Journal of the Operations Research Society of Japan
Vol. 62, No. 2, April 2019, pp. 83–107

ESTIMATING FORWARD LOOKING DISTRIBUTION WITH THE ROSS RECOVERY THEOREM

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(Received June 29, 2018; Revised December 3, 2018)

Abstract Ross (2015) introduced a remarkable theorem, named the "Recovery Theorem." It enables us to estimate the real world distribution from the risk neutral distribution derived from option prices under a particular assumption about a representative investor's risk preferences. The real world distribution estimated with the Recovery Theorem is suitable for many financial problems such as market risk management and portfolio optimization due to its forward looking nature. However, it is not easy to derive the appropriate estimators because of an ill-posed problem in the estimation process. We propose a new method to derive the accurate solution by formulating the regularization term involving prior information. Previous studies propose methods to estimate the real world distribution, but they do not investigate the estimation accuracy. We show the effectiveness of the proposed method through the numerical analysis with hypothetical data.

Keywords: Finance, estimation, Recovery Theorem, regularization

1. Introduction

In general, we need to estimate asset distributions to solve financial problems such as market risk management and optimal asset allocation. A common approach is to estimate the distribution from historical data. However, the financial market is quite volatile, and utilizing a forward looking distribution in implied option prices is more desirable and suitable than a backward looking distribution derived using historical data.

The payoff of an option is determined by the future price of the underlying asset and therefore the option prices contain forward looking information. The forward looking risk neutral distribution can be derived from option prices under the assumption of a complete market (Breeden and Litzenberger [7]). The risk neutral distribution generally differs from the real world distribution which expresses market participants' consensus. In particular, the expected return (mean) of the risk neutral distribution must be equal to the risk free rate and the expected return of the real world distribution does not have to be. Previous studies proposed methods to adjust a risk neutral distribution to a real world distribution (risk adjustment methods). Bliss and Panigirtzoglou [5] adjust the distribution assuming CARA utility or CRRA utility as a representative investor's preference. Fackler and King [10] proposed the method that uses a beta distribution as a calibration function. Shackleton et al. [22] proposed the nonparametric method that uses kernel density estimation as a calibration function. However, these methods do not offer a completely forward looking real

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world distribution because the adjustment parameters are estimated from historical data[1].

Ross [21][2] introduced a remarkable theorem, named the "Recovery Theorem." It enables us to estimate a completely forward looking real world distribution from option prices under a particular assumption about a representative investor's risk preferences. Our paper discusses the method of estimating the distribution using the Recovery Theorem. Ross [21] shows quite a simple estimation procedure. However, many improvements are required for practical use. Spears [23], Audrino et al. [1], Backwell [2], Jackwerth and Menner [12] and Jensen et al. [13] have developed the practical methodology for estimating the real world distribution from option prices using the Recovery Theorem[3]. Spears [23] indicates that estimators derived by the simple method of Ross [21] are intuitively inaccurate, and compares the estimators under various constraints. Audrino et al. [1] point out that it is necessary to solve an ill-posed problem in the estimation process, and propose application of the Tikhonov method, which is a standard regularization method for ill-posed problems. In addition, they estimate a real world distribution from thirteen years of S&P500 option data and investigate the effectiveness of a simple investment strategy based on the moments of the distribution. Backwell [2] shows that the time-homogeneity of state prices, which is hypothesized when estimating a real world distribution, cannot be realized in the market. The estimation method is also proposed to reduce the bias. Jackwerth and Menner [12] estimate real world distributions of the S&P500 index under the economic constraiants and find that such distributions are incompatible with realized returns. They showed one of the reason is numerical instabilities of the recovery method. Jensen et al. [13] generalize the Recovery Theorem by removing the assumption of time-homogeneity. Moreover, they estimate a real world distribution from S&P500 option data and verify the predictive power of the moments. However, the uniqueness of the estimated distribution is not guaranteed.

As shown in Audrino et al. [1], the regularization method is required to derive the appropriate estimators of the real world distribution because there is an ill-posed problem in the estimation process. However, there are no previous studies which use prior information to solve the ill-posed problem and evaluate the accuracy of the estimation method.

Our contribution is summarized in the following two points.

1. We propose a new method to derive a more accurate solution by formulating the regularization term involving the prior information. Our proposed method provides clear interpretation of the relationship between the regularization parameter and the estimators.
2. We conduct numerical analysis on the estimation accuracy with hypothetical data to show the effectiveness of the proposed method and the criteria for selecting a regularization parameter. We find the following four points from the results.

(1) The divergence of the distribution estimated by the method of Ross [21] from true

[1]Kiriu and Hibiki [14] compare the investment performance of the real world distribution risk-adjusted based on historical data and that of the risk neutral distribution in the asset allocation framework. They show that the risk adjustment using historical data deteriorates investment performance because the appropriate parameter is not obtained from historical data.

[2]The working paper version of this paper was published in 2011.

[3]There are other studies related to the Recovery Theorem. Carr and Yu [8], Dubynskiy and Goldstein [9], Walden [24], Park [19], and Qin and Linetsky [20] studied the theoretical extension into a continuous time or infinite state. These studies mainly focus on the conditions in which the real world probability can be recovered from the risk neutral probability. Martin and Ross [17] apply the Recovery Theorem to the long bond, which is a zero coupon bond with infinite maturity, and investigate the result. Borovička et al. [6] and Bakshi et al. [3] criticize the assumptions of the Recovery Theorem, but we do not discuss this problem.

distribution becomes larger than that of the risk neutral distribution due to the numerical instabilities. In other words, risk adjustment with the basic recovery method deteriorates the estimation accuracy.
(2) Stabilizing the solution by introducing a regularization term increases the estimation accuracy.
(3) The proposed method can estimate a real world distribution more accurately than the Tikhonov method.
(4) The criteria for selecting a regularization parameter offers the solution whose divergence from the true distribution is smaller than that of the risk neutral distribution in most cases.

This paper proceeds as follows. Section 2 summarizes the Recovery Theorem of Ross [21]. Section 3 shows the procedure for estimating the real world distribution from option prices by the Recovery Theorem and proposes the new method. In Section 4, we show the results of the numerical analysis and examine the effectiveness of the proposed method. The final section describes our conclusion and future work.

2. Recovery Theorem

In this section, we summarize the Recovery Theorem of Ross [21]. We assume an arbitrage free and complete market in discrete time with a finite state, one period model. Market states $\theta_i\,(i = 1, \ldots, n)$ are defined by r_i, the underlying stock index returns from time 0. $P := (p_{i,j})$ is an $n \times n$ transition state price matrix. $p_{i,j}$ is a state price from θ_i to θ_j[4]. We similarly define an $n \times n$ transition risk neutral probability matrix $Q := (q_{i,j})$ and an $n \times n$ transition real world probability matrix $F := (f_{i,j})$. We also describe the notation Q as "risk neutral probability" and F as "real world probability" depending on the context. Matrix P is assumed to be irreducible[5], and therefore matrix Q and matrix F are also irreducible. In this section, we suppose that matrix P is known because it can be estimated from option prices[6]. Matrix Q is easily derived from matrix P, since $q_{i,j}$ is expressed as follows:

$$q_{i,j} = \frac{p_{i,j}}{\sum_{k=1}^{n} p_{i,k}} \quad (i, j = 1, \ldots, n). \tag{1}$$

On the other hand, it is difficult to derive matrix F because the state price is simultaneously a function of both a real world probability and market risk preferences. However, Ross [21] showed that matrix F can be derived from matrix P under the assumption that there is a representative investor with Time Additive Intertemporal Expected Utility Theory preferences over consumption (TAIEUT investor). A utility function of the TAIEUT investor is given by

$$u(c_i) + \delta \sum_{j=1}^{n} f_{i,j} u(c_j) \quad (i = 1, \ldots, n) \tag{2}$$

where c_i is the consumption at θ_i, $u(c)$ is a utility for the consumption c and $\delta(> 0)$ is the discount factor of the utility. We assume that $u(c)$ holds the nonsatiation condition

[4]The state price $p_{i,j}$ shows the price of the security at θ_i which pays one dollar if the next state becomes θ_j and nothing otherwise.
[5]Irreducibility is defined as the existing $k \in \mathbb{N}$ which satisfies $(P^k)_{i,j} > 0$ for all i, j. This assumption is very likely to be held.
[6]This is explained in Section 3 in detail.

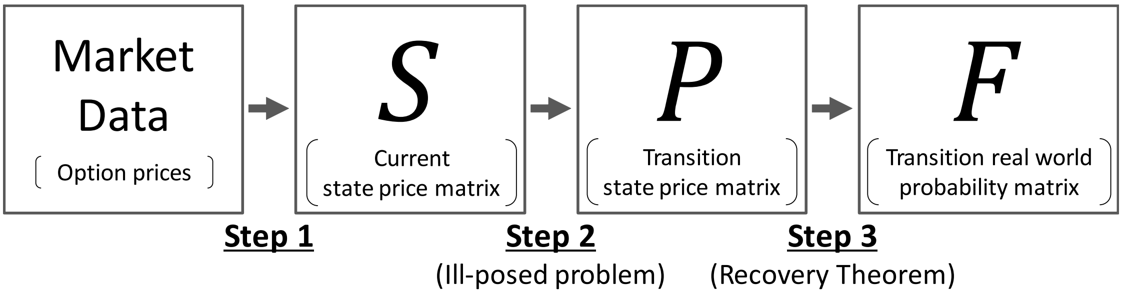


Figure 1: Estimation steps

$u'(c) > 0$ but do not restrict its parametric form. The relationship between $f_{i,j}$ and $p_{i,j}$ is expressed as

$$f_{i,j} = \frac{1}{\delta}\frac{u'(c_i)}{u'(c_j)}p_{i,j} \quad (i, j = 1, \dots, n). \tag{3}$$

The ratio of $p_{i,j}$ to $f_{i,j}$ is called pricing kernel, and it is expressed as

$$\phi_{i,j} := \frac{p_{i,j}}{f_{i,j}} = \delta\frac{u'(c_j)}{u'(c_i)} \quad (i, j = 1, \dots, n). \tag{4}$$

Pricing kernel is dependent on investor's risk preferences.

Since matrix P is non-negative and irreducible, the Perron-Frobenius Theorem asserts that matrix P has a unique strictly positive eigenvector $\boldsymbol{v}$ associated with the maximum eigenvalue λ. The Recovery Theorem says that

$$\delta = \lambda \tag{5}$$
$$u'(c_i) = v_i^{-1} \quad (i = 1, \dots, n) \tag{6}$$

hold, where v_i denotes the i-th element of $\boldsymbol{v}$.

