CHARACTERIZING DELAUNAY GRAPHS
VIA FIXED POINT THEOREM: A SIMPLE PROOF

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Abstract   This paper discusses the problem of determining whether a given plane graph is a Delaunay graph, i.e., whether it is topologically equivalent to a Delaunay triangulation. There exist theorems which characterize Delaunay graphs and yield polynomial time algorithms for the problem only by solving some linear inequality systems. A polynomial time algorithm proposed by Hodgson, Rivin and Smith solves a linear inequality system given by Rivin, which is based on sophisticated arguments about hyperbolic geometry. Independently, Hiroshima, Miyamoto and Sugihara gave another linear inequality system and a polynomial time algorithm. Although their discussion is based on primitive arguments on Euclidean geometry, their proofs are long and intricate, unfortunately. In this paper, we give a simple proof of the theorem shown by Hiroshima et al. by employing the fixed point theorem.

1.   Introduction

The two-dimensional Delaunay triangulation and its dual, the Voronoi diagram, are fundamental concepts in computational geometry, and have many practical applications such as interpolation and mesh generation [1, 4, 9]. It is also important to recognize Delaunay triangulations. The recognition problem can be divided into two types: geometric and combinatorial. The geometric problem is to judge whether a given drawing is a Delaunay triangulation. The combinatorial problem, which is discussed in this paper, determines whether a given embedded graph is topologically equivalent to a Delaunay triangulation. The combinatorial problem is important not only theoretically but also practically because it is closely related to the design of a topologically consistent algorithm for constructing the Delaunay/Voronoi diagram in finite-precision arithmetic [8, 12, 13].

Hodgson, Rivin and Smith [6] constructed a strongly polynomial time algorithm for judging whether a given 2-connected plane graph is topologically equivalent to a Delaunay triangulation. Their results are based on Rivin's inscribability criterion [10, 11] of 3-connected planar graphs, which uses sophisticated arguments about hyperbolic geometry. We briefly review their results in Subsection 2.2.

Hiroshima et al. independently proposed a polynomial time algorithm [5], which solves a linear inequality system different from that appearing in [6]. Although their discussion is based on primitive arguments on Euclidean geometry, their proof is long and intricate. In this paper, we give a simple proof of their theorem in [5] by employing the fixed point theorem. After making preparations in Section 2, we discuss some properties of Delaunay graphs in Section 3 and describe our proof in Section 4.

2. Preliminaries

In this paper, for any finite set S, $\mathbb{R}^S$ denotes the set of real vectors whose components are indexed by S.

2.1. Notations and definitions

First, we briefly review a notion of a Delaunay triangulation. Given a finite set of mutually distinct points $P \subseteq \mathbb{R}^2$, a *Delaunay triangulation* of P is commonly defined as a triangulation of P satisfying the property that the circumcircle of each inner cell (triangle) contains no point of P in its interior. A Delaunay triangulation of P is also known as the planar dual of the *Voronoi diagram* of P. A Delaunay triangulation is called *non-degenerate* if and only if it satisfies the conditions that no three vertices on the outermost cell are collinear, and no four vertices lie on a common circle that circumscribes an inner cell.

Next, we give a definition of *combinatorial triangulation*. Let $G' = (V, E)$ be a given undirected graph with vertex set V and edge set E. We assume that G' is a 2-connected plane graph (planar graph embedded in the 2-dimensional plane) without self-loops and parallel edges. The *outermost cell* of G' is unbounded while the other cells, called *inner cells*, are bounded. We also assume that all the inner cells are bounded by exactly three edges. A vertex in V is called an *outer vertex* if it is on the boundary of the outermost cell, and an *inner vertex* otherwise. Similarly, an edge in E is called an *outer edge* if it is on the boundary of the outermost cell, and an *inner edge* otherwise. In the rest of this paper, we denote the set of outer vertices and the set of inner vertices by V^{outer} and V^{inner}, respectively. For each inner cell, we associate a directed 3-cycle which is obtained from an undirected 3-cycle of G' forming boundary of the cell by directing the edges counterclockwise. Let C be the set of all the directed 3-cycles corresponding to all the inner cells of G'. A combinatorial triangulation is defined by a triplet (V, E, C). In the rest of this paper, we write $G = (V, E, C)$ and concentrate our attention on the topological structure of G; we do not care about the actual positions at which the vertices are placed. If a given undirected graph $G' = (V, E)$ is 2-connected, we say that a combinatorial triangulation $G = (V, E, C)$ is *2-connected*. When a given combinatorial triangulation is 2-connected, we can define the set of outer (inner) edges and the set of outer (inner) vertices without any ambiguity.

