DYNAMIC INVENTORY CONTROL MODEL WITH FLEXIBLE SUPPLY NETWORK

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Abstract   In this paper, we consider the inventory problem of a firm facing disruption probability in its supply chain which consists of multiple suppliers and multiple demand nodes. The firm wishes to minimize its total expected cost in a finite time horizon setting. In order to manage the supply chain disruption, we introduce flexibility into the supply chain network of our inventory management problem. The problem is formulated as a Markov decision process, and a state-dependent optimal threshold policy is derived. We show that the expected cost function is monotonic in the convex ordering of the demand distribution and that the optimal policy can be characterized with the ratio of the ordering cost and the disruption probability of supply. We also numerically demonstrate that the flexibility of the supply chain network reduces the total expected cost.

1. Introduction

The management of supply chain disruption has drawn attention as firms have become more interested in diversifying their production facilities via outsourcing. Such work has been reviewed in Tang [14] and Synder [13]. This paper focuses on the disruption in the supply chain, and models this disruption by considering a state dependent supply network and a state dependent demand distribution whose state follows a Markov chain. The importance of properly managing such disruptions as well as the various sources that potentially affect the supply chain has been discussed in Tang [15]. Among the various sources, we focus on the disruptions that occur due to "complete failure of transportation from a facility" and/or "complete loss of production of a facility". There are many ways to deal with such a disruption, and we attempt to create resilience in the supply chain by considering flexibility in the transportation network of products rather than altering the production capacity, supply and/or demand of final goods*.

The proper management of transportation failure risks when the demand distribution is stochastic, does not only prevent disruptions in the downstream of the supply chain, but may also increase the overall efficiency of production at the firm level as well as the industry level. In short, this paper attempts to manage the supply chain disruption probability by introducing flexibility in the supply transportation network instead of optimizing production capacity, etc., when there is a disruption in the delivery process of products. Because we only consider a complete shutdown from a particular facility, the theoretical treatment of failure in the transportation and production become indistinguishable in our case.

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*In such a situation, the yield uncertainty does not become a major issue and will not be considered in this paper.

The practical importance of managing transportation disruption probability in the supply chain, has drawn increasing attention in Japan, especially after the 2011 East Japan Earthquake. Many domestic corporations have become aware of the importance and have become interested in diversifying their production facilities in order to accommodate the product supply via constructing a flexible supply network and/or allocating multiple potential suppliers. In the inventory management literature, the supply chain disruption probability may be categorized as a partial disruption probability defined as the lack of supply due to human error or product defects. However, the disruption probability considered in this paper is defined as the complete loss of delivery of products due to loss of transportation infrastructure and/or strike.

2. Literature Review

Many papers have analyzed the supply chain and its disruption probability. Mieghem [16] deals with the production risk by altering the demand via adjusting the price of final goods to be delivered. Jordan and Graves [6] discuss about the process flexibility and how to add production capacity when facing potential production failure. When potential failure may be correlated, Sak and Haksöz [11] deal with this phenomenon by using delivery contracts, but we do not use such contracts to mitigate risk[†]. The proposed model in this paper differs from these paper in the sense that it dose not try to achieve the total cost minimization by either altering the supply production capacity and/or adjusting the price so that the demand meets the supply. The risk originating from the disruption is mitigated by rerouting the excess supply production capacity created by the transportation disruption. Thus, there is no incentive to share or hedge such risk using some kind of supply contract[‡].

Tang [15] provides a list of various risks that may affect the supply chain, and among these risks, this paper is most related to the transportation disruption. The paper contributes to the literature in the sense that it fills in the gap of supply chain disruption probability by focusing on the transportation flexibility and/or complete loss of production at a particular facility. The originality lies in the way we model the transportation network, which is considered to depend on a state variable that follows a Markov process with a finite time horizon. This allows us to analyze the transient policy instead of the steady state policy, which is more valuable in practice. In addition, the model does not consider production capacity that is dedicated to fulfill the demand shortfall (backup capacity). Instead, we may use any production facility to serve as a backup supplier, which may be useful in practical applications. By using transportation flexibility to deal with such supply chain disruptions, we derive an easy to implement risk adjusted cost index that allows us to construct a policy that introduces flexibility in a supply chain facing disruption probability. This index shares some features with the GBE index proposed in Saghafian and van Oyen [10]. The the cost and disruption probability term become the same when (i) the demand at each point are the same, (ii) a single backup supplier only supplies products to demand points. In other words, there are no the main suppliers. (iii) there is no limitation of backup capacity. Unlike Saghafian and van Oyen [10], the production capacity is not filled in by a backup supplier, but rather satisfied by reallocating the excess capacity generated by the failure in the transportation network. Thus, the policy can be easily implemented by altering the

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[†]If the failure of several transportation arcs are correlated, it is theoretically possible to deal with such models in our proposed framework by orthogonalizing (create uncorrelated supply nodes) the supply node arcs.

[‡]If such a contract exists, this risk does not need to be dealt with and the hedged capacity may be simply removed from the original network in our model.

transportation network. In addition, because the transportation network itself is assumed to be state dependent, the policy can be solved in a dynamic setting (transient state), in comparison to the steady state behavior derived in Saghafian and van Oyen [10]. The model also provides a solution (threshold policy) on how to introduce transportation flexibility in the supply chain network rather than attempting to fill in the shortfall of demand through backup production capacity (or altering the final demand to match the insufficient supply).

Gallego and Hu [2] focus on the case when there are partial disruptions in supply, and the demand follows a distribution that depends on a finite time horizon Markov chain. They derive an optimal order policy in a multiple period inventory model exposed to Markov modulated demand/supply with finite capacity. Schmitt and Snyder [12] consider the case when there is a partial and complete disruption in the supply chain, and consider the impact of adding a backup supplier to increase the flexibility in the supply chain. This paper studies the case when there are multiple suppliers and one firm that needs multiple products when there are only complete disruptions in supply. We also consider a backup supplier and analyze whether to supply from this backup supplier or not as well as the quantity of the inventory to be ordered. This model enables us not only to deal with the backup supplier but also consider a complete disruption in the supply chain by setting up multiple supply routes, which differs from Gallego and Hu [2] and Schmitt and Snyder [12].

In comparison, we consider potential backup suppliers (main suppliers) and analyze whether to supply from these backup suppliers or not, and derive the optimal quantity to be ordered. The proposed model enables us not only to deal with potential backup suppliers but also consider complete disruptions in the supply chain. Such disruption is dealt by setting up multiple supply routes.

We consider a supply chain constituting two stages with multiple supply nodes and multiple demand nodes, this allows us to formulate the probabilistic inventory model as a Markov decision process. In this model, we consider the case where the demand distribution is affected by exogenous factors such as weather hazards and/or business conditions. In the following, we derive an optimal policy under each risk factor that minimizes the total cost in a multi-period framework.

The originality of the paper lies in the way we model the supply chain disruption. The disruption is considered to occur in the delivery process which is dealt with creating a flexible procurement network. Thus, the decision is to choose the optimal supply chain network given the failure (total disruption) in the network. The uncertainty is in the demand of the product which is shown to be monotone in the stochastic order of demand. This implies a unique switching threshold policy that can be easily implemented according to the state of the network when the demand uncertainty increases.

The paper is structured as the following. Section 3 introduces the notation and definitions. Section 4 and 5 explain the single and multiperiod version of the model, followed by the conclusion in Section 6.

3. Notation and Definitions

The time horizon of the problem is set to be T where each time period t satisfies $t \in [1, \cdots, T]$. There are K supply nodes which deliver L goods to the demand nodes via the supply chain network. The demand process of each node and the supply chain network can be formulated as a finite Markov chain $X(t)$, $t \geq 0$. The state space of the Markov chain has N states such that the state space S can be defined as $S = \{1, 2, \cdots N\}$.

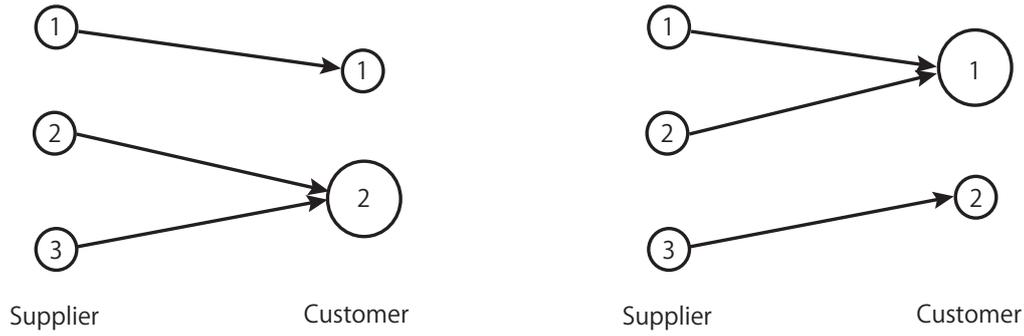This formulation follows that of Saghafian and van Oyen [10] and Gallego and Hu [2],

Figure 1: Supply chain network for state 1. The demand of node 2 is larger than node 1

Figure 2: Supply chain network for state 2. The demand of node 1 is larger than node 2
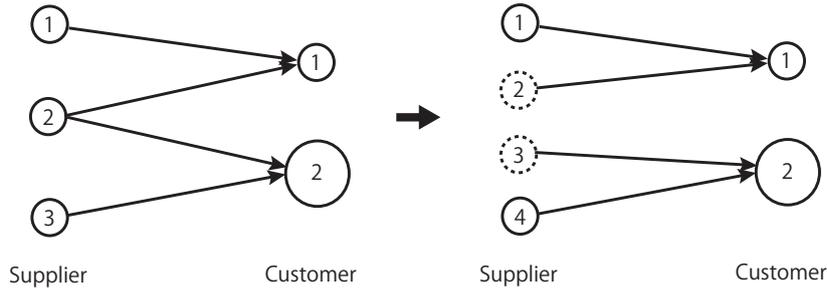

Figure 3: Splitting supply nodes

which further allows us to analyze the problem as a Markov decision problem[§].