We can get matrix F from matrix P with the Recovery Theorem as follows. We solve the eigenvalue problem of matrix P and derive the maximum eigenvalue λ and the corresponding eigenvector $\boldsymbol{v}$. Then, we can calculate the elements of matrix F as

$$f_{i,j} = \frac{1}{\lambda}\frac{v_j}{v_i}p_{i,j} \quad (i, j = 1, \dots, n). \tag{7}$$

In addition, Ross [21] proves that the real world probability becomes equal to the risk neutral probability, or $F = Q$, when the sum of the row elements of matrix P is the same for each row, and it is a special case of the Recovery Theorem.

3. Implementation of the Recovery Theorem

In this section, we describe the process of estimating the real world distribution with the Recovery Theorem. The process is divided into three steps as referenced by Spears [23] in Figure 1.

An $n \times m$ current state price matrix is defined as $S := (s_{j,\tau})$, where $s_{j,\tau}$ is a current state price for $\tau(= 1, \dots, m)$ periods transition from current state θ_{i_0} to next state θ_j. Matrix S is estimated from option prices in Step 1. In Step 2, we estimate the $n \times n$ transition state price matrix P from the $n \times m$ matrix S. This section discusses Step 1 and Step 2 because Step 3 simply applies the Recovery Theorem as mentioned in Section 2.

For simplicity, we assume that (1) the number of states n is odd, (2) the current state is the center state ($r_{i_0} = 0$ where $i_0 = (n+1)/2$), and (3) the return of each state is symmetrical from the center state ($r_{i_0-k} = r_{i_0+k}, (k = 0, \ldots, i_0 - 1)$).

The i_0-th row vector of the matrix P, $\boldsymbol{p}_{i_0}$, represents a state price distribution of the current state. We denote $\boldsymbol{p}_{i_0}$ as the state price distribution. Likewise, we denote $\boldsymbol{q}_{i_0}$ as the risk neutral distribution and $\boldsymbol{f}_{i_0}$ as the real world distribution.

3.1. Step 1: from option prices to matrix S

Breeden and Lizenberger [7] showed that a state price is calculated from option prices. The state price function $s(k, \tau)$ is represented as follows:

$$s(k,\tau) = \frac{\partial^2 c(k,\tau)}{\partial k^2} \quad (\tau = 1, \ldots, m) \tag{8}$$

where k is the strike price and $c(k, \tau)$ is the call option price function. We can get matrix S by discretizing $s(k, \tau)$. Although this equation is quite simple, there are many methods of estimating the state price function, such as the method of assuming mixed log-normal distribution (Melick and Thomas [18]), the method of using polynomial approximation (Malz [16]), the method of using a smoothing spline (Bliss and Panigirtzoglou [4]) and the method of using a neural network (Ludwig [15]).

However, in our numerical analysis of the estimation accuracy, we generate matrix S from hypothetical data to eliminate the effect of the estimation method of Step 1.

3.2. Step 2: from matrix S to matrix P

In Step 2, we estimate the $n \times n$ matrix P from the $n \times m$ matrix S assuming that state transitions follow a time-homogeneous Markov chain. We assume that $n \leq m$, which means that the number of equations is greater than the number of estimation variables, except for the analysis in Section 4.4.4.

3.2.1. Basic method (Ross [21])

We explain the basic method of Ross [21] to estimate matrix P. Denote the first column vector of matrix S by $\boldsymbol{s}_1$. This vector corresponds to the one-period state price distribution of the current state from its definition. It is formulated as,

$$\boldsymbol{s}_1 = \boldsymbol{p}_{i_0}^{\intercal}. \tag{9}$$

Because matrix P represents the one-period state transition, we have the following relationship among $\boldsymbol{s}_\tau, \boldsymbol{s}_{\tau+1}$, and P.

$$\boldsymbol{s}_{\tau+1}^{\intercal} = \boldsymbol{s}_\tau^{\intercal} P \quad (\tau = 1, \ldots, m-1) \tag{10}$$

Denote the $(m-1) \times n$ matrix transposed from the $n \times m$ matrix S except the last column and the first column respectively by matrix $S_{-\mathrm{col}.m}^{\intercal}$ and $S_{-\mathrm{col}.1}^{\intercal}$. Equation (10) can be expressed as,

$$S_{-\mathrm{col}.m}^{\intercal} P = S_{-\mathrm{col}.1}^{\intercal}. \tag{11}$$

Matrix P should be estimated by minimizing the differences of both sides of Equation (11) under the no-arbitrage conditions $p_{i,j} \geq 0\,(i, j = 1, \ldots, n)$ and Equation (9). The mathematical formulation is

$$\min_{P} \quad ||S^{\mathsf{T}}_{-\mathrm{col}.m}P - S^{\mathsf{T}}_{-\mathrm{col}.1}||_2^2 \tag{12}$$

$$\text{subject to} \quad \boldsymbol{s}_1 = \boldsymbol{p}_{i_0}^{\mathsf{T}} \tag{13}$$

$$p_{i,j} \geq 0 \quad (i, j = 1, \ldots, n). \tag{14}$$

3.2.2. Tikhonov method (Audrino et al. [1])

Audrino et al. [1] indicate that the average condition number[7] of 11×11 matrix $S^{\mathsf{T}}_{-\mathrm{col}.m}$ estimated from S&P 500 option data is very large, and therefore the problem of Equations (12-14) is ill-posed. The ill-posed problem has a set of candidates of optimal solutions whose objective function values are almost the same due to low independence of the equations. Consequently, it has the awkward characteristic that the solution is highly sensitive to a small noise. They propose to use the Tikhonov method, which is a standard regularization method for ill-posed problems. The regularization method is formulated by adding the regularization term to the objective function to stabilize the solution. Specifically, the objective function is reformulated as,

$$\min_{P} \quad ||S^{\mathsf{T}}_{-\mathrm{col}.m}P - S^{\mathsf{T}}_{-\mathrm{col}.1}||_2^2 + \zeta||P||_2^2 \tag{16}$$

$$\text{subject to} \quad (13) \text{ and } (14).$$

The second term is a regularization term and $||\cdot||_2$ denotes the Euclidean norm. ζ is called a regularization parameter and controls the trade-off between fitting and stability. Equation (16) can be transformed using an $n \times n$ unit matrix I and an $n \times n$ null matrix O.

$$\min_{P} \quad \left\| \begin{bmatrix} S^{\mathsf{T}}_{-\mathrm{col}.m} \\ \sqrt{\zeta}I \end{bmatrix} P - \begin{bmatrix} S^{\mathsf{T}}_{-\mathrm{col}.1} \\ O \end{bmatrix} \right\|_2^2 \tag{17}$$

Because the coefficient of matrix P determines the stability of the solution, we focus on the condition number of the matrix created by combining matrices $S^{\mathsf{T}}_{-\mathrm{col}.m}$ and $\sqrt{\zeta}I$ vertically. Figure 2 shows the condition number for the hypothetical data explained in Section 4 (base case, $\sigma = 0\%$). The condition number of the original problem (Equation (12) which corresponds to $\zeta = 0$ in the Tikhonov method) is very large, at 1.3×10^{17}. The condition number decreases as ζ increases.

In the regularization method, we stabilize the solution by adding a regularization term to the objective function. In other words, we derive the solution of the original problem under certain prior information, which is ancillary information about the solution, to solve the problem. It is expected that a solution is derived accurately under appropriate prior information. However, it seems the Tikhonov method does not give the appropriate prior information of this recovery problem. The Tikhonov method stabilizes the solution by the prior information that matrix P is closed to the null matrix as shown in the regularization term of Equation (16). However, the state prices are expected to be unimodal with extreme

[7] A condition number κ is difined as the maximum ratio of a relative error in P of Equation (11) to a relative error in $S^{\mathsf{T}}_{-\mathrm{col}.1}$. It is written as

$$\kappa = \frac{\lambda_{\max}(S^{\mathsf{T}}_{-\mathrm{col}.m})}{\lambda_{\min}(S^{\mathsf{T}}_{-\mathrm{col}.m})}, \tag{15}$$

where $\lambda_{\max}(S^{\mathsf{T}}_{-\mathrm{col}.m})$ and $\lambda_{\min}(S^{\mathsf{T}}_{-\mathrm{col}.m})$ are maximal and minimal singular values of $S^{\mathsf{T}}_{-\mathrm{col}.m}$ respectively.

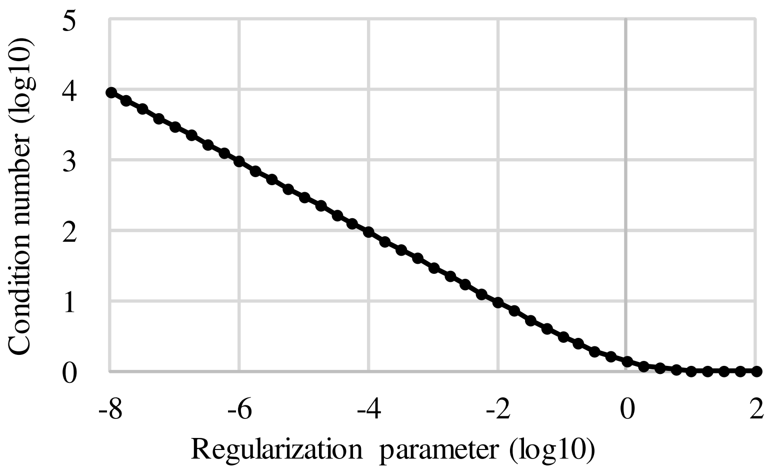


Figure 2: Condition number with respect to regularization parameter

value around main diagonal because the probability of moving to the same state should be higher than the probability of moving to some faraway state. In addition, if we set ζ as infinity, $P = O$ is derived. In this case, we cannot get the real world distribution because the Recovery Theorem can apply only to the irreducible matrix[8]. Therefore, the solution of Step 2 (matrix P) is likely to become stable as ζ increases, whereas the solution of Step 3 (matrix F) is likely to become unstable. According to these properties, the prior information of the Tikhonov method is poorly related with the Recovery Theorem.

Audrino et al. [1] also proposed the selection method of optimal ζ, which minimizes the discrepancy between original state price matrix (S^O) and the state price matrix implied in matrix P (S^P). The τ-th column vector of S^P is equal to the transposed vector of the i_0-th row of P^τ which is P to the τ-th power($\tau = 1, \dots, m$). They use the generalized Kullback-Leibler (KL) divergence as a measure of the discrepancy between two matrices. They propose the selection criteria which minimizes the following function $h_A(\zeta)$ defined as

$$h_A(\zeta) := \sum_{i=1}^{n} \sum_{\tau=1}^{m} s^O_{i,\tau} \ln \left(\frac{s^O_{i,\tau}}{s^P_{i,\tau}} \right) - \sum_{i=1}^{n} \sum_{\tau=1}^{m} s^O_{i,\tau} + \sum_{i=1}^{n} \sum_{\tau=1}^{m} s^P_{i,\tau}. \tag{18}$$

Optimal ζ is derived by iterative calculation. However, they do not evaluate the estimation accuracy. Therefore the effectiveness of this selection criteria function has not been examined. We examine the effectiveness in Section 4.