In the following, we introduce some notions related to a problem for judging whether a given combinatorial triangulation is obtained from a Delaunay triangulation. Given a combinatorial triangulation $G = (V, E, C)$, we seek an injection $\psi : V \to \mathbb{R}^2$ satisfying that the set of points $\{\psi(v) \in \mathbb{R}^2 \mid v \in V\}$ and the set of line segments between pairs in $\{\{\psi(u), \psi(v)\} \mid \{u, v\} \in E\}$ define a Delaunay triangulation. A map ψ satisfying the above conditions is called a *Delaunay realization*, if it exists. When a combinatorial triangulation G has a Delaunay realization, we say that G is a *Delaunay graph*. Especially, if a corresponding Delaunay triangulation is non-degenerate, then G is called a *non-degenerate Delaunay graph*.

2.2. Previous works

Hodgson, Rivin and Smith [6] proposed a strongly polynomial time algorithm for judging whether a given 2-connected combinatorial triangulation is a Delaunay graph or not. In this subsection, we briefly review their results in the non-degenerate case.

The stellation G_s of a given 2-connected combinatorial triangulation $G = (V, E, C)$ is a graph obtained by introducing an artificial vertex s and connecting it to all the outer vertices. The following theorem gives a necessary and sufficient condition that a given 2-connected combinatorial triangulation is a non-degenerate Delaunay graph.

Theorem 1 ([10] Theorem 13.1, [3] Lemma 2.2)**.** A 2-connected combinatorial triangulation $G = (V, E, C)$ is a non-degenerate Delaunay graph if and only if its stellation G_s is of

inscribable type, i.e., G_s has a (combinatorially equivalent) realization as the edges and vertices of the convex hull of a non-coplanar set of points on the surface of a sphere in 3-dimensional space.

The following theorem gives Rivin's inscribability criterion [10, 11].

Theorem 2 ([10, 11], [6] p.247, [3] Theorem 2.1)**.** A graph is of inscribable type if and only if it is a 3-connected planar graph and weights w can be assigned to its edges such that: (a) for each edge e, $0 < w(e) < 1/2$, (b) for each vertex v, the sum of the weights of all edges incident to v is equal to 1, and (c) for each non-coterminous cutset Γ, the sum of the weights of edges in Γ is greater than 1. (Here we note that a cutset is *non-coterminous* if its edges do not have a common endpoint.)

The above theorem says that we can judge whether a given 2-connected combinatorial triangulation is a non-degenerate Delaunay graph or not by checking the feasibility of the linear inequality system defined in Theorem 2. Although the number of inequalities in the linear inequality system grows exponentially in $|V|$ (in the worst case), we can check the feasibility in strongly polynomial time (with respect to $|V|$) by employing the ellipsoid method (see [6] Theorem 3).

3. Characterizing Delaunay Graphs

In this section, we briefly review a set of conditions, proposed in [5], which characterizes (non-degenerate) Delaunay graphs. Given a combinatorial triangulation $G = (V, E, C)$, we denote the elements of C by $c_0, c_1, \ldots, c_{|C|-1}$. For each (counterclockwise) directed cycle $c_i \in C$, we introduce three variables $x_{3i+1}, x_{3i+2}, x_{3i+3}$ assigned to the three vertices in c_i. In the rest of this paper, we interpret these variables as inner angles in degrees at the corresponding corners of a triangle defined by cycle c_i. We call these variables *angle variables*. Let v_α, v_β, v_γ be the three vertices of c_i corresponding to the three angle variables $x_{3i+1}, x_{3i+2}, x_{3i+3}$, respectively. We say that $x_{3i+1}, x_{3i+2}, x_{3i+3}$ are *facing* variables of undirected edges $\{v_\beta, v_\gamma\}, \{v_\gamma, v_\alpha\}, \{v_\alpha, v_\beta\} \in E$, respectively. There are $3|C|$ angle variables and we denotes the index set of angle variables by $J := \{1, 2, \ldots, 3|C|\}$. For example, a combinatorial triangulation shown in Figure 1 has nine angle variables $x_1, x_2, \ldots, x_9$.
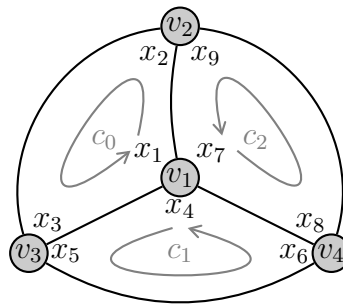

Figure 1: A combinatorial triangulation and its angle variables.

If a given combinatorial triangulation $G = (V, E, C)$ is a Delaunay graph, there exists a vector of angle variables $\boldsymbol{x} \in \mathbb{R}^J$ obtained by a Delaunay realization, which satisfies the following conditions.

C1 For each cycle in C, the sum of the associated three angle variables is equal to 180.
C2 For each inner vertex, the sum of all the associated angle variables is equal to 360.

C3 For each outer vertex, the sum of all the associated angle variables is at most 180.

C4 For each inner edge, the sum of the associated pair of the facing angle variables is at most 180.

C5 Each angle variable is positive.

