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ON FUNDAMENTAL OPERATIONS FOR MULTIMODULAR FUNCTIONS

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Abstract Multimodular functions, primarily used in the literature of queueing theory, discrete-event systems, and operations research, constitute a fundamental function class in discrete convex analysis. The objective of this paper is to clarify the properties of multimodular functions with respect to fundamental operations such as permutation and scaling of variables, projection (partial minimization) and convolution. It is shown, in particular, that the class of multimodular functions is stable under projection under a certain natural condition on the variables to be minimized, and the convolution of two multimodular functions is not necessarily multimodular, even in the special case of the convolution of a multimodular function with a separable convex function.

Keywords: Discrete optimization, discrete convex analysis, multimodular function, L-convex function, projection, infimal convolution

1. Introduction

Multimodular functions, due to Hajek [6], have been used as a fundamental tool in the literature of queueing theory, discrete-event systems, and operations research [1–3, 5, 8, 9, 20–23]. In connection to discrete convex analysis [4, 13, 14, 18], multimodularity can be regarded as a variant of L$^\natural$-convexity in the sense that a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is multimodular if and only if it can be represented as $f(x) = g(x_1,\ x_1 + x_2,\ \ldots,\ x_1 + \cdots + x_n)$ for some L$^\natural$-convex function g [15].

Various operations can be defined for discrete functions $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$. With changes of variables we can define operations such as an origin shift $f(x) \mapsto f(x + b)$, a sign inversion of variables $f(x) \mapsto f(-x)$, a permutation of variables $f(x) \mapsto f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$, and a scaling of variables $f(x) \mapsto f(sx)$ with a positive integer s. With arithmetic or numerical operations on function values we can define nonnegative multiplication of function values $f(x) \mapsto af(x)$ with $a \geq 0$, addition of a linear function $f(x) \mapsto f(x) + \sum_{i=1}^{n} c_i x_i$ with $c \in \mathbb{R}^n$, projection* (partial minimization) $f(x) \mapsto \inf_z f(y, z)$, sum $f_1 + f_2$ of two functions f_1 and f_2, convolution $(f_1 \,\square\, f_2)(x) = \inf\{f_1(y) + f_2(z) \mid x = y + z,\ y, z \in \mathbb{Z}^n\}$ of two functions f_1 and f_2, etc.

Stability of discrete convexity under these operations has been investigated for many function classes in discrete convex analysis, such as L$^\natural$-convex functions, M$^\natural$-convex functions, and integrally convex functions [7, 10–12, 14, 19]. For multimodular functions, however, no systematic study has been made, though there are results and observations scattered in the literature.

The objective of this paper is to investigate fundamental operations for multimodular functions with particular interest in their connection to those for L$^\natural$-convex functions. By compiling known and new results we shall arrive at a complete comparison of various kinds of discrete convexity with respect to fundamental operations, as presented in Table 1 at the end of the paper.

This paper is organized as follows. Section 2 is a review of relevant results on multimodular functions. Section 3 deals with operations defined by changes of variables and Section 4 treats operations defined by arithmetic or numerical operations on functions values, such as restriction,

*Here $x = (y, z)$ up to a permutation of components. See (4.8) in Section 4.3 for the precise meaning of the notation.

projection, and convolution. In Section 5 we conclude the paper with a table to compare the major classes of discrete convex functions.

2. Multimodular Functions

We consider functions defined on integer lattice points, $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and the function may possibly take $+\infty$. The *effective domain* of f means the set of x with $f(x) < +\infty$ and is denoted by $\mathrm{dom}\, f = \{x \in \mathbb{Z}^n \mid f(x) < +\infty\}$.

A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is said to be *submodular* if it satisfies

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$$

for all $x, y \in \mathbb{Z}^n$, where $x \vee y$ and $x \wedge y$ denote, respectively, the vectors of componentwise maximum and minimum of x and y, i.e.,

$$(x \vee y)_i = \max(x_i, y_i), \quad (x \wedge y)_i = \min(x_i, y_i) \qquad (i = 1, 2, \ldots, n).$$

Let e^i denote the ith unit vector for $i = 1, 2, \ldots, n$, and $\mathcal{F} \subseteq \mathbb{Z}^n$ be the set of vectors defined by

$$\mathcal{F} = \{-e^1, e^1 - e^2, e^2 - e^3, \ldots, e^{n-1} - e^n, e^n\}. \tag{2.1}$$