3.2.3. Proposed method

We propose a new method which modifies the regularization term of the Tikhonov method considering the characteristics of this recovery problem and which has a clear interpretation of the relation between regularization parameter ζ and final estimated value, $\boldsymbol{f}_{i_0}$. Specifically, we modify the regularization term of the Tikhonov method by taking the following two different kinds of prior information into account.

PI 1. The state dependent discount factor of the current state is close to those of other states.

The sums of the row elements of the matrix P represent the state dependent discount factors, namely, $\sum_{k=1}^{n} p_{i,k}$ are close to $\sum_{k=1}^{n} p_{i_0,k} (i = 1, \dots, n)$.

[8]To be precise, all of the components of the matrix except the i_0-th row become zero because of Constraint (13). Even in this case, matrix P is not irreducible.

PI 2. The risk neutral distribution of the current state is close to those of the other states.

$\boldsymbol{q}_{i_0}$ is determined because of Constraint (13)[9], while $\boldsymbol{q}_i (i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n)$ are unknown. Because it is difficult to set the appropriate values as prior information, we simply assume that $\boldsymbol{q}_{i_0}$ is close to $\boldsymbol{q}_i$, and the risk neutral probabilities which have the same distance from the initial state are close to each other[10], namely, $q_{i_0,j}$ are close to $q_{i_0+k,j+k} (j = 1, \ldots, n; k \in \mathbb{Z}, 1 \leq i_0 + k \leq n, 1 \leq j + k \leq n)$. If $\boldsymbol{q}_{i_0}$ is unimodal with an extreme value around θ_{i_0}, this information means that we expect the risk neutral probability matrix Q is also unimodal with the extreme value around the main diagonal.

We modify the regularization term based on the two kinds of prior information as follows,

$$\min_{P} \quad ||S^{\mathsf{T}}_{-\mathrm{col}.m} P - S^{\mathsf{T}}_{-\mathrm{col}.1}||_2^2 + \zeta ||P - \bar{P}||_2^2 \tag{19}$$

$$\Leftrightarrow \min_{P} \quad \left\| \begin{bmatrix} S^{\mathsf{T}}_{-\mathrm{col}.m} \\ \sqrt{\zeta} I \end{bmatrix} P - \begin{bmatrix} S^{\mathsf{T}}_{-\mathrm{col}.1} \\ \sqrt{\zeta} \bar{P} \end{bmatrix} \right\|_2^2 \tag{20}$$

where,

$$\bar{P} = \begin{array}{r} \\ 1 \\ \vdots \\ i_0-1 \\ i_0 \\ i_0+1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & & i_0-1 & i_0 & i_0+1 & & n-1 & n \end{array} \\ \begin{pmatrix} \sum_{k=1}^{i_0} s_{k,1} & s_{i_0+1,1} & \cdots & s_{n-1,1} & s_{n,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \sum_{k=1}^{2} s_{k,1} & s_{3,1} & \cdots & s_{i_0,1} & s_{i_0+1,1} & s_{i_0+2,1} & \cdots & s_{n,1} & 0 \\ s_{1,1} & s_{2,1} & \cdots & s_{i_0-1,1} & s_{i_0,1} & s_{i_0+1,1} & \cdots & s_{n-1,1} & s_{n,1} \\ 0 & s_{1,1} & \cdots & s_{i_0-2,1} & s_{i_0-1,1} & s_{i_0,1} & \cdots & s_{n-2,1} & \sum_{k=n-1}^{n} s_{k,1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & s_{1,1} & s_{2,1} & \cdots & s_{i_0-1,1} & \sum_{k=i_0}^{n} s_{k,1} \end{pmatrix} \end{array}. \tag{21}$$

The problem is subject to the constraints (13-14). The values are accumulated in the first column and last column of matrix $\bar{P}$, and we set zero to the elements which cannot be determined by PI 2. The proposed method [11] attempts to find the solution which is close to $\bar{P}$ from the feasible set. The condition number of the problem is the same as that of the Tikhonov method as shown in Figure 2 because the same coefficient of matrix P is used. Therefore, when we use the same ζ in both methods, the sensitivities of the solution to a small noise are also the same for each other.

The proposed method has a clear interpretation about the relation between regularization parameter ζ and the final estimated value, $\boldsymbol{f}_{i_0}$. PI 1 makes all column sums of matrix $\bar{P}$ equal. As we stabilize the solution of Step 2 by increasing ζ, the estimated real world distribution $\boldsymbol{f}_{i_0}$ becomes close to the risk neutral distribution $\boldsymbol{q}_{i_0}$ because the real world distribution becomes equal to the risk neutral distribution when the sum of the row elements of matrix P is the same for each row. In other words, the proposed method uses the risk

[9]A risk neutral distribution $\boldsymbol{q}_{i_0}$ is easily calculated from a state price distribution $\boldsymbol{p}_{i_0}$ by Equation (1).

[10]For example, we assume there are only three states where the returns are $-5\%, 0\%, +5\%$ for each state. The return of a transition from -5% to 0% ($(1 + 0)/(1 - 0.05) - 1 = +5.3\%$) is different from that of a transition from 0% to $+5\%$ ($(1 + 0.05)/(1 + 0) - 1 = +5\%$). However, both returns are regarded as $+5\%$ approximately.

[11]Mathematical formulation involving Equation (19) is called generalized Tikhonov regularization. The proposed method is a special case in which the matrix $\bar{P}$ is defined as Equation (21), while we set $\bar{P} = O$ in the ordinary Tikhonov method.

neutral distribution as the basis of estimation. This is methodologically reasonable because the Recovery Theorem is used to get the real world distribution from the risk neutral distribution.

To investigate the relation between the prior information and the estimation accuracy in Section 4.4.3, we formulate the estimation method which assumes only PI 1 or only PI 2, respectively[12].

When we solve the problem assuming only PI 1, we replace the objective function (12) with the following function.

$$\min_{P,x} \quad ||S^{\mathsf{T}}_{-\mathrm{col}.m}P - S^{\mathsf{T}}_{-\mathrm{col}.1}||_2^2 + \zeta||P - \bar{P}(x)||_2^2 \tag{22}$$

where,

$$\bar{P}(x) = \begin{array}{cc} & \begin{array}{ccccccccc} 1 & 2 & & i_0-1 & i_0 & i_0+1 & & n-1 & n \end{array} \\ \begin{array}{c} 1 \\ \\ i_0-1 \\ i_0 \\ i_0+1 \\ \\ n \end{array} & \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,i_0-1} & x_{1,i_0} & x_{1,i_0+1} & \cdots & x_{1,n-1} & x_{1,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{i_0-1,1} & x_{i_0-1,2} & \cdots & x_{i_0-1,i_0-1} & x_{i_0-1,i_0} & x_{i_0-1,i_0+1} & \cdots & x_{i_0-1,n-1} & x_{i_0-1,n} \\ s_{1,1} & s_{2,1} & \cdots & s_{i_0-1,1} & s_{i_0,1} & s_{i_0+1,1} & \cdots & s_{n-1,1} & s_{n,1} \\ x_{i_0+1,1} & x_{i_0+1,2} & \cdots & x_{i_0+1,i_0-1} & x_{i_0+1,i_0} & x_{i_0+1,i_0+1} & \cdots & x_{i_0+1,n-1} & x_{i_0+1,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,i_0-1} & x_{n,i_0} & x_{n,i_0+1} & \cdots & x_{n,n-1} & x_{n,n} \end{pmatrix} \end{array} \tag{23}$$

$$\text{subject to} \quad \sum_{k=1}^{n} x_{i,k} = \sum_{k=1}^{n} s_{k,1} \quad (i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n). \tag{24}$$

$x_{i,j}$ $(i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n;\ j = 1, \ldots, n)$ are intermediate variables. When we solve the problem assuming only PI 2, we set the objective function (22) and replace the matrix $\bar{P}(x)$ with,

$$\bar{P}(x) = \begin{array}{cc} & \begin{array}{ccccccccc} 1 & 2 & & i_0-1 & i_0 & i_0+1 & & n-1 & n \end{array} \\ \begin{array}{c} 1 \\ \\ i_0-1 \\ i_0 \\ i_0+1 \\ \\ n \end{array} & \begin{pmatrix} x_1 \sum_{k=1}^{i_0} s_{k,1} & x_1 s_{i_0+1,1} & \cdots & x_1 s_{n-1,1} & x_1 s_{n,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{i_0-1} \sum_{k=1}^{2} s_{k,1} & x_{i_0-1} s_{3,1} & \cdots & x_{i_0-1} s_{i_0,1} & x_{i_0-1} s_{i_0+1,1} & x_{i_0-1} s_{i_0+2,1} & \cdots & x_{i_0-1} s_{n,1} & 0 \\ s_{1,1} & s_{2,1} & \cdots & s_{i_0-1,1} & s_{i_0,1} & s_{i_0+1,1} & \cdots & s_{n-1,1} & s_{n,1} \\ 0 & x_{i_0+1} s_{1,1} & \cdots & x_{i_0+1} s_{i_0-2,1} & x_{i_0+1} s_{i_0-1,1} & x_{i_0+1} s_{i_0,1} & \cdots & x_{i_0+1} s_{n-2,1} & x_{i_0+1} \sum_{k=n-1}^{n} s_{k,1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_n s_{1,1} & x_n s_{2,1} & \cdots & x_n s_{i_0-1,1} & x_n \sum_{k=i_0}^{n} s_{k,1} \end{pmatrix} \end{array}. \tag{25}$$

x_i $(i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n)$ are intermediate variables. Note that in the case of assuming either PI 1 or PI 2, the sensitivity of the solution of the Tikhonov method is not the same as that of the proposed method even if we use the same value of ζ because the matrix $\bar{P}(x)$ contains variables. Table 1 summarizes the prior information and corresponding formulations with the objective function (22) for each recovery method.

The selection method of ζ is also important in finding the accurate solution. The objective function of the optimization problem in Step 2 is Equation (19), and it consists of two terms. The first term shows the fitting error. We denote it by y_{fit}. The second term except ζ, $(||P - \bar{P}||_2^2)$, shows the deviation between the matrices P and $\bar{P}$. We denote it by y_{reg}.

