L-CONVEXITY ON GRAPH STRUCTURES

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Abstract   In this paper, we study classes of discrete convex functions: submodular functions on modular semilattices and L-convex functions on oriented modular graphs. They were introduced by the author in complexity classification of minimum 0-extension problems. We clarify the relationship to other discrete convex functions, such as k-submodular functions, skew-bisubmodular functions, $L^\natural$-convex functions, tree submodular functions, and UJ-convex functions. We show that they actually can be viewed as submodular/L-convex functions in our sense. We also prove a sharp iteration bound of the steepest descent algorithm for minimizing our L-convex functions. The underlying structures, modular semilattices and oriented modular graphs, have rich connections to projective and polar spaces, Euclidean building, and metric spaces of global nonpositive curvature (CAT(0) spaces). By utilizing these connections, we formulate an analogue of the Lovász extension, introduce well-behaved subclasses of submodular/L-convex functions, and show that these classes can be characterized by the convexity of the Lovász extension. We demonstrate applications of our theory to combinatorial optimization problems that include multicommodity flow, multiway cut, and related labeling problems: these problems had been outside the scope of discrete convex analysis so far.

1.   Introduction

Discrete Convex Analysis (DCA) is a theory of "convex" functions on integer lattice $\mathbf{Z}^n$, developed by K. Murota [45, 46] and his collaborators (including S. Fujishige, A. Shioura, and A. Tamura), and aims to provide a unified theoretical framework to well-solvable combinatorial optimization problems related to network flow and matroid/submodular optimization; see also [13, Chapter VII]. In DCA, two classes of discrete convex functions, *M-convex functions* and *L-convex functions*, play primary roles; the former originates from the base exchange property of matroids, the latter abstracts potential energy in electric circuits, and they are in relation of conjugacy. They generalize several known concepts in matroid/submodular optimization by means of analogy of convex analysis in continuous optimization. Besides its fundamental position in combinatorial optimization and operations research, the scope of DCA has been enlarging over past 20 years, and several DCA ideas and concepts have been successfully and unexpectedly applied to other areas of applied mathematics that include mathematical economics, game theory [53], inventory theory [52] and so on; see recent survey [47].

The present article addresses a new emerging direction of DCA, which might be called a theory of discrete convex functions on graph structures, or *DCA beyond* $\mathbf{Z}^n$. Our central subjects are graph-theoretic generalizations of $L^\natural$-convex function. An $L^\natural$-*convex function* [16] is an essentially equivalent variant of L-convex function, and is defined as a function g on $\mathbf{Z}^n$ satisfying a discrete version of the convexity inequality, called the *discrete midpoint*

convexity:

$$g(x) + g(y) \geq g(\lfloor (x+y)/2 \rfloor) + g(\lceil (x+y)/2 \rceil) \quad (x, y \in \mathbf{Z}^n), \tag{1.1}$$

where $\lfloor \cdot \rfloor$ (resp. $\lceil \cdot \rceil$) denotes an operation on $\mathbf{R}^n$ that rounds down (resp. up) the fractional part of each component. L$^\natural$-convex functions have several fascinating properties for optimization and algorithm design; see [46, Chapters 7 and 10]. They are submodular functions on each cube $x + \{0,1\}^n$ ($x \in \mathbf{Z}^n$), and are extended to convex functions on $\mathbf{R}^n$ via the Lovász extension. The global optimality is guaranteed by the local optimality (*L-optimality criterion*), which is checked by submodular function minimization (SFM). This fact brings about a simple descent algorithm, called the *steepest descent algorithm (SDA)*, to minimize L$^\natural$-convex functions through successive application of an SFM algorithm. The number of iterations (= calls of the SFM algorithm) of SDA is estimated by the l_∞-diameter of domain [41, 48].

Observe that the discrete midpoint convexity (1.1) is still definable if $\mathbf{Z}^n$ is replaced by the Cartesian product P^n of n directed paths P, and L$^\natural$-convex functions are also definable on P^n. Starting from this observation, Kolmogorov [40] considered a generalization of L$^\natural$-convex functions defined on the product of rooted binary trees, and discussed an SDA framework, where SFM is replaced by bisubmodular function minimization. One may ask: *Can we define analogues of L$^\natural$-convex functions on more general graph structures, and develop a similar algorithmic framework to attack combinatorial optimization problems beyond the current scope of DCA?*

This question was answered affirmatively in the study [30] of *the minimum 0-extension problem* (alias *multifacility location problem*) [38]—the problem of finding locations x_i ($i = 1, 2, \ldots, n$) of n facilities in a graph Γ such that the weighed sum $\sum_v \sum_i b_{vi} d_\Gamma(v, x_i) + \sum_{i,j} c_{ij} d_\Gamma(x_i, x_j)$ of their distances is minimum. This problem is viewed as an optimization over the graph $\Gamma \times \Gamma \times \cdots \times \Gamma$, and motivates us to consider "convex" functions on graphs. The 0-extension problem was known to be NP-hard unless Γ is an *orientable modular graph*, and known to be polynomial time solvable for special subclasses of orientable modular graphs [10, 38, 39]. In [30], we revealed several intriguing structural properties of orientable modular graphs: they are amalgamations of modular lattices and *modular semilattices* in the sense of [5]. On the basis of the structure, we introduced two new classes of discrete convex functions: *submodular functions* on modular semilattices and *L-convex functions* on oriented modular graphs (called *modular complexes* in [30]). Here an *oriented modular graph* is an orientable modular graph endowed with a specific edge-orientation. We established analogues of local submodularity, L-optimality criterion and SDA for the new L-convex functions, and proved the VCSP-tractability [42, 54, 58] of the new submodular functions. Finally we showed that the 0-extension problem on an orientable modular graph can be formulated as an L-convex function minimization on an oriented modular graph, and is solvable in polynomial time by SDA. This completes the complexity classification of the 0-extension problem. In the subsequent work [28, 29], by developing analogous theories of L-convex functions on certain graph structures, we succeeded in designing efficient combinatorial polynomial algorithms for some classes of multicommodity flow problems for which such algorithms had not been known.

Although these discrete convex functions on graph structures brought algorithmic developments, their relations and connections to other discrete convex functions (in particular, original L$^\natural$-convex functions) were not made clear. In this paper, we continue to study submodular functions on modular semilattices and L-convex functions on oriented modular graphs. The main objectives are (i) to clarify their relations to other classes of discrete

convex functions, (ii) to introduce several new concepts and pursue further L-convexity properties, and (iii) to present their occurrences and applications in actual combinatorial optimization problems.

A larger part of our investigation is built on *Metric Graph Theory (MGT)* [4], which studies graph classes from distance properties and provides a suitable language for analysis if orientable modular graphs and, more generally, *weakly modular graphs.* Recently MGT [9, 12] explored rich connections among weakly modular graphs, incidence geometries [56], Euclidean building [1, 55], and metric spaces of nonpositive curvature [7]. To formulate and prove our results, we will utilize some of the results and concepts from these fields of mathematics, previously unrelated to combinatorial optimization. We hope that this will provide fruitful interactions among these fields, explore new perspective, and trigger further evolution of DCA.

The rest of the paper is organized as follows. In Section 2, we introduce basic notation, and summarize some of the concepts and results from MGT [9] that we will use. We introduce modular semilattices, *polar spaces*, orientable modular graphs, *sweakly modular graphs* (*swm-graphs*) [9], and *Euclidean building of type C*. They are underlying structures of discrete convex functions considered in the subsequent sections. In particular, polar spaces turn out to be appropriate generalizations of underlying structures of bisubmodular and *k-submodular functions* [33]. A Euclidean building is a generalization of tree, and is a simplicial complex having an amalgamated structure of Euclidean spaces, which naturally admits an analogue of discrete midpoint operators. We introduce continuous spaces, *CAT(0) metric spaces* [7] and *orthoscheme complexes* [6]. They play roles of continuous relaxations of orientable modular graphs, analogous to continuous relaxation $\mathbf{R}^n$ of $\mathbf{Z}^n$, and enable us to formulate the Lovász extension and its convexity for our discrete convex functions.

In Section 3, we study submodular functions on modular semilattices. After reviewing their definition and basic properties, we focus on submodular functions on polar spaces, and establish their relationship to k-submodular functions and *skew-bisubmodular functions* [35], and also show that submodular functions on polar spaces are characterized by the convexity of the Lovász extension.

In Section 4, we study L-convex functions on oriented modular graphs. We give definition and basics established by [30]. We then present a sharp l_∞-iteration bound of the steepest descent algorithm. We introduce notions of *L-extendability* and *L-convex relaxations*, extending those considered by [28] for the product of trees. We study L-convex functions on a Euclidean building (of type C). After establishing a characterization by the discrete midpoint convexity and the convexity of the Lovász extension, we explain how our framework captures original $L^\natural$-convex functions, *UJ-convex functions* [14], *strongly-tree submodular functions* [40], and *alternating L-convex functions* [28].

In Section 5, we present applications of our theory to combinatorial optimization problems including multicommodity flow, multiway cut, and related labeling problems. We see that our L-convex/submodular functions arise as the dual objective functions of several multiflow problems or half-integral relaxations of multiway cut problems.

The beginning of Sections 3, 4, and 5 includes a more detailed summary of results. The proofs of all results in Sections 3 and 4 are given in Section 6. Some of the results which we present in this paper were announced in [25, 28–30]. An expository article [31] provides an overview of the present theory and results.

2. Preliminaries

2.1. Basic notation

Let $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$, and $\mathbf{Z}_+$ denote the sets of reals, nonnegative reals, integers, and nonnegative integers, respectively. The ith unit vector in $\mathbf{R}^n$ is denoted by e_i. Let ∞ denote the infinity element treated as $a + \infty = \infty$, $a < \infty$ for $a \in \mathbf{R}$, and $\infty + \infty = \infty$. Let $\overline{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$. For a function $f : X \to \overline{\mathbf{R}}$, let dom f denote the set of elements x with $f(x) < \infty$.

By a graph G we mean a connected simple undirected graph. An edge joining vertices x and y is denoted by xy. We do not assume that G is a finite graph. For notational simplicity, the vertex set $V(G)$ of G is also denoted by G. A path or cycle is written by a sequence $(x_1, x_2, \ldots, x_k)$ of vertices. If G has an edge-orientation, then G is called an *oriented* graph, whereas paths, cycles, or distances are considered in the underlying graph. We write $x \to y$ if edge xy is oriented from x to y. The *(Cartesian) product* of two graphs G, G' is the graph on $V(G) \times V(G')$ with an edge given between (x, x') and (y, y') if $x = y$ and $x'y'$ is an edge of G', or $x' = y'$ and xy is an edge of G. The product of G, G' is denoted by $G \times G'$. In the case where both G and G' have edge-orientations, the edge-orientation of $G \times G'$ is defined by $(x, x') \to (y, y')$ if $x \to y$ and $x' = y'$, or $x' \to y'$ and $x = y$.