A finite-valued function $f : \mathbb{Z}^n \to \mathbb{R}$ is said to be *multimodular* if it satisfies

$$f(z + d) + f(z + d') \geq f(z) + f(z + d + d') \tag{2.2}$$

for all $z \in \mathrm{dom}\, f$ and all distinct $d, d' \in \mathcal{F}$ [1, 6]. It is known [6, Proposition 2.2] that $f : \mathbb{Z}^n \to \mathbb{R}$ is multimodular if and only if the function $\tilde{f} : \mathbb{Z}^{n+1} \to \mathbb{R}$ defined by

$$\tilde{f}(x_0, x) = f(x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}) \qquad (x_0 \in \mathbb{Z}, x \in \mathbb{Z}^n) \tag{2.3}$$

is submodular in $n + 1$ variables. This characterization enables us to define multimodularity for a function that may take the infinite value $+\infty$. That is, we say that a function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ with $\mathrm{dom}\, f \neq \emptyset$ is multimodular if the function $\tilde{f} : \mathbb{Z}^{n+1} \to \overline{\mathbb{R}}$ associated with f by (2.3) is submodular.

A function $g : \mathbb{Z}^n \to \overline{\mathbb{R}}$ with $\mathrm{dom}\, g \neq \emptyset$ is said to be *L*$^\natural$*-convex*† if it has the property called "discrete midpoint convexity," i.e., if it satisfies

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) \tag{2.4}$$

for all $p, q \in \mathbb{Z}^n$, where, for $z \in \mathbb{R}$ in general, $\lceil z \rceil$ denotes the smallest integer not smaller than z (rounding-up to the nearest integer) and $\lfloor z \rfloor$ the largest integer not larger than z (rounding-down to the nearest integer), and this operation is extended to a vector by componentwise applications. It is known [14] that $g : \mathbb{Z}^n \to \overline{\mathbb{R}}$ with $\mathrm{dom}\, g \neq \emptyset$ is L$^\natural$-convex if and only if the function $\tilde{g} : \mathbb{Z}^{n+1} \to \overline{\mathbb{R}}$ defined by

$$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1}) \qquad (p_0 \in \mathbb{Z}, p \in \mathbb{Z}^n) \tag{2.5}$$

is submodular in $n+1$ variables, where $\mathbf{1} = (1, 1, \ldots, 1)$. A function $h(q_0, q_1, \ldots, q_n)$ with $\mathrm{dom}\, h \neq \emptyset$ is called L-*convex* if it is submodular on $\mathbb{Z}^{n+1}$ and there exists $r \in \mathbb{R}$ such that

$$h(q + \mathbf{1}) = h(q) + r \tag{2.6}$$

for all $q = (q_0, q_1, \ldots, q_n) \in \mathbb{Z}^{n+1}$. If h is L-convex, the function $h(0, q_1, \ldots, q_n)$ is L$^\natural$-convex, and any L$^\natural$-convex function arises in this way. The function $\tilde{g}$ in (2.5) derived from an L$^\natural$-convex function g is an L-convex function, and we have $g(p) = \tilde{g}(0, p)$.

Multimodularity and L$^\natural$-convexity have the following close relationship.

†"L$^\natural$-convex" should be read "ell natural convex."

Theorem 1 ([15, 17])**.** A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is multimodular if and only if the function $g : \mathbb{Z}^n \to \overline{\mathbb{R}}$ defined by

$$g(p) = f(p_1,\ p_2 - p_1,\ p_3 - p_2, \ldots, p_n - p_{n-1}) \qquad (p \in \mathbb{Z}^n) \tag{2.7}$$

is L$^\natural$-convex.

Proof. By definition, the multimodularity of f is equivalent to the submodularity of $\tilde{f}$ in (2.3). Since $\tilde{f}$ satisfies (2.6) for $r = 0$, the submodularity of $\tilde{f}$ is equivalent to the L-convexity of $\tilde{f}$. On the other hand, since $\tilde{f}(p_0, p) = g(p - p_0\mathbf{1})$, the L-convexity of $\tilde{f}$ is equivalent to the L$^\natural$-convexity of g (cf., (2.5)). □

Note that the relation (2.7) between f and g can be rewritten as

$$f(x) = g(x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3, \ldots, x_1 + \cdots + x_n) \qquad (x \in \mathbb{Z}^n). \tag{2.8}$$