For example, if we consider the combinatorial triangulation in Figure 1, the above conditions give the following linear inequality system;

$$
\begin{array}{lll}
\text{(C1) defined by } c_0: & x_1 + x_2 + x_3 = 180, \\
\text{(C1) defined by } c_1: & x_4 + x_5 + x_6 = 180, \\
\text{(C1) defined by } c_2: & x_7 + x_8 + x_9 = 180, \\
\text{(C2) defined by } v_1: & x_1 + x_4 + x_7 = 360, \\
\text{(C3) defined by } v_2: & x_2 + x_9 \leq 180, \\
\text{(C3) defined by } v_3: & x_3 + x_5 \leq 180, \\
\text{(C3) defined by } v_4: & x_6 + x_8 \leq 180, \\
\text{(C4) defined by } \{v_1, v_2\}: & x_3 + x_8 \leq 180, \\
\text{(C4) defined by } \{v_1, v_3\}: & x_2 + x_6 \leq 180, \\
\text{(C4) defined by } \{v_1, v_4\}: & x_5 + x_9 \leq 180, \\
\text{(C5)}: & x_1, x_2, \ldots, x_9 > 0.
\end{array}
$$

Figure 2 gives an example of a Delaunay realization of the combinatorial triangulation in Figure 1.
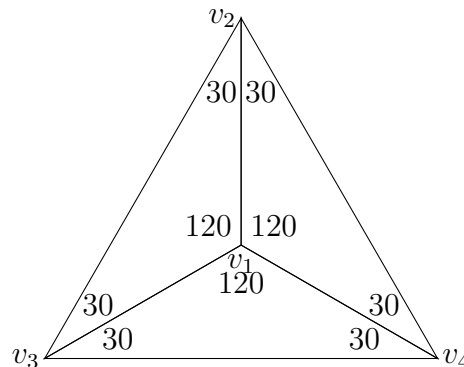

Figure 2: Realized Delaunay triangulation.

Unfortunately, the values of the angle variables satisfying all the conditions C1–C5 do not necessarily correspond to a Delaunay triangulation. For example, the combinatorial triangulation in Figure 1 has a vector of angle variables defined by

$$x_1 = x_4 = x_7 = 120, \; x_2 = x_5 = x_8 = 32, \; x_3 = x_6 = x_9 = 28,$$

which satisfies C1–C5, but it does not correspond to any triangulations. If we try to draw the diagram using these angle values, we come across an inconsistency as shown in Figure 3. In order to avoid such inconsistency we need still other conditions described below.

Let $c \in C$ be an inner cell with three vertices v_α, v_β, v_γ, and x_i, x_j, x_k be three angle variables corresponding to the three vertices, respectively. We say that x_j is *cc-facing* (meaning "facing counterclockwise") around v_α and x_k is *c-facing* (meaning "facing clockwise") around v_α. In Figure 4, for example, x_2, x_5, x_8 are cc-facing around v_1 while x_3, x_9, x_6 are c-facing around v_1.
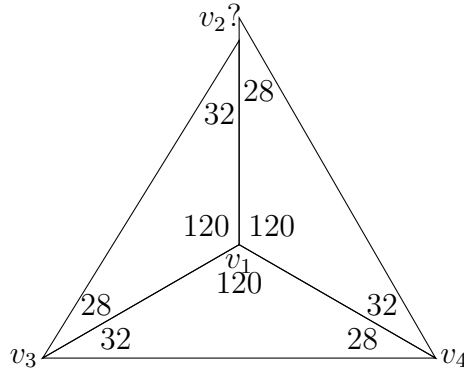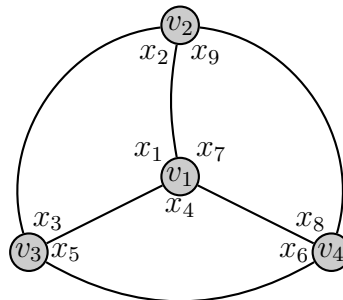
Figure 3: Angle variables satisfying C1–C5.

Figure 4: Variables x_3, x_6, x_9 are c-facing and x_2, x_5, x_8 are cc-facing around v_1.

For any inner vertex $v \in V^{\text{inner}}$, let $X_v^{\text{CC}} \subseteq J$ be indices of cc-facing angle variables around v, and $X_v^{\text{C}} \subseteq J$ be indices of c-facing angle variables around v. Furthermore, we introduce a function

$$F_v(\boldsymbol{x}) := \frac{\displaystyle\prod_{j \in X_v^{\text{CC}}} \sin x_j}{\displaystyle\prod_{j \in X_v^{\text{C}}} \sin x_j}, \tag{3.1}$$

where $\boldsymbol{x} \in \mathbb{R}^J$ is a vector of angle variables (in degrees). We only consider angle variables satisfying $0 < x_j < 180$ ($\forall j \in J$), and hence we get $0 < F_v(\boldsymbol{x}) < \infty$.

Now we describe a necessary and sufficient condition that a combinatorial triangulation becomes a Delaunay graph.