Table 1: Prior information and corresponding formulations for each recovery method

Name	Prior Information	ζ	$\bar{P}$	Constraints
Basic	None	$= 0$	-	Eqs. (13), (14)
Tikhonov	P is close to null matrix	> 0	Null matrix	Eqs. (13), (14)
PI 1	PI 1	> 0	Eq. (23)	Eqs. (13), (14), (24)
PI 2	PI 2	> 0	Eq. (25)	Eqs. (13), (14)
Proposed	PI 1 and PI 2	> 0	Eq. (21)	Eqs. (13),(14)

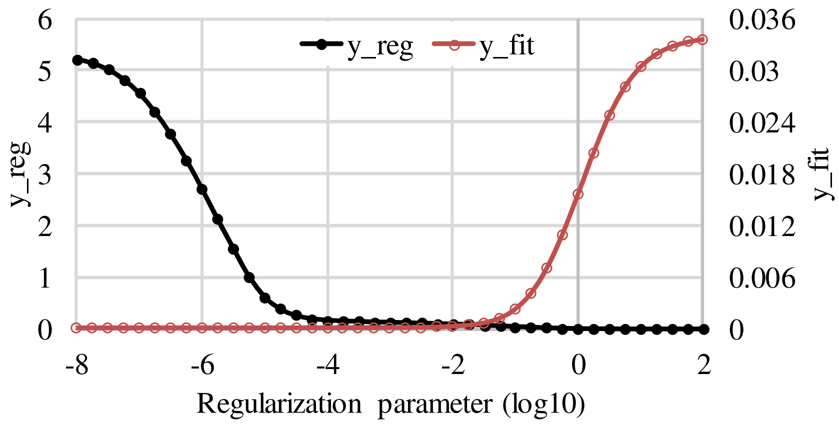


Figure 3: Decomposition of the objective function: y_{fit} and y_{reg} with respect to the regularization parameter

Table 3 shows y_{fit} and y_{reg} for the hypothetical data which we explain in Section 4 (base case, $\sigma = 1\%$) for various values of ζ.

As ζ increases, y_{fit} increases and y_{reg} decreases monotonically. Both y_{fit} and y_{reg} have the domain in which the values greatly change. For example, the value of y_{fit} greatly increases around $\log_{10}\zeta = 0$, and y_{reg} decreases around $\log_{10}\zeta = -6$. This is one of the characteristics of the ill-posed problem. The purpose of adding the regularization term to the objective function is to find the stable solution based on the prior information rather than a degenerate solution. We propose a method of selecting ζ by minimizing a selection criteria function $h_K(\zeta)$ defined as,

$$\min_{\zeta} \quad h_K(\zeta) := \frac{y_{fit}(\zeta) - y_{fit}(0)}{y_{fit}(\infty) - y_{fit}(0)} + \frac{y_{reg}(\zeta) - y_{reg}(\infty)}{y_{reg}(0) - y_{reg}(\infty)}. \tag{26}$$

$y_{fit}(\zeta)$ and $y_{reg}(\zeta)$ are functions of ζ as shown in Figure 3. $h_K(\zeta)$ is the sum of the normalized values of y_{fit} and y_{reg}. $y_{fit}(0)$ and $y_{reg}(0)$ are the values without the regularization term and $y_{fit}(\infty)$ and $y_{reg}(\infty)$ are the values derived under the condition $P = \bar{P}$. Therefore, $y_{reg}(\infty) = 0$ must hold. In addition, $h_K(0) = 1$ and $h_K(\infty) = 1$ must hold because both $y_{fit}(\zeta)$ and $y_{reg}(\zeta)$ are monotonic functions. We obtain different values of $h_K(\zeta)$ by solving the optimization problems for different values of ζ, and then we adopt ζ that minimizes $h_K(\zeta)$.

[12]We can formulate the estimation method which involves the respective regularization parameters for PI 1 and PI 2. However, we omit it since the analysis is very complicated.

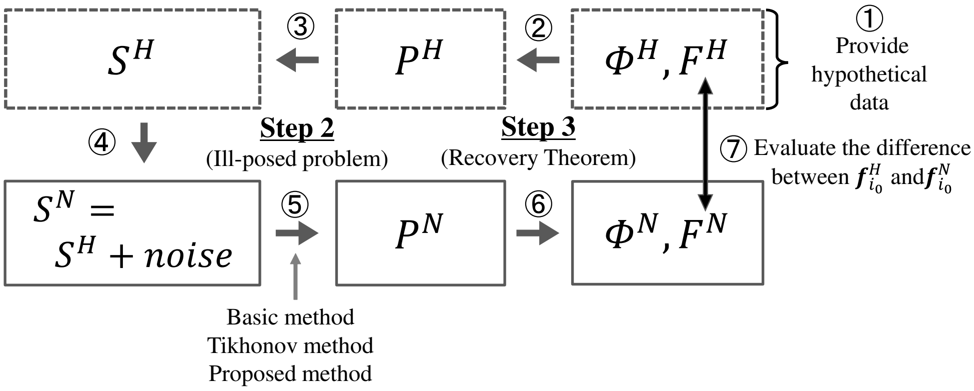


Figure 4: Overview of analysis

4. Numerical Analysis of Estimation Accuracy

4.1. Overview

We analyze the estimation accuracy to examine the effectiveness of the proposed method by comparing a preset hypothetical distribution and a distribution which is estimated from the data with noise. We use hypothetical data because it is difficult to specify the true distribution from market data. Using hypothetical data enables us to evaluate estimation accuracy independently of the estimation method of Step 1. Figure 4 shows the overview of the analysis.

A specific procedure of the analysis is described as follows. Each number corresponds to the number in Figure 4.

① First, we provide two hypothetical matrices: hypothetical real world probability matrix F^H and hypothetical pricing kernel matrix Φ^H.

② A transition state price matrix P^H is calculated backward from the matrices F^H and Φ^H.

③ A current state price matrix S^H is calculated backward from the matrix P^H.

④ We generate a current state price matrix with noise S^N by adding white noise to the matrix S^H. We assume the noise $e_{i,j}$ follows a normal distribution with mean 0 and standard deviation σ. Each component of the matrix S^N is expressed as,

$$s^N_{i,j} = s^H_{i,j}(1 + e_{i,j}) \quad (i, j = 1, \dots, n). \tag{27}$$

⑤ We estimate a matrix P^N from the matrix S^N (Step 2) using the basic method, Tikhonov method, and proposed method.

⑥ A matrix F^N is derived by applying the Recovery Theorem for the matrix P^N.

⑦ If the estimated real world distribution obtained from matrix F^N or $\boldsymbol{f}^N_{i_0}$ is close to $\boldsymbol{f}^H_{i_0}$, the preset real world distribution obtained from the matrix F^H, we evaluate the estimation results as having a high accuracy. A specific evaluation criteria of the estimation accuracy is described in Section 4.2.

4.2. Setting

We explain the base setting of the analysis which includes the definition of the state, the number of maturities, evaluation criteria, and comparison of methods.

• **State**
A market state is defined by the return from time 0. We set 31 returns (states) in total, placed by 2% symmetrically from the return of 0%. Specifically, $r_1 = -30\%$, $r_{16} = 0\%$, and $r_{31} = 30\%$.

• **Number of Maturities**
We can apply any number of maturities m because we calculate the matrix S^H backward from hypothetical data. We set the number of maturities m as equal to n in the base case for simplicity. On the other hand, the number of maturities of options traded in the market is likely to become smaller than the number of states in practice when we estimate a matrix S from market data. We analyze the practical case where $m < n$ in Section 4.4.4[13].

• **Evaluation criteria**
The estimation accuracy is evaluated by the Kullback-Leibler divergence (KL divergence) of the estimated distribution $\boldsymbol{f}_{i_0}^N$ from the preset hypothetical distribution $\boldsymbol{f}_{i_0}^H$. KL divergence is a measure of the difference between the two distributions and is defined as[14]

$$D_{KL}(\boldsymbol{f}_{i_0}^H || \boldsymbol{f}_{i_0}^N) := \sum_{j=1}^{n} f_{i_0,j}^N \ln \left(\frac{f_{i_0,j}^N}{f_{i_0,j}^H} \right). \tag{28}$$

If $\boldsymbol{f}_{i_0}^N$ is exactly equal to $\boldsymbol{f}_{i_0}^H$, $D_{KL}(\boldsymbol{f}_{i_0}^H || \boldsymbol{f}_{i_0}^N) = 0$ holds. We also use Euclidean distance instead of KL divergence as a measure of the estimation accuracy, but we arrive at the same conclusion. Therefore, we show only the result using KL divergence hereafter[15].

• **Comparison of methods**
We compare the estimation accuracy of five methods: "risk neutral (RN) method," "perfect method," and three methods we demonstrated in Section 3.2 (basic method, Tikhonov method, and proposed method).

The RN method uses the risk neutral distribution $\boldsymbol{q}_{i_0}^N$ as an approximation of the real world distribution $\boldsymbol{f}_{i_0}^N$. Risk adjustment by the Recovery Theorem affects the estimation accuracy both positively and negatively. The positive effect is that the risk preference of investors to the distribution can be reflected. The negative effects are that the estimation accuracy is affected by the noise due to the ill-posed problem and it is biased by prior information. From the comparison of the estimation accuracy calculated by each method and the RN method, we evaluate which effect is larger, positive or negative.

The perfect method uses the transition state price matrix P^H calculated from hypothetical data as the prior information $\bar{P}$. In this method, the solution is estimated under the perfectly accurate prior information. Therefore, it is expected that the estimation accuracy monotonically improves as ζ increases. Unfortunately, we cannot use this method practically because we cannot know the true state price matrix P^H when the distribution is estimated from market data. We add this method as a benchmark for comparison to evaluate the estimation accuracy.

[13]We omit the result of the case where $m > n$, because the conclusion is almost the same as the case where $m = n$.
[14]We add a very small value (10^{-20}) to each component of $\boldsymbol{f}_{i_0}^H$ and $\boldsymbol{f}_{i_0}^N$ to prevent its anti-logarithm from being zero and avoid dividing it by zero. However, this procedure has no impact on the result.
[15]We note that the estimation accuracy of the state price distribution $\boldsymbol{s}_{i_0}$ is the same among all methods because of the constraint (13).

4.3. Hypothetical data

The hypothetical data Φ^H and F^H should be generated as appropriately as possible to reproduce the ill-posed problem. We explain the setting of the hypothetical data.

- **Pricing kernel matrix**

We assume that the TAIEUT investor has a CRRA utility function $U(c) = c^{1-\gamma}/(1-\gamma)$ with a relative risk aversion γ. Pricing kernel ϕ is decomposed into U' and δ as shown in Equation (4). We denote the (i, j) element of matrix Φ^H by

$$\phi^H_{i,j} = \delta \left(\frac{1 + r_j}{1 + r_i} \right)^{-\gamma} \quad (i, j = 1, \ldots, n). \tag{29}$$

The parameters of $\gamma = 3$ and $\delta = 0.999$ are used in the base case[16].