    Given the supply chain network structure, we assume the following decision structure at each time period.

1. The state $X(t) = i$ is observed at the beginning of each time period.

2. The quantity of goods from each supply node to each demand node is determined by the supply chain network at state i and the demand node distribution. We assume that there is no lead time in production[¶].

3. At the end of each time period, the demand quantity is determined and the inventory is consumed accordingly, and the unsatisfied demand is backlogged.

In the following, we illustrate the notation using an example in Figure 1 and Figure 2 which depicts the case for $K = 3$ and $L = 2$. The state $i = 1$ corresponds to the case when the demand at node 2 is expected to be larger than node 1. In this situation, goods are shipped to demand node 2 from supply node 2 and 3. However, if the state is $i = 2$, then the demand for node 1 is fulfilled by supply node 1 and 2.

    The demand at demand node l in state i is a non-negative random variable $D_l(i)$ with a cumulative distribution function $F_{i,l}(\cdot)$. The supply chain network matrix in state i is defined as $\boldsymbol{C}(i) \in \mathbb{R}^{K \times L}$ where $P_{ij}(t)$ is the transition probability from state i to j with $i, j \in S$. The element of $\boldsymbol{C}(i)$ is $C_{kl}(i)$, which satisfies $C_{kl}(i) = 1$ if goods are shipped from supply node k to demand node l, and $C_{kl}(i) = 0$ otherwise, $i = 1, \cdots, N$, $k = 1, \cdots, K$,

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[§]Even though the computational complexity of the problem may increase rapidly as the number of demand/supply nodes and states of the network increases, it is possible to reduce the computational burden by applying approximations proposed in Federgruen and Zipkin [1], Kunnumkai and Topaloglu [7], [8].

[¶]We follow the assumptions in Saghafian and van Oyen [10], which argues that supply lead times and production cycles are negligible in comparison with the review period.

$l = 1, \cdots, L$. Let $\boldsymbol{C}_l(i)$ be the lth vector of supply chain matrix $\boldsymbol{C}(i)$. The supply chain network matrix $\boldsymbol{C}(i)$ for Figure 1 and Figure 2 become the following.

$$\boldsymbol{C}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{C}(2) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In our model, we make two assumptions on the demand and the supply nodes. (i) There are at most two suppliers (main supplier and backup supplier) for each demand node, but each supply node can ship goods to multiple demand nodes. (ii) There is no upper limit on the production capacity. This assumption enables us to alter the supply chain network $C_{kl}(i)$ by splitting the supply node so that each supply node only ships goods to one demand node when the supply nodes need to ship goods to multiple demand nodes, i.e. node 2 in Figure 3 is splitted. The two conditions are satisfied if the following Assumption 3.1 holds.

Assumption 3.1. *For all $k = 1, \cdots, \overline{K}_i$ and $i = 1, \cdots, N$,*

(i) $\overline{\boldsymbol{C}}_k'(i)\boldsymbol{1} \leq 2$.

(ii) $\sum_{l=1}^{L} \overline{C}_{kl}(i) \leq 1$.

$\overline{K}_i(\geq K)$ represents the total number of supply nodes at state i after rearranging the network so it satisfies Assumption 3.1, and $\overline{\boldsymbol{C}}(i) \in \mathbb{R}^{\overline{K}_i \times L}$ is the connection matrix of the network. The value of random variable R_k describes whether supply node k can be potentially supplied or not. If $R_k = 1$, the node can be supplied and R_k is set to 0 otherwise. Then, R_k, $k = 1, ..., K$ becomes an i.i.d. Bernoulli random variable, where q_k is the probability of reliability. Thus, $P(R_k = 1) = q_k$ and $P(R_k = 0) = 1 - q_k$ hold. R_k and $D_l(i)$ are assumed to be independent.

Let the order quantity of each supply node be vector $\boldsymbol{s}^i \in \mathbb{R}^{\overline{K}_i}$, and the production cost $\boldsymbol{\rho}^i \in \mathbb{R}_+^{\overline{K}_i}$. Accordingly, define the inventory level of the demand node as $\boldsymbol{y} \in \mathbb{R}^L$, and the inventory/penalty cost to be $\boldsymbol{h} \in \mathbb{R}_+^L$, $\boldsymbol{p} \in \mathbb{R}_+^L$ respectively. The transportation cost function from supply node k to demand node l is $a_{kl} \geq 0$. Because goods are only shipped to a single demand node for each supply node by Assumption 3.1, we can write $a_{kl} = a_k$. In addition, since the supply node k which supplies goods to two demand nodes, $\boldsymbol{C}_k'(i)\boldsymbol{1} = 2$, is split into node k_1 and k_2, we assume that $q_{k_1} = q_{k_2}$, $\rho_{k_1} = \rho_{k_2}$ and $a_{k_1} = a_{k_2}$.

4. Single Period Model

4.1. Formulation

We first prove the single period case so that we can provide a detailed explanation of the optimal threshold policy using a numerical example. This section formulates the one period ($T = 1$) model, and derives the optimal order policy. If the state is i, the inventory level for the demand node is $\boldsymbol{y}$ and the order amount is $\boldsymbol{s}$ at the beginning of the period, the total expected cost function $J(i, \boldsymbol{s}, \boldsymbol{y})$ becomes the following.

$$J(i, \boldsymbol{s}, \boldsymbol{y}) = \sum_{k=1}^{\overline{K}_i} (\rho_k + a_k)s_k + E\left[\sum_{l=1}^{L} g_l\left(\sum_{k=1}^{\overline{K}_i} \overline{C}_{kl}(i)R_k s_k, D_l(i), y_l \right) \right], \tag{4.1}$$

where function $g_l(x, D, y) \equiv h_l(x + y - D)^+ + p_l(D - x - y)^+$ is the inventory cost for demand node l, where $(a)^+ = \max\{a, 0\}$. The first term in equation (4.1) is the production and transportation cost and the second term is the total inventory cost. For the rest of the paper, we define c_k as the ordering cost which is the sum of the production cost and

transportation cost, $c_k \equiv \rho_k + a_k$. The objective of this paper is to derive the the optimal order policy s which minimizes the total expected cost J.

$$V_1(i, \boldsymbol{y}) = \min_s J(i, \boldsymbol{s}, \boldsymbol{y}). \tag{4.2}$$

The following lemma is useful to show that the cost function J is convex.

Lemma 4.1 (Gallego and Hu, [2])**.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex, then for any constant vector $\boldsymbol{e} \in \mathbb{R}^n$ and any scalar d, $\psi(\boldsymbol{y}) \equiv \phi(\boldsymbol{e}'\boldsymbol{y} - d) : \mathbb{R}^n \to \mathbb{R}$ is also convex.*

Lemma 4.2. *(i) $J(i, \boldsymbol{s}, \boldsymbol{y})$ is convex in $(\boldsymbol{y}, \boldsymbol{s})$.*

(ii) $V_1(i, \boldsymbol{y})$ is convex in $\boldsymbol{y}$

Proof. (i) Since convexity is preserved by positive parts and by expectations, it is easy to see from Lemma 4.1 that $J(i, \boldsymbol{s}, \boldsymbol{y})$ is convex in $(\boldsymbol{y}, \boldsymbol{s})$.

(ii) The minimization in (4.2) is taken over a convex set and $J(i, \boldsymbol{s}, \boldsymbol{y})$ is convex in $(\boldsymbol{y}, \boldsymbol{s})$. Thus, by Heyman and Sobel [5], $V_1(i, \boldsymbol{y})$ is convex in $\boldsymbol{y}$. $\qquad\square$

Definition 4.1 (Müller and Stoyan [9], p.16)**.** *Let X and Y be random variables with finite means. Then we say that*

(i) X is less than Y in convex order (written $X \leq_{cx} Y$), if $E[f(X)] \leq E[f(Y)]$ for all real convex functions f such that the expectations exist.

(ii) X is less than Y in increasing convex order (written $X \leq_{icx} Y$), if $E[f(X)] \leq E[f(Y)]$ for all increasing convex functions f such that the expectations exist.

Proposition 4.1. *(i) For any l and $i, i' \in S$ $(i \leq i')$, if $D_l(i) \leq_{cx} D_l(i')$, then $V_1(i, \cdot) \leq V_1(i', \cdot)$.*

(ii) For any k, if $p_l = 0$ and $\hat{R}_k \leq_{icx} R_k$, then $V_1(i, \cdot : \hat{R}_k) \leq V_1(i, \cdot : R_k)$.

Proof. By definition, $g_l(x, D, y)$ is a convex function of D for each x and y. For fixed R_k, $k = 1, ..., \overline{K}_i$, and $D_l(i')$, $i' \neq i$, if $D_l(i) \leq_{cx} D_l(i')$, then we have $J(i, \boldsymbol{s}, \boldsymbol{y}) \leq J(i', \boldsymbol{s}, \boldsymbol{y})$. Hence, we have the result. For part (ii), since R_k and $\hat{R}_k$ follow a Bernoulli distribution, the relation $\hat{R}_k \leq_{icx} R_k$ holds. By Theorem 1.5.7 of Müller and Stoyan [9], we have $E[h_l(\hat{R}_k s_k + y - D)^+] \leq E[h_l(R_k s_k + y - D)^+]$. Thus, a similar argument with part (i) leads to the result. $\qquad\square$

By Theorem 1.5.3 and Corollary 1.5.4 of Müller and Stoyan [9], if A and B are random variables and $A \leq_{cx} B$, then $E[A] = E[B]$ and $Var(A) \leq Var(B)$. Thus, Proposition 4.1(i) implies that less demand variability always gives lower expected cost. For Proposition 4.1(ii), when there is no stockout penalty, a disruption probability with higher mean (i.e. q_k is low) is guaranteed to lower expected cost. This occurs because the inventory level increases when the disruption probability is high. A similar effect of supply uncertainty on expected cost (or profit) is also shown in Gerchak and Parlar [3] and Gupta and Cooper [4]. Gerchak and Parlar [3] show that a yield rate with a fixed (or larger) mean and lower variance ensures higher expected profit and a higher order quantity in the EOQ environment. Gupta and Cooper [4] show that a yield rate that is smaller in the convex order ensures higher expected profit.