Theorem 3 ([5]). *A 2-connected combinatorial triangulation $G = (V, E, C)$ is a Delaunay graph (i.e., G has a Delaunay realization) if and only if there exists a vector of angle variables satisfying the set of conditions C1–C6, where*

C6 $F_v(\boldsymbol{x}) = 1$ *for any inner vertex $v \in V^{\text{inner}}$.*

It is not so difficult to prove the above theorem. For example, Hiroshima, Miyamoto and Sugihara gave a short and elementary proof of Theorem 3 in their paper [5], which gives a method to construct a Delaunay realization. If we restrict the Delaunay triangulations to non-degenerate ones, the conditions C3 and C4 are respectively changed in the following way.

C3' For each outer vertex, the sum of all the associated angle variables is *less than* 180.

C4' For each inner edge, the sum of the associated pair of the angle values facing the edge is *less than* 180.

A non-degenerate version of Theorem 3 is as follows.

Theorem 4 ([5]). A 2-connected combinatorial triangulation $G = (V, E, C)$ is a non-degenerate Delaunay graph (i.e., G has a Delaunay realization corresponding to a non-degenerate Delaunay triangulation) if and only if there exists a vector of angle variables satisfying the set of conditions C1, C2, C3', C4', C5 and C6.

Thus, we get a necessary and sufficient condition for a combinatorial triangulation to be a (non-degenerate) Delaunay graph. However, the conditions stated in Theorems 3 and 4 are not useful for the recognition of a Delaunay graph, because we do not know any finite-step algorithm for finding a vector of angle variables satisfying these conditions.

4. Main Result

Now we describe a linear inequality system shown in [5] which yields an efficient method for recognizing Delaunay graphs. (The method finds neither a Delaunay realization nor a vector of angle variables satisfying set of conditions C1–C6.) The following theorem says that when we only need to judge whether a given combinatorial triangulation is a Delaunay graph (or not), we can drop condition C6, surprisingly.

Theorem 5 ([5]). A 2-connected combinatorial triangulation $G = (V, E, C)$ is a Delaunay graph (i.e., G has a Delaunay realization) if and only if there exists a vector of angle variables satisfying the set of conditions C1–C5.

It is obvious that if G is a Delaunay graph, there exists a vector of angle variables satisfying the set of conditions C1–C5. To show the inverse implication, we only need to prove that "if there exists a vector $\boldsymbol{x} \in \mathbb{R}^J$ of angle variables satisfying C1–C5, then there also exists a vector $\boldsymbol{x'} \in \mathbb{R}^J$ of angle variables satisfying C1–C6." By employing Theorem 3, the existence of $\boldsymbol{x'} \in \mathbb{R}^J$ satisfying C1–C6 implies that G is a Delaunay graph.

We can judge the satisfiability of the set of conditions C1–C5 in finite steps because the conditions C1 through C5 are linear in the variables and the method for checking their satisfiability has been established in linear programming (see [5] for detail). Especially, the obtained linear programming problem satisfies that all the non-zero coefficients are $+1$ or -1, and thus it is solvable in strongly polynomial time [14].

The following theorem deals with the non-degenerate case.

Theorem 6 ([5]). A 2-connected combinatorial triangulation $G = (V, E, C)$ is a non-degenerate Delaunay graph (i.e., G has a Delaunay realization corresponding to a non-degenerate Delaunay triangulation) if and only if there exists a vector of angle variables satisfying the set of conditions C1, C2, C3', C4', C5.

We employ the fixed point theorem and give simple proofs of Theorem 5 and Theorem 6.

Theorem 7 (Fixed Point Theorem [2]).

Every continuous map $f : B^m \to B^m$ defined on an m-dimensional closed ball B^m has a fixed point (a point $\boldsymbol{x} \in B^m$ with $f(\boldsymbol{x}) = \boldsymbol{x}$).

It is well known that we can extend the above theorem to a continuous map defined on a convex compact set.

Before describing our proof, we give a sketch of an important procedure, which transforms a feasible solution of the linear inequality system defined by C1–C5. Let us recall a vector of angle variables shown in Figure 3, that satisfies conditions C1–C5, but not C6. Now we construct a (new) vector by increasing angle variables c-facing around the inner vertex v_1

by α degree, and decreasing angle variables cc-facing around v_1 by α degree. After this procedure, conditions C1–C4 are preserved. When we set $\alpha = 2$, the obtained vector of angle variables, shown in Figure 2, satisfies conditions C1–C6.

Now we describe the above procedure precisely. Given a non-negative vector $\boldsymbol{x} \geq \boldsymbol{0}$ of angle variables satisfying C1–C4, an inner vertex v and a real number α, we introduce a vector $\boldsymbol{x}(v, \alpha)$ defined by

$$
x(v, \alpha)_j = \begin{cases} x_j + \alpha, & j \in X_v^{\mathrm{CC}}, \\ x_j - \alpha, & j \in X_v^{\mathrm{C}}, \\ x_j, & \text{otherwise.} \end{cases} \tag{4.1}
$$

The following lemma shows some properties of $\boldsymbol{x}(v, \alpha)$.