Using a bidiagonal matrix $D = (d_{ij} \mid 1 \le i, j \le n)$ defined by

$$d_{ii} = 1 \quad (i = 1, 2, \ldots, n), \qquad d_{i+1,i} = -1 \quad (i = 1, 2, \ldots, n-1), \tag{2.9}$$

we can express (2.7) and (2.8) more compactly as $g(p) = f(Dp)$ and $f(x) = g(D^{-1}x)$, respectively. The matrix D is unimodular, and its inverse D^{-1} is an integer matrix with $(D^{-1})_{ij} = 1$ for $i \ge j$ and $(D^{-1})_{ij} = 0$ for $i < j$. For $n = 4$, for example, we have

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \qquad D^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Remark 2.1. The *indicator function* of a set $S \subseteq \mathbb{Z}^n$ is the function $\delta_S : \mathbb{Z}^n \to \{0, +\infty\}$ defined by $\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S). \end{cases}$ A set S is called an *L$^\natural$-convex set* if its indicator function δ_S is L$^\natural$-convex. Similarly, let us call a set S a *multimodular set* if its indicator function δ_S is multimodular. A multimodular set S can be represented as $S = \{Dp \mid p \in T\}$ for some L$^\natural$-convex set T, where T is uniquely determined from S as $T = \{D^{-1}x \mid x \in S\}$. It follows from (2.7) that the effective domain of a multimodular function is a multimodular set. ■

Remark 2.2. For functions in two variables, multimodularity is the same as M$^\natural$-convexity. That is, a function $f : \mathbb{Z}^2 \to \overline{\mathbb{R}}$ is multimodular if and only if it is M$^\natural$-convex. This fact follows easily from the definition or from Theorem 1 and the relation between L$^\natural$-convex and M$^\natural$-convex functions for $n = 2$. See [14] for the definition of M$^\natural$-convex functions. ■

A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ in $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ is called *separable* (*discrete*) *convex* if it can be represented as $f(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n)$ with univariate functions $\varphi_i : \mathbb{Z} \to \overline{\mathbb{R}}$ satisfying $\varphi_i(t-1) + \varphi_i(t+1) \ge 2\varphi_i(t)$ for all $t \in \mathbb{Z}$.

Proposition 2. A separable convex function is multimodular.

Proof. For $f(x) = \sum_{i=1}^n \varphi_i(x_i)$ the function g in (2.7) is given as $g(p) = \varphi_1(p_1) + \sum_{i=2}^n \varphi_i(p_i - p_{i-1})$. It is known [14, 18] that such function is L$^\natural$-convex. □

A quadratic function admits a simple characterization of multimodularity in terms of its coefficient matrix.

Proposition 3. A quadratic function $f(x) = x^\top Ax$ is multimodular if and only if

$$a_{ij} - a_{i,j+1} - a_{i+1,j} + a_{i+1,j+1} \le 0 \qquad (0 \le i < j \le n), \tag{2.10}$$

where $A = (a_{ij} \mid i, j = 1, 2, \ldots, n)$ and $a_{ij} = 0$ if $i = 0$ or $j = n + 1$.

Proof. The inequality (2.2) for $d = e^i - e^{i+1}$ and $d' = e^j - e^{j+1}$, where $e^0 = e^{n+1} = \mathbf{0}$ by convention, is equivalent to $(e^i - e^{i+1})^\top A(e^j - e^{j+1}) \leq 0$. This is further equivalent to (2.10). □

Remark 2.3. Here is an alternative proof of Proposition 3 via L$^\natural$-convexity. Let $\mathcal{L}$ denote the set of all $n \times n$ symmetric matrices $B = (b_{ij})$ such that $b_{ij} \leq 0$ for all $i \neq j$ and $b_{ii} \geq \sum_{j \neq i} |b_{ij}|$ for all i. It is known [14] that $g(p) = p^\top Bp$ is L$^\natural$-convex if and only if B belongs to $\mathcal{L}$. Then, by Theorem 1, $f(x) = x^\top Ax$ is multimodular if and only if $D^\top AD$ belongs to $\mathcal{L}$. This latter condition is equivalent to (2.10). ■

The following nice properties of multimodular functions are worth mentioning, though we do not use them in this paper.