- **Real world probability matrix**

The real world probability matrix F^H is generated based on the S&P500 historical daily price data. We set a reference date and calculate twelve returns in the periods from the reference date to the dates which come every 30 calendar days. If it is a holiday, the return until the day before a holiday is calculated. A matrix is generated by counting the number of state transitions of the return sequence in one period. We denote the return of state θ_j by r_j in the matrix, which is discretely set every 2%. When a real historical return is between $r_j - 1\%$ and $r_j + 1\%$, it is assigned to state θ_j. For example, suppose that a return is 12.5%. It is between 11%(12% − 1%) and 13%(12% + 1%), and therefore 12% is assigned to the return. A return greater than or equal to 29% (less than or equal to −29%) is assigned to 30% (−30%). This is repeated daily by changing the reference date from Jan 3, 1950 to Jan 3, 2014. Then, all the matrices are summed up. Finally, each element of the summed matrix is divided by each sum of the row elements to make it a probability matrix. The generated matrix F^H is shown in Table 2. F^H is almost unimodal with extreme values around the main diagonal because the probability of moving from current state to some close state should be higher than the probability of moving from current state to some faraway state. The real world distributions $\boldsymbol{f}_i$ are positively skewed. These are well-known stylized facts about the real world probability.

4.4. Result

4.4.1. Characteristics of each estimation method

We check the characteristics of each estimation method. Figure 5(a) illustrates the hypothetical transition state price matrix P^H and Figure 5(b-f) illustrate the transition state price matrix P^N estimated with each estimation method. The noise parameter is set as $\sigma = 1\%$ and the regularization parameter is set as $\zeta = 10^{-3}$ for each method. Figure 6 illustrates the hypothetical pricing kernel (True) and the estimated pricing kernels estimated with each estimation method. Risk neutral distribution $\boldsymbol{p}^H_{i_0}$ is also included as RND in Figure 6.

As for the basic method, the estimated transition state price matrix is largely disturbed by the noise because of the ill-posed nature of estimation. As a result, the estimated pricing kernel is also largely disturbed by the noise. This result is consistent with Jackwerth

[16]Bliss and Panigirtzoglou [5] estimate a risk aversion parameter γ implied in S&P500 option data and historical price data from 1993 to 2010. The estimated value is dependent on maturity. The minimum value is 3.37 and the maximum value is 9.52. Therefore, we use $\gamma = 3$ in the base case and $\gamma = 10$ in the robustness check of 4.4.5.

Table 2: Hypothetical real world probability matrix F^H generated from the S&P500 historical data (Values that are lower than 10^{-2} are replaced into blanks)

$r_i \setminus r_j$	-30%	-28%	-26%	-24%	-22%	-20%	-18%	-16%	-14%	-12%	-10%	-8%	-6%	-4%	-2%	0%	2%	4%	6%	8%	10%	12%	14%	16%	18%	20%	22%	24%	26%	28%	30%
-30%	0.76	0.07	0.06	0.04	0.03	0.03	0.01																								
-28%	0.31	0.07	0.12	0.18	0.13	0.08	0.06	0.02		0.01	0.01																				
-26%	0.24	0.12	0.06	0.13	0.15	0.12	0.07	0.05	0.03	0.01	0.01																				
-24%	0.20	0.08	0.07	0.06	0.12	0.16	0.15	0.08	0.05	0.02		0.01																			
-22%	0.10	0.04	0.06	0.07	0.12	0.20	0.14	0.15	0.08	0.04	0.01																				
-20%	0.05	0.03	0.04	0.07	0.12	0.15	0.17	0.16	0.11	0.06	0.01	0.01																			
-18%	0.03	0.03	0.03	0.05	0.09	0.13	0.15	0.16	0.15	0.11	0.03	0.02	0.01																		
-16%	0.01	0.01	0.03	0.04	0.05	0.09	0.13	0.16	0.16	0.17	0.08	0.03	0.02	0.01	0.01																
-14%	0.01	0.01	0.02	0.02	0.03	0.05	0.08	0.13	0.18	0.17	0.15	0.07	0.04	0.01	0.01		0.01														
-12%			0.01	0.02	0.02	0.04	0.04	0.08	0.13	0.16	0.20	0.14	0.08	0.03	0.01	0.01															
-10%					0.02	0.02	0.03	0.06	0.09	0.14	0.18	0.18	0.14	0.07	0.03	0.02	0.01														
-8%						0.01	0.01	0.03	0.06	0.10	0.15	0.17	0.20	0.14	0.07	0.03	0.02	0.01													
-6%							0.01	0.01	0.02	0.06	0.10	0.15	0.19	0.19	0.13	0.07	0.03	0.01	0.01												
-4%								0.01	0.01	0.02	0.06	0.09	0.15	0.18	0.20	0.15	0.07	0.04	0.02	0.01											
-2%									0.01	0.01	0.02	0.05	0.09	0.15	0.18	0.21	0.16	0.08	0.03	0.02	0.01										
0%											0.01	0.02	0.04	0.08	0.14	0.19	0.23	0.16	0.07	0.03	0.01										
2%												0.01	0.02	0.04	0.08	0.14	0.20	0.23	0.17	0.07	0.02	0.01									
4%														0.02	0.04	0.08	0.14	0.20	0.24	0.16	0.06	0.03	0.01								
6%														0.01	0.02	0.04	0.08	0.14	0.22	0.24	0.15	0.06	0.02	0.01							
8%															0.01	0.02	0.04	0.08	0.15	0.20	0.23	0.14	0.08	0.02	0.01						
10%																0.01	0.02	0.04	0.08	0.14	0.20	0.22	0.15	0.08	0.03	0.01					
12%																	0.01	0.02	0.03	0.08	0.14	0.18	0.21	0.18	0.09	0.04	0.01				
14%																		0.01	0.01	0.03	0.08	0.13	0.17	0.22	0.17	0.10	0.04	0.01			
16%																			0.01	0.01	0.03	0.07	0.13	0.19	0.21	0.18	0.09	0.04	0.02	0.01	
18%																					0.02	0.04	0.08	0.14	0.19	0.20	0.16	0.10	0.04	0.02	0.01
20%																					0.01	0.02	0.04	0.08	0.14	0.19	0.19	0.17	0.10	0.04	0.02
22%																						0.01	0.02	0.05	0.08	0.13	0.18	0.18	0.16	0.11	0.08
24%																							0.01	0.03	0.04	0.09	0.13	0.18	0.18	0.15	0.19
26%																								0.01	0.03	0.04	0.08	0.13	0.18	0.18	0.35
28%																									0.02	0.03	0.04	0.08	0.11	0.17	0.53
30%																											0.01	0.02	0.03	0.06	0.86

and Menner [12]. They report that the state price matrix obtained from real option data with the basic method is largely disturbed by the noise. The Tikhonov method offers smooth transition state price matrix by stabilizing the solution with regularization method. However, large state prices is found not only around main diagonal but also around sub diagonal, and estimated pricing kernel is U-shape. These are due to the bias introduced by regularization terms of the Tikhonov method. In contrast, the proposed method offers smooth transition state price matrix that has large state prices only around main diagonal. In addition, the estimated pricing kernel decreases monotonically and takes similar value as the hypothetical pricing kernel especially in the range where the risk neutral probability is positive. The pricing kernel estimated under PI 1 is smooth because PI 1 makes state dependent discount factors stable. However, the state price matrix estimated under PI 1 is not smooth and takes large values for states that are far away from the current state. The state price matrix estimated under PI 2 does not take large values for states that are far away from the current state, but pricing kernel under PI 2 is not smooth and is largely different from the hypothetical pricing kernel.

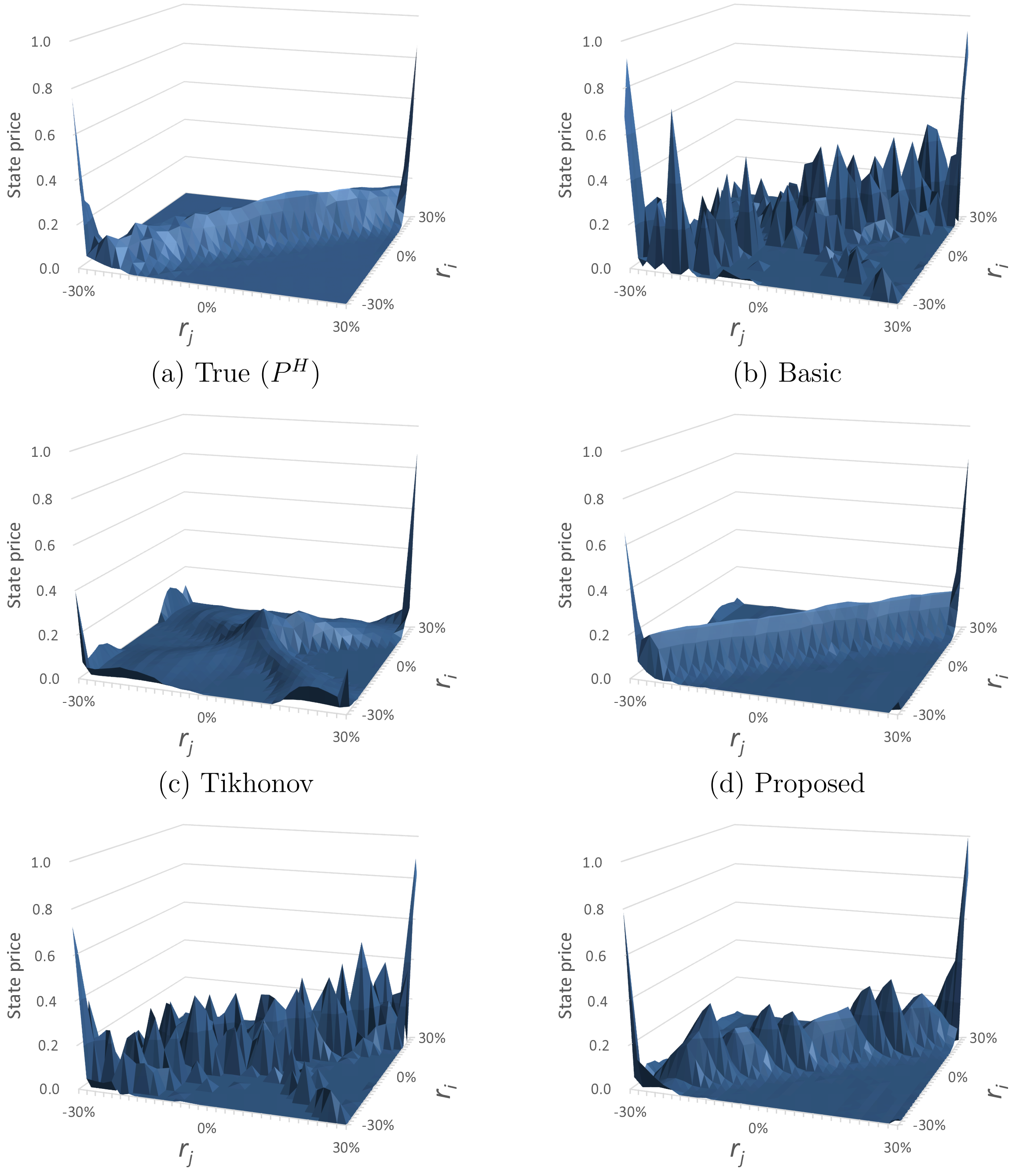
As we mentioned in Section 3, the proposed method assumes the two sets of prior information, PI 1 and PI 2. PI 1 makes state dependent discount factors stable. This also makes pricing kernel stable. PI 2 gives large state prices around main diagonal and small state prices around sub diagonal. Therefore, the proposed method offers more accurate solution than the other methods.