Lemma 8. Let $\boldsymbol{x} \geq \boldsymbol{0}$ be a non-negative vector of angle variables satisfying C1–C4 and $\boldsymbol{x}(v, \alpha)$ be a vector defined by (4.1) w.r.t. an inner vertex $v \in V^{\mathrm{inner}}$. For any $\alpha \in \mathbb{R}$, vector $\boldsymbol{x}(v, \alpha)$ satisfies conditions C1–C4. We define

$$
\begin{aligned}
\alpha_{\max} &= \max\{\alpha \in \mathbb{R} \mid \boldsymbol{x}(v, \alpha) \geq \boldsymbol{0}\}, \\
\alpha_{\min} &= \min\{\alpha \in \mathbb{R} \mid \boldsymbol{x}(v, \alpha) \geq \boldsymbol{0}\}.
\end{aligned}
$$

If $\alpha_{\min} < \alpha_{\max}$, then a real valued function $F_v(\boldsymbol{x}(v, \alpha))$ w.r.t. α defined on an open interval $(\alpha_{\min}, \alpha_{\max})$ is a continuous monotonically increasing function.

Proof. It is easy to show that $\boldsymbol{x}(v, \alpha)$ satisfies conditions C1–C4. The continuity of $F_v(\boldsymbol{x}(v, \alpha))$ with respect to α is obvious. Let $C(v) \subseteq C$ be a set of cycles including v. For each cycle $c' \in C(v)$, angle variable $x_{c'}^{\mathrm{CC}}$ ($x_{c'}^{\mathrm{C}}$) denotes associated cc-facing (c-facing) angle variable around v. Condition C1, non-negativity of $\boldsymbol{x}$, and the inequality $\alpha_{\min} < \alpha_{\max}$ imply that $0 < x_{c'}^{\mathrm{CC}} + x_{c'}^{\mathrm{C}} \leq 180$ ($\forall c' \in C(v)$). We transform the following differentiation and obtain that

$$
\begin{aligned}
\frac{\mathrm{d} \log F_v(\boldsymbol{x}(v, \alpha))}{\mathrm{d}\alpha} &= \sum_{j \in X_v^{\mathrm{CC}}} \frac{\mathrm{d} \log \sin(x_j + \alpha)}{\mathrm{d}\alpha} - \sum_{j \in X_v^{\mathrm{C}}} \frac{\mathrm{d} \log \sin(x_j - \alpha)}{\mathrm{d}\alpha} \\
&= \sum_{j \in X_v^{\mathrm{CC}}} \frac{\cos(x_j + \alpha)}{\sin(x_j + \alpha)} + \sum_{j \in X_v^{\mathrm{C}}} \frac{\cos(x_j - \alpha)}{\sin(x_j - \alpha)} \\
&= \sum_{c' \in C(v)} \left(\frac{\cos(x_{c'}^{\mathrm{CC}} + \alpha)}{\sin(x_{c'}^{\mathrm{CC}} + \alpha)} + \frac{\cos(x_{c'}^{\mathrm{C}} - \alpha)}{\sin(x_{c'}^{\mathrm{C}} - \alpha)} \right) \\
&= \sum_{c' \in C(v)} \frac{\sin(x_{c'}^{\mathrm{CC}} + x_{c'}^{\mathrm{C}})}{\sin(x_{c'}^{\mathrm{CC}} + \alpha) \sin(x_{c'}^{\mathrm{C}} - \alpha)} > 0,
\end{aligned}
$$

where the last inequality is derived from the facts that (1) $\forall \alpha \in (\alpha_{\min}, \alpha_{\max})$, $\forall c' \in C(v)$, $\sin(x_{c'}^{\mathrm{CC}} + \alpha) \sin(x_{c'}^{\mathrm{C}} - \alpha) > 0$ (2) $\forall c' \in C(v)$, $\sin(x_{c'}^{\mathrm{CC}} + x_{c'}^{\mathrm{C}}) \geq 0$, and (3) $\exists c' \in C(v)$, $\sin(x_{c'}^{\mathrm{CC}} + x_{c'}^{\mathrm{C}}) > 0$ (obtained from C2). Thus, both $\log F_v(\boldsymbol{x}(v, \alpha))$ and $F_v(\boldsymbol{x}(v, \alpha))$ are monotonically increasing in α. $\square$

In our proof of Theorem 5, we adopt the above procedure around all inner vertices simultaneously.

Proof. (Proof of Theorem 5.) From Theorem 3, we only need to show that once we obtain a vector of angle variables satisfying C1–C5, there exists a vector of angle variables satisfying C1–C6.