- An integer point $x \in \mathrm{dom}\, f$ is a (global) minimizer of a multimodular function f if and only if it is a local minimizer in the sense that $f(x) \leq f(x \pm d)$ for all $d \in \mathcal{T}$, where $\mathcal{T}$ is the set of vectors of the form $e^{i_1} - e^{i_2} + \cdots + (-1)^{k-1}e^{i_k}$ for some increasing sequence of indices $i_1 < i_2 < \cdots < i_k$ [15, Theorem 3.1].
- A multimodular function f can be extended to a convex function in a specific manner [1, Theorem 2.1]. Furthermore, a multimodular function is integrally convex [17, Section 14.6]; see [14] for the definition of integrally convex functions.
- A discrete separation theorem holds for multimodular functions [15, Theorem 4.1]. Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be functions such that f and $-g$ are multimodular, and assume that $f(x_0)$ and $g(x_0)$ are finite for some $x_0 \in \mathbb{Z}^n$. If $f(x) \geq g(x)$ for all $x \in \mathbb{Z}^n$, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^n$ such that $f(x) \geq \alpha^* + \langle p^*, x\rangle \geq g(x)$ for all $x \in \mathbb{Z}^n$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of vectors. Moreover, if f and g are integer-valued, there exist integer-valued $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbb{Z}^n$.

3. Operations via Change of Variables

In this section we consider multimodularity of functions induced by changes of variables such as an origin shift, a sign inversion of variables, a permutation of variables, and a scaling of variables. We consistently adopt the proof strategy to translate the operations for multimodular functions to those for L$^\natural$-convex functions, so that we can better understand the connection between multimodularity and L$^\natural$-convexity. In the proofs we use notations f for a given multimodular function, $\tilde{f}$ for the function resulting from the operation, and

$$g(p) = f(p_1,\ p_2 - p_1,\ p_3 - p_2, \ldots, p_n - p_{n-1}) = f(Dp), \tag{3.1}$$

$$\tilde{g}(p) = \tilde{f}(p_1,\ p_2 - p_1,\ p_3 - p_2, \ldots, p_n - p_{n-1}) = \tilde{f}(Dp), \tag{3.2}$$

which imply

$$f(x) = g(x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3, \ldots, x_1 + \cdots + x_n) = g(D^{-1}x), \tag{3.3}$$

$$\tilde{f}(x) = \tilde{g}(x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3, \ldots, x_1 + \cdots + x_n) = \tilde{g}(D^{-1}x). \tag{3.4}$$

We start with an origin shift and a sign inversion of variables.

Proposition 4. For a multimodular function f and an integer vector b, the function $\tilde{f}(x) = f(x+b)$ is multimodular.

Proof. By (3.3) and (3.4), we can translate $\tilde{f}(x) = f(x + b)$ to $\tilde{g}(p) = g(p + c)$ with $c = (b_1, b_1 + b_2, b_1+b_2+b_3, \ldots, b_1+\cdots+b_n)$, where g is L$^\natural$-convex. Then $\tilde{g}$ is also L$^\natural$-convex, since L$^\natural$-convexity is stable under an origin shift. □

Proposition 5. For a multimodular function f, the function $\tilde{f}(x) = f(-x)$ is multimodular.

Proof. By (3.3) and (3.4), we can translate $\tilde{f}(x) = f(-x)$ to $\tilde{g}(p) = g(-p)$, where g is L$^\natural$-convex. Then $\tilde{g}$ is also L$^\natural$-convex, since L$^\natural$-convexity is stable under a sign inversion of variables. □

It is known that reversing the ordering of variables preserves multimodularity [6, Remarks (1)]. It is emphasized that this is not obvious since the definition of multimodularity depends on the ordering of variables.

Proposition 6 ([6])**.** For a multimodular function f, the function $\tilde{f}$ defined by $\tilde{f}(x_1, x_2, \ldots, x_n) = f(x_n, \ldots, x_2, x_1)$ is multimodular.