4.2. Optimal policy in the single period model

In order to derive the optimal ordering policy, we first need to define the set of supply nodes. Let the set of supply nodes that only ship goods to a single demand node l:

$$\mathcal{K}_I^i(l) := \{k : \overline{C}_{kl} = 1, \overline{\boldsymbol{C}}_l'(i)\mathbf{1} = 1\}.$$

Let the set of supply nodes that ship goods to a single demand node l by multiple suppliers:

$$\mathcal{K}_{II}^i(l) := \{k : \overline{C}_{kl} = 1, \overline{\boldsymbol{C}}_l'(i)\mathbf{1} = 2\}$$

where $\mathbf{1} = (1, \cdots, 1)' \in \mathbb{R}^{\overline{K}_i}$ holds. Moreover, we define $\boldsymbol{s}_l^i$ as the vector of order quantity ordering by demand node l in state i, that is, $\boldsymbol{s}_l^i = \{s_k^i\}$ for $k \in \mathcal{K}_I^i(l)$ and $\boldsymbol{s}_l^i = (s_{k_1}^i \ s_{k_2}^i)'$ for $k \in \mathcal{K}_{II}^i(l)$. In order to derive the sufficient conditions for a threshold policy to exist, we make the following assumptions for each parameter.

Assumption 4.1. *For any l, $\{k_1, k_2\} \in \mathcal{K}_{II}^i(l)$, $c_{k_2} \le c_{k_1}$ and $q_{k_2} \le q_{k_1}$.*

Assumption 4.1 assures the that the ordering cost for node k_2 is larger than node k_1, and the possibility of disruption is small.

Assumption 4.2. *For any l, when $k \in \mathcal{K}_I^i(l)$,*

$$q_k p_l - c_k > 0,$$

when $\{k_1, k_2\} \in \mathcal{K}_{II}^i(l)$,

$$q_{k_1} p_l - c_{k_1} > q_{k_2} p_l - c_{k_2} > 0.$$

Assumption 4.2 relates the penalty for no inventory and ordering cost. If this inequality does not hold, there will be no incentive for corporations to hold inventory. When these assumptions hold, there will exist a threshold type order policy.

Lemma 4.3. *Under Assumptions 3.1, 4.1 and 4.2, for any i, l, k, there exists a threshold $\tilde{y}_l(1, i, k)$ as follows; For $k \in \mathcal{K}_I^i(l)$, we have*

$$\tilde{y}_l(1, i, k) = F_{i,l}^{-1}\left(\frac{p_l q_k - c_k}{q_k(h_l + p_l)} \right), \tag{4.3}$$

for $\{k_1, k_2\} \in \mathcal{K}_{II}^i(l)$, we have

$$\tilde{y}_l(1, i, k_1) = F_{i,l}^{-1}\left(\frac{p_l q_{k_1} - c_{k_1} - q_{k_1}(p_l q_{k_2} - c_{k_2})}{q_{k_1}(1 - q_{k_2})(h_l + p_l)} \right), \tag{4.4}$$

$$\tilde{y}_l(1, i, k_2) = F_{i,l}^{-1}\left(\frac{p_l q_{k_2} - c_{k_2} - q_{k_2}(p_l q_{k_1} - c_{k_1})}{q_{k_2}(1 - q_{k_1})(h_l + p_l)} \right). \tag{4.5}$$

Proof. The total expected cost function in equation (4.1) can be rewritten as the following.

$$\begin{aligned} J(i, \boldsymbol{s}, \boldsymbol{y}) = \ & \sum_{k=1}^{\overline{K}_i} c_k s_k + E\left[\sum_{l=1}^{L} \left\{ h_l \int_0^{y_l + B_l(\boldsymbol{s})} (y_l + B_l(\boldsymbol{s}) - x) dF_{i,l}(x) \right. \right. \\ & \left. \left. + p_l \int_{y_l + B_l(\boldsymbol{s})}^{\infty} (x - y_l - B_l(\boldsymbol{s})) dF_{i,l}(x) \right\} \right], \end{aligned}$$

where $B_l(\boldsymbol{s}) = \sum_{k=1}^{\overline{K}_i} \overline{C}_{kl}(i) R_k s_k$.

We define $\Theta_{s_k}(i, \boldsymbol{s}, y_l) \equiv \frac{\partial}{\partial s_k} J(i, \boldsymbol{s}, \boldsymbol{y})$, and it is given by

$$\Theta_{s_k}(i, \boldsymbol{s}, y_l) = \begin{cases} c_k - p_l q_k + (h_l + p_l) E[R_k F_{i,l}(y_l + R_k s_k)], & \text{if } k \in \mathcal{K}_I^i(l), \\ c_k - p_l q_k + (h_l + p_l) E[R_k F_{i,l}(y_l + \sum_{u=1}^{\overline{K}_i} \overline{C}_{ul}(i) R_u s_u)], & \text{if } k \in \mathcal{K}_{II}^i(l). \end{cases} \quad (4.6)$$

For given l and $k \in \mathcal{K}_I^i(l)$, $\tilde{y}_l(1, i, k)$ is directly obtained from equation (4.6) with $s_k = 0$. For given l and $k_1 \in \mathcal{K}_{II}^i(l)$, by Assumption 3.1, the equation (4.6) can be rewritten as the following:

$$\begin{aligned} \Theta_{s_{k_1}}(i, \boldsymbol{s}_l, y_l) &= c_{k_1} - p_l q_{k_1} + (h_l + p_l) E[R_{k_1} F_{i,l}(y_l + R_{k_1} s_{k_1} + R_{k_2} s_{k_2})] \\ &= c_{k_1} - p_l q_{k_1} + (h_l + p_l)\{q_{k_1} q_{k_2} F_{i,l}(y_l + s_{k_1} + s_{k_2}) \\ &\quad + q_{k_1}(1 - q_{k_2}) F_{i,l}(y_l + s_{k_1})\}, \end{aligned} \quad (4.7)$$

where $k_2 \in \mathcal{K}_{II}^i(l)$. Reversing the role of k_1 and k_2, we have

$$\begin{aligned} \Theta_{s_{k_2}}(i, \boldsymbol{s}_l, y_l) &= c_{k_2} - p_l q_{k_2} + (h_l + p_l)\{q_{k_2} q_{k_1} F_{i,l}(y_l + s_{k_1} + s_{k_2}) \\ &\quad + q_{k_2}(1 - q_{k_1}) F_{i,l}(y_l + s_{k_2})\}. \end{aligned} \quad (4.8)$$

Substituting $s_{k_1} = 0$ into $\Theta_{s_{k_1}}(i, \boldsymbol{s}_l, y_l)$ and $\Theta_{s_{k_2}}(i, \boldsymbol{s}_l, y_l)$, we have

$$\begin{cases} c_{k_1} - p_l q_{k_1} + (h_l + p_l)\{q_{k_1} q_{k_2} F_{i,l}(\tilde{y}_l + s_{k_2}) + q_{k_1}(1 - q_{k_2}) F_{i,l}(\tilde{y}_l) = 0, \\ c_{k_2} - p_l q_{k_2} + (h_l + p_l) q_{k_2} F_{i,l}(\tilde{y}_l + s_{k_2}) = 0. \end{cases}$$

The solution to the above system is given by

$$F_{i,l}(\tilde{y}_l(1, i, k_1)) = \frac{p_l q_{k_1} - c_{k_1} - q_{k_1}(p_l q_{k_2} - c_{k_2})}{q_{k_1}(1 - q_{k_2})(h_l + p_l)}.$$

From Assumptions 4.1 and 4.2, we obtain

$$0 \le \frac{p_l q_{k_1} - c_{k_1} - q_{k_1}(p_l q_{k_2} - c_{k_2})}{q_{k_1}(1 - q_{k_2})(h_l + p_l)} \le 1.$$

This gives equation (4.4). Similarly, equation (4.5) is obtained by setting $s_{k_2} = 0$ in equations (4.7) and (4.8). $\qquad \square$

Lemma 4.4. *If $c_{k_2}/q_{k_2} \le (\ge) c_{k_1}/q_{k_1}$, then $\tilde{y}_l(1, i, k_1) \le (\ge) \tilde{y}_l(1, i, k_2)$.*

Proof. For $k_1, k_2 \in \mathcal{K}_{II}^i(l)$, we define

$$\begin{aligned} I_1 &\equiv \frac{p_l q_{k_1} - c_{k_1} - q_{k_1}(p_l q_{k_2} - c_{k_2})}{q_{k_1}(1 - q_{k_2})(h_l + p_l)}, \\ I_2 &\equiv \frac{p_l q_{k_2} - c_{k_2} - q_{k_2}(p_l q_{k_1} - c_{k_1})}{q_{k_2}(1 - q_{k_1})(h_l + p_l)}. \end{aligned}$$

Then, we have

$$I_1 - I_2 = \frac{1 - q_{k_1} q_{k_2}}{(1 - q_{k_1})(1 - q_{k_2})(h_l + p_l)} \left(\frac{c_{k_2}}{q_{k_2}} - \frac{c_{k_1}}{q_{k_1}} \right).$$

If $c_{k_2}/q_{k_2} \le (\ge) c_{k_1}/q_{k_1}$, we obtain $I_1 \le (\ge) I_2$. Since $F^{-1}(x)$ is increasing in x, we have $\tilde{y}_l(1, i, k_1) \le (\ge) \tilde{y}_l(1, i, k_2)$. $\qquad \square$

Although parameters c_k and q_k, $k = k_1, k_2$, satisfy Assumption 4.1, the ratio c_k/q_k may vary depending on the value of each parameter. We further refer to the ratio c_k/q_k as the *risk-adjusted ordering cost index* in this paper. This index will be used to determine the optimal threshold policy discussed in Section 4.3.