Let $\boldsymbol{b} \in \mathbb{R}^J$ be a vector of angle variables satisfying conditions C1–C5. We define a matrix M whose rows are indexed by $J = \{1, 2, \ldots, 3|C|\}$ (the index set of angle variables), columns are indexed by the vertex set V, and each entry m_{iv} is defined as follows:

$$m_{iv} = \begin{cases} 1, & \text{if angle variable } x_i \text{ is cc-facing around } v, \\ -1, & \text{if angle variable } x_i \text{ is c-facing around } v, \\ 0, & \text{otherwise.} \end{cases}$$

Figure 5 shows a matrix M corresponding to Figure 1.

	v_1	v_2	v_3	v_4
x_1	0	−1	1	0
x_2	1	0	−1	0
x_3	−1	1	0	0
x_4	0	0	−1	1
x_5	1	0	0	−1
x_6	−1	0	1	0
x_7	0	1	0	−1
x_8	1	−1	0	0
x_9	−1	0	0	1

Figure 5: A matrix M corresponding to Figure 1.

Let $\widetilde{M}$ be a column submatrix of M corresponding to inner vertices V^{inner}. It is easy to see that the vector of angle variables $\widetilde{M}\boldsymbol{y}+\boldsymbol{b}$ is obtained from $\boldsymbol{b}$ by decreasing angle variables c-facing around the inner vertex v by y_v, and increase angle variables cc-facing around v by y_v, for each inner vertex $v \in V^{\text{inner}}$. The definition of $\widetilde{M}$ directly implies that for any vector $\boldsymbol{y} \in \mathbb{R}^{V^{\text{inner}}}$, a vector of angle variables $\widetilde{M}\boldsymbol{y} + \boldsymbol{b}$ also satisfies conditions C1–C4.

We introduce a closed subset $\Omega \subseteq \mathbb{R}^{V^{\text{inner}}}$ defined by

$$\Omega := \left\{ \boldsymbol{y} \in \mathbb{R}^{V^{\text{inner}}} \,\Big|\, \widetilde{M}\boldsymbol{y} + \boldsymbol{b} \geq \boldsymbol{0} \right\}.$$

Here, we briefly show the boundedness of Ω by contradiction. Motzkin [7] showed that the set of all the solutions of a given linear inequality system can be written as the Minkowski sum of a (bounded) polytope and a cone. From Motzkin's theorem, the unboundedness of Ω implies the existence of a ray vector $\boldsymbol{y}' \in \mathbb{R}^{V^{\text{inner}}}$ of Ω satisfying $\widetilde{M}\boldsymbol{y}' \geq \boldsymbol{0}$. We define $V^{\pm} = \{v \in V^{\text{inner}} \mid y_v' \neq 0\}$. Since $V^{\pm} \subseteq V^{\text{inner}}$, there exists an inner edge connecting a vertex $v \in V^{\pm}$ and a vertex in $V \setminus V^{\pm}$, and thus vertex v has a pair of angle variables x and x' which are c-facing and cc-facing around v, respectively. If y_v' is positive (negative), then a component of $\widetilde{M}\boldsymbol{y}'$ indexed by x (x') is negative, which contradicts with the assumption $\widetilde{M}\boldsymbol{y}' \geq \boldsymbol{0}$. From the above, Ω is a compact convex set.

For any pair $(\boldsymbol{y}, v) \in \Omega \times V^{\text{inner}}$, we define following two values:

$$\alpha_{\max}(\boldsymbol{y}, v) := \max\{\alpha \in \mathbb{R} \mid \boldsymbol{y} + \alpha \boldsymbol{e}_v \in \Omega\},$$
$$\alpha_{\min}(\boldsymbol{y}, v) := \min\{\alpha \in \mathbb{R} \mid \boldsymbol{y} + \alpha \boldsymbol{e}_v \in \Omega\},$$

where $\boldsymbol{e}_v \in \{0, 1\}^{V^{\text{inner}}}$ is a unit vector whose entry is equal to 1 if and only if the corresponding index is equal to v. (Here we note that both the maximum and the minimum

always exist, because Ω is a bounded closed set and is nonempty; clearly $\boldsymbol{b} \in \Omega$.) Since $\boldsymbol{y} \in \Omega$, inequalities $\alpha_{\min}(\boldsymbol{y}, v) \leq 0 \leq \alpha_{\max}(\boldsymbol{y}, v)$ hold. When $\alpha_{\min}(\boldsymbol{y}, v) < \alpha_{\max}(\boldsymbol{y}, v)$, we have that

$$\lim_{\alpha \to \alpha_{\max}(\boldsymbol{y}, v)} F_v(\boldsymbol{y} + \alpha \boldsymbol{e}_v) = +\infty, \tag{4.2}$$

$$\lim_{\alpha \to \alpha_{\min}(\boldsymbol{y}, v)} F_v(\boldsymbol{y} + \alpha \boldsymbol{e}_v) = +0, \tag{4.3}$$