Proof. We give an alternative proof via L$^\natural$-convexity in accordance with our strategy. Let $R = (r_{ij})$ denote the permutation matrix representing the reversal of the ordering, i.e., $r_{i,n+1-i} = 1$ for $i = 1, 2, \ldots, n$ and other entries being zero. Then we have $\tilde{f}(x) = f(Rx)$. By (3.3) and (3.4), we can translate $\tilde{f}(x) = f(Rx)$ to $\tilde{g}(D^{-1}x) = g(D^{-1}Rx)$, that is, $\tilde{g}(p) = g(D^{-1}RDp)$. A direct calculation shows that the matrix $T = (t_{ij}) = D^{-1}RD$ is given by: $t_{in} = 1$ $(i = 1, 2, \ldots, n)$, $t_{i,n-i} = -1$ $(i = 1, 2, \ldots, n-1)$, and $t_{ij} = 0$ for other (i, j). For $n = 4$, for example, we have $T = D^{-1}RD = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then we obtain[‡]

$$\tilde{g}(p) = g(-(p_{n-1}, p_{n-2}, \ldots, p_1, 0) + p_n\mathbf{1}). \tag{3.5}$$

The L$^\natural$-convexity of $\tilde{g}$ can be seen as follows. Define $h : \mathbb{Z}^{n+1} \to \overline{\mathbb{R}}$ by

$$h(p_0, p_1, p_2, \ldots, p_n) = g(-(p_{n-1}, p_{n-2}, \ldots, p_1, p_0) + p_n\mathbf{1})$$

and $g^{\mathrm{rev}} : \mathbb{Z}^n \to \overline{\mathbb{R}}$ by

$$g^{\mathrm{rev}}(p_0, p_1, \ldots, p_{n-2}, p_{n-1}) = g(-p_{n-1}, -p_{n-2}, \ldots, -p_1, -p_0).$$

The function h is L-convex, since g^{rev} is L$^\natural$-convex and the function derived from g^{rev} by (2.5) coincides with h. Then the relation $\tilde{g}(p) = h(0, p_1, p_2, \ldots, p_n)$ in (3.5) means that $\tilde{g}$ is obtained from an L-convex function by restriction. Therefore, $\tilde{g}$ is L$^\natural$-convex. □

Not every permutation of variables preserves multimodularity.

Example 3.1. The quadratic function $f(x) = x^\top Ax$ with $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is multimodular, whereas $\tilde{f}(x_1, x_2, x_3) = f(x_2, x_1, x_3)$ arising from a transposition is not multimodular. Indeed we have $\tilde{f}(x) = x^\top \tilde{A}x$ for $\tilde{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, for which the condition (2.10) fails for $(i, j) = (1, 3)$. Referring to Remark 2.3 we also note that $B = D^\top AD = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \in \mathcal{L}$ and $\tilde{B} = D^\top \tilde{A}D = \begin{bmatrix} 1 & -1 & \boxed{1} \\ -1 & 2 & -1 \\ \boxed{1} & -1 & 1 \end{bmatrix} \notin \mathcal{L}$. A cyclic permutation of variables $f(x_3, x_1, x_2)$ is not multimodular, either, since it coincides with $x^\top \tilde{A}x$. ■

A scaling of variables preserves multimodularity.

[‡]It is somewhat surprising that the order reversal of variables corresponds to the transformation (3.5) for L$^\natural$-convex functions.

Proposition 7. For a multimodular function f and a positive integer s, the function $\tilde{f}(x) = f(sx)$ is multimodular.

Proof. By (3.3) and (3.4), we can translate $\tilde{f}(x) = f(sx)$ to $\tilde{g}(p) = g(sp)$, where g is L$^\natural$-convex. Then $\tilde{g}$ is also L$^\natural$-convex, since L$^\natural$-convexity is stable under a scaling of variables [14]. □

4. Operations Relating to Function Values

In this section we consider multimodularity of functions resulting from operations such as nonnegative multiplication of function values, addition of a linear function, projection (partial minimization), sum of two functions, and convolution of two functions. We continue with the proof strategy of translating the operations for multimodular functions to those for L$^\natural$-convex functions.

4.1. Multiplication and addition

We start with simple operations, for which the following statements are obvious.