We will use the standard terminology of posets (partially ordered sets), which basically follows [9, Section 2.1.3]. The partial order of a poset $\mathcal{P}$ is denoted by $\preceq$, where $p \prec q$ means $p \preceq q$ and $p \neq q$. The join and meet of elements p, q, if they exist, are denoted by $p \vee q$ and $p \wedge q$, respectively. The rank function of a meet-semilattice is denoted by r. An element of rank 1 is called an *atom*. Lattices and semilattices are supposed to have finite rank. The minimum element of a meet-semilattice is denoted by 0. For $p \preceq q$, define the *interval* $[p, q]$ by $[p, q] := \{u \in \mathcal{P} \mid p \preceq u \preceq q\}$. The *principal filter* $\mathcal{F}_p$ of $p \in \mathcal{P}$ is the set of elements u with $p \preceq u$, and is regarded as a poset by the restriction of $\preceq$. The *principal ideal* $\mathcal{I}_p$ of p is the set of elements u with $u \preceq p$, and regarded as a poset by the reverse of the restriction of $\preceq$ (so that p is the minimum element). We say that p *covers* q if $p \succ q$ and there is no u with $p \succ u \succ q$. The *Hasse diagram* of poset $\mathcal{P}$ is an oriented graph on $\mathcal{P}$ such that elements p, q have an edge $p \to q$ if p covers q.

2.2. Modular semilattices and polar spaces

In this section, we introduce modular semilattices and polar spaces, which are underlying structures of submodular functions in Section 3. A lattice $\mathcal{L}$ is called *modular* if for every $x, y, z \in \mathcal{L}$ with $x \succeq z$ it holds $x \wedge (y \vee z) = (x \wedge y) \vee z$. A semilattice $\mathcal{L}$ is called *modular* [5] if every principal ideal is a modular lattice, and for every $x, y, z \in \mathcal{L}$ the join $x \vee y \vee z$ exists provided $x \vee y$, $y \vee z$, and $z \vee x$ exist. A *complemented* modular (semi)lattice is a modular (semi)lattice such that every element is the join of atoms.

Example 1 ($\mathcal{S}_k{}^n$)**.** For a nonnegative integer $k \geq 0$, let $\mathcal{S}_k$ denote the $k + 1$ element set containing a special element 0. The partial order on $\mathcal{S}_k$ is given by $0 \leftarrow u$ for $u \in \mathcal{S}_k \setminus \{0\}$; other pairs are incomparable. Clearly, $\mathcal{S}_k$ is a complemented modular semilattice, and so is the product $\mathcal{S}_k{}^n$. Notice that $\mathcal{S}_2{}^n = \{-1, 0, 1\}^n$ is the domain of bisubmodular functions. More generally, $\mathcal{S}_k{}^n$ is the domain of k-submodular functions.

A canonical example of a modular semilattice is a polar space. A *polar space* $\mathcal{L}$ of rank n is a poset that is the union of subposets, called *polar frames*, satisfying the following axiom.[*]

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[*]Our definition of a polar space is not standard; see [56] for the definition of a polar space as an incidence geometry. It is known that polar spaces and spherical buildings of type C constitute the same object [55]. Our conditions reformulate the axiom that the order complex of $\mathcal{L} \setminus \{0\}$ is a spherical building of type C, where the order complex of $\mathcal{S}_2{}^n \setminus \{0\}$ is a Coxeter complex of type C, and apartments are subcomplexes corresponding to polar frames.

P0: Each polar frame is isomorphic to $\mathcal{S}_2{}^n = \{-1, 0, 1\}^n$.

P1: For two chains C, D in $\mathcal{L}$, there is a polar frame $\mathcal{F}$ containing them.

P2: If polar frames $\mathcal{F}, \mathcal{F}'$ both contain two chains C, D, then there is an isomorphism $\mathcal{F} \to \mathcal{F}'$ being identity on C and D.

One can see from P0 and P1 that a polar space is a semilattice.

Proposition 1 ([9, Lemma 2.19]). *A polar space is a complemented modular semilattice.*

In Section 3, we define submodular functions on polar spaces, which turn out to be appropriate generalizations of bisubmodular and k-submodular functions.

Example 2. For nonnegative integer $k \geq 2$, $\mathcal{S}_k$ is a polar space of rank 1, where polar frames are $\{0, a, b\} \simeq \mathcal{S}_2$ for distinct $a, b \in \mathcal{S}_k \setminus \{0\}$. The product $\mathcal{S}_k{}^n$ is also a polar space (of rank n). We give another example. For nonnegative integers $k, l \geq 2$, let $\mathcal{S}_{k,l}$ denote the poset on $\mathcal{S}_k \times \mathcal{S}_l$ with partial order: $(a, 0) \to (a, b) \leftarrow (0, b)$ and $(a, b) \to (0, 0)$ for $a \in \mathcal{S}_k \setminus \{0\}$ and $b \in \mathcal{S}_l \setminus \{0\}$. Notice that this partial order is different from the one in the direct product. Then $\mathcal{S}_{k,l}$ is a polar space of rank 2, where polar frames take the form of $\{0, a, a'\} \times \{0, b, b'\}$ with $a \neq a'$ and $b \neq b'$.

2.3. Orientable modular graphs and swm-graphs

In this section, we introduce orientable modular graphs and swm-graphs, which are the underlying structures of L-convex and L-extendable functions in Section 4. For a graph G, let $d = d_G$ denote the shortest path metric with respect to a specified uniform positive edge-length of G. The *metric interval* $I(x, y)$ of vertices x, y is the set of points z satisfying $d(x, z) + d(z, y) = d(x, y)$. We regard G as a metric space with this metric d. A nonempty subset Y of vertices is said to be *d-convex* if $I(x, y) \subseteq Y$ for every $x, y \in Y$.

2.3.1. Modular graphs and orientable modular graphs

A *modular graph* is a graph such that for every triple x, y, z of vertices, the intersection $I(x, y) \cap I(y, z) \cap I(z, x)$ is nonempty. An edge-orientation of a modular graph is called *admissible* [38] if for every 4-cycle (x_1, x_2, x_3, x_4), $x_1 \to x_2$ implies $x_4 \to x_3$. A modular graph may or may not have an admissible orientation. A modular graph is said to be *orientable* [38] if it has an admissible orientation. A modular graph endowed with an admissible orientation is called an *oriented modular graph*. Any admissible orientation is acyclic [30], and induces a partial order $\preceq$ on the set of vertices. In this way, an oriented modular graph is viewed as a poset.

A canonical example of an oriented modular graph is the Hasse diagram of a modular lattice. More generally, the following is known.

Theorem 2.1 ([5, Theorem 5.4]; see also [30]). *A semilattice is modular if and only if its Hasse diagram is oriented modular.*

Define a binary relation $\sqsubseteq$ on oriented modular graph Γ by: $p \sqsubseteq q$ if $p \preceq q$ and $[p, q]$ is a complemented modular lattice. Notice that $\sqsubseteq$ is not a partial order ($\sqsubseteq$ is not transitive) but $p \sqsubseteq q$ and $p \preceq p' \preceq q' \preceq q$ imply $p' \sqsubseteq q'$ (since any interval of a complemented modular lattice is a complemented modular lattice). For a vertex p, the *principal $\sqsubseteq$-ideal* $\mathcal{I}'_p$ (resp. *$\sqsubseteq$-filter* $\mathcal{F}'_p$) are the set of vertices q with $p \sqsupseteq q$ (resp. $p \sqsubseteq q$). In particular, $\mathcal{I}'_p \subseteq \mathcal{I}_p$ and $\mathcal{F}'_p \subseteq \mathcal{F}_p$ hold.

Proposition 2 ([30, Proposition 4.1, Theorem 4.2]). *Let Γ be an oriented modular graph.*

(1) *Every principal ideal (filter) is a modular semilattice and d-convex. In particular, every interval is a modular lattice.*

(2) *Every principal $\sqsubseteq$-ideal (filter) is a complemented modular semilattice and d-convex.*

The product of oriented modular graphs is an oriented modular graph [9, Proposition 2.16]. The behavior of relation $\sqsubseteq$ under product is given as follows.

Lemma 1 ([30, Lemma 4.6])**.** *For oriented modular graphs Γ, Γ', it holds $(x, x') \sqsubseteq (y, y')$ in $\Gamma \times \Gamma'$ if and only if it holds $x \sqsubseteq y$ in Γ and $x' \sqsubseteq y'$ in Γ'.*

An oriented modular graph is said to be *well-oriented* if $\preceq$ and $\sqsubseteq$ are the same, i.e., every interval is a complemented modular lattice. If Γ is well-oriented, then $\mathcal{I}'_p = \mathcal{I}_p$ and $\mathcal{F}'_p = \mathcal{F}_p$.

Example 3 (linear and alternating orientations of the grid)**.** The set $\mathbf{Z}$ of integers is regarded as a graph (path) by adding an edge to each pair $x, y \in \mathbf{Z}$ with $|x - y| = 1$. Any orientation of $\mathbf{Z}$ is admissible. There are two canonical orientations. The *linear orientation* is the edge-orientation such that $x \leftarrow y$ if $y = x + 1$. Let $\vec{\mathbf{Z}}$ denote the oriented modular graph $\mathbf{Z}$ endowed with the linear orientation. The *alternating orientation* of $\mathbf{Z}$ is the edge-orientation such that $x \to y$ if x is even. Let $\check{\mathbf{Z}}$ denote the oriented modular graph $\mathbf{Z}$ endowed with the alternating orientation. Observe that $\vec{\mathbf{Z}}$ is not well-oriented and $\check{\mathbf{Z}}$ is well-oriented. The products $\vec{\mathbf{Z}}^n$ and $\check{\mathbf{Z}}^n$ are oriented modular graphs, where $\check{\mathbf{Z}}^n$ is well-oriented. In $\vec{\mathbf{Z}}^n$, each principal filter is isomorphic to $\mathbf{Z}^n_+$ (with respect to the vector order) and each principal $\sqsubseteq$-filter is isomorphic to $\mathcal{S}_1^n = \{0, 1\}^n$. In $\check{\mathbf{Z}}^n$, the principal filter of vector p is isomorphic to $\mathcal{S}_2^k = \{-1, 0, 1\}^k$, where k is the number of odd components of p.

We introduce a structure, known as a *Euclidean building of type C*, which is an amalgamation of several $\check{\mathbf{Z}}^n$. As in the definition of polar spaces, we reformulate the definition in a poset terminology. Consider a poset $\mathcal{P}$ that is the union of subposets, called *apartments*, satisfying the following axiom:

B0: Each apartment is isomorphic to $\check{\mathbf{Z}}^n$ (as a poset).

B1: For any two chains C, D in $\mathcal{P}$, there is an apartment Σ containing them.

B2: If Σ and Σ' are apartments containing two chains C, D, then there is an isomorphism $\Sigma \to \Sigma'$ being identity on C and D.