Theorem 4.1. *Suppose that Assumptions 3.1, 4.1 and 4.2 hold.*
(i) When the state is i and the inventory level of demand point l is y_l, the optimal order quantity for supplier $k \in \mathcal{K}_I^i(l)$ is given by

$$\tilde{s}_k(1, i, y_l) = \begin{cases} \hat{s}_k(1, i, y_l), & \text{if } y_l < \tilde{y}_l(1, i, k), \\ 0, & \text{if } \tilde{y}_l(1, i, k) \le y_l, \end{cases} \tag{4.9}$$

where

$$\hat{s}_k(1, i, y_l) = -y_l + F_{il}^{-1}\left(\frac{p_l q_k - c_k}{q_k(h_l + p_l)}\right), \tag{4.10}$$

and $\tilde{y}_l(1, i, k)$ is given by equation (4.3).
(ii) For $k_1, k_2 \in \mathcal{K}_{II}^i(l)$, there exists a unique value $(\hat{s}_{k_1}(1, i, y_l), \hat{s}_{k_2}(1, i, y_l))$ satisfying

$$\Theta_{s_{k_1}}(i, \boldsymbol{s}_l, y_l) = \Theta_{s_{k_2}}(i, \boldsymbol{s}_l, y_l) = 0,$$

where $\Theta_{s_k}(i, \boldsymbol{s}, y_l) = \partial J(i, \boldsymbol{s}, \boldsymbol{y})/\partial s_k$. In addition, when $c_{k_2}/q_{k_2} \le c_{k_1}/q_{k_1}$, the optimal order quantity for supplier $k_1, k_2 \in \mathcal{K}_{II}^i(l)$ is given by

$$\{\tilde{s}_{k_1}(1, i, y_l), \tilde{s}_{k_2}(1, i, y_l)\} = \begin{cases} \{\hat{s}_{k_1}(1, i, y_l), \hat{s}_{k_2}(1, i, y_l)\}, & \text{if } y_l \le \tilde{y}_l(1, i, k_1), \\ \{0, \hat{s}_{k_2}(1, i, y_l)\}, & \text{if } \tilde{y}_l(1, i, k_1) \le y_l \le \tilde{y}_l(1, i, k_2), \\ \{0, 0\}, & \text{if } \tilde{y}_l(1, i, k_2) \le y_l, \end{cases} \tag{4.11}$$

and, when $c_{k_1}/q_{k_1} \le c_{k_2}/q_{k_2}$, it is given by

$$\{\tilde{s}_{k_1}(1, i, y_l), \tilde{s}_{k_2}(1, i, y_l)\} = \begin{cases} \{\hat{s}_{k_1}(1, i, y_l), \hat{s}_{k_2}(1, i, y_l)\}, & \text{if } y_l \le \tilde{y}_l(1, i, k_2), \\ \{\hat{s}_{k_1}(1, i, y_l), 0\}, & \text{if } \tilde{y}_l(1, i, k_2) \le y_l \le \tilde{y}_l(1, i, k_1), \\ \{0, 0\}, & \text{if } \tilde{y}_l(1, i, k_1) \le y_l, \end{cases} \tag{4.12}$$

where $\tilde{y}_l(1, i, k_1)$, $\tilde{y}_l(1, i, k_2)$ are from equation (4.4) and equation (4.5), respectively.

Proof. (i) By equation (4.6), $\Theta_{s_k}(i, s_k, y_l)$ is increasing in s_k for $k \in \mathcal{K}_I^i(l)$. Since $\Theta_{s_k}(i, s_k, y_l) = c_k + h_l q_k \ge 0$ as $s_k \to +\infty$ and $\Theta_{s_k}(i, s_k, y_l) = c_k - p_l q_k \le 0$ as $s_k \to -\infty$ by Assumption 4.2, there exists a unique value $\hat{s}_k$ such that $\Theta_{s_k}(i, \hat{s}_k, y_l) = 0$. Hence, from Lemma 4.3, the base-stock policy for supplier $k \in \mathcal{K}_I^i(l)$ is obtained from (4.9). Equation (4.10) can be derived by the first-order condition $\Theta_{s_k}(i, s_k, y_l) = 0$.
(ii) From equations (4.7) and (4.8), $\Theta_{s_{k_1}}(i, \boldsymbol{s}_l, y_l) - \Theta_{s_{k_2}}(i, \boldsymbol{s}_l, y_l) = 0$ simplifies to

$$s_{k_1} = F_{i,l}^{-1}\left[\frac{q_{k_2}(1 - q_{k_1})}{q_{k_1}(1 - q_{k_2})}F_{i,l}(y_l + s_{k_2}) + \frac{c_{k_2} - c_{k_1} + p_l(q_{k_1} - q_{k_2})}{h_l + p_l}\right] - y_l.$$

Since $F_{i,l}(\cdot)$ and $F_{i,l}^{-1}(\cdot)$ are increasing functions, s_{k_1} is increasing in s_{k_2}. Thus, for each i and l, we have $\Theta_{s_{k_2}}(i, \boldsymbol{s}_l, y_l) = c_{k_2} + h_l q_{k_2} > 0$ as $s_{k_2} \to +\infty$, and $\Theta_{s_{k_2}}(i, \boldsymbol{s}_l, y_l) = c_{k_2} - p_l q_{k_2} < 0$ as $s_{k_2} \to -\infty$. Hence, there exists a unique value $\hat{s}_{k_2}$ such as $\Theta_{s_{k_2}}(i, \tilde{\boldsymbol{s}}_l, y_l) = 0$, and so we obtain $(\tilde{s}_{k_1}, \tilde{s}_{k_2})$. In addition, from Lemma 4.3 and Lemma 4.4, we obtain the state-dependent base-stock policy (4.11). $\square$

If there is only one supplier at a demand node, the base-stock policy becomes the optimal order policy. If there are multiple suppliers for a demand node, the order policy will change depending on the inventory level. When the inventory level is at a very low level, it is optimal to procure goods from both supply nodes (the first case in equation (4.12)). But if the inventory level is at a moderate level (the second case in equation (4.12)), both the ordering cost and disruption probability affect the choice of the supplier. It becomes optimal to only procure from the supplier with the smaller risk-adjusted ordering cost index. In other words, when the ordering cost (resp. disruption probability) of the main supplier is almost the same as that of the backup one, the supplier with the smaller disruption probability (resp. ordering cost) should be selected.

4.3. Numerical example

This section calculates the optimal procurement policy numerically for the case in Figure 1 where demand node 2 receives goods from supply node 2 and 3. Production costs are set as $\rho_2 = 3$, $\rho_3 = 2.5$, the probability of reliability are set as $q_2 = 0.95$, $q_3 = 0.9$, and the transportation cost are set as $a_{22} = a_{32} = 0$. Although there are no particular distribution assumptions required for the demand distribution, for numerical illustration, we assume that the demand at node 2, D_2, follows a normal distribution with mean 13 and standard deviation $\sigma_2 = 4$. The inventory cost and penalty are $h_2 = 5$, $p_2 = 15$ respectively. Because $c_3/q_3 = 2.78 \leq c_2/q_2 = 3.16$ holds by Assumption 4.1, we have $k_1 = 2$, $k_2 = 3$. In addition, we assume that the parameters satisfy Assumption 4.2. When the initial inventory level of demand node 2 is $y_2 = 0$, the optimal procurement policy for supply node 2 becomes $\{\tilde{s}_2(1, 1, 0), \tilde{s}_3(1, 1, 0)\} = \{8.204, 6.718\}$. In this case, the inventory level is low, and despite the fact that the production cost is high for supply node 2, a large amount of goods are shipped from supply node 2 that also has a low probability of disruption compared to that of node 3 to avoid inventory shortage at demand node 2.

Figure 4 shows how the optimal threshold value changes as a function of the probability of reliability for demand node 3 and Figure 5 shows how the order quantity changes as the probability of reliability changes. As the probability of reliability increases at supply node 3, the order quantity of this supply node 3 increases as well.

Since the threshold of node 3 is larger than that of node 2 for $q_3 \geq 0.8$ in Figure 4, goods are only shipped from supply node 3 when the inventory level is at a moderate level because the production cost of supplier 3 is lower than that of supplier 2 [∥]. However, when the probability of reliability at node 3 is smaller than $q_3 = 0.8$, goods are only shipped from supply node 2 due to low probability of reliability at node 3 [**]. It is optimal to supply from node 2 and 3 when the inventory level lies in the gray area of Figure 4.

Figure 6 and Figure 7 show how the optimal threshold level and order quantity change as a function of the demand risk at node 2 (σ_2). Figure 6 tells us inventory level increases as σ_2 increases for supply node 3, but the inventory level decreases as σ_2 increases for supply node 2. This occurs because supply node 3 serves as the main supplier (i.e. risk adjusted cost index is larger). Figure 7 tells us that when demand risk is high, the order quantity increases for supply node 3 which has a smaller production cost, and the order quantity decreases for supply node 2 which has a higher production cost. This means that when the demand risk is high, the procurement is based on the supplier that has a lower production cost/lower reliability.

[∥]Shaded area in Figure 4. This corresponds to the second case in equation (4.11).