Proposition 8 ([1])**.** Let f, f_1, f_2 be multimodular functions.
(1) For any $a \geq 0$, $\tilde{f}(x) = af(x)$ is multimodular.
(2) For any $c \in \mathbb{R}^n$, $\tilde{f}(x) = f(x) + \sum_{i=1}^{n} c_i x_i$ is multimodular.
(3) For any separable convex function $\varphi(x)$, $\tilde{f}(x) = f(x) + \varphi(x)$ is multimodular.
(4) Sum $\tilde{f}(x) = f_1(x) + f_2(x)$ is multimodular. ■

4.2. Restriction

Let $N = \{1, 2, \ldots, n\}$. For a function $f : \mathbb{Z}^N \to \overline{\mathbb{R}}$ and a subset $U \subseteq N$, the *restriction* of f to U is a function $f_U : \mathbb{Z}^U \to \overline{\mathbb{R}}$ defined by[§]

$$f_U(y) \quad = \quad f(y, \mathbf{0}_{N \setminus U}) \qquad (y \in \mathbb{Z}^U), \tag{4.1}$$

where $\mathbf{0}_{N\setminus U}$ denotes the zero vector in $\mathbb{Z}^{N\setminus U}$. The notation $(y, \mathbf{0}_{N\setminus U})$ means the vector whose ith component is equal to y_i for $i \in U$ and to 0 for $i \in N \setminus U$; for example, if $N = \{1, 2, 3\}$ and $U = \{1, 3\}$, $(y, \mathbf{0}_{N\setminus U})$ means $(y_1, 0, y_3)$.

The restriction of a multimodular function is known to be multimodular [1, Lemma 2.3] (see also [2, Lemma 3]).

Proposition 9 ([1])**.** For a multimodular function f and any subset U, the restriction f_U is multimodular, provided that $\mathrm{dom}\, f_U \neq \emptyset$.

Proof. We give an alternative proof in accordance with our strategy. It suffices to consider the case where $N \setminus U = \{k\}$ for some $k \in N$. Define $\tilde{f}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) = f(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n)$. Then $\tilde{f}$ is multimodular if and only if the inequality (2.2) holds for f for all $z \in \mathbb{Z}^n$ and all distinct elements d, d' of

$$\tilde{\mathcal{F}} = \mathcal{F} \setminus \{e^{k-1} - e^k, e^k - e^{k+1}\} \cup \{e^{k-1} - e^{k+1}\},$$

where $e^0 = e^{n+1} = \mathbf{0}$. We use notation $\psi(x) = (x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3, \ldots, x_1 + \cdots + x_n)$ for the transformation $x \mapsto p$ in (2.8), i.e., $f(x) = g(\psi(x))$. If $k = 1$, we have

$$\psi(-e^2) = (0, -1, \ldots, -1) = e^1 - \mathbf{1},$$
$$\psi(e^i - e^{i+1}) = e^i \qquad (i \in \{2, \ldots, n\})$$

for the elements of $\tilde{\mathcal{F}}$, and therefore, $\tilde{f}$ is multimodular if and only if

$$g(p + e^i) + g(p + e^j) \geq g(p) + g(p + e^i + e^j), \tag{4.2}$$
$$g(p + e^i) + g(p + e^1 - \mathbf{1}) \geq g(p) + g(p + e^i + e^1 - \mathbf{1}), \tag{4.3}$$

[§]For any $z \in \mathbb{Z}^{N\setminus U}$ we may consider a function $f(y, z)$ in $y \in \mathbb{Z}^U$. For simplicity we choose $z = \mathbf{0}_{N\setminus U}$.

where $i, j \in \{2, \dots, n\}$ and $i \neq j$. If $2 \le k \le n$, we have

$$\begin{aligned} \psi(-e^1) &= (-1, -1, \dots, -1) = -\mathbf{1}, \\ \psi(e^i - e^{i+1}) &= e^i \qquad (i \in \{1, \dots, k-2\} \cup \{k+1, \dots, n\}), \\ \psi(e^{k-1} - e^{k+1}) &= e^{k-1} + e^k \end{aligned}$$

for the elements of $\tilde{\mathcal{F}}$, and therefore, $\tilde{f}$ is multimodular if and only if

$$g(p + e^i) + g(p + e^j) \ge g(p) + g(p + e^i + e^j), \tag{4.4}$$
$$g(p + e^i) + g(p + e^{k-1} + e^k) \ge g(p) + g(p + e^i + e^{k-1} + e^k), \tag{4.5}$$
$$g(p + e^i) + g(p - \mathbf{1}) \ge g(p) + g(p + e^i - \mathbf{1}), \tag{4.6}$$
$$g(p + e^{k-1} + e^k) + g(p - \mathbf{1}) \ge g(p) + g(p + e^{k-1} + e^k - \mathbf{1}), \tag{4.7}$$

where $i, j \in \{1, \dots, k-2\} \cup \{k+1, \dots, n\}$ and $i \neq j$. We finally observe that inequalities (4.2)–(4.7) hold by the discrete midpoint convexity (2.4) of g. □