A Euclidean building of type C is an abstract simplicial complex consisting of all chains of such a poset $\mathcal{P}$. See [1, Definitions 4.1 and 11.1] for the usual definition, where apartments are subcomplexes isomorphic to the Euclidean Coxeter complex of type C that is identical with the simplicial complex of all chains of $\check{\mathbf{Z}}^n$. The Hasse diagram of poset $\mathcal{P}$ satisfying B0, B1, and B2 is simply called the Hasse diagram of a Euclidean building of type C.

Theorem 2.2 ([9, Theorem 6.23])**.** *The Hasse diagram of a Euclidean building of type C is well-oriented modular.*

2.3.2. Weakly modular graphs and swm-graphs

A *weakly modular graph* is a connected graph satisfying the following conditions:

TC: For every triple of vertices x, y, z with $d(x, y) = 1$ and $d(x, z) = d(y, z)$, there exists a common neighbor u of x, y with $d(u, z) = d(x, z) - 1$.

QC: For every quadruple of vertices x, y, w, z with $d(x, y) = 2$, $d(w, y) = d(w, x) = 1$ and $d(w, z) - 1 = d(x, z) = d(y, z)$, there exists a common neighbor u of x, y with $d(u, z) = d(x, z) - 1$.

Modular graphs are precisely bipartite weakly modular graphs.

In a graph G, a nonempty subset X of vertices is said to be *d-gated* (or *gated*) if for every vertex x there is (unique) $y \in X$ such that $d(x, z) = d(x, y) + d(y, z)$ for every $z \in X$. A d-gated set is always d-convex. In our case, the following simple characterization of gated sets may be regarded as the definition.

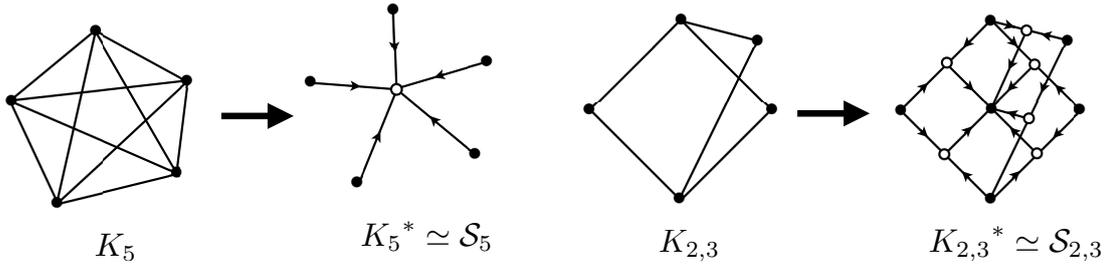$$K_5 \qquad K_5{}^* \simeq \mathcal{S}_5 \qquad K_{2,3} \qquad K_{2,3}{}^* \simeq \mathcal{S}_{2,3}$$

Figure 1: Barycentric subdivision

Lemma 2 (Chepoi's lemma; [9, Lemma 2.2]). *For a weakly modular graph G, a nonempty subset X is d-gated if and only if X induces a connected subgraph and for $x, y \in X$ every common neighbor of x, y belongs to X. If G is modular, then d-gated sets and d-convex sets are the same.*

A *sweakly modular graph* (*swm-graph* for short) [9] is a weakly modular graph having no induced K_4^--subgraph and isometric $K_{3,3}^-$-subgraph, where K_4^- and $K_{3,3}^-$ are graphs obtained from K_4 and $K_{3,3}$, respectively, by removing one edge. Since $K_{3,3}^-$ is not orientable, an orientable modular graph cannot have induced $K_{3,3}^-$. Therefore, any orientable modular graph is an swm-graph. A canonical example of an swm-graph is a *dual polar space* [8], which is a graph such that vertices are maximal elements of a polar space and two vertices p, q are adjacent if $p \to p \wedge q \leftarrow q$.

Theorem 2.3 ([9, Theorem 5.2]). *A graph is a dual polar space if and only if it is a thick swm-graph of finite diameter.*

Here a weakly modular graph is said to be *thick* if for every pair of vertices x, y with $d(x, y) = 2$ there are at least two common neighbors z, w of x, y with $d(z, w) = 2$.

For an swm-graph G, a nonempty set X of vertices is called *Boolean-gated* if X is gated and induces a connected thick subgraph. In a weakly modular graph the subgraph induced by a gated set is weakly modular. Thus, by Theorem 2.3, Boolean-gated sets are precisely gated sets inducing dual polar spaces. Also the nonempty intersection of Boolean-gated sets is again Boolean-gated. In this way, an swm-graph is viewed as an amalgamation of dual polar graphs. The set of all Boolean-gated sets of G is denoted by $\mathcal{B}(G)$. The partial order on $\mathcal{B}(G)$ is defined as the reverse inclusion order. Let G^* denote the Hasse diagram of $\mathcal{B}(G)$, called the *barycentric subdivision* of G. The edge-length of G^* is defined as one half of the edge-length of G.

The barycentric subdivision G^* serves as a *half-integral relaxation* of (problems on) G, and is used to define the notion of L-extendability in Section 4.

Example 4. Consider the path $\mathbf{Z}$. The barycentric subdivision $\mathbf{Z}^*$ is naturally identified with the set $\mathbf{Z}/2$ of half-integers with edge xy given if $|x - y| = 1/2$ and oriented $x \leftarrow y$ if $y \in \mathbf{Z}$. The product $\mathbf{Z}^n$ is the n-dimensional grid graph. A Boolean-gated set is exactly a vertex subset inducing a cube-subgraph, which is equal to $\{z \in \mathbf{Z}^n \mid x_i \le z_i \le y_i, i = 1, 2, \ldots, n\}$ for some $x, y \in \mathbf{Z}^n$ with $|x_i - y_i| \le 1$ ($i = 1, 2, \ldots, n$). The barycentric subdivision $(\mathbf{Z}^*)^n$ is isomorphic to the grid graph on the half-integral lattice $(\mathbf{Z}/2)^n$ with the alternating orientation. Note that $\check{\mathbf{Z}}^n$ is isomorphic to the half-integer grid $(\mathbf{Z}^*)^n$ by $x \mapsto x/2$.

Example 5. Trees, cubes, complete graphs, and complete bipartite graphs are all swm-graphs. The last three are dual polar spaces. In the case of a tree, Boolean-gated sets are all the singletons, and all the edges (pairs of vertices inducing edges). Hence the barycentric subdivision is just the edge-subdivision, where the original vertices are sources (i.e., have

no entering edges). Consider now the case of a cube. Boolean-gated sets are vertex subsets inducing cube-subgraphs. Therefore the barycentric subdivision is the facial subdivision of the cube, and is equal to $\mathcal{S}_2^n$. Suppose that G is a complete graph K_k of vertices $v_1, v_2, \ldots, v_k$. Boolean-gated sets are $\{v_1\}, \{v_2\}, \ldots, \{v_k\}$, and $\{v_1, v_2, \ldots, v_k\}$. Hence G^* is a star with k leaves, and is isomorphic to the polar space $\mathcal{S}_k$ (Example 1); see the left of Figure 1. Suppose that G is the n-product K_k^n of complete graph K_k. In this case, G^* is isomorphic to $\mathcal{S}_k^n$. Suppose that G is a complete bipartite graph $K_{k,l}$. Boolean-gated sets are all the singletons, all the edges, and the whole set of vertices. Therefore G^* is obtained by subdividing each edge $e = xy$ into a path $xv_e, v_e y$ and joining v_e to a new vertex corresponding to the whole set. Then G^* is isomorphic to the polar space $\mathcal{S}_{k,l}$; see the right of Figure 1.

An oriented modular graph is viewed as an amalgamation of complemented modular lattices as follows.

Lemma 3 ([9, Lemma 6.7]). *For an oriented modular graph Γ, a vertex set X is Boolean-gated if and only if $X = [x, y]$ for some vertices x, y with $x \sqsubseteq y$. Hence Γ^* is the poset of all intervals $[x, y]$ with $x \sqsubseteq y$, where the partial order is the reverse inclusion order.*

Any singleton $\{v\}$ of a vertex v is Boolean-gated. Thus we can regard $G \subseteq G^*$. Moreover G is an isometric subspace of G^*.

Theorem 2.4 ([9, Theorem 6.9] and [30, Proposition 4.5]). *For an swm-graph G, the barycentric subdivision G^* is oriented modular. Moreover, G is isometrically embedded into G^* by $x \mapsto \{x\}$, i.e., $d_G(x, y) = d_{G^*}(\{x\}, \{y\})$. In addition, if G is an oriented modular, then it holds*

$$d_{G^*}([p, q], [p', q']) = (d_G(p, p') + d_G(q, q'))/2 \quad ([p, q], [p', q'] \in G^*).$$

The barycentric subdivision G^* of a dual polar graph G is identical with the Hasse diagram of the polar space corresponding to G. Thus the barycentric subdivision of a general swm-graph is obtained by replacing each dual polar subgraph induced by a Boolean-gated set with the Hasse diagram of the corresponding polar space.

Lemma 4 ([9, Proposition 6.10]). *For an swm-graph G, the principal filter of every vertex of G^* is a polar space. Hence G^* is well-oriented.*

The *thickening* G^Δ of an swm-graph G is the graph obtained from G by joining all pairs of vertices belonging to a common Boolean-gated set [9, Section 6.5]. The shortest path metric with respect to G^Δ is denoted by $d^\Delta(:= d_{G^\Delta})$, where the edge-length of new edges are the same as in G. A path in G^Δ is called a Δ-*path*. In Section 4, the metric d^Δ is used to estimate the number of iterations of the steepest descent algorithm (SDA). The thickening is a kind of l_∞-metrization. The relation between G and G^Δ is similar to that between l_1- and l_∞-metrics.

Example 6. For a grid graph $G = \mathbf{Z}^n$, the thickening G^Δ is obtained from G by joining each pair of vertices x, y with $\|x - y\|_\infty \leq 1$. The distances of G and G^Δ are given as $d(x, y) = \|x - y\|_1$ and $d^\Delta(x, y) = \|x - y\|_\infty$. Here d-convex sets (d-gated sets) in G are precisely box subsets. Any l_∞-ball is a box subset, and is d-convex.

This property of l_∞-balls is generalized as follows. For a vertex x and a nonnegative integer r, let $B_r^\Delta(x)$ denote the set of vertices y with $d^\Delta(x, y) \leq r$, i.e., the ball of center x and radius r in G^Δ.