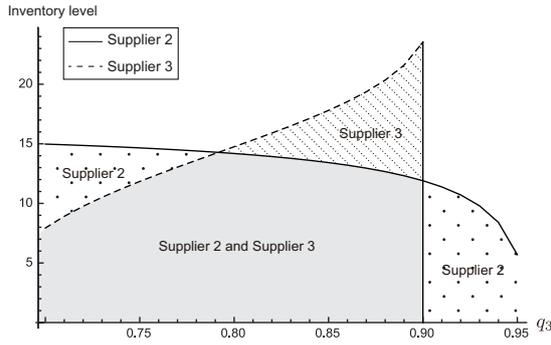[**]Dotted area in Figure 4. This corresponds to the second case in equation (4.12).

Figure 4: The optimal policy threshold as a function of the probability of reliability at supply node 3
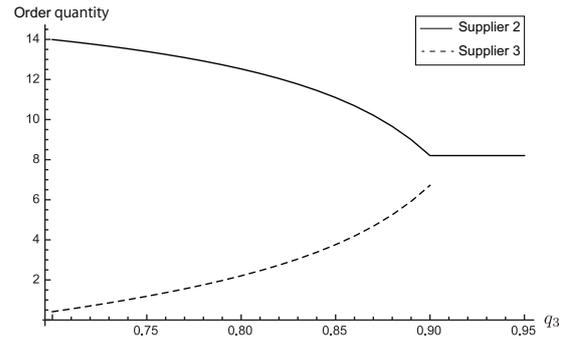

Figure 5: The optimal order quantity as a function of the probability of reliability at supply node 3
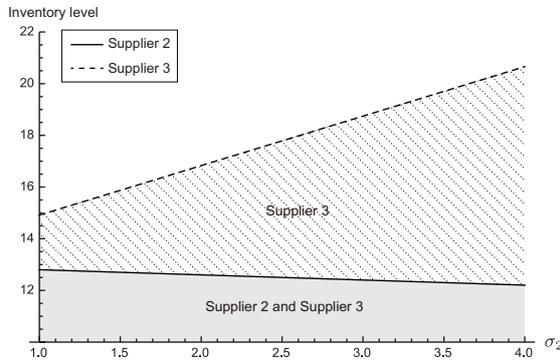

Figure 6: The optimal policy threshold as a function of the demand risk at supply node 2
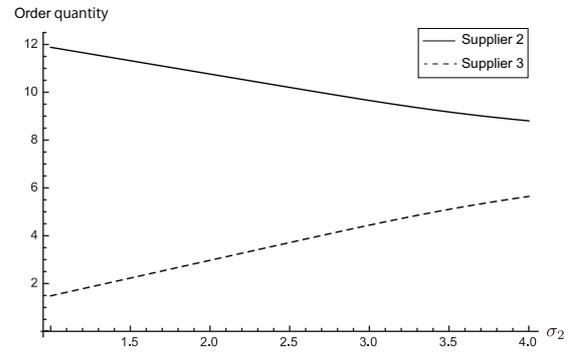

Figure 7: The optimal order quantity as a function of the demand risk at supply node 2

5. Multiperiod Model

5.1. The formulation of the multiperiod model

This section extends the single period model to a multiperiod model in discrete time where the time to the planning horizon t is $t = 0, 1, ..., T$, i.e. $t = 0$ is the last period and $t = T$ corresponds to the first period. If the state is i at the beginning of period t, the inventory level of the demand node is $\boldsymbol{y}$, and the order quantity is $\boldsymbol{s}$, the total expected cost function becomes

$$v_t(i, \boldsymbol{s}, \boldsymbol{y}) = J(i, \boldsymbol{s}, \boldsymbol{y}) + \sum_{i \neq j} P_{ij}(t-1)h_N + K_t(i, \boldsymbol{s}, \boldsymbol{y}), \tag{5.1}$$

$$K_t(i, \boldsymbol{s}, \boldsymbol{y}) = \sum_{j=1}^{N} P_{ij}(t-1)E[V_{t-1}(j, \boldsymbol{y} + \boldsymbol{C}'(i)(\boldsymbol{R} \cdot \boldsymbol{s}) - \boldsymbol{D}(i))], \tag{5.2}$$

where function J is the one period total cost function given by equation (4.1) and $\boldsymbol{R} \in \mathbb{R}^{\overline{K}_i}$. $h_N \geq 0$ is the cost to switch the supply chain network in state N, the second term in function K is the minimum total expected cost beyond period $t - 1$. Thus, the order quantity that minimizes the total expected cost becomes the optimal policy:

$$V_t(i, \boldsymbol{y}) = \min_{\boldsymbol{s}} v_t(i, \boldsymbol{s}, \boldsymbol{y}), \ t = 1, ..., T,$$

where $V_0(i, \boldsymbol{y}) = 0$.

Lemma 5.1. *For fixed i and t, we have*
(i) $v_t(i, \boldsymbol{s}, \boldsymbol{y})$ is convex in $(\boldsymbol{s}, \boldsymbol{y})$.
(ii) $V_t(i, \boldsymbol{y})$ is convex in $\boldsymbol{y}$.

Proof. The proof for this lemma follows the proof of Lemma 4 in Gallego and Hu [2]. The proof is by induction on t. For $t = 1$, the assertions clearly hold. We assume for $t - 1$. By Lemma 4.1, since v_{t-1} is convex, $K_t(i, \cdot, \cdot)$ is convex in $(\boldsymbol{s}, \boldsymbol{y})$. Thus, $v_t(i, \cdot, \cdot)$ is convex in $(\boldsymbol{s}, \boldsymbol{y})$. $\square$

Lemma 5.2. *Suppose that Assumptions 3.1, 4.1 and 4.2 hold. For any $l = 1, ..., L$ and $k \in \mathcal{K}_I^i(l)$, there exists a unique value $\tilde{y}_l(t, i, k)$ such that*

$$\Phi_{s_k}(t, i, \boldsymbol{s}_l, \tilde{y}_l(t, i, k))|_{s_k=0} = 0, \tag{5.3}$$

where $\Phi_{s_k}(t, i, \boldsymbol{s}, y_l) \equiv \partial v_t(i, \boldsymbol{s}_l, \boldsymbol{y})/\partial s_k$, and, for $k_1, k_2 \in \mathcal{K}_{II}^i(l)$, there exist values $(\tilde{y}_l(t, i, k_1), \tilde{y}_l(t, i, k_2))$ where $\tilde{y}_l(t, i, k_1)$ is satisfying

$$\Phi_{s_{k_1}}(t, i, \boldsymbol{s}_l, \tilde{y}_l(t, i, k_1))|_{s_{k_1}=0} = \Phi_{s_{k_2}}(t, i, \boldsymbol{s}_l, \tilde{y}_l(t, i, k_2))|_{s_{k_1}=0} = 0. \tag{5.4}$$

and $\tilde{y}_l(t, i, k_2)$ is satisfying

$$\Phi_{s_{k_1}}(t, i, \boldsymbol{s}_l, \tilde{y}_l(t, i, k_1))|_{s_{k_2}=0} = \Phi_{s_{k_2}}(t, i, \boldsymbol{s}_l, \tilde{y}_l(t, i, k_2))|_{s_{k_2}=0} = 0.$$

Proof. For $k \in \mathcal{K}_I^i(l)$, from equation (4.6), we obtain

$$\Phi_{s_k}(t, i, \boldsymbol{s}, y_l)|_{s_k=0} = c_k - p_l q_k + (h_l + p_l)F_{i,l}(y_l) + \Xi_{s_k}(t, i, \boldsymbol{s}, y_l)|_{s_k=0},$$

where $\Xi_{s_k}(t, i, \boldsymbol{s}_l, y_l) \equiv \partial K_t(i, \boldsymbol{s}, \boldsymbol{y})/\partial s_k$. From equation (5.2), we have

$$\Xi_{s_k}(t, i, \boldsymbol{s}_l, y_l) = \sum_{j=1}^N P_{ij}(t-1)\left\{\frac{\partial}{\partial s_k}E[J(j, \hat{\boldsymbol{s}}^j, \overline{\boldsymbol{y}})] + \frac{\partial}{\partial s_k}E[K_{t-1}(j, \hat{\boldsymbol{s}}^j, \overline{\boldsymbol{y}})]\right\},$$

where $\overline{\boldsymbol{y}} \equiv \boldsymbol{y} + \boldsymbol{C}'(i)(\boldsymbol{R}_i \cdot \boldsymbol{s}^i) - \boldsymbol{D}_i$. Here, we have

$$
\begin{aligned}
\frac{\partial}{\partial s_k}E[J(j, \hat{\boldsymbol{s}}^j, \overline{\boldsymbol{y}})] &= E\left[\frac{\partial}{\partial s_k}J(j, \hat{\boldsymbol{s}}^j, \overline{\boldsymbol{y}})\right] \\
&= E\left[\frac{\partial}{\partial \overline{y}_l}J(j, \hat{\boldsymbol{s}}^j, \overline{\boldsymbol{y}})\frac{\partial \overline{y}_l}{\partial s_k}\right] \\
&= E[\{(h_l + p_l)F_{i,l}(\overline{y}_l + B_l(\hat{\boldsymbol{s}}^j)) - p_l\}R_k]
\end{aligned}
$$

where $\overline{y}_l$ is the lth component of the vector $\overline{\boldsymbol{y}}$, that is, $\overline{y}_l = y_l + \sum_{u=1}^{\overline{K}_i} C_{ul}(i)R_u s_u - D_l$. Thus, we have

$$\lim_{y_l \to +\infty} \frac{\partial}{\partial s_k}E[J(j, \boldsymbol{s}_j^*, \overline{\boldsymbol{y}})]\bigg|_{s_k=0} = h_l q_k.$$