4.3. Projection

For a function $f : \mathbb{Z}^N \to \overline{\mathbb{R}}$ and a subset $U \subseteq N$, the *projection* of f to U means a function $f^U : \mathbb{Z}^U \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$f^U(y) = \inf\{f(y, z) \mid z \in \mathbb{Z}^{N \setminus U}\} \qquad (y \in \mathbb{Z}^U), \tag{4.8}$$

where the notation (y, z) means the vector whose ith component is equal to y_i for $i \in U$ and to z_i for $i \in N \setminus U$; for example, if $N = \{1, 2, 3, 4\}$ and $U = \{2, 3\}$, $(y, z) = (z_1, y_2, y_3, z_4)$. We assume $f^U > -\infty$. The projection is sometimes called *partial minimization*.

A subset U of $N = \{1, 2, \dots, n\}$ is said to be an *interval* if it consists of consecutive numbers. The projection of a multimodular function to an interval is multimodular.

Proposition 10. For a multimodular function f and an interval U, the projection f^U is multimodular.

Proof. We first consider the case of $U = N \setminus \{n\}$. By (4.8) and (2.8) we obtain

$$\begin{aligned} & f^U(x_1, x_2, \dots, x_{n-1}) \\ &= \inf_{z \in \mathbb{Z}} f(x_1, x_2, \dots, x_{n-1}, z) \\ &= \inf_{z \in \mathbb{Z}} g(x_1, x_1 + x_2, \dots,\ x_1 + \cdots + x_{n-1},\ x_1 + \cdots + x_{n-1} + z) \\ &= g^U(x_1, x_1 + x_2, \dots, x_1 + \cdots + x_{n-1}), \end{aligned}$$

where g^U denotes the projection of g to U. Here g^U is L$^\natural$-convex, since the projection of an L$^\natural$-convex function is known [14, Theorem 7.11] to be L$^\natural$-convex. Therefore, f^U is multimodular.

The case of $U = N \setminus \{1\}$ can be reduced to the above case by Proposition 6, which allows us to reverse the ordering of variables. For a general interval U, we repeat eliminating variables from both ends of $\{1, 2, \dots, n\}$. □

The projection of a multimodular function to an arbitrary subset U is not necessarily multimodular.

Example 4.1. The quadratic function $f(x) = x^\top Ax$ with $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ is multimodular, whereas its projection f^U to $U = \{1,2,4\}$ is not. Indeed we have $f^U(y) = y^\top \tilde{A} y$ for $\tilde{A} = \frac{1}{2}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, where $\tilde{A} = (\tilde{a}_{ij} \mid i,j = 1,2,4)$ is obtained from A by the usual sweep-out operation: $\tilde{a}_{ij} = a_{ij} - a_{i3}a_{3j}/a_{33}$ $(i, j \in \{1,2,4\})$. The matrix $\tilde{A}$ violates the condition (2.10) for $(i, j) = (1,2)$. Referring to Remark 2.3 we also note that $B = D_4^\top A D_4 = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{L}$ and $\tilde{B} = D_3^\top \tilde{A} D_3 = \frac{1}{2}\begin{bmatrix} 3 & \boxed{1} & -1 \\ \boxed{1} & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \notin \mathcal{L}$, where D_4 and D_3 are 4×4 and 3×3 matrices defined as in (2.9). ■

4.4. Convolution

The (infimal) *convolution* of two functions $f_1, f_2 : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is defined by

$$(f_1 \Box f_2)(x) = \inf\{f_1(y) + f_2(z) \mid x = y + z,\ y, z \in \mathbb{Z}^n\} \qquad (x \in \mathbb{Z}^n), \tag{4.9}$$

where it is assumed that the infimum is bounded from below (i.e., $\neq -\infty$) for every $x \in \mathbb{Z}^n$. The *Minkowski sum* of two sets $S_1, S_2 \subseteq \mathbb{Z}^n$ is defined by

$$S_1 + S_2 = \{y + z \mid y \in S_1, z \in S_2\}. \tag{4.10}$$

The indicator function of the Minkowski sum coincides with the convolution of the respective indicator functions, i.e., $\delta_{S_1+S_2} = \delta_{S_1} \Box \delta_{S_1}$.

