

and the associated base polyhedron by

$$\mathrm{B}(g) = \{x \in \mathrm{P}(g) \mid x(E) = g(E)\}. \tag{A.7}$$

For a submodular function $f : 2^E \to \mathbb{R}$ with $f(\emptyset) = 0$ the *dual* supermodular function $f^\# : 2^E \to \mathbb{R}$ is defined by

$$f^\#(X) = f(E) - f(E \setminus X) \qquad (\forall X \subseteq E). \tag{A.8}$$

We have $\mathrm{B}(f) = \mathrm{B}(f^\#)$. Note that $(f^\#)^\# = f$.

For a submodular function $f : 2^E \to \mathbb{R}$ and a supermodular function $g : 2^E \to \mathbb{R}$ with $f(\emptyset) = g(\emptyset) = 0$, if we have

$$f(X) - g(Y) \geq f(X \setminus Y) - g(Y \setminus X) \qquad (\forall X, Y \subseteq E), \tag{A.9}$$

then the polyhedron $Q(f, g) \equiv \mathrm{P}(f) \cap \mathrm{P}(g)$ is called a *generalized polymatroid*. Every polymatroid is a generalized polymatroid.

When f and g are integer-valued, all the polyhedra $\mathrm{P}(f)$, $\mathrm{P}(g)$, $\mathrm{B}(f)$, and $Q(f, g)$ are integral. Moreover, given another integer-valued submodular f' and supermodular g', the intersections $\mathrm{P}(f) \cap \mathrm{P}(f')$, $\mathrm{P}(g) \cap \mathrm{P}(g')$, $\mathrm{B}(f) \cap \mathrm{B}(f')$, and $Q(f, g) \cap Q(f', g')$, if nonempty, are integral polyhedra.

For any generalized polymatroid $Q(f, g)$, letting $\hat{e}$ be a new element and putting $\hat{E} = E \cup \{\hat{e}\}$, for an arbitrary $t \in \mathbb{R}$ define $\hat{f} : \hat{E} \to \mathbb{R}$ by $\hat{f}(\hat{E}) = t$ and

$$\hat{f}(X) = \begin{cases} f(X) & \text{if } \hat{e} \notin X \\ g(\hat{E} \setminus X) & \text{if } \hat{e} \in X \end{cases} \qquad (\forall X \subset \hat{E}). \tag{A.10}$$

Then $\hat{f}$ is a submodular function and the projection of the base polyhedron $\mathrm{B}(\hat{f}) \subset \mathbb{R}^{\hat{E}}$ along the axis $\hat{e}$ into the coordinate subspace $\mathbb{R}^E$ is a generalized polymatroid $Q(f, g)$. Every generalized polymatroid is obtained in this way and vice versa. This is an isomorphic correspondence.

For a submodular function f, a supermodular function g, and vectors $l \in (\mathbb{R} \cup \{-\infty\})^E$ and $u \in (\mathbb{R} \cup \{+\infty\})^E$ with $l(e) \leq u(e)$ for all $e \in E$ we have the following three facts:

Fact 1. $\mathrm{P}(f)^u \equiv \{x \in \mathrm{P}(f) \mid x \leq u\}$ is a submodular polyhedron.

Fact 2. $\mathrm{P}(g)_l \equiv \{x \in \mathrm{P}(g) \mid x \geq l\}$ is a supermodular polyhedron.

Fact 3. $\mathrm{B}(f)_l^u \equiv \{x \in \mathrm{B}(f) \mid l \leq x \leq u\}$, if nonempty, is a base polyhedron. (In particular, this implies that for a generalized polymatroid $Q(f, g)$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$, $Q(f, g)_\alpha^\beta \equiv \{x \in Q(f, g) \mid \alpha \leq x(E) \leq \beta\}$, if nonempty, is a generalized polymatroid, due to the isomorphic correspondence between base polyhedra and generalized polymatroids.)

These polyhedra are integral if f and g are integer-valued and if finite $l(e)$s and $u(e)$s are integers.

For any family $\mathcal{F}$ of subsets of E denote by $\mathrm{Conv}(\mathcal{F})$ the convex hull of characteristic vectors $\chi_X \in \mathbb{R}^E$ for all $X \in \mathcal{F}$, where $\chi_X(e) = 1$ if $e \in X$ and $= 0$ if $e \in E \setminus X$.

Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid with a rank function ρ. Then we have

$$\mathrm{Conv}(\mathcal{I}) = \mathrm{P}(\rho) \cap [0, 1]^E, \tag{A.11}$$

which is called a *matroid polytope* and denoted by $\mathrm{P}_{(+)}(\rho)$. Let $\mathcal{S}$ be the set of spanning sets of $\mathbf{M}$. Then,

$$\mathrm{Conv}(\mathcal{S}) = \mathrm{P}(\rho^{\#}) \cap [0,1]^E, \tag{A.12}$$

where $\rho^{\#}$ is the dual supermodular function of ρ. Define $\bar{\mathcal{I}} = \{E \setminus X \mid X \in \mathcal{I}\}$, which is the family of *co-spanning* sets of $\mathbf{M}$, i.e., the family of spanning sets of the dual matroid $\mathbf{M}^*$. Then we have

$$\mathrm{Conv}(\bar{\mathcal{I}}) = \mathrm{P}((\rho^*)^{\#}) \cap [0,1]^E, \tag{A.13}$$

where ρ^* is the rank function of the dual matroid $\mathbf{M}^*$. It follows from (A.3) and (A.8) that

$$(\rho^*)^{\#}(X) = |X| - \rho(X) \qquad (\forall X \subseteq E). \tag{A.14}$$

Finally, for any positive integer k define

$$\mathcal{I}^k = \{X_1 \cup \cdots \cup X_k \mid \forall j \in \{1, \cdots, k\} : X_j \in \mathcal{I}\}, \tag{A.15}$$

where note that imposing the condition that $X_j \in \mathcal{I}$ $(j = 1, \cdots, k)$ are disjoint gives the same $\mathcal{I}^k$. The pair $(E, \mathcal{I}^k)$ is a matroid, called a *union matroid* of k copies of $\mathbf{M}$, which we denote by $\mathbf{M}^k$. The rank function ρ^k of $\mathbf{M}^k$ is given by

$$\rho^k(X) = \min\{|E \setminus X| + k\rho(X) \mid X \subseteq E\}. \tag{A.16}$$

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