4.4.2. Base analysis

Base analysis compares the estimation accuracy under the setting in Section 4.2 and hypothetical data in Section 4.3. The optimization problem in Step 2 is still ill-posed because the condition number of matrix A^H calculated backward from the matrices Φ^H and F^H is very large, and 1.3×10^{17}. The results for the specific random numbers are shown hereafter, but we obtain the same conclusions for the different random seeds.

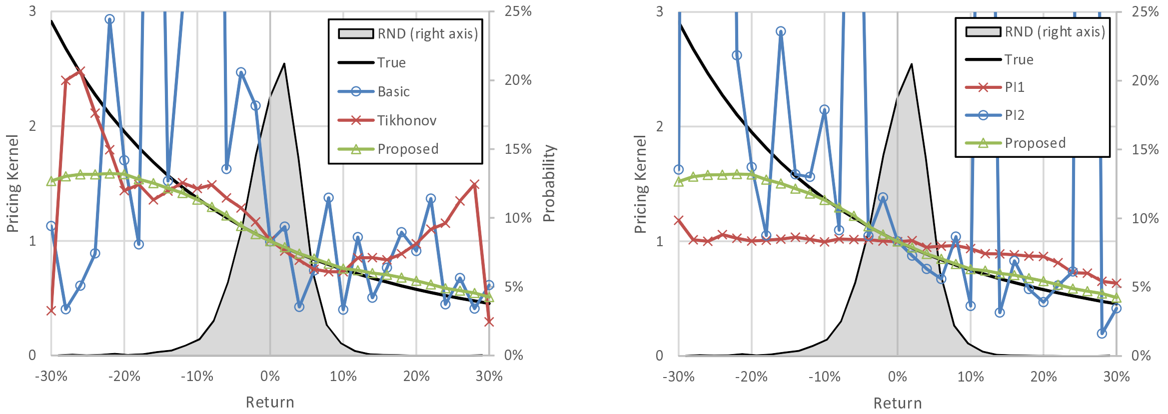
Figure 7 displays the KL divergence of $\boldsymbol{f}_{i_0}^N$ from $\boldsymbol{f}_{i_0}^H$ for various values.

First, we discuss the result of the case of $\sigma = 0\%$ where the matrix S^N is generated without noise. Theoretically, the KL divergence of the basic method is equal to zero.



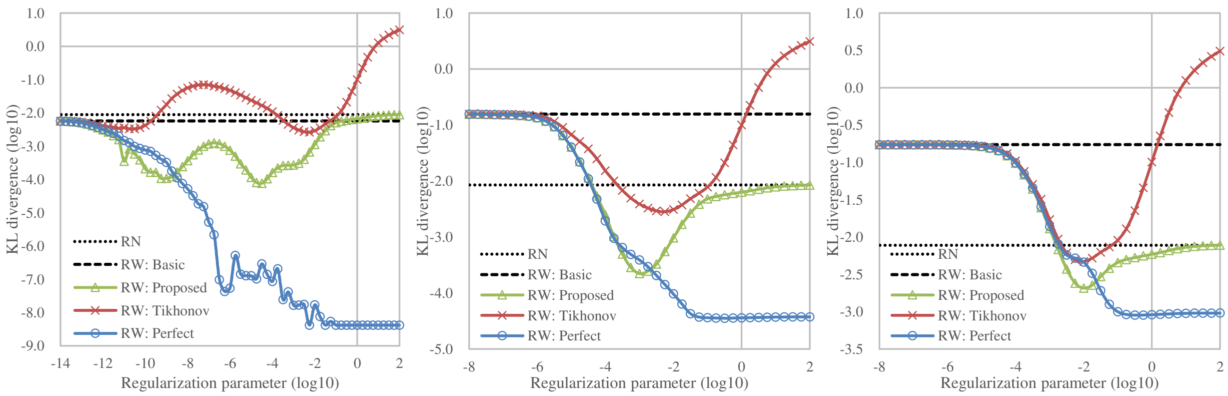
(a) True (P^H) (b) Basic
(c) Tikhonov (d) Proposed
(e) PI 1 (f) PI 2

Figure 5: State price matrices P for different estimation methods ($\sigma = 1\%,\ \zeta = 10^{-3}$)



(a) Basic, Tikhonov, Proposed (b) PI 1, PI 2, Proposed

Figure 6: Pricing kernels for different estimation methods ($\sigma = 1\%,\ \zeta = 10^{-3}$)



(a) $\sigma = 0\%$ (b) $\sigma = 1\%$ (c) $\sigma = 5\%$

Figure 7: Base case: KL divergences with respect to regularization parameters

However, the calculated KL divergence of the basic method is 4.4×10^{-3} due to the estimation error. This shows how it is difficult to get an accurate estimator of the ill-posed problem. The estimation accuracy of the proposed method is better than that of the basic method in the range where $\log_{10}\zeta$ is less than 1.5. This result shows that the proposed method is effective to increase the estimation accuracy even without noise. As ζ gets larger, the graph of the proposed method approaches that of the RN method. This is because the real world distribution estimated with the proposed method where $\zeta = \infty$ equals the risk neutral distribution. The estimation accuracy of the Tikhonov method is worse than that of the basic method. The regularization term of the Tikhonov method introduces the bias because it is not formulated with the prior information appropriately. The estimation accuracy of the perfect method is the highest, as we expected. The graph of the perfect method between $\log_{10}\zeta = -6$ and $\log_{10}\zeta = -2$ is distorted by numerical error because the objective value of the optimization problem is very small.

We check the two cases with noise ($\sigma = 1\%$ and $\sigma = 5\%$). The estimation accuracy of the basic method is worse than that of the RN method because of the ill-posed problem. The KL divergence of the Tikhonov method and that of the proposed method are U-shaped in the graph. This indicates that introducing the regularization term is effective to estimate the real world distribution accurately, but the value of ζ needs to be selected appropriately.

Table 3: Base case: Common logarithm of KL divergences ($\log_{10} D_{KL}$) for each selection criteria of regularization parameters

Volatility of noise	Selection criteria of ζ	RN	RW				
			Basic	Proposed		Tikhonov	
$\sigma = 1\%$	min $h_K(\zeta)$	−2.07	−0.81	−3.18	(−2.23)	−2.07	(−0.89)
	min $h_A(\zeta)$			−1.72	(−4.65)	−0.91	(−5.55)
	min KL			−3.65	(−2.97)	−2.55	(−2.23)
$\sigma = 5\%$	min $h_K(\zeta)$	−2.11	−0.76	−2.57	(−2.39)	−2.03	(−0.97)
	min $h_A(\zeta)$			−1.92	(−2.97)	−0.97	(−3.98)
	min KL			−2.68	(−1.98)	−2.33	(−2.05)

∗ The common logarithm of $\zeta(\log_{10} \zeta)$ selected by each criteria is in the parenthesis.

The estimation accuracy of the proposed method is better than that of the Tikhonov method because the regularization term of the proposed method is formulated involving the prior information more appropriately.

We next examine the effectiveness of the selection function of ζ. Table 3 shows the common logarithm of KL divergence ($\log_{10} D_{KL}$) for each selection criteria of ζ.

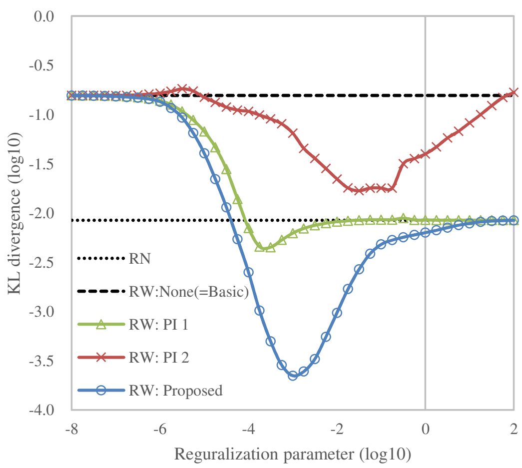
The value of "min $h_K(\zeta)$" is calculated by our selection criteria of ζ (Equation (26)) and the value of "min $h_A(\zeta)$" is calculated by the selection criteria proposed by Audrino et al. [1] (Equation(18)). The values of "min KL" are derived minimizing KL divergence[17]. We find the optimal value of ζ using the golden section search method in the range from $\log_{10} \zeta = -8$ to $\log_{10} \zeta = 2$. The estimation accuracies of the real world distribution estimated using our proposed method and the Tikhonov method are dependent on the effectiveness of the regularization term of the prior information and the selection criteria of ζ. We can evaluate the effectiveness of the regularization term by comparing the KL divergence of "min KL" of the methods introducing the regularization term with that of the RN method because the value of "min KL" is not dependent on the selection criteria of ζ. As shown in Table 3, the KL divergence of "min KL" is smaller than that of the RN method, and we find that the regularization term can be effective if we select the appropriate criteria of ζ[18]. Next, we evaluate the selection criteria.

The selection function $h_K(\zeta)$ gives more appropriate ζ than $h_A(\zeta)$ in both the proposed method and the Tikhonov method. The result indicates that it is effective to select ζ based on the normalized value of the residual term and the normalized value of the regularization term. We can derive a more accurate real world distribution by only utilizing the combination of the proposed method and selection function $h_K(\zeta)$ rather than the RN method.

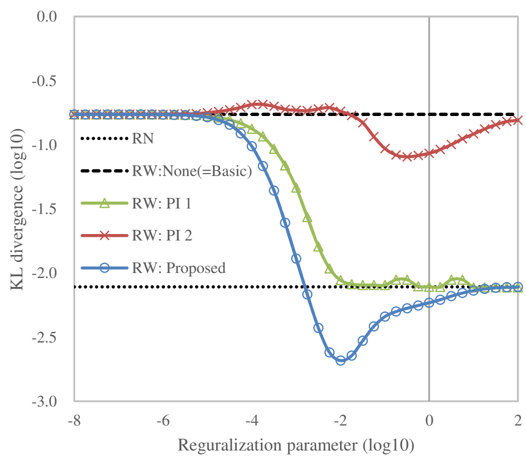
We have the following three findings obtained in the base analysis: (1) the proposed method and the Tikhonov method are effective to improve the estimation accuracy, (2) we can obtain a more accurate solution by the proposed method than the Tikhonov method, and (3) we can select an appropriate regularization parameter by minimizing the function $h_K(\zeta)$ in the proposed method. We check the robustness of these findings in Appendix A.

[17] The parameter ζ for the value of "min KL" is the best because the KL divergence is evaluated as a selection criteria.

[18] If the KL divergence of "min KL" is larger than that of the RN method, we need to formulate the appropriate regularization term regardless of the selection criteria.