Lemma 5 ([9, Proposition 6.15]). *$B_r^\Delta(x)$ is d-gated.*

2.4. CAT(0) spaces and orthoscheme complexes

Here we introduce continuous spaces/relaxations into which the discrete structures introduced in Sections 2.2 and 2.3 are embedded, analogously to $\mathbf{Z}^n \hookrightarrow \mathbf{R}^n$. Let X be a metric space with metric $d : X \times X \to \mathbf{R}_+$. A *path* in X is a continuous map γ from $[0, 1]$ to X. The length of a path γ is defined as $\sup \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))$, where the supremum is taken over all sequences $t_0, t_1, \ldots, t_n$ in $[0, 1]$ with $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ $(n > 0)$. A path γ *connects* $x, y \in X$ if $\gamma(0) = x$ and $\gamma(1) = y$. A *geodesic* is a path γ satisfying $d(\gamma(s), \gamma(t)) = d(\gamma(0), \gamma(1))|s - t|$ for every $s, t \in [0, 1]$. If every pair of points in X can be connected by a geodesic, then X is called a *geodesic metric space*. If every pair of points can be connected by a unique geodesic, then X is said to be *uniquely geodesic*.

2.4.1. CAT(0) spaces

We introduce a class of uniquely-geodesic metric spaces, called CAT(0) spaces; see [3, 7]. Let X be a geodesic metric space. For points x, y in X, let $[x, y]$ denote the image of some geodesic γ connecting x, y (though such a geodesic is not unique). For $t \in [0, 1]$, the point p on $[x, y]$ with $d(x, p)/d(x, y) = t$ is denoted by the formal sum $(1 - t)x + ty$. A *geodesic triangle* of vertices $x, y, z \in X$ is the union of $[x, y]$, $[y, z]$, and $[z, x]$. A *comparison triangle* with $x, y, z \in X$ is the union of three segments $[\bar{x}, \bar{y}]$, $[\bar{y}, \bar{z}]$, and $[\bar{z}, \bar{x}]$ in the Euclidean plane $\mathbf{R}^2$ such that $d(x, y) = \|\bar{x} - \bar{y}\|_2$, $d(y, z) = \|\bar{y} - \bar{z}\|_2$, and $d(z, x) = \|\bar{z} - \bar{x}\|_2$. For $p \in [x, y]$, the *comparison point* $\bar{p}$ is the point on $[\bar{x}, \bar{y}]$ with $d(x, p) = \|\bar{x} - \bar{p}\|_2$.

A geodesic metric space is called CAT(0) if for every geodesic triangle $\Delta = [x, y] \cup [y, z] \cup [z, x]$ and every $p, q \in \Delta$, it holds $d(p, q) \leq \|\bar{p} - \bar{q}\|_2$. Intuitively this says that triangles in X are thinner than Euclidean plane triangles.

Proposition 3 ([7, Proposition 1.4]). *A* CAT(0) *space is uniquely geodesic.*

By this property, several convexity concepts are defined naturally in CAT(0) spaces. Let X be a CAT(0) space. A subset C of X is said to be *convex* if for arbitrary $x, y \in C$ the geodesic $[x, y]$ is contained in C. A function $f : X \to \overline{\mathbf{R}}$ on X is said to be *convex* if for every $x, y \in X$ and $t \in [0, 1]$ it holds

$$(1 - t)f(x) + tf(y) \geq f((1 - t)x + ty). \tag{2.1}$$

The Euclidean space is an obvious example of a CAT(0) space. Other examples include a *metric tree* (1-dimensional contractible complex endowed with the length metric), and the product of metric trees.

2.4.2. Orthoscheme complex and Lovász extension

An n-dimensional *orthoscheme* is a simplex in $\mathbf{R}^n$ with vertices

$$0, \ e_1, \ e_1 + e_2, \ e_1 + e_2 + e_3, \ \ldots, \ e_1 + e_2 + \cdots + e_n,$$

where e_i is the ith unit vector. An orthoscheme complex, introduced by Brady and McCammond [6] in the context of geometric group theory, is a metric simplicial complex obtained by gluing orthoschemes as in Figure 2. In our view, an orthoscheme complex is a generalization of the well-known simplicial subdivision of a cube $[0, 1]^n$ on which the Lovász extension of $f : \{0, 1\}^n \to \mathbf{R}$ is defined via the piecewise interpolation; see Example 7 below.

Let us introduce it formally. Let $\mathcal{P}$ be a *graded* poset, that is, a poset having a function $r : \mathcal{P} \to \mathbf{Z}$ with $r(q) = r(p) + 1$ whenever $q \to p$. Let $K(\mathcal{P})$ denote the set of all formal convex combinations $\sum_{p \in \mathcal{P}} \lambda(p)p$ of elements in $\mathcal{P}$ such that the nonzero support $\{p \mid \lambda(p) > 0\}$ forms a chain. The set of all formal combinations of some chain C is called a *simplex* of
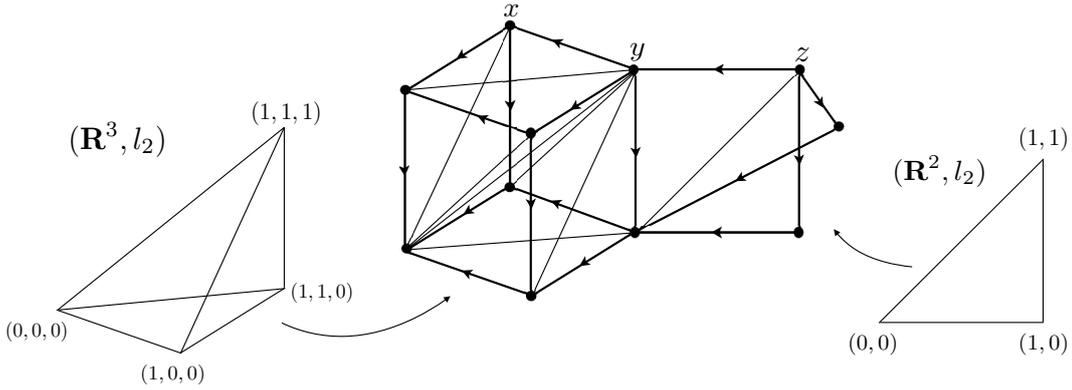
Figure 2: Orthoscheme complex

$K(\mathcal{P})$. For a simplex σ corresponding to a chain $C = p_0 \prec p_1 \prec \cdots \prec p_k$, define a map φ_σ from σ to the $(r(p_k) - r(p_0))$-dimensional orthoscheme by

$$\varphi_\sigma(x) = \sum_{i=1}^{k} \lambda_i(e_1 + e_2 + \cdots + e_{r(p_i)-r(p_0)}), \qquad (2.2)$$

where $x = \sum_{i=0}^{k} \lambda_k p_k \in \sigma$. For each simplex σ, a metric d_σ on σ is defined as

$$d_\sigma(x, y) := \|\varphi_\sigma(x) - \varphi_\sigma(y)\|_2 \quad (x, y \in \sigma).$$

The length of a path γ in $K(\mathcal{P})$ is defined as $\sup \sum_{i=0}^{m-1} d_{\sigma_i}(\gamma(t_i), \gamma(t_{i+1}))$, where sup is taken over all $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ $(m \geq 1)$ such that $\gamma([t_i, t_{i+1}])$ belongs to a simplex σ_i for each i. Then the metric on $K(\mathcal{P})$ is (well-)defined as above. The resulting metric space $K(\mathcal{P})$ is called the *orthoscheme complex* of $\mathcal{P}$ [6]. If the lengths of chains are uniformly bounded, then $K(\mathcal{P})$ is a (complete) geodesic metric space [7, Theorem 7.19].

By considering the orthoscheme complex, we can define an analogue of the Lovász extension for a function defined on any graded poset $\mathcal{L}$. For a function $f : \mathcal{L} \to \overline{\mathbf{R}}$, the *Lovász extension* $\overline{f}$ of f is a function on the orthoscheme complex $K(\mathcal{L})$ defined by

$$\overline{f}(x) := \sum_i \lambda_i f(p_i) \quad \left(x = \sum_i \lambda_i p_i \in K(\mathcal{L}) \right). \qquad (2.3)$$

In the case where $K(\mathcal{L})$ is CAT(0), we can discuss the convexity property of $\overline{f}$ with respect to the CAT(0)-metric. The following examples of $K(\mathcal{L})$ show that our Lovász extension actually generalizes the Lovász extension for functions on $\{0, 1\}^n$ and on $\{-1, 0, 1\}^n$.

Example 7. Let $\mathcal{L}$ be a Boolean lattice with atoms $a_1, a_2, \ldots, a_n$. Then $K(\mathcal{L})$ consists of points $p = \sum_{k=0}^{n} \lambda_k(a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_k})$ for some permutation $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ and nonnegative coefficients λ_k whose sum is equal to one. The map $p = \sum_{k=0}^{n} \lambda_k(a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_k}) \mapsto \sum_{k=0}^{n} \lambda_k(e_{i_1} + e_{i_2} + \cdots + e_{i_k})$ is an isometry from $K(\mathcal{L})$ to cube $[0, 1]^n$ with respect to l_2-metric [9, Lemma 7.7]. In particular, $K(\mathcal{L})$ is viewed as the well-known simplicial subdivision of cube $[0, 1]^n$ consisting of vertices $0, e_{i_1}, e_{i_1} + e_{i_2}, \ldots, e_{i_1} + e_{i_2} + \cdots + e_{i_n}$ for all permutations $(i_1, i_2, \ldots, i_n)$. Let $\mathcal{L}$ be a distributive lattice of rank n. By the Birkhoff representation theorem, $\mathcal{L}$ is the poset of principal ideals of a poset $\mathcal{P}$ of n elements $a_1, a_2, \ldots, a_n$. Then $K(\mathcal{L})$ is isometric to the well-known simplicial subdivision of the *order polytope* $\{x \in [0, 1]^n \mid x_i \geq x_j \ (i, j : a_i \prec a_j)\}$ into simplices of the above form [9, Proposition 7.7].
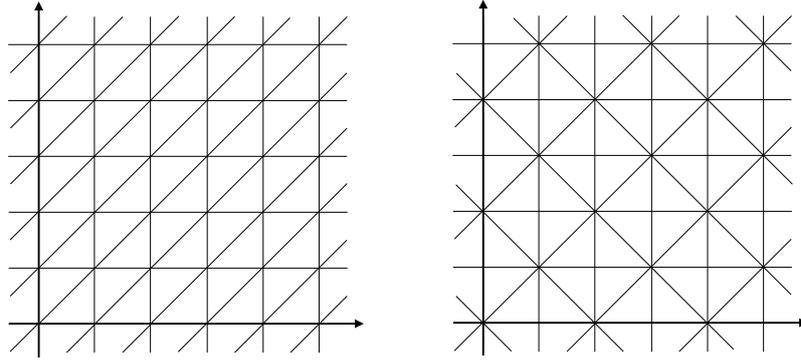
Figure 3: $K'(\vec{\mathbf{Z}}^2)$ (left) and $K(\check{\mathbf{Z}}^2)$ (right)

Example 8. Consider $\mathcal{S}_2 = \{-1, 0, 1\}$ and the product $\mathcal{S}_2{}^n$ (Example 1). Then $K(\mathcal{S}_2{}^n)$ is CAT(0), isomorphic to the simplicial subdivision of the cube $[-1, 1]^n$ consisting of simplices of vertices $0, s_{i_1}, s_{i_1} + s_{i_2}, \ldots, s_{i_1} + s_{i_2} + \cdots + s_{i_n}$, where $(i_1, i_2, \ldots, i_n)$ is a permutation of $\{1, 2, \ldots, n\}$ and $s_i \in \{e_i, -e_i\}$ for $i = 1, 2, \ldots, n$.