and thus Lemma 8 and the intermediate value theorem imply that there exists a unique value α^* in the open interval $(\alpha_{\min}(\boldsymbol{y}, v), \alpha_{\max}(\boldsymbol{y}, v))$ satisfying equality $F_v(\boldsymbol{y} + \alpha^* \boldsymbol{e}_v) = 1$. Now we introduce a map $f_v : \Omega \to \Omega$ for each $v \in V^{\mathrm{inner}}$ defined by

$$f_v(\boldsymbol{y}) = \begin{cases} \boldsymbol{y}, & \text{if } \alpha_{\min}(\boldsymbol{y}, v) = \alpha_{\max}(\boldsymbol{y}, v) = 0, \\ \boldsymbol{y} + \alpha^* \boldsymbol{e}_v, & \text{if } \alpha_{\min}(\boldsymbol{y}, v) < \alpha_{\max}(\boldsymbol{y}, v), \end{cases}$$

where α^* is a unique value satisfying $F_v(\boldsymbol{y} + \alpha^* \boldsymbol{e}_v) = 1$. It is obvious that for each inner vertex v, the corresponding map f_v is continuous.

Lastly, we define a map $f : \Omega \to \Omega$ as:

$$f(\boldsymbol{y}) := \frac{1}{|V^{\mathrm{inner}}|} \sum_{v \in V^{\mathrm{inner}}} f_v(\boldsymbol{y}),$$

where $f(\boldsymbol{y})$ is the gravity center of vectors $\{f_v(\boldsymbol{y}) \mid v \in V^{\mathrm{inner}}\}$. Since f_v is continuous for each inner vertex v, f is also continuous.

Now we apply the fixed point theorem to the continuous map f and obtain a result that there exists a fixed point $\boldsymbol{y}^* \in \Omega$, i.e., $\boldsymbol{y}^*$ satisfies $f(\boldsymbol{y}^*) = \boldsymbol{y}^*$.

Every fixed point $\boldsymbol{y}^*$ satisfies that

$$\begin{aligned} &\forall v \in V^{\mathrm{inner}}, \\ &\alpha_{\min}(\boldsymbol{y}^*, v) = \alpha_{\max}(\boldsymbol{y}^*, v) \text{ or } F_v(\boldsymbol{y}^*) = 1. \end{aligned} \tag{4.4}$$

Otherwise, there exists at least one inner vertex v' satisfying $\alpha_{\min}(\boldsymbol{y}^*, v') < \alpha_{\max}(\boldsymbol{y}^*, v')$ and $F_{v'}(\boldsymbol{y}^*) \neq 1$. Then v' also satisfies $f_{v'}(\boldsymbol{y}^*) \neq \boldsymbol{y}^*$, which implies $f(\boldsymbol{y}^*) \neq \boldsymbol{y}^*$. It is a contradiction.

It is obvious that every fixed point $\boldsymbol{y}^* \in \Omega$ satisfies C1–C4. Next we discuss condition C5, i.e., we show that $\widetilde{M}\boldsymbol{y}^* + \boldsymbol{b} > \boldsymbol{0}$ for *any* fixed point $\boldsymbol{y}^*$.

If a vertex v satisfies $\alpha_{\min}(\boldsymbol{y}^*, v) = \alpha_{\max}(\boldsymbol{y}^*, v)$, then $\alpha_{\min}(\boldsymbol{y}^*, v) \leq 0 \leq \alpha_{\max}(\boldsymbol{y}^*, v)$ implies $\alpha_{\min}(\boldsymbol{y}^*, v) = \alpha_{\max}(\boldsymbol{y}^*, v) = 0$. When a vertex v satisfies $\alpha_{\min}(\boldsymbol{y}^*, v) < \alpha_{\max}(\boldsymbol{y}^*, v)$, property (4.4) gives the equality $F_v(\boldsymbol{y}^*) = 1$. In this case, (4.2) and (4.3) imply $0 < \alpha_{\max}(\boldsymbol{y}^*, v)$ and $\alpha_{\min}(\boldsymbol{y}^*, v) < 0$, respectively. From the above discussion, we have shown that for any $v \in V^{\mathrm{inner}}$,

$$\alpha_{\min}(\boldsymbol{y}^*, v) = 0 \text{ if and only if } \alpha_{\max}(\boldsymbol{y}^*, v) = 0. \tag{4.5}$$

Let $\boldsymbol{x}^* = \widetilde{M}\boldsymbol{y}^* + \boldsymbol{b}$. If an inner vertex v has a cc-facing angle variable x_j satisfying $x_j^* = 0$, then $\alpha_{\min}(\boldsymbol{y}^*, v) = 0$ and thus $\alpha_{\max}(\boldsymbol{y}^*, v) = 0$, which implies that v also has a c-facing angle variable $x_{j'}$ satisfying $x_{j'}^* = 0$. Similarly, when an inner vertex v has a c-facing angle variable x_j satisfying $x_j^* = 0$, then $\alpha_{\max}(\boldsymbol{y}^*, v) = 0$ and thus $\alpha_{\min}(\boldsymbol{y}^*, v) = 0$, which implies that v also has a cc-facing angle variable $x_{j'}$ satisfying $x_{j'}^* = 0$.