So, we obtain

$$\lim_{y_l \to +\infty} \Xi_{s_k}(t, i, \boldsymbol{s}_l, y_l)|_{s_k=0} = h_l q_k + \lim_{y_l \to +\infty}\sum_{j=1}^N P_{ij}(t-1)\frac{\partial}{\partial s_k}E[K_{t-1}(j, \hat{\boldsymbol{s}}^j, \overline{\boldsymbol{y}})]\bigg|_{s_k=0}.$$

By applying the argument above repeatedly, we obtain

$$\lim_{y_l \to +\infty} \Xi_{s_k}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_k=0} = (t-1)h_l q_k. \tag{5.5}$$

Thus, we have

$$\lim_{y_l \to +\infty} \Phi_{s_k}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_k=0} = c_k + t h_l q_k > 0.$$

Similarly, we have

$$\lim_{y_l \to -\infty} \Phi_{s_k}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_k=0} = c_k - t p_l q_k < c_k - p_l q_k < 0.$$

Thus, there exists $\tilde{y}_l(t, i, k)$, $k \in \mathcal{K}_I^i(l)$, satisfying equation (5.3). To show that $\tilde{y}_l(t, i, k)$, $k \in \mathcal{K}_I^i(l)$, is unique, suppose there exists $\tilde{y}_l^1 < \tilde{y}_l^2$ such that

$$\Phi_{s_k}(t, i, \boldsymbol{s}_l, \boldsymbol{y}_l^1) \mid_{s_k=0} = \Phi_{s_k}(t, i, \boldsymbol{s}_l, \boldsymbol{y}_l^2) \mid_{s_k=0} = 0. \tag{5.6}$$

For $\tilde{y}_l^2 > 0$, we have $F_{i,l}(\tilde{y}_l^2) > F_{i,l}(\tilde{y}_l^1)$. It follows from equation (4.6) that $\Theta_{s_{k_1}}(i, \boldsymbol{s}, \tilde{y}_l^2) \mid_{s_k=0} > \Theta_{s_{k_1}}(i, \boldsymbol{s}, \tilde{y}_l^1) \mid_{s_k=0}$. Since the function $V_{t-1}(i, \cdot)$ is convex in $\boldsymbol{y}$ by the induction hypothesis, $\partial V_{t-1}(i, \cdot)/\partial y_l$ is increasing in y_l. Thus, $\Xi_{s_k}(t, i, \boldsymbol{s}_l, y_l)$ is also increasing in y_l which implies

$$\Phi_{s_k}(t, i, \boldsymbol{s}_l, y_l^1) \mid_{s_k=0} < \Phi_{s_k}(t, i, \boldsymbol{s}_l, y_l^2) \mid_{s_k=0} .$$

This is a contradiction to equation (5.6). For $\tilde{y}_l^2 \leq 0$, we have $F_{i,l}(\tilde{y}_l^1) = F_{i,l}(\tilde{y}_l^2) = 0$. Thus, we have $\Theta_{s_{k_1}}(i, \boldsymbol{s}_l, y_l^1) \mid_{s_k=0} = \Theta_{s_{k_1}}(i, \boldsymbol{s}_l, y_l^2) \mid_{s_k=0} = c_k - p_l q_k$. Since $\Phi_{s_k}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_k=0} = \Theta_{s_k}(i, \boldsymbol{s}_l, y_l) \mid_{s_k=0} + \Xi_{s_k}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_k=0}$, the first equality of equation (5.6) is equivalent to

$$\Xi_{s_k}(t, i, \boldsymbol{s}_l, y_l^1) = \Xi_{s_k}(t, i, \boldsymbol{s}_l, y_l^2) = p_l q_k - c_k > 0.$$

Hence, this is a contradiction to equation (5.6).

For $k_1, k_2 \in \mathcal{K}_{II}^i(l)$, by equations (4.7) and (4.8), we obtain

$$
\begin{aligned}
\Phi_{s_{k_1}}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_{k_1}=0} = {} & c_{k_1} - p_l q_{k_1} + (h_l + p_l)\{q_{k_1} q_{k_2} F_{i,l}(y_l + s_{k_2}) \\
& + q_{k_1}(1 - q_{k_2}) F_{i,l}(y_l)\} + \Xi_{s_{k_1}}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_{k_1}=0} = 0, \tag{5.7} \\
\Phi_{s_{k_2}}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_{k_1}=0} = {} & c_{k_2} - p_l q_{k_2} + (h_l + p_l) q_{k_2} F_{i,l}(y_l + s_{k_2}) \\
& + \Xi_{s_{k_2}}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_{k_1}=0} = 0. \tag{5.8}
\end{aligned}
$$

Substituting equation (5.7) into equation (5.8), we have $A_1(y_l) = 0$, where

$$
\begin{aligned}
A_1(y_l) \equiv {} & c_{k_1} - c_{k_2} q_{k_1} - p_l q_{k_1}(1 - q_{k_2}) + (h_l + p_l) q_{k_1}(1 - q_{k_2}) F_{i,l}(y_l) \\
& + \Xi_{s_{k_1}}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_{k_1}=0} - q_{k_1} \Xi_{s_{k_2}}(t, i, \boldsymbol{s}_l, y_l) \mid_{s_{k_1}=0} . \tag{5.9}
\end{aligned}
$$

We will show the existence of a value $\tilde{y}_l$ satisfying $A_1(\tilde{y}_l) = 0$. From equation (5.5), we obtain

$$\lim_{y_l \to +\infty} A_1(y_l) = c_{k_1} - c_{k_2} q_{k_1} + t h_l q_{k_1}(1 - q_{k_2}) > 0.$$

The last inequality follows from Assumption 4.1. Similarly, we have

$$\lim_{y_l \to -\infty} A_1(y_l) = c_{k_1} - c_{k_2} q_{k_1} - t p_l q_{k_1}(1 - q_{k_2}) < 0.$$

The last inequality follows from Assumption 4.2. Thus, there exists $\tilde{y}_l(t, i, k)$, $k \in \mathcal{K}_{II}^i(l)$, satisfying equation (5.4) (that is, $A_1(\tilde{y}_l) = 0$). To show that $\tilde{y}_l(t, i, k)$ is unique, suppose there exist $\tilde{y}_l^1 < \tilde{y}_l^2$ such that

$$A_1(y_l^1) = A_1(y_l^2) = 0. \tag{5.10}$$

Here, we have

$$\Xi_{s_{k_1}}(t, i, \boldsymbol{s}_l, y_l)\,|_{s_{k_1}=0} - q_{k_1}\Xi_{s_{k_2}}(t, i, \boldsymbol{s}_l, y_l)\,|_{s_{k_1}=0}$$
$$= \sum_{j=1}^{N} P_{i,j}(t-1)E\left[\frac{\partial V_{t-1}(j, \overline{\boldsymbol{y}})}{\partial s_{k_1}} - q_{k_1}\frac{\partial V_{t-1}(j, \overline{\boldsymbol{y}})}{\partial s_{k_2}}\right]$$
$$= \sum_{j=1}^{N} P_{i,j}(t-1)E\left[q_{k_1}\frac{\partial V_{t-1}(j, \overline{\boldsymbol{y}})}{\partial \overline{\boldsymbol{y}}} - q_{k_1}q_{k_2}\frac{\partial V_{t-1}(j, \overline{\boldsymbol{y}})}{\partial \overline{\boldsymbol{y}}}\right]$$
$$= q_{k_1}(1 - q_{k_2}) \sum_{j=1}^{N} P_{i,j}(t-1)E\left[\frac{\partial V_{t-1}(j, \overline{\boldsymbol{y}})}{\partial \overline{\boldsymbol{y}}}\right].$$

Thus, from Lemma 5.1(ii), $V_{t-1}(j, \overline{\boldsymbol{y}})$ is convex in y_l, and so $A_1(y_l)$ is also increasing in y_l. Hence, there exists a value $\tilde{y}_l(t, i, k_1)$ such as $\Phi_{s_{k_1}}(t, i, \boldsymbol{s}_l, \tilde{y}_l)|_{s_{k_1}=0} = 0$. Similarly, we can also show the existence of $\tilde{y}_l(t, i, k_2)$ satisfying equation (5.4) by setting $s_{k_2} = 0$. In other words, $\tilde{y}_l(t, i, k_2)$ can be obtained as a solution to $A_2(\tilde{y}_l) = 0$, where

$$A_2(y_l) \equiv c_{k_2} - c_{k_1}q_{k_2} - p_lq_{k_2}(1 - q_{k_1}) + (h_l + p_l)q_{k_2}(1 - q_{k_1})F_{i,l}(y_l)$$
$$+ \Xi_{s_{k_2}}(t, i, \boldsymbol{s}, \boldsymbol{y}_l)\,|_{s_{k_2}=0} - q_{k_2}\Xi_{s_{k_1}}(t, i, \boldsymbol{s}, \boldsymbol{y}_l)\,|_{s_{k_2}=0}. \tag{5.11}$$

$\square$

Lemma 5.3. *When* $c_{k_2}/q_{k_2} \leq (\geq)c_{k_1}/q_{k_1}$, *we have* $\tilde{y}_l(t, i, k_1) \leq (\geq)\tilde{y}_l(t, i, k_2)$.