In these examples, $K(\mathcal{L})$ is isometric to a convex polytope in the Euclidean space, and obviously is CAT(0). In the case of a modular lattice $\mathcal{L}$, the orthoscheme complex $K(\mathcal{L})$ cannot be realized as a convex polytope of a Euclidean space. However $K(\mathcal{L})$ still has the CAT(0) property.

Theorem 2.5 ([9, Theorem 7.2]). *The orthoscheme complex of a modular lattice is* CAT(0).

This property holds for every polar space.

Theorem 2.6 ([9, Proposition 7.4]). *The orthoscheme complex of a polar space is* CAT(0).

By using these theorems, in Section 3, we characterize a submodular function on a modular lattice/polar space $\mathcal{L}$ as a function on $\mathcal{L}$ such that its Lovász extension is convex on $K(\mathcal{L})$.

An oriented modular graph $\varGamma$ is graded (as a poset) [9, Theorem 6.2]. Thus we can consider $K(\varGamma)$. A special interest lies in the subcomplex $K'(\varGamma)$ of $K(\varGamma)$ consisting of simplices that correspond to a chain $p_0 \prec p_1 \prec \cdots \prec p_k$ with $p_0 \sqsubseteq p_k$. Figure 2 depicts $K'(\varGamma)$, where $x \prec y \prec z$ but $x \not\sqsubseteq z$, hence the simplex corresponding to $\{x, y, z\}$ does not exist. If $\varGamma$ has a uniform edge-length s, we multiply φ_σ by s in (2.2).

Example 9. Consider linearly-oriented grid graph $\vec{\mathbf{Z}}^n$ (Example 4). Then $x \sqsubseteq y$ if and only if $x \leq y$ and $\|x - y\|_\infty \leq 1$. Therefore the orthoscheme complex $K'(\vec{\mathbf{Z}}^n)$ is the simplicial subdivision of $\mathbf{R}^n$ into simplices of vertices $x, x + e_{i_1}, x + e_{i_1} + e_{i_2}, \ldots, x + e_{i_1} + e_{i_2} + \cdots + e_{i_n}$ over all $x \in \mathbf{Z}^n$ and all permutations $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$; see the left of Figure 3. Next consider $\check{\mathbf{Z}}^n$. Then $x \sqsubseteq y$ if and only if $\|x - y\|_\infty \leq 1$ and $x_i \notin 2\mathbf{Z}$ implies $x_i = y_i$. The orthoscheme complex $K'(\check{\mathbf{Z}}^n) = K(\check{\mathbf{Z}}^n)$ is the simplicial subdivision of $\mathbf{R}^n$ into simplices of vertices $x, x + s_{i_1}, x + s_{i_1} + s_{i_2}, \ldots, x + s_{i_1} + s_{i_2} + \cdots + s_{i_n}$ over all $x \in (2\mathbf{Z})^n$, all permutations $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, and $s_i \in \{e_i, -e_i\}$ for $i = 1, 2, \ldots, n$; see the right of Figure 3. The simplicial complex $K(\check{\mathbf{Z}}^n)$ is known as the *Euclidean Coxeter complex of type C* in building theory [1], and is called the *Union-Jack division* in [14].

Suppose that $\varGamma$ is the Hasse diagram of a Euclidean building of type C. In this case, the orthoscheme complex $K(\varGamma)$ is the same as the standard metrization of the building, and is known to be CAT(0); see [1, Section 11].

Theorem 2.7 (see [1, Theorem 11.16]). *Let $\varGamma$ be the Hasse diagram of a Euclidean building of type C. The orthoscheme complex $K(\varGamma)$ is CAT(0).*
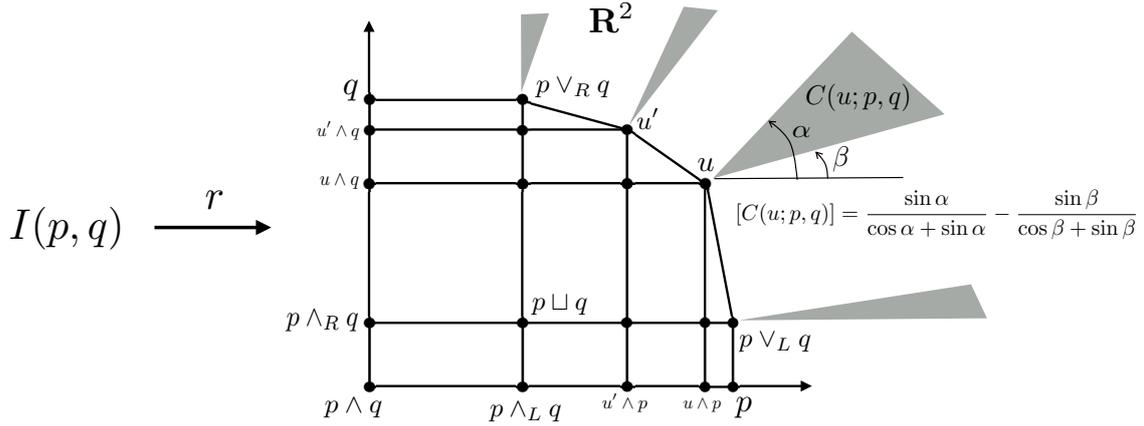
Figure 4: Construction of the fractional join

By using this theorem, in Section 4, we characterize L-convex functions on a Euclidean building by means of the convexity of the Lovász extension.

3. Submodular Functions on Modular Semilattices

In this section, we study submodular functions on modular semilattices. We quickly review the (rather complicated) definition and basic results established in [30]. Then we focus on submodular functions on polar spaces (Section 3.2). In a polar space, the defining inequality for submodularity is quite simple (Theorems 3.2 and 3.3). We show that the submodularity is characterized by the convexity of the Lovász extension (Section 3.2.1), and establish the relation to k-submodular and α-bisubmodular functions (Sections 3.2.2 and 3.2.3).

3.1. Basics

3.1.1. Definition

Let $\mathcal{L}$ be a modular semilattice and d denote the path metric of the Hasse diagram of $\mathcal{L}$. To define "join" of $p, q \in \mathcal{L}$, the previous work [30] introduced the following construction.[†] Figure 4 is a conceptual diagram.

(i) The metric interval $I(p,q)$ is equal to the set of elements u that is represented as $u = a \vee b$ for some $(a, b) \in [p \wedge q, p] \times [p \wedge q, q]$, where such a representation is unique, and (a, b) equals $(u \wedge p, u \wedge q)$ [30, Lemma 2.15].

(ii) For $u \in I(p,q)$, let $r(u; p, q)$ be the vector in $\mathbf{R}_+^2$ defined by

$$r(u; p, q) = (r(u \wedge p) - r(p \wedge q), r(u \wedge q) - r(p \wedge q)). \tag{3.1}$$

(iii) Let $\operatorname{Conv} I(p,q)$ denote the convex hull of vectors $r(u; p, q)$ for all $u \in I(p,q)$.

(iv) Let $\mathcal{E}(p,q)$ be the set of elements u in $I(p,q)$ such that $r(u; p, q)$ is a maximal extreme point of $\operatorname{Conv} I(p,q)$. Then $\mathcal{E}(p,q) \ni u \mapsto r(u; p, q)$ is injective [30, Lemma 3.1].

(vi) For $u \in \mathcal{E}(p,q)$, let $C(u; p, q)$ denote the nonnegative normal cone at $r(u; p, q)$:

$$C(u; p, q) := \left\{ w \in \mathbf{R}_+^2 \mid \langle w, r(u; p \wedge q) \rangle = \max_{x \in \operatorname{Conv} I(p,q)} \langle w, x \rangle \right\}.$$

(vii) For a convex cone C in $\mathbf{R}_+^2$ represented as

$$C = \{ (x, y) \in \mathbf{R}_+^2 \mid y \cos \alpha \leq x \sin \alpha, y \cos \beta \geq x \sin \beta \}$$

[†]The argument in [30] works even if $|\mathcal{L}| = \infty$.

for some $0 \leq \beta \leq \alpha \leq \pi/2$, let

$$[C] := \frac{\sin\alpha}{\sin\alpha + \cos\alpha} - \frac{\sin\beta}{\sin\beta + \cos\beta}.$$

(viii) The *fractional join* of $p, q \in \mathcal{L}$ is defined as the formal sum $\sum_{u \in \mathcal{E}(p,q)} [C(u; p, q)]u$.

We use an alternative form of the fractional join. Let $\mathcal{E}(\mathcal{L})$ denote the set of all binary operations $\theta : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that (i) $\theta(p, q)$ belongs to $\mathcal{E}(p, q)$ for $p, q \in \mathcal{L}$ and (ii) the cone

$$C(\theta) := \bigcap_{p,q \in \mathcal{L}} C(\theta(p, q); p, q)$$

has an interior point, i.e., $[C(\theta)] > 0$. Then it is shown in [30, Proposition 3.3] that the following equality holds:

$$\sum_{\theta \in \mathcal{E}(\mathcal{L})} [C(\theta)]\theta(p, q) = \sum_{u \in \mathcal{E}(p,q)} [C(u; p, q)]u \quad (p, q \in \mathcal{L}). \tag{3.2}$$

The *fractional join operation* of $\mathcal{L}$, denoted by the formal sum $\sum_{\theta \in \mathcal{E}(\mathcal{L})} [C(\theta)]\theta$, is a function on $\mathcal{L} \times \mathcal{L}$ defined by $(p, q) \mapsto \sum_{\theta \in \mathcal{E}(\mathcal{L})} [C(\theta)]\theta(p, q)$.

We are now ready to define submodular functions on $\mathcal{L}$. A function $f : \mathcal{L} \to \overline{\mathbf{R}}$ is called *submodular* if it satisfies

$$f(p) + f(q) \geq f(p \wedge q) + \sum_{\theta \in \mathcal{E}(\mathcal{L})} [C(\theta)]f(\theta(p, q)) \quad (p, q \in \mathcal{L}), \tag{3.3}$$

where $\sum_{\theta \in \mathcal{E}(\mathcal{L})} [C(\theta)]f(\theta(p, q))$ may be replaced by $\sum_{u \in \mathcal{E}(p,q)} [C(u; p, q)]f(u)$. If $\mathcal{L}$ is a (modular) lattice, then the fractional join is equal to the join $\vee$, and our definition of submodularity coincides with the usual definition. The class of our submodular functions includes all constant functions, and is closed under nonnegative combinations and direct sums [30, Lemma 3.7]. Here the *direct sum* of two functions $g : X \to \overline{\mathbf{R}}$ and $h : Y \to \overline{\mathbf{R}}$ is the function on $X \times Y$ defined by $(x, y) \mapsto g(x) + h(y)$. A canonical example of submodular functions is the distance function of a modular semilattice.