Example 4.2 below shows the following facts. Recall that a multimodular set means a set whose indicator function is multimodular (Remark 2.1) and that a separable convex function is multimodular (Proposition 2).

- The Minkowski sum of a multimodular set and an integer interval (box) is not necessarily a multimodular set.
- The convolution $f \Box \varphi$ of a multimodular function f and a separable convex function φ is not necessarily a multimodular function.
- The convolution $f_1 \Box f_2$ of two multimodular functions f_1 and f_2 is not necessarily a multimodular function.

Example 4.2. Let $S_1 = \{(0,0,0), (1,0,-1)\}$ and $S_2 = \{(0,0,0), (0,1,0)\}$, where S_2 is an integer interval. Both S_1 and S_2 are multimodular, but their Minkowski sum $S_1 + S_2 = \{(0,0,0), (1,0,-1), (0,1,0), (1,1,-1)\}$ is not multimodular. We can check this directly or via transformation to $T_i = \{D^{-1}x \mid x \in S_i\}$ for $i = 1,2$. We have $T_1 = \{(0,0,0), (1,1,0)\}$ and $T_2 = \{(0,0,0), (0,1,1)\}$, which are easily seen to be L$^\natural$-convex. But their Minkowski sum $T_1 + T_2 = \{(0,0,0), (0,1,1), (1,1,0), (1,2,1)\}$ is not L$^\natural$-convex, since for $p = (0,1,1)$ and $q = (1,1,0)$ in $T_1 + T_2$, we have $\lceil (p+q)/2 \rceil = (1,1,1) \notin T_1 + T_2$ and $\lfloor (p+q)/2 \rfloor = (0,1,0) \notin T_1 + T_2$. Since $T_1 + T_2 = \{D^{-1}x \mid x \in S_1 + S_2\}$, this means that $S_1 + S_2$ is not multimodular. It it mentioned that this example is based on the example for L$^\natural$-convex sets given in [14, Note 5.11] and [19, Example 3.11]. ■

5. Concluding Remarks

Multimodular functions have been used as a fundamental tool to analyze recurrence relations in the literature of queueing theory, discrete-event systems, and operations research. In some analysis, propagation or stability of multimodularity through recurrence formulas plays a critical role.

Table 1: Fundamental operations on discrete convex functions

Discrete convexity	Variables: Permut.	Variables: Scaling	Restriction	Projection	Addition: $f+\varphi$	Addition: f_1+f_2	Convolution: $f\square\varphi$	Convolution: $f_1\square f_2$	Reference
Separable conv	Y	Y	Y	Y	Y	Y	Y	Y	
Integrally conv	Y	***N***	Y	Y	Y	***N***	Y	***N***	[10, 11, 19]
$\mathrm{L}^\natural$-convex	Y	Y	Y	Y	Y	Y	Y	***N***	[14]
L-convex	Y	Y	***N***	Y	Y	Y	Y	***N***	[14]
$\mathrm{M}^\natural$-convex	Y	***N***	Y	Y	Y	***N***	Y	Y	[14]
M-convex	Y	***N***	Y	***N***	Y	***N***	Y	Y	[14]
			Y		Y	Y			[1]
Multimodular	***N***	Y: Prop.7	alt. proof	***N***			***N***	***N***	this paper
	Y*:Prop.6								[6]
	alt. proof			Y*:Prop.10					this paper
Globally d.m.c.	Y	Y	Y	Y	Y	Y	***N***	***N***	[12]
Locally d.m.c.	Y	Y	Y	Y	Y	Y	***N***	***N***	[12]
M-conv (jump)	Y	***N***	Y	Y	Y	***N***	Y	Y	[7, 16]

d.m.c.: discrete midpoint convex, φ: separable convex

Y: Discrete convexity (of that kind) is preserved, ***N***: Not preserved

Y*: Discrete convexity (of that kind) is preserved in some cases

A recurrence formula consists of various kinds of operations, some of which preserve multimodularity and others not. The projection operation (partial minimization) is closely related to the Bellman equation in dynamic programming, and the assumption of U being an interval (consecutive variables) in Proposition 10 is quite natural in this interpretation. The reversal of the ordering of variables in Proposition 6 corresponds to the reversal of "time" in recurrence relations. It is hoped that the results of this paper will find applications in concrete problems in operations research.

The known facts about fundamental operations on discrete convex functions, including those obtained in this paper, are summarized in Table 1.

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