Proof. By equation (5.9), we have $A_1(\tilde{y}_l(t, i, k_1)) = 0$, that is,

$$q_{k_1}(1 - q_{k_2})\{p_l - (h_l + p_l)F_{i,l}(\tilde{y}_l) - M(\tilde{y}_l)\} = c_{k_1} - c_{k_2}q_{k_1},$$

where $M(y) = \sum_{j=1}^{N} P_{i,j}(t-1)E[\partial V_{t-1}(i, \overline{\boldsymbol{y}})/\partial \overline{\boldsymbol{y}}]$. Substituting the above equation into equation (5.11), we have

$$A_2(\tilde{y}_l(t, i, k_1)) = c_{k_2} - c_{k_1}q_{k_2} - q_{k_2}(1 - q_{k_1})\{p_l - (h_l + p_l)F_{i,l}(\tilde{y}_l(t, i, k_1)) - M(\tilde{y}_l(t, i, k_1))\}$$
$$= \frac{1 - q_{k_1}q_{k_2}}{q_{k_1}(1 - q_{k_2})}(q_{k_1}c_{k_2} - q_{k_2}c_{k_1}).$$

Thus, if $c_{k_2}/q_{k_2} \leq (\geq)c_{k_1}/q_{k_1}$, we have $A_2(\tilde{y}_l(t, i, k_1)) \leq (\geq)A_1(\tilde{y}_l(t, i, k_1)) = 0$. Since $A_1(y_l)$ and $A_2(y_l)$ are increasing in y_l, we obtain $\tilde{y}_l(t, i, k_1) \leq (\geq)\tilde{y}_l(t, i, k_2)$. $\square$

Lemma 5.4. *Suppose that Assumptions 3.1, 4.1 and 4.2 hold. For fixed t and y_l, when* $k \in \mathcal{K}_I^i(l)$, *there exists a value $\hat{s}_k(t, i, y_l)$ such that $\Phi_{s_k}(t, i, \boldsymbol{s}_l, y_l) = 0$. Moreover, when $k \in \mathcal{K}_{II}^i(l)$, there exist a value $(\hat{s}_{k_1}(t, i, y_l), \hat{s}_{k_2}(t, i, y_l))$ such that $\Phi_{s_{k_1}}(t, i, \boldsymbol{s}_l, y_l) = \Phi_{s_{k_2}}(t, i, \boldsymbol{s}_l, y_l) = 0$.*

Proof. For $k \in \mathcal{K}_I^i(l)$, we have $\Phi_{s_k}(t, i, \boldsymbol{s}, y_l) \to c_k + q_k h_l t > 0$ as $s_k \to +\infty$ and $\Phi_{s_k}(t, i, \boldsymbol{s}, y_l) \to c_k - q_k p_l t < 0$ as $s_k \to -\infty$. By induction, $\Phi_{s_k}(t, i, \boldsymbol{s}_l, y_l)$ is increasing in s_k. Thus, there exists a value $\hat{s}_k(t, i, y_l)$.

For $k \in \mathcal{K}_{II}^i(l)$, from equations $\Theta_{s_{k_1}}(i, \boldsymbol{s}_l, y_l) = 0$ and $\Theta_{s_{k_2}}(i, \boldsymbol{s}_l, y_l) = 0$, we obtain $B(s_{k_1}, s_{k_2}) = 0$, where

$$
\begin{aligned}
B_{i,l}(s_{k_1}, s_{k_2}) &\equiv \Theta_{s_{k_1}}(i, \boldsymbol{s}_l, y_l) - \Theta_{s_{k_2}}(i, \boldsymbol{s}_l, y_l) \\
&= c_{k_1} - c_{k_2} - p_l(q_{k_1} - q_{k_2}) \\
&\quad + (h_l + p_l)\{q_{k_1}(1 - q_{k_2})F_{i,l}(y_l + s_{k_1}) - q_{k_2}(1 - q_{k_1})F_{i,l}(y_l + s_{k_2})\} \\
&\quad + (q_{k_1} - q_{k_2}) \sum_{j=1}^{N} P_{i,j}(t - 1) E\left[\frac{\partial V_{t-1}(j, \overline{\boldsymbol{y}})}{\partial \overline{\boldsymbol{y}}}\right].
\end{aligned}
$$

For given s_{k_2}, we have

$$
\frac{\partial B_{i,l}}{\partial s_{k_1}} = (h_l + p_l)q_{k_1}(1 - q_{k_2})f_{i,l}(y_l + s_{k_1}) + q_{k_1}(q_{k_1} - q_{k_2}) \sum_{j=1}^{N} P_{i,j} E\left[\frac{\partial^2 V_{t-1}(j, \overline{\boldsymbol{y}})}{\partial \overline{\boldsymbol{y}}^2}\right] \geq 0.
$$

The last inequality follows from the convexity of $V_{t-1}(j, \cdot)$. As $s_{k_1} \to +\infty$, we have

$$
\begin{aligned}
B_{i,l}(s_{k_1}, s_{k_2}) &= c_{k_1} - c_{k_2} - p_l(q_{k_1} - q_{k_2}) \\
&\quad + (h_l + p_l)\{q_{k_1}(1 - q_{k_2}) - q_{k_2}(1 - q_{k_1})F_{i,l}(y_l + s_{k_2})\} \\
&\quad + (q_{k_1} - q_{k_2})(t - 1)h_l q_{k_1} \\
&\geq c_{k_1} - c_{k_2} + h_l(q_{k_1} - q_{k_2})(1 + (t - 1)q_{k_1}) \\
&\geq 0.
\end{aligned}
$$

In addition, as $s_{k_1} \to -\infty$, we have

$$
\begin{aligned}
B_{i,l}(s_{k_1}, s_{k_2}) &= c_{k_1} - c_{k_2} - p_l(q_{k_1} - q_{k_2}) \\
&\quad - q_{k_2}(h_l + p_l)(1 - q_{k_1})F_{i,l}(y_l + s_{k_2}) - (q_{k_1} - q_{k_2})(t - 1)q_{k_1}p_l \\
&\leq 0
\end{aligned}
$$

Thus, for given s_{k_2}, there exists a unique $\hat{s}_{k_1}$ such that $B_{i,l}(s_{k_1}, s_{k_2}) = 0$. Similarly, for given s_{k_1}, we can show that there exists a value $\hat{s}_{k_2}$. Hence, there exists a pair $(\hat{s}_{k_1}, \hat{s}_{k_2})$ satisfying $\Phi_{s_{k_1}}(t, i, \boldsymbol{s}_l, y_l) = \Phi_{s_{k_2}}(t, i, \boldsymbol{s}_l, y_l) = 0$. $\square$

Theorem 5.1. *Suppose that Assumptions 3.1, 4.1 and 4.2 hold. (i) When the state is i and the inventory level of demand point l is y_l, the optimal order quantity for supplier $k \in \mathcal{K}_I^i(l)$ is given by*

$$
\tilde{s}_k(t, i, y_l) = \begin{cases} \hat{s}_k(t, i, y_l), & \text{if } y_l < \tilde{y}_l(t, i, k), \\ 0, & \text{if } y_l \geq \tilde{y}_l(t, i, k). \end{cases}
$$

(ii) When the state is i and the inventory level of demand point l is y_l, if $c_{k_2}/q_{k_2} \leq c_{k_1}/q_{k_1}$, then the optimal order quantity for supplier $k_1, k_2 \in \mathcal{K}_{II}^i(l)$ is given by

$$
\{\tilde{s}_{k_1}(t, i, y_l), \tilde{s}_{k_2}(t, i, y_l)\} = \begin{cases} \{\hat{s}_{k_1}(t, i, y_l), \hat{s}_{k_2}(t, i, y_l)\}, & \text{if } y_l < \tilde{y}_l(t, i, k_1), \\ \{0, \hat{s}_{k_2}(t, i, y_l)\}, & \text{if } \tilde{y}_l(1, i, k_1) < y_l < \tilde{y}_l(t, i, k_2), \\ \{0, 0\}, & \text{if } \tilde{y}_l(t, i, k_2) < y_l. \end{cases}
$$

and if $c_{k_1}/q_{k_1} \geq c_{k_2}/q_{k_2}$, we have

$$\{\tilde{s}_{k_1}(t,i,y_l), \tilde{s}_{k_2}(t,i,y_l)\} = \begin{cases} \{\hat{s}_{k_1}(t,i,y_l), \hat{s}_{k_2}(t,i,y_l)\}, & \text{if } y_l < \tilde{y}_l(t,i,k_2), \\ \{\hat{s}_{k_1}(t,i,y_l), 0\}, & \text{if } \tilde{y}_l(1,i,k_2) < y_l < \tilde{y}_l(t,i,k_1), \\ \{0,0\}, & \text{if } \tilde{y}_l(t,i,k_1) < y_l. \end{cases}$$

Proof. The proof directly follows from Lemma 5.1-5.4. □

We see that the structure of the optimal policy for the multiperiod case is the same as that for the single-period case.

Next, we show the monotonicity of the expected cost function for the multi period case. Here, we assume that a finite Markov chain X is a homogeneous Markov process with state space $S = \{1,..,N\}$. Let $P_{i,j}$ denote the transition probability that satisfies the following assumption.

Assumption 5.1. *For each $k \in S$, $\sum_{j=k}^{N} P_{i,j}$ is non-decreasing in i.*

Definition 5.1 (Müller and Stoyan [9], p.98). *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be n-dimensional random vectors with finite expectations. Then we say that*

(i) $\boldsymbol{X}$ is less than $\boldsymbol{Y}$ in convex order (written $\boldsymbol{X} \leq_{cx} \boldsymbol{Y}$), if $E[f(\boldsymbol{X})] \leq E[f(\boldsymbol{Y})]$ for all convex functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the expectations exist.

(i) $\boldsymbol{X}$ is less than $\boldsymbol{Y}$ in increasing convex order (written $\boldsymbol{X} \leq_{icx} \boldsymbol{Y}$), if $E[f(\boldsymbol{X})] \leq E[f(\boldsymbol{Y})]$ for all increasing convex functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the expectations exist.

Proposition 5.1. *Suppose that Assumption 5.1 holds.*

(i) Define $\boldsymbol{O}(i) \equiv \boldsymbol{C}'(i)(\boldsymbol{R} \cdot \boldsymbol{s})$. For any l and $i \leq i'$, $i, i' \in S$, if $\boldsymbol{D}(i) \leq_{cx} \boldsymbol{D}(i')$ and $\boldsymbol{O}(i) \leq_{cx} \boldsymbol{O}(i')$, then $V_t(i, \cdot) \leq V_t(i', \cdot)$.