(a) $\sigma = 1\%$



(b) $\sigma = 5\%$

Figure 8: Effect of the prior information: KL divergences with respect to the regularization parameters

Table 4: Effect of the prior information: Common logarithm of minimum KL divergence ($\log_{10} D_{KL}$)

	$\sigma = 1\%$	$\sigma = 5\%$
RN	-2.07	-2.11
RW: None (= Basic method)	-0.81	-0.76
RW: PI 1	-2.35	-2.11
RW: PI 2	-1.77	-1.09
RW: PI 1 and PI 2 (= Proposed method)	-3.65	-2.68

4.4.3. Effect of the prior information

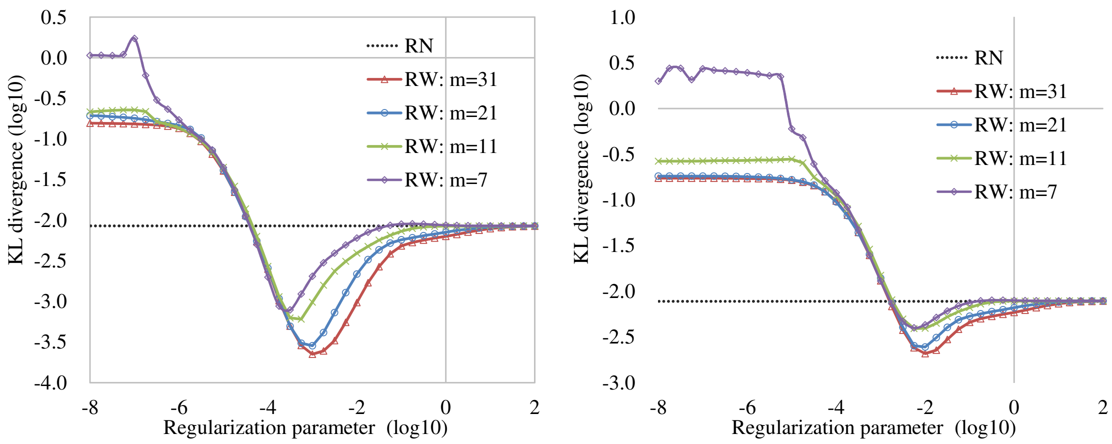
This section analyzes the contribution of the prior information to the estimation accuracy. In particular, we compare the estimation accuracy among five formulations: four methods without the Tikhonov method shown in Table 1 and the RN method. Figure 8 shows KL divergences with respect to ζ in the setting of the base case.

In both the case of $\sigma = 1\%$ and 5%, the graph of "PI 1" approaches the risk neutral distribution. The KL divergence of "PI 1" is less than that of RN method around $\log_{10} \zeta = -3.5$ and the KL divergence of "PI 2" is less than that of RN method around $\log_{10} \zeta = -2$. We must note that the sensitivities of the solutions of the three formulations ("PI 1", "PI 2", and "PI 1 and PI 2") are different because the variables are included in the regularization terms of "PI 1" and "PI 2." Therefore we evaluate the estimation accuracy by the minimum KL divergence as shown in Table 4.

The formulations can be ranked in the estimation accuracy as "PI 1 and PI 2", "PI 2", "PI 1", "RN", and "None". This result shows that both "PI 1" and "PI 2" contribute to the improvement of estimation accuracy and the proposed method can estimate more accurate solutions by the combination of "PI 1" and "PI 2."

4.4.4. Effect of insufficient data

We have analyzed the estimation accuracy with $m = 31$, where m is the number of option maturities traded in the market. In practice, the number of option maturities is less than 31. For example, the number of S&P500 option maturities traded monthly in the CBOE



(a) $\sigma = 1\%$ (b) $\sigma = 5\%$

Figure 9: Effect of insufficient data: KL divergences with respect to the regularization parameters

Table 5: Effect of insufficient data: Common logarithm of KL divergences ($\log_{10} D_{KL}$) for each selection criteria of regularization parameters

Volatility of noise	Selection criteria of ζ	RN	RW $m = 31$	RW $m = 21$	RW $m = 11$	RW $m = 7$
$\sigma = 1\%$	min $h_K(\zeta)$	-2.07	-3.18	-3.32	-3.16	-2.67
	min $h_A(\zeta)$		-1.72	-1.14	-0.79	-0.18
	min KL		-3.65	-3.57	-3.24	-3.13
$\sigma = 5\%$	min $h_K(\zeta)$	-2.11	-2.57	-2.07	-1.23	-0.88
	min $h_A(\zeta)$		-1.92	-1.24	-0.56	0.00
	min KL		-2.68	-2.63	-2.42	-2.40

is twelve. The number of Nikkei 225 options in the Osaka Exchange is nine. In addition, the number m becomes smaller because long-term options are likely to have low liquidity. We conduct an analysis for the case where the number of maturities (equations) m is less than the number of states (estimated variables) n. Specifically, we estimate the real world distribution where $n = 31$ and there are four kinds of the numbers of column of S^H ($m = 7, 11, 21, 31$) by the proposed method and calculate the KL divergences. Figure 9 shows KL divergences with respect to the regularization parameters.

Usually, it is difficult to get an accurate estimator because the solution is not determined uniquely when the number of equations m is less than that of estimated parameters n. Figure 9 shows that the estimation accuracy decreases as m decreases. However, the KL divergence of the proposed method where $m = 7$ is less than that of the RN method around $\log_{10} \zeta = -3.5$ for $\sigma = 1\%$ and $\log_{10} \zeta = -2$ for $\sigma = 5\%$. This is because the prior information included in the regularization term offsets the insufficient information. In other words, the necessary information to estimate the real world distribution is almost included in the state price matrix of seven maturities.

In addition, we evaluate the estimation accuracy with respect to the selection of ζ. Table 5 shows the KL divergence for each selection criteria.

In the case of $\sigma = 1\%$, the KL divergence of the proposed method with $h_K(\zeta)$ is less than that of the RN method even if $m = 7$. On the other hand, the KL divergence of $h_A(\zeta)$

increases by the risk adjustment because $h_A(\zeta)$ cannot select an appropriate ζ. In the case of $\sigma = 5\%$, the KL divergence of the proposed method with $h_K(\zeta)$ is not less than that of the RN method except for $m = 31$ which has a sufficient number of equations. However, it is expected that the accurate solution is obtained by setting more appropriate selection criteria because the minimum KL divergence is less than the KL divergence of the RN method.

4.4.5. Robustness check

The proposed method can be used to derive the real world distribution under two sets of prior information (PI 1 and PI 2). Therefore, it is expected that the estimation accuracy decreases in the case in which the hypothetical data does not reflect the prior information used in the proposed method. We check the robustness using such hypothetical data.

We explain how we provide the hypothetical data which does not reflect the prior information of the proposed method. First, PI 1 is the information that the real world distribution becomes equal to the risk neutral distribution when $\zeta = \infty$. In other word, PI 1 assumes the representative investor is risk neutral $(\gamma = 0)$. As risk aversion increases, the difference between the risk neutral distribution and the real world distribution gets larger. We set a larger risk aversion parameter γ as 10 to check robustness.

Next, PI 2 is the information that the risk neutral distribution of the current state is close to the risk neutral distributions of the other states. A typical example that is different from this information is to set a local volatility, which is the case in which the volatilities of the distributions are largely different in each state. We set the hypothetical data assuming that the real world distributions of each state $\boldsymbol{f}_i (i = 1, \ldots, n)$ follow discretized log-normal distributions which have different volatilities $LN(\mu_i, \sigma_i^2)(i = 1, \ldots, n)$. In this case, if the real world probability has a local volatility then the risk neutral probability also has a local volatility. It is a well-known stylized fact that volatilities calculated using the Black Scholes formula have different values for each strike price, which is called volatility smile. Therefore, we can generate the hypothetical data with a local volatility by setting the parameter $\sigma_i (i = 1, \ldots, n)$ based on volatilities implied in market data. We set each mean parameter $\mu_i (i = 1, \ldots, n)$ assuming the risk premium is proportional to volatility. We explain the method specifically.

To check robustness with a high local volatility setting, we generate the hypothetical data based on the implied volatilities on the date when Skew Index (SI), which is the index of the skewness of the S&P500 risk neutral distribution calculated by the Chicago Board Options Exchange (CBOE), takes the highest value. We make the list of dates 30 days before S&P500 option maturities (the third Friday in every month) from Jan 4, 2000 to Jan 12, 2004. Then, we select the date when the highest SI value is shown in the list. The selected date is Jan 22, 2014 and the SI value is 138.92. We calculate implied volatilities from the OTM option with 30-day maturity on Jan 22, 2014. We get an implied volatility $\sigma_i (i = 1, \ldots, n)$ of each state by interpolating with a cubic spline. The implied volatilities are extrapolated with the largest available value because option prices with more than $+10\%$ return are not available. Figure 10 shows the implied volatilities σ_i which is related with the return r_i. The volatility is set at a range that is at most five times.

We assume the risk premium $(\mu_i - r_f)$ is proportional to volatility σ_i as

$$\mu_i - r_f = k\sigma_i \quad (i = 1, \ldots, n), \tag{30}$$

where r_f is a risk free rate and k is a proportionality coefficient. We calculate k

$$k = \frac{\bar{\mu} - \bar{r}_f}{\bar{\sigma}} = 0.02087 \tag{31}$$

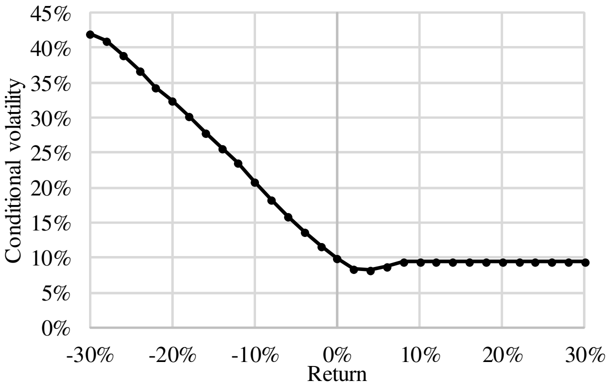


Figure 10: Implied volatility σ_i calculated from OTM option on Jan 22, 2014.