Proposition 4 ([30, Theorem 3.6]). *Let $\mathcal{L}$ be a modular semilattice. The distance function d is submodular on $\mathcal{L} \times \mathcal{L}$.*

Consider the important case where a modular semilattice $\mathcal{L}$ in question is the product of (smaller) n modular semilattices $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$. For binary operations $\theta_i : \mathcal{L}_i \times \mathcal{L}_i \to \mathcal{L}_i$ for $i = 1, 2, \ldots, n$, the *componentwise extension* $(\theta_1, \theta_2, \ldots, \theta_n)$ is the binary operation $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by

$$(\theta_1, \theta_2, \ldots, \theta_n)(p, q) := (\theta_1(p_1, q_1), \theta_2(p_2, q_2), \ldots, \theta_n(p_n, q_n))$$

for $p = (p_1, p_2, \ldots, p_n), q = (q_1, q_2, \ldots, q_n) \in \mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$. Then the fractional join of $\mathcal{L}$ is decomposed as follows.

Proposition 5 ([30, Proposition 3.4]).

$$\sum_{\theta \in \mathcal{E}(\mathcal{L})} [C(\theta)]\theta = \sum_{\theta_1, \theta_2, \ldots, \theta_n} [C(\theta_1) \cap C(\theta_2) \cap \cdots \cap C(\theta_n)](\theta_1, \theta_2, \ldots, \theta_n),$$

where the summation over θ_i is taken over all binary operations in $\mathcal{E}(\mathcal{L}_i)$ for $i = 1, 2, \ldots, n$. Moreover, if $\mathcal{L}_i = \mathcal{L}_j$ and $\theta_i \neq \theta_j$ for some i, j, then $[C(\theta_1) \cap C(\theta_2) \cap \cdots \cap C(\theta_n)] = 0$.

3.1.2. Left join $\vee_L$, right join $\vee_R$, and pseudo join $\sqcup$

We introduce three canonical operations. For $p, q \in \mathcal{L}$, there exists a unique maximal element $u \in [p \wedge q, q]$ such that p and u have the join $p \vee u$. Indeed, suppose that for $u, u' \in [p \wedge q, q]$ both joins $u \vee p$ and $u' \vee p$ exist. By the definition of a modular semilattice, $u \vee u' \vee p$ exists. Define the *left join* $p \vee_L q$ as the join of p and the maximal element u. Also define the *right join* $p \vee_R q$ as the join of q and the unique maximal element $v \in [p \wedge q, p]$ with $v \vee q \in \mathcal{L}$. Define the *pseudo join* $\sqcup$ as the meet of the left join and the right join:

$$p \sqcup q := (p \vee_L q) \wedge (p \vee_R q) \quad (p, q \in \mathcal{L}). \tag{3.4}$$

Example 10 ($\vee_L$, $\vee_R$, $\sqcup$ in $\mathcal{S}_k{}^n$)**.** In $\mathcal{S}_k{}^n$, the left join $p \vee_L q$ is obtained from p by replacing p_i with q_i for each i with $q_i \neq 0 = p_i$. Then $p \sqcup q$ is given by

$$(p \sqcup q)_i := \begin{cases} p_i \vee q_i & \text{if } p_i \preceq q_i \text{ or } p_i \preceq q_i, \\ 0 & \text{if } 0 \neq p_i \neq q_i \neq 0. \end{cases} \tag{3.5}$$

Hence $\sqcup$ here equals the $\sqcup$ used in [33], where $\wedge$ here equals the $\sqcap$ there. The left and right joins are represented as $p \vee_L q = (p \sqcup q) \sqcup p$ and $p \vee_R q = (p \sqcup q) \sqcup q$.

3.1.3. Submodularity with respect to valuation

A *valuation* of a modular semilattice $\mathcal{L}$ is a function $v : \mathcal{L} \to \mathbf{R}$ such that

(1) $v(p) < v(q)$ for $p, q \in \mathcal{L}$ with $p \prec q$, and
(2) $v(p) + v(q) = v(p \wedge q) + v(p \vee q)$ for $p, q \in \mathcal{L}$ with $p \vee q \in \mathcal{L}$.

The rank function r is a valuation.

For a valuation v, let $v(u; p, q)$ be defined by replacing r with v in $r(u; p, q)$ in (3.1). Then Conv $I(p, q)$, $\mathcal{E}(p, q)$, $C(u; p, q)$, $C(\theta)$, and $\mathcal{E}(\mathcal{L})$ are defined by replacing $r(u; p, q)$ with $v(u; p, q)$. In this setting with a general valuation v, the fractional join and the fractional join operation are also defined, and (3.2) holds. The corresponding submodular functions are called submodular functions *with respect to valuation* v.

Suppose that $\mathcal{L}$ is the product of modular semilattices $\mathcal{L}_i$ for $i = 1, 2, \ldots, n$ and v is a valuation of $\mathcal{L}$. Then there exist valuations v_i on $\mathcal{L}_i$ for $i = 1, 2, \ldots, n$ such that v is represented as $v(x) = v_1(x_1) + v_2(x_2) + \cdots + v_n(x_n)$ for $x = (x_1, x_2, \ldots, x_n) \in \mathcal{L}$. Then Proposition 5 holds, where $C(\theta_i)$ and $\mathcal{E}(\mathcal{L}_i)$ are defined with respect to the valuation v_i, and by $\mathcal{L}_i = \mathcal{L}_j$ we mean that $\mathcal{L}_i = \mathcal{L}_j$ (as a poset) and $v_i = v_j$.

3.1.4. Minimization

Here we consider the problem of minimizing submodular functions on the product of finite modular semilattices $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$. We do not know whether this problem is tractable in the value-oracle model. We consider the VCSP situation, where submodular function in question is given by the sum of submodular functions with a small number of variables. Let us formulate the *Valued Constraint Satisfaction Problem (VCSP)* more precisely. An instance/input of the problem consists of finite sets (domains) $D_1, D_2, \ldots, D_n$, functions $f_i : D_{i_1} \times D_{i_2} \times \cdots \times D_{i_{k_i}} \to \overline{\mathbf{R}}$ with $i = 1, 2, \ldots, m$ and $1 \leq i_1 < i_2 < \cdots < i_{k_i} \leq n$, where each f_i is given as the table of all function values. The objective is to find $p = (p_1, p_2, \ldots, p_n) \in D_1 \times D_2 \times \cdots \times D_n$ that minimizes $\sum_{i=1}^m f_i(p_{i_1}, p_{i_2}, \ldots, p_{i_{k_i}})$. In this situation, the size of the input is $O(nN + mN^k)$, where $N := \max_i |D_i|$ and $k := \max_i k_i$. As a consequence of a tractability criterion of VCSP by Thapper and Živný [54] (see also [42]), we have:

Theorem 3.1 ([30, Theorem 3.9])**.** *Suppose that each D_i is a modular semilattice and f_i is submodular on $D_{i_1} \times D_{i_2} \times \cdots \times D_{i_{k_i}}$. Then VCSP is solved in strongly polynomial time.*

Remark 1. Well-known oracle-minimizable classes of our submodular functions are those on $\mathcal{L} = \mathcal{S}_1^n$ (corresponding to the ordinary submodular functions) and on $\mathcal{S}_2^n$ (corresponding to bisubmodular functions). Fujishige, Tanigawa and Yoshida [17] and Huber and Krokhin [34] showed the oracle tractability of α-bisubmodular functions (= submodular functions on $\mathcal{S}_2^n$ with valuations); see Section 3.2.3. Consider submodular functions on the product $\mathcal{L}$ of complemented modular lattices of rank 2 (*diamonds*). Following a pioneering work of Kuivinen [43] on NP $\cap$ co-NP characterization, Fujishige, Király, Makino, Takazawa, and Tanigawa [15] proved the oracle-tractability of this class of submodular functions.

3.2. Submodular functions on polar spaces

Here we consider submodular functions on polar spaces; see Section 2.2 for the definition of polar spaces. It turns out that they are natural generalization of bisubmodular functions. We first show the explicit formula of the fractional join in a polar space.

Theorem 3.2. *In a polar space, the fractional join operation is equal to* $(1/2)\vee_L + (1/2)\vee_R$.

Thus submodular functions on polar space $\mathcal{L}$ are functions satisfying

$$f(p) + f(q) \geq f(p \wedge q) + \frac{1}{2}f(p \vee_L q) + \frac{1}{2}f(p \vee_R q) \quad (p, q \in \mathcal{L}). \tag{3.6}$$

We present an alternative characterization of submodularity using pseudo join $\sqcup$.

Theorem 3.3. *Let $\mathcal{L}$ be a polar space. For a function $f : \mathcal{L} \to \overline{\mathbf{R}}$, the following conditions are equivalent:*

(1) *f is submodular on $\mathcal{L}$.*
(2) *$f(p) + f(q) \geq f(p \wedge q) + f(p \sqcup q)$ holds for $p, q \in \mathcal{L}$.*
(3) *f is bisubmodular on each polar frame $\mathcal{F} \simeq \mathcal{S}_2^n$.*

3.2.1. Convexity of the Lovász extension

We discuss the convexity property of Lovász extensions with respect to the metric on the orthoscheme complex. As seen in Example 7, the orthoscheme complex of a distributive lattice is isometric to the order polytope in $[0, 1]^n$. It is well-known that a submodular function on a distributive lattice can be characterized by the convexity of its Lovász extension on the order polytope.

Theorem 3.4 ([44]; see [13])**.** *Let $\mathcal{L}$ be a distributive lattice. A function $f : \mathcal{L} \to \overline{\mathbf{R}}$ is submodular if and only if the Lovász extension $\overline{f} : K(\mathcal{L}) \to \overline{\mathbf{R}}$ is convex.*

We first establish an analogy of this characterization for submodular functions on modular lattices. Recall Theorem 2.5 that the orthoscheme complex of a modular lattice is CAT(0).

Theorem 3.5. *Let $\mathcal{L}$ be a modular lattice. A function $f : \mathcal{L} \to \overline{\mathbf{R}}$ is submodular if and only if the Lovász extension $\overline{f} : K(\mathcal{L}) \to \overline{\mathbf{R}}$ is convex.*

Consider the case where $\mathcal{L}$ is $\mathcal{S}_2^n = \{-1, 0, 1\}^n$. In this case, submodular functions on $\mathcal{S}_2^n$ in our sense are bisubmodular functions; see Section 3.2.2. Also $K(\mathcal{S}_2^n)$ is isometric to $[-1, 1]^n$. Thus the Lovász extension of a function $f : \mathcal{S}_2^n \to \overline{\mathbf{R}}$ is the same as that given by Qi [49].

Theorem 3.6 ([49])**.** *A function $f : \mathcal{S}_2^n \to \overline{\mathbf{R}}$ is bisubmodular if and only if $\overline{f} : K(\mathcal{S}_2^n) \to \overline{\mathbf{R}}$ is convex.*

Recall Example 1 that $\mathcal{S}_k^n$ is a polar space. By Theorem 2.6, the orthoscheme complex of a polar space is CAT(0). The generalization of Theorem 3.6 is the following.