Let us consider a directed graph H whose incident matrix is $M^\top$: i.e., a directed graph obtained from G by substituting a pair of parallel arcs with opposite direction for each inner edge in E and directing outer edges in E appropriately (see Figure 6). Digraph H has vertex set V and arc set J, which is an index set of angle variables. Each angle variable x_j corresponds to an arc in H from u to v where x_j is a c-facing variable around u and cc-facing variable around v.
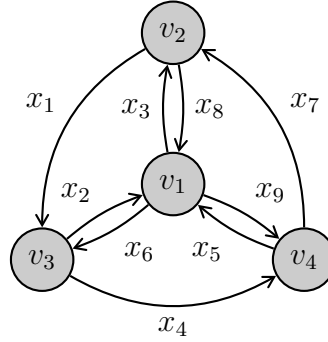

Figure 6: A directed graph $H = (V, J)$ corresponding to Figure 1.

If an arc a of H satisfies that the corresponding angle variable, denoted by x_a, satisfies $x_a^* = 0$, we say that a is a *critical* arc. In the directed graph, each inner vertex v satisfies that v has an incoming critical arc if and only if v has an outgoing critical arc. It is because, property (4.5) implies that v has a cc-facing angle variable x_j satisfying $x_j^* = 0$ if and only if v has c-facing angle variable $x_{j'}$ satisfying $x_{j'}^* = 0$, and x_j and $x_{j'}$ correspond to an incoming critical arc and an outgoing critical arc of v, respectively.

Now we show $\boldsymbol{x}^* > \boldsymbol{0}$. Assume on the contrary that there exists an angle variable x_j satisfying $x_j^* = 0$. Then, there exists a critical arc in H. Let A_0 be the set of all the critical arcs. From the above discussion, a digraph defined by (V, A_0) has either (Case 1) "a directed elementary path Γ_1 connecting a pair of outer vertices and passing only inner vertices" or (Case 2) "a directed elementary cycle Γ_2 consisting of inner vertices." (Here we note that a path is called elementary if no vertices appear more than once in it. An elementary cycle is a closed walk with no repeated vertices or edges aside from the necessary repetition of the start and end vertex.)

<u>Case 1.</u> Let $\chi_1 \in \{0, 1\}^J$ be a characteristic vector of the set of arcs in path Γ_1. Since Γ_1 consists of critical edges, $\chi_1^\top \boldsymbol{x}^* = 0$ hold. Every inner vertex v is not a terminal vertex of Γ_1, and thus v has an incoming arc in Γ_1 if and only if v has an outgoing arc in Γ_1. Accordingly, the equality $\chi_1^\top \widetilde{M} = \boldsymbol{0}$ hold. Thus we have that

$$0 = \chi_1^\top \boldsymbol{x}^* = \chi_1^\top (\widetilde{M} \boldsymbol{y}^* + \boldsymbol{b}) = \chi_1^\top \widetilde{M} \boldsymbol{y}^* + \chi_1^\top \boldsymbol{b} = \chi_1^\top \boldsymbol{b} > 0.$$

Contradiction.

<u>Case 2.</u> Let $\chi_2 \in \{0, 1\}^J$ be a characteristic vector of the set of arcs in cycle Γ_2. Since Γ_2 consists of inner vertices and critical edges, both $\chi_2^\top \widetilde{M} = \boldsymbol{0}$ and $\chi_2^\top \boldsymbol{x}^* = 0$ hold. Thus we have that

$$0 = \chi_2^\top \boldsymbol{x}^* = \chi_2^\top (\widetilde{M} \boldsymbol{y}^* + \boldsymbol{b}) = \chi_2^\top \widetilde{M} \boldsymbol{y}^* + \chi_2^\top \boldsymbol{b} = \chi_2^\top \boldsymbol{b} > 0.$$

Contradiction.

Lastly, we show that every fixed point $\boldsymbol{y}^*$ satisfies C6. Since every fixed point $\boldsymbol{y}^*$ satisfies C5, i.e., $\boldsymbol{x}^* = \widetilde{M} \boldsymbol{y}^* + \boldsymbol{b} > \boldsymbol{0}$, every inner vertex v satisfies $\alpha_{\min}(\boldsymbol{y}^*, v) < 0 < \alpha_{\max}(\boldsymbol{y}^*, v)$.

From property (4.4), equality $F_v(\boldsymbol{y}^*) = 1$ holds at every inner vertex v. As a consequence, condition C6 is satisfied.   □

A proof of Theorem 6 is almost the same. Actually, we only have to replace C3 and C4 with C3' and C4' respectively in our proofs of Lemma 8 and Theorem 5.

Acknowledgment

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