Table 6: Hypothetical real world probability matrix F^H with a local volatility (Values that are lower than 10^{-2} are replaced into blanks)

$r_i \setminus r_j$	-30%	-28%	-26%	-24%	-22%	-20%	-18%	-16%	-14%	-12%	-10%	-8%	-6%	-4%	-2%	0%	2%	4%	6%	8%	10%	12%	14%	16%	18%	20%	22%	24%	26%	28%	30%
-30%	0.52	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.01	0.01																			
-28%	0.42	0.09	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.01	0.01																		
-26%	0.33	0.09	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.01	0.01																	
-24%	0.24	0.09	0.10	0.10	0.09	0.09	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.01	0.01																
-22%	0.15	0.08	0.09	0.10	0.10	0.10	0.09	0.08	0.06	0.05	0.04	0.02	0.02	0.01	0.01																
-20%	0.09	0.06	0.08	0.09	0.10	0.11	0.10	0.09	0.08	0.06	0.05	0.03	0.02	0.02	0.01	0.01															
-18%	0.04	0.04	0.06	0.08	0.09	0.11	0.11	0.11	0.09	0.08	0.06	0.05	0.03	0.02	0.01	0.01															
-16%	0.02	0.02	0.03	0.05	0.08	0.10	0.11	0.12	0.11	0.10	0.08	0.06	0.04	0.03	0.02	0.01	0.01														
-14%		0.01	0.02	0.03	0.05	0.08	0.10	0.12	0.12	0.12	0.10	0.08	0.06	0.04	0.03	0.02	0.01														
-12%			0.01	0.01	0.03	0.05	0.08	0.10	0.12	0.13	0.13	0.11	0.08	0.06	0.04	0.02	0.01	0.01													
-10%					0.01	0.02	0.04	0.08	0.11	0.14	0.15	0.14	0.11	0.08	0.06	0.03	0.02	0.01													
-8%							0.02	0.04	0.07	0.11	0.15	0.16	0.15	0.12	0.08	0.05	0.03	0.01													
-6%								0.01	0.03	0.06	0.11	0.16	0.18	0.17	0.13	0.08	0.04	0.02	0.01												
-4%										0.02	0.05	0.11	0.18	0.21	0.18	0.13	0.07	0.03	0.01												
-2%											0.01	0.04	0.11	0.19	0.24	0.20	0.13	0.06	0.02												
0%													0.03	0.10	0.21	0.27	0.22	0.12	0.04	0.01											
2%														0.01	0.08	0.22	0.31	0.24	0.10	0.03											
4%															0.01	0.08	0.22	0.31	0.24	0.10	0.02										
6%																0.02	0.09	0.21	0.29	0.23	0.11	0.03	0.01								
8%																0.01	0.03	0.10	0.20	0.26	0.22	0.12	0.04	0.01							
10%																	0.01	0.03	0.10	0.20	0.26	0.22	0.12	0.05	0.01						
12%																		0.01	0.03	0.10	0.20	0.25	0.21	0.12	0.05	0.01					
14%																			0.01	0.04	0.10	0.20	0.25	0.21	0.12	0.05	0.01				
16%																				0.01	0.04	0.10	0.20	0.25	0.21	0.12	0.05	0.02			
18%																					0.01	0.04	0.11	0.19	0.24	0.21	0.13	0.05	0.02		
20%																						0.01	0.04	0.11	0.19	0.24	0.21	0.13	0.06	0.02	0.01
22%																							0.01	0.04	0.11	0.19	0.23	0.20	0.13	0.06	0.03
24%																								0.01	0.04	0.11	0.19	0.23	0.20	0.13	0.09
26%																									0.01	0.04	0.11	0.19	0.23	0.20	0.22
28%																										0.01	0.05	0.11	0.19	0.22	0.42
30%																											0.01	0.05	0.11	0.18	0.64

where $\bar{\mu}$, $\bar{r}_f$, and $\bar{\sigma}$ are the average of monthly returns of the S&P500, the average of one-month LIBOR, and the average of the CBOE Volatility Index (VIX), respectively. The average is calculated using the monthly data from Jan, 1990 to Dec, 2014. Next, we calculate μ_i from Equation (30) where r_f is the one-month LIBOR on Jan 22, 2014. A matrix F^H is generated by discretizing the log-normal distribution, and it is shown in Table 6. We conduct the analysis for the three new cases shown in Table 7. We show the results for the three cases.

- **Case A**

In case A, the pricing kernel matrix Φ^H where $\gamma = 10$ and the real world probability matrix F^H generated based on historical data are used as the hypothetical data for the analysis. Table 8 shows the KL divergences for the three types of selection criteria of ζ. The estimation accuracy of the combination of the Tikhonov method and the selection function $h_A(\zeta)$ is inferior to that of the RN method. On the other hand, the estimation accuracy of the combination of the proposed method and the selection function $h_K(\zeta)$ is superior to that of the RN method. In case A, the three findings obtained from the base case are observed.

Table 7: Robustness check: Setting of the hypothetical data

		Real world probability matrix F^H	
		Table 2	Table 6
Pricing kernel matrix Φ^H	$\gamma = 3$	Base case	Case B
	$\gamma = 10$	Case A	Case C

- **Case B**

We utilize the hypothetical real world probability from the matrix generated with a local volatility in case B, instead of the matrix generated based on historical data. The pricing kernel matrix is the same as that of the base case. Table 9 shows the KL divergences for the three selection criteria of ζ. The combination of the proposed method and the selection function $h_K(\zeta)$ can estimate a more accurate solution than the others in most cases, but the estimation accuracy of this combination is worse than that of the RN method for $\sigma = 5\%$. However, it is expected that a more accurate solution can be derived by improving the selection function because the minimum KL divergence of the proposed method is lower than the KL divergence of the RN method.

- **Case C**

In case C, we use the matrix Φ^H where $\gamma = 10$ and the matrix F^H showed in Table 6 as hypothetical data. Table 10 shows the KL divergences for the three selection criteria of ζ. The combination of the proposed method and the selection function $h_K(\zeta)$ gives a more accurate solution than the RN method and the combination of the Tikhonov method and the selection function $h_A(\zeta)$.

Why can the proposed method give an accurate estimator under the hypothetical data, regardless of the inappropriate prior information of the proposed method? This is because the first term of the objective function (19) weighs more heavily than the second term associated with the prior information if the proposed method cannot use the appropriate prior information to derive the solution stably. Therefore, we can derive the accurate estimator. Even in this case, it is important to involve the appropriate regularization term considering the characteristics of the Recovery Theorem because the estimation accuracy of the proposed method is superior to that of the Tikhonov method.

We cannot fully show the robustness of the results because of the limited hypothetical datasets. However, we obtain almost the same results as the base analysis even if we use the hypothetical data which does not reflect the prior information of the proposed method. It is expected that we will get similar results in other cases. The further analysis is our future work.

5. Conclusion

The Recovery Theorem of Ross [21] enables us to estimate the real world distribution from the risk neutral distribution. However, it is not easy to derive an appropriate estimator because there is an ill-posed problem in the estimation process. We proposed a new method to derive the appropriate estimator by formulating the regularization term involving the prior information. The estimated real world distribution of the proposed method approaches the risk neutral distribution by increasing the regularization parameter, and therefore the solution can be stable. It is important to interpret the relationship between the regularization parameter and the estimator clearly from a practical perspective when we utilize the estimation method.

Table 8: Case A: Common logarithm of KL divergences ($\log_{10} D_{KL}$) for each selection criteria of regularization parameters

Volatility of noise	Selection criteria of ζ	RN	RW Basic	RW Proposed		RW Tikhonov	
$\sigma = 1\%$	min $h_K(\zeta)$	-0.95	-0.81	-1.35	(-1.46)	-0.65	(-0.14)
	min $h_A(\zeta)$			-1.55	(-5.06)	-0.81	(-7.39)
	min KL			-1.98	(-4.47)	-1.52	(-4.88)
$\sigma = 5\%$	min $h_K(\zeta)$	-0.96	-0.76	-1.38	(-1.72)	-0.71	(-0.38)
	min $h_A(\zeta)$			-0.97	(-4.14)	-0.78	(-5.32)
	min KL			-1.53	(-3.06)	-1.04	(-3.39)

∗ The common logarithm of $\zeta(\log_{10} \zeta)$ selected by each criteria is in the parenthesis.

Table 9: Case B: Common logarithm of KL divergences ($\log_{10} D_{KL}$) for each selection criteria of regularization parameters

Volatility of noise	Selection criteria of ζ	RN	RW Basic	RW Proposed		RW Tikhonov	
$\sigma = 1\%$	min $h_K(\zeta)$	-2.45	-0.20	-2.50	(-3.06)	-1.78	(-0.97)
	min $h_A(\zeta)$			-0.26	(-5.73)	-0.21	(-6.89)
	min KL			-4.13	(-0.48)	-2.47	(-3.73)
$\sigma = 5\%$	min $h_K(\zeta)$	-2.49	-0.19	-0.93	(-3.19)	-1.72	(-1.15)
	min $h_A(\zeta)$			-0.30	(-4.39)	-0.19	(-8.50)
	min KL			-2.90	(-0.38)	-2.07	(-2.52)

∗ The common logarithm of $\zeta(\log_{10} \zeta)$ selected by each criteria is in the parenthesis.

Table 10: Case C: Common logarithm of KL divergences ($\log 10DKL$) for each selection criteria of regularization parameters

Volatility of noise	Selection criteria of ζ	RN	RW Basic	RW Proposed		RW Tikhonov	
$\sigma = 1\%$	min $h_K(\zeta)$	-1.38	-0.19	-2.53	(-1.98)	-1.32	(-0.41)
	min $h_A(\zeta)$			-0.26	(-5.39)	-0.20	(-7.45)
	min KL			-2.54	(-1.82)	-1.32	(-0.38)
$\sigma = 5\%$	min $h_K(\zeta)$	-1.42	-0.19	-1.89	(-2.13)	-1.36	(-0.48)
	min $h_A(\zeta)$			-0.24	(-4.14)	-0.19	(-11.96)
	min KL			-2.30	(-1.25)	-1.36	(-0.48)

∗ The common logarithm of $\zeta(\log_{10} \zeta)$ selected by each criteria is in the parenthesis.

We compared the estimation accuracy of the proposed method with that of the Tikhonov method in our numerical analysis. From the result, we found the following four points:
(1) The divergence of the distribution estimated by the method of Ross [21] from true distribution becomes larger than that of the risk neutral distribution due to the numerical instabilities.
(2) Stabilizing the solution by introducing a regularization term increases the estimation accuracy.
(3) The proposed method can estimate a real world distribution more accurately than the Tikhonov method.
(4) Our criteria for selecting a regularization parameter offers the solution whose divergence from the true distribution is smaller than that of the risk neutral distribution in most cases.

Flint and Mare [11] implemented our proposed method based on the working paper version of this study. They estimate the real world distribution with our proposed method from South African stock futures options and examine the investment performance of a simple investment strategy of Audrino et al. [1]. They show that the investment performance of the real world distribution estimated with our proposed method is superior to that of the risk neutral distribution. However, they do not conduct the statistical test.

Our future work is to estimate the real world distribution from the market option price, and examine the investment performance and the empirical effectiveness of the proposed method by statistical test. Zdorovenin and Pézier [25] and Kiriu and Hibiki [14] derive the real world distribution with risk adjustment which uses historical data and compare the investment performance of the real world distribution with that of the risk neutral distribution. Forward looking risk adjustment using the Recovery Theorem will enhance the investment performance of backward looking risk adjustment using historical data.

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