(ii) For any k, if $p_l = 0$ and $\hat{\boldsymbol{R}} \leq_{icx} \boldsymbol{R}$, then $V_t(i, \cdot : \hat{\boldsymbol{R}}) \leq V_t(i, \cdot : \boldsymbol{R})$.

Proof. By equation (5.1), for $i \leq i'$, we have

$$\begin{aligned} v_t(i, \boldsymbol{s}, \boldsymbol{y}) &= J(i, \boldsymbol{s}, \boldsymbol{y}) + \sum_{i \neq j} P_{ij} h_N + \sum_{j=1}^{N} P_{ij} E[V_{t-1}(j, \overline{\boldsymbol{y}}_i)], \\ &\leq J(i, \boldsymbol{s}, \boldsymbol{y}) + \sum_{i \neq j} P_{ij} h_N + \sum_{j=1}^{N} P_{ij} E[V_{t-1}(j, \overline{\boldsymbol{y}}_{i'})], \\ &\leq J(i', \boldsymbol{s}, \boldsymbol{y}) + \sum_{i \neq j} P_{i'j} h_N + \sum_{j=1}^{N} P_{i'j} E[V_{t-1}(j, \overline{\boldsymbol{y}}_{i'})], \\ &= v_t(i', \boldsymbol{s}, \boldsymbol{y}), \end{aligned}$$

where $\overline{\boldsymbol{y}} = \boldsymbol{y} + \boldsymbol{C}'(i)(\boldsymbol{R} \cdot \boldsymbol{s}) - \boldsymbol{D}(i)$. The first inequality follows from the convexity of $V_t(i, \boldsymbol{y})$ with respect to $\boldsymbol{y}$ given in Lemma 5.1(ii). The second inequality is obtained by the proof of Proposition 4.1(ii) and Assumption 5.1. This result leads to the assertion. Similarly, we can show part (ii) by the convexity of $V_t(i, \boldsymbol{y})$ with respect to $\boldsymbol{y}$. □

5.2. Numerical example

This section examines how network flexibility impacts the total expected cost by numerically analyzing a supply chain consisted of 3 supply nodes and 2 demand nodes (e.g. Figure 1

and Figure 2), The planning horizon is set to $T = 3$ and the inventory level at the beginning of time period 3 is $y_1 = y_2 = 0$, and the transition probability matrix is given as

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}.$$

The transportation cost is set to $\boldsymbol{a} = \boldsymbol{0}$. Although there are no particular distribution assumptions required for the demand distribution, for numerical illustration, we assume that demand $D_l(i)$ follows a binomial distribution with parameters in Table 1. The binomial distribution will be denoted as $B(\text{trials}, \text{successes})$. In order to evaluate the performance of flexibility residing in the supply chain network, we will compare the following two cases. The first case will apply the network for state 1 regardless of the realized state of supply (no switching in the network). The second case switches the network depending on the state of the supply such that the total cost is minimized. In both cases, the demand will follow a probability distribution that depends on the state of the network.

Figure 8 shows the minimum total cost for the planning horizon, i.e. 3 periods, when the disruption probability q_1 is set to $0.9 - 0.99$. Here, the bold line and dotted line represent the switching case and no switching case, respectively. For each case, there are two lines with a different initial state for the planning horizon (i.e. the beginning of the 3rd period). When there is no switching in the network, the disruption in supply node 1 completely stops the supply to demand node 1. Thus, as the disruption probability increases (i.e. q_1 decreases), the cost increases. Accordingly, when the network is in state 2, demand node 1 receives supply from supply node 1 and supply node 2 simultaneously. As a consequence, when there is switching in the network, the total cost is smaller than the case for no switching, i.e. the cost becomes smaller. In addition, Figure 9 illustrates the percentage of cost of switching the network compared to the cost of non-switching. Since the cost improvement is positive regardless of the network state and disruption probability, we can say that the flexibility in supply reduces the total expected cost when disruption probability is present.

Figure 10 compares the minimum total cost when the transition probability from state 1 to state 1 is altered. We can easily observe that when the transition probability p_{11} is smaller, the state of market demand is more likely to shift from state 1 to state 2. By observing the dotted lines, the cost increases if market demand is more likely to shift to state 2 (i.e. p_{11} is smaller) when there is no switching in the supply chain network. This phenomenon occurs because the demand variance of demand node 1 in state 2 is larger than that of state 1. Figure 11 shows the cost improvement by flexibility. When the initial state is in state 1, the effect of flexibility on the improvement is lower as the possibility of remaining in state 1 becomes higher. However, we can say that the network switching is effective in a case when the demand environment is more volatile.

6. Conclusion

This paper explicitly models disruption probability in the supply chain with state dependent uncertain demand. We derive an optimal threshold policy when there are two supply nodes and one demand node that needs multiple products. The policy is derived for the single period case and multiperiod case, and in addition, we show that the expected cost function is monotonic in the convex ordering of the demand distribution. The results differ from the past literature in the sense that we deal with the disruption probability via altering the state dependent supply chain network, and derive the optimal threshold policy.

The contribution of this paper lies in showing a general direction to corporations that may need to diversify their production facilities by modifying their supply chain network.

Table 1: Parameters

probability of reliability	supply node 1	supply node2	supply node3
	$q_1 = 0.95$	$q_2 = 0.95$	$q_3 = 0.9$
production cost	supply node1	supply node2	supply node3
	$\rho_1 = 2$	$\rho_2 = 3$	$\rho_3 = 2$
inventory cost	demand node1	demand node2	
	$h_1 = 2$	$h_2 = 5$	
penalty cost	demand node1	demand node2	
	$p_1 = 10$	$p_2 = 15$	
demand	demand node1	demand node2	
state 1	B(1, 0.5)	B(2, 0.5)	
state 2	B(3, 0.5)	B(2, 0.5)	

The numerical examples show that the potential improvement of the total expected cost may run up to 17% to 30% depending on the parameters. This numerically verifies the results of our theorems and justifies the importance of using our proposed model when introducing flexibility in the supply chain.

We believe this model can cope with the practical needs of corporations because of its simplicity. The supply chain disruptions cause the network configuration to become state dependent, which significantly reduces the number of situations to be considered. In addition, the risk adjusted cost index provides a good indicator on how to switch suppliers to fulfill the unmet demand.

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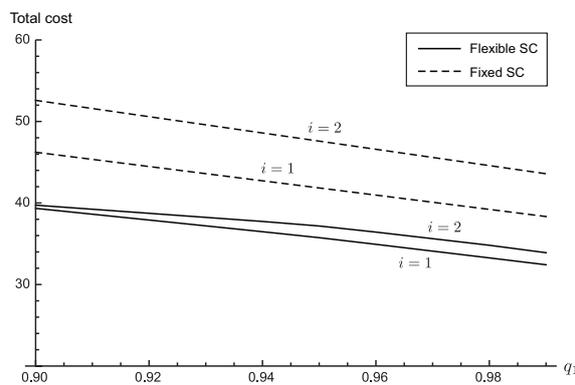
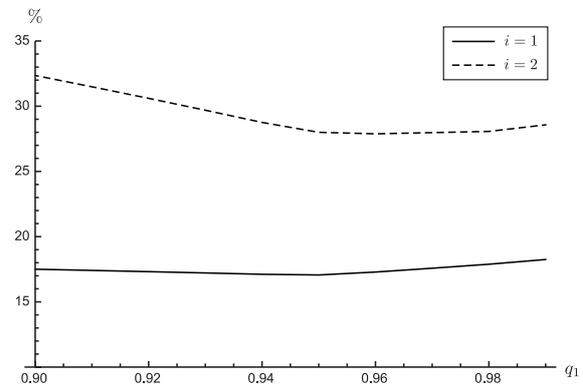Figure 8: Reliability probability at supply node 1 and minimum total expected cost

Figure 9: Reliability probability at supply node 1 and improvement due to flexibility
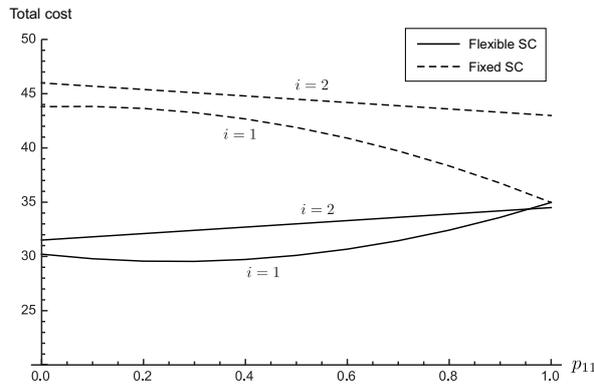
Figure 10: Transition probability from state 1 to state 1 and minimum total expected cost
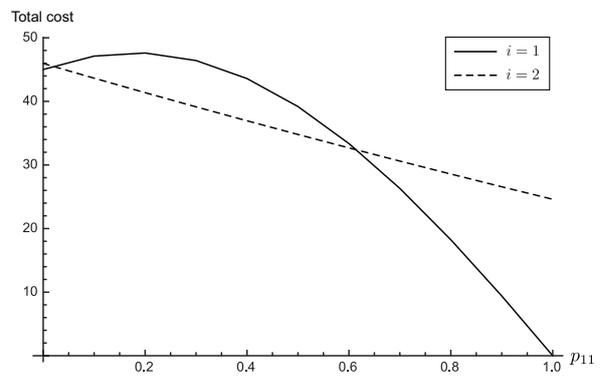
Figure 11: Transition probability from state 1 to state 1 and improvement due to flexibility

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