Theorem 3.7. *Let $\mathcal{L}$ be a polar space. A function $f : \mathcal{L} \to \overline{\mathbf{R}}$ is submodular if and only if the Lovász extension $\overline{f} : K(\mathcal{L}) \to \overline{\mathbf{R}}$ is convex.*

It would be interesting to develop minimization algorithms based on this CAT(0) convexity; see [22] for a related attempt. Notice that submodularity and convexity are not equal in general, even if $K(\mathcal{L})$ is CAT(0); consider a modular semilattice $\mathcal{L} = \{0, a, b, c, a'\}$ such that a, b, a' are atoms, and c covers a, b and has no relation with a'.

3.2.2. k-submodular functions

We discuss k-submodular functions of Huber and Kolmogorov [33] from our viewpoint. Let k, n be nonnegative integers with $k \geq 2$ and $n \geq 1$. Then $\mathcal{S}_k{}^n$ is a polar space. Recall the operation $\sqcup$ (Example 10). A function $f : \mathcal{S}_k{}^n \to \overline{\mathbf{R}}$ is called k-submodular [33] if

$$f(p) + f(q) \geq f(p \wedge q) + f(p \sqcup q) \quad (p, q \in \mathcal{S}_k{}^n). \tag{3.7}$$

By Theorem 3.3 we have:

Theorem 3.8. $f : \mathcal{S}_k{}^n \to \overline{\mathbf{R}}$ *is k-submodular if and only if f is submodular on $\mathcal{S}_k{}^n$.*

Remark 2. Submodular functions on polar space $\mathcal{S}_{k,l}{}^n$ (Example 1) will be a next natural class to be investigated, and called (k, l)-*submodular*. In Section 5, we see that a $(2, k)$-submodular function arises from the node-multiway cut problem.

3.2.3. $\boldsymbol{\alpha}$-bisubmodular functions

Here we study α-bisubmodular functions introduced by Huber, Krokhin, and Powell [35], and their generalization by Fujishige, Tanigawa, and Yoshida [17]. Let $\mathcal{S}_2 = \{-, 0, +\}$. In addition to $\sqcup$, define a new binary operation $\sqcup_+$ on $\mathcal{S}_2$ by

$$x \sqcup_+ y = \begin{cases} + & \text{if } \{x, y\} = \{-, +\}, \\ x \sqcup y & \text{otherwise,} \end{cases} \tag{3.8}$$

and extend it to an operation on $\mathcal{S}_2{}^n$ componentwise. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-tuple of positives satisfying $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1$. For $i = 0, 1, 2, \ldots, n$, define operation $\sqcup^i$ by $\sqcup^i := (\underbrace{\sqcup_+, \sqcup_+, \ldots, \sqcup_+}_{i}, \sqcup, \sqcup, \ldots, \sqcup)$. A function $f : \mathcal{S}_2{}^n \to \overline{\mathbf{R}}$ is called $\boldsymbol{\alpha}$-*bisubmodular* [17] if

$$f(p) + f(q) \geq f(p \wedge q) + \sum_{i=0}^{n} (\alpha_{i+1} - \alpha_i) f(p \sqcup^i q) \quad (p, q \in \mathcal{S}_2{}^n), \tag{3.9}$$

where we let $\alpha_0 := 0$ and $\alpha_{n+1} := 1$. Notice that $(\alpha, \alpha, \ldots, \alpha)$-bisubmodularity coincides with α-bisubmodularity in the sense of [35].

Our framework captures $\boldsymbol{\alpha}$-bisubmodularity by using valuations; see Section 3.1.3. For $i = 1, 2, \ldots, n$, define a valuation v_i on $\mathcal{S}_2$ by $v_i(0) := 0$, $v_i(+) := 1$, and $v_i(-) := \alpha_i$. Define a valuation $v_{\boldsymbol{\alpha}}$ on $\mathcal{S}_2{}^n$ by

$$v_{\boldsymbol{\alpha}}(x) := v_1(x_1) + v_2(x_2) + \cdots + v_n(x_n) \quad (x = (x_1, x_2, \ldots, x_n) \in \mathcal{S}_2{}^n).$$

Theorem 3.9. $f : \mathcal{S}_2{}^n \to \overline{\mathbf{R}}$ *is $\boldsymbol{\alpha}$-bisubmodular if and only if f is submodular on the modular semilattice $\mathcal{S}_2{}^n$ with respect to the valuation $v_{\boldsymbol{\alpha}}$.*

Theorem 3.9 can be obtained from an explicit formula of the fractional join operation of $\mathcal{S}_2{}^n$. For $i = 0, 1, 2, \ldots, n$, define operations $\vee_L^i$ and $\vee_R^i$ by

$$\vee_L^i := (\underbrace{\sqcup_+, \sqcup_+, \ldots, \sqcup_+}_{i}, \vee_L, \vee_L, \ldots, \vee_L), \quad \vee_R^i := (\underbrace{\sqcup_+, \sqcup_+, \ldots, \sqcup_+}_{i}, \vee_R, \vee_R, \ldots, \vee_R).$$

Proposition 6. *The fractional join operation of $\mathcal{S}_2{}^n$ with respect to the valuation $v_{\boldsymbol{\alpha}}$ is equal to*

$$\sum_{i=0}^{n-1} \left(\frac{1}{1 + \alpha_i} - \frac{1}{1 + \alpha_{i+1}} \right) (\vee_L^i + \vee_R^i) + \frac{1 - \alpha_n}{1 + \alpha_n} \sqcup^n. \tag{3.10}$$

4. L-convex Functions on Oriented Modular Graphs

In this section, we study L-convex functions on oriented modular graphs. In Section 4.1, we define L-convex functions and establish some basic properties (closedness under nonnegative summation, local characterization, L-optimality criterion, steepest descent algorithm (SDA)). A central result in this section is a sharp iteration bound of SDA (Theorem 4.3). Then we introduce L-extendable functions on swm-graphs via barycentric subdivision, and establish the persistency property (Theorem 4.4). In Section 4.2, we study L-convex functions on (oriented modular graphs arising from) Euclidean buildings. In a Euclidean building, the discrete midpoint operators are naturally defined. We show in Theorem 4.5 that L-convex functions are characterized by the discrete midpoint convexity as well as the convexity of the Lovász extension. Then we explain how our framework captures other discrete convex functions, such as L$^{\natural}$-convex functions, UJ-convex functions, strongly-tree submodular functions, and alternating L-convex functions.

4.1. Basics

4.1.1. Definition

Let Γ be an oriented modular graph. We define L-convex functions on Γ, following basically [30]. Here we allow Γ to be an infinite graph, and functions on Γ to take an infinite value, while [30] assumed that the domain is finite and function values are finite. We utilize notions in Section 2.3, such as the relation $\sqsubseteq$, the principal ideal $\mathcal{I}_x$, the principal $\sqsubseteq$-ideal $\mathcal{I}'_x$, and the barycentric subdivision.

We first introduce a connectivity concept for subsets of vertices in Γ. A sequence $x = x_0, x_1, \ldots, x_m = y$ of vertices is called a Δ'-*path* if $x_i \sqsubseteq x_{i+1}$ or $x_{i+1} \sqsubseteq x_i$ for $i = 0, 1, 2, \ldots, m-1$. A subset X of vertices of Γ is said to be Δ'-*connected* if every pair of distinct vertices in X is connected by a Δ'-path in X.

Next we introduce the local structure at each vertex of Γ via barycentric subdivision. Let $\Gamma^*(= \{[x,y] \mid x,y \in \Gamma : x \sqsubseteq y\})$ be the barycentric subdivision. Recall that Γ is isometrically embedded into Γ^* by $x \mapsto \{x\}$ (Theorem 2.4). We regard $\Gamma \subseteq \Gamma^*$. By Lemma 4, Γ^* is well-oriented. For a vertex x of Γ, the *neighborhood semilattice* $\mathcal{I}_x^*$ is defined by

$$\mathcal{I}_x^* := \mathcal{I}_{\{x\}}(\Gamma^*) = \mathcal{I}'_{\{x\}}(\Gamma^*).$$

By Proposition 2, $\mathcal{I}_x^*$ is a (complemented) modular semilattice.

For a function $g : \Gamma \to \overline{\mathbf{R}}$, define $g^* : \Gamma^* \to \overline{\mathbf{R}}$ by

$$g^*([x,y]) := (g(x) + g(y))/2 \quad ([x,y] \in \Gamma^*). \tag{4.1}$$

A function $g : \Gamma \to \overline{\mathbf{R}}$ is called *L-convex* if dom g is Δ'-connected, and for every vertex x, the restriction of g^* to $\mathcal{I}_x^* \subseteq \Gamma^*$ is submodular.

The class of L-convex functions includes all constant functions, and is closed under a direct sum; the Δ'-connectivity of the domain of direct sum follows from Lemma 1 (2). Here the closedness under nonnegative sum is nontrivial.

Lemma 6. *A nonnegative sum of two L-convex functions is L-convex.*

Two basic examples of L-convex functions are given. The *indicator function* $[X]$ of a vertex subset X is defined by $[X](x) = 0$ if $x \in X$ and $[X](x) = \infty$ otherwise..

Lemma 7. *The indicator function of a d_Γ-convex set is L-convex on Γ.*

Theorem 4.1 ([30, Theorem 4.8]). *The distance function d_Γ is L-convex on $\Gamma \times \Gamma$.*

L-convex functions are locally submodular in the following sense.

Proposition 7 ([30, Lemma 4.10]). *Let g be an L-convex function on Γ. For every vertex x, the restrictions of g to the principal $\sqsubseteq$-filter $\mathcal{F}'_x$ and to the principal $\sqsubseteq$-ideal $\mathcal{I}'_x$ are submodular.*

The proofs of the above two results in [30] work for our non-finite setting. We show that the converse of Proposition 7 holds if Γ is well-oriented.

Proposition 8. *Suppose that Γ is well-oriented. Then $g : \Gamma \to \overline{\mathbf{R}}$ is L-convex if and only if $\operatorname{dom} g$ is Δ'-connected, and for every vertex x, the restrictions of g to the principal $\sqsubseteq$-filter $\mathcal{F}'_x$ and to the principal $\sqsubseteq$-ideal $\mathcal{I}'_x$ are submodular.*

4.1.2. L-optimality criterion and steepest descent algorithm

We present a local-to-global optimality criterion for minimization of L-convex functions, which is a direct analogue of the L-optimality condition in DCA. Let Γ be an oriented modular graph. Here we consider an L-convex function g on Γ such that g has discrete image, i.e., there is $\epsilon > 0$ such that $[g(x) - \epsilon, g(x) + \epsilon] \cap g(\Gamma) = \{g(x)\}$ for every $x \in \operatorname{dom} g$. This property holds when Γ (or $\operatorname{dom} g$) is finite.

Theorem 4.2. *A vertex $x \in \operatorname{dom} g$ is a minimizer of g over Γ if and only if x is a minimizer of g over the union of the principal $\sqsubseteq$-filter $\mathcal{F}'_x$ and the principal $\sqsubseteq$-ideal $\mathcal{I}'_x$ of x.*

Since g is submodular on $\mathcal{F}'_x$ and on $\mathcal{I}'_x$, the optimality can be checked by submodular function minimization. This leads us to the following algorithm, which is also a direct analogue of the *steepest descent algorithm* in DCA [46, Section 10.3.1].

Steepest Descent Algorithm (SDA)
Input: An L-convex function $g : \Gamma \to \overline{\mathbf{R}}$ and a vertex x in $\operatorname{dom} g$.
step 1: Let y be a minimizer of g over $\mathcal{F}'_x \cup \mathcal{I}'_x$.
step 2: If $g(x) = g(y)$, then x is minimizer, and stop.
step 3: $x := y$, and go to step 1.

In many applications, the oriented modular graph in question is a product of finite oriented modular graphs Γ_i $(i = 1, 2, \ldots, n)$, and the L-convex function g is a sum of L-convex functions g_j of small number of variables. In this case, the step 1 is reduced to VCSP for submodular functions, which can be solved efficiently by Theorem 3.1.

To obtain a complexity bound, we need to estimate the number of iterations of the algorithm. In the case of $L^\natural$-convex functions, the number of iterations is bounded by the l_∞-diameter of the effective domain [41], and exactly equals a certain directed l_∞-distance between the initial point and the minimizers [48]. We show that an analogous bound holds when Γ is well-oriented. Let Γ^Δ be the thickening of Γ. By definition, SDA yields a Δ'-path, which is also a Δ-path (a path in Γ^Δ). Let $\operatorname{opt}(g)$ denote the set of minimizers of g. Obviously the total number of the iterations is at least $d^\Delta(x, \operatorname{opt}(g)) := \min_{y \in \operatorname{opt}(g)} d^\Delta(x, y)$. The next theorem says that this bound is tight; special cases are given in [28, 29].

Theorem 4.3. *The total number N of the iterations of SDA with initial vertex x is at most $d^\Delta(x, \operatorname{opt}(g)) + 2$. In addition, if the initial vertex x satisfies $g(x) = \min_{y \in \mathcal{F}'_x} g(y)$ or $g(x) = \min_{y \in \mathcal{I}'_x} g(y)$, then N is equal to $d^\Delta(x, \operatorname{opt}(g))$.*

Remark 3. The assumption that Γ is well-oriented is not restrictive. Indeed, consider the barycentric subdivision Γ^*, and the L-convex function g^* instead of g (see Proposition 9). Apply SDA to g^*. Since Γ^* is well-oriented, Theorem 4.3 is applicable. For a minimizer $u = [x, y]$ of g^*, both x and y are minimizers of g.

4.1.3. L-extendable functions

Next we introduce the concept of L-extendability, which aims at capturing well-behaved NP-hard problems having half-integral relaxations or *k-submodular relaxations* [21, 36]. L-extendable functions have been introduced in [28] for the product of trees.

Let H be an swm-graph. Then the barycentric subdivision H^* is a well-oriented modular graph (Lemma 4). A function $h : H \to \overline{\mathbf{R}}$ is called *L-extendable* if there exists an L-convex function g on H^* such that the restriction of g to H coincides with h. Then g is called an *L-convex relaxation* of h. An L-convex relaxation g of h is said to be *exact* if the minimum value of g is equal to that of h. Recall that an orientable modular graph Γ is also an swm-graph. The class of L-extendable functions contains all L-convex functions with regard to all admissible orientations of Γ.

Proposition 9. *Let Γ be an oriented modular graph. Any L-convex function $g : \Gamma \to \overline{\mathbf{R}}$ is L-extendable, and $g^* : \Gamma^* \to \overline{\mathbf{R}}$ is an exact L-convex relaxation of g.*

The next result, called *persistency*, says that there exists a minimizer of h reasonably close to any minimizer of the relaxation g.

Theorem 4.4. *Let H be an swm-graph, $h : H \to \overline{\mathbf{R}}$ an L-extendable function, and $g : H^* \to \overline{\mathbf{R}}$ an L-convex relaxation of h. Suppose that g has discrete image. For any minimizer x^* of g (over H^*) there exists a minimizer of h (over H) in $H \cap \mathcal{F}_{x^*}(H^*)$.*

A canonical example of L-extendable functions is the distance function, which is a direct consequence of Theorems 2.4 and 4.1.

Proposition 10. *Let H be an swm-graph. The distance function d_H is L-extendable on $H \times H$, and an L-convex relaxation is given by the distance function d_{H^*} of H^**

Example 11. A *k-submodular relaxation* [21, 36] of a function h on $\{1, 2, \dots, k\}^n$ is a k-submodular function g on $\{0, 1, 2, \dots, k\}^n \simeq \mathcal{S}_k{}^n$ such that the restriction of the function g to $\{1, 2, \dots, k\}^n$ is equal to h. Identify $\{1, 2, \dots, k\}^n$ with the n-product $K_k{}^n$ of a complete graphs K_k. Then the barycentric subdivision $(K_k{}^n)^*$ is isomorphic to $\mathcal{S}_k{}^n$ (Example 5). L-convex functions and $(k\text{-})$submodular functions are the same on $(K_k{}^n)^* \simeq \mathcal{S}_k{}^n$. Thus L-extendable functions on $K_k{}^n$ are exactly those functions which admit k-submodular relaxations. See also [32] for k-submodular relaxation.

4.2. L-convex functions on Euclidean buildings

The Hasse diagram Γ of a Euclidean building is well-oriented modular (Theorem 2.2). Here we consider L-convex functions on Γ. We first define the discrete midpoint operator. For two vertices x, y, there is an apartment Σ containing them (by B1 in the definition). Now $\Sigma \simeq \check{\mathbf{Z}}^n$. The vertices x, y are regarded as integral vectors in $\mathbf{R}^n$. The midpoint $(x + y)/2$ is defined, and is a half-integral vector. Let $\lceil (x + y)/2 \rceil$ be defined as the integral vector obtained from $(x+y)/2$ by rounding each non-integral component to the nearest even integer, and let $\lfloor (x + y)/2 \rfloor$ be defined as the integral vector obtained by rounding each non-integral component to the nearest odd integer. They are actually well-defined. Indeed, since $K(\Gamma)$ is CAT(0) (Theorem 2.7), any pair of points in $K(\Gamma)$ can be joined by the unique geodesic (Proposition 3). Since the subcomplex $K(\Sigma)$ is isometric (convex) in $K(\Gamma)$ [1, Theorem 11.16 (4)], the unique geodesic $[x, y]$ belongs to $K(\Sigma) \simeq \mathbf{R}^n$. The midpoint $(x + y)/2$ in the geodesic coincides with the above construction. Now $\lfloor (x + y)/2 \rfloor \sqsubseteq \lceil (x + y)/2 \rceil$ and $(x + y)/2 = (\lfloor (x + y)/2 \rfloor + \lceil (x + y)/2 \rceil)/2$ (in $\mathbf{R}^n$). This implies that $\lfloor (x + y)/2 \rfloor$ and $\lceil (x+y)/2 \rceil$ are vertices of a uniquely-determined 1-dimensional simplex in $K(\Gamma)$ containing $(x + y)/2$ as a relative interior. Thus $\lceil (x + y)/2 \rceil$ and $\lfloor (x + y)/2 \rfloor$ are well-defined. Then L-convex functions are characterized as follows, where the property (3) is called the *discrete midpoint convexity*.

Theorem 4.5. *Let Γ be the Hasse diagram of a Euclidean building of type C. For a function $g : \Gamma \to \overline{\mathbf{R}}$, the following conditions are equivalent:*

(1) *g is L-convex.*

(2) *Lovász extension $\overline{g} : K(\Gamma) \to \overline{\mathbf{R}}$ is convex.*

(3) $g(x) + g(y) \geq g(\lfloor (x+y)/2 \rfloor) + g(\lceil (x+y)/2 \rceil)$ *holds for any* $x, y \in \Gamma(K)$.

Example 12 (UJ-convex function)**.** Fujishige [14] introduced a *UJ-convex function*, which is defined as a function g on $\mathbf{Z}^n$ such that its Lovász extension with respect to Union-Jack division $K(\check{\mathbf{Z}}^n)$ is convex. Here $\check{\mathbf{Z}}^n$ itself is a building of a single apartment. By Theorem 4.5 (1) $\Leftrightarrow$ (2), UJ-convex functions are the same as L-convex functions on $\check{\mathbf{Z}}^n$ (in our sense).

Example 13 (Alternating L-convex function)**.** Let T be an infinite tree without degree-one vertices (leaves). Regard T as a bipartite graph. Let B, W denote the color classes of T. For $x, y \in T$, there is a unique pair (u, v) of vertices such that $d(x, y) = d(x, u) + d(u, v) + d(v, y)$, $d(u, v) \leq 1$, and $d(x, u) = d(v, y)$. Define $(x \bullet y, x \circ y) := (u, v)$ if $(u, v) \in B \times W$, (v, u) if $(v, u) \in B \times W$, and (u, u) if $u = v$ (i.e., $u, v \in B$ or $u, v \in W$). Consider the direct product T^n, and extend operations $\bullet$ and $\circ$ componentwise. An *alternating L-convex function* [28] is a function $g : T^n \to \overline{\mathbf{R}}$ satisfying the inequality

$$g(x) + g(y) \geq g(x \bullet y) + g(x \circ y) \quad (x, y \in T^n).$$

Here T^n becomes a Euclidean building with respect to the orientation: $x \to y$ if $x \in B$ and $y \in W$. This orientation is called the *zigzag orientation*. Apartments are given by $P_1 \times P_2 \times \cdots \times P_n$ for all possible n simple paths $P_1, P_2, \ldots, P_n$ of infinite length. Then points $(x \bullet y)$ and $(x \circ y)$ are equal to $\lceil (x+y)/2 \rceil$ and $\lfloor (x+y)/2 \rfloor$, respectively, when vertices in B are associated with even integers. Thus alternating L-convex functions coincide with L-convex functions on Euclidean building T^n.

As seen in the examples above, the product of zigzag oriented trees forms a Euclidean building. We here consider a slightly more general situation where orientations are arbitrary. Let $T_1, T_2, \ldots, T_n$ be trees, where each tree has an edge-orientation. For two vertices x, y in T_i, there exists a unique pair (u, v) of vertices such that $d(x, y) = d(x, u) + d(u, v) + d(v, y)$, $d(u, v) \leq 1$, and $d(x, u) = d(v, y)$. Define $(\lceil (x+y)/2 \rceil, \lfloor (x+y)/2 \rfloor) := (u, v)$ if $u \to v$, (v, u) if $v \to u$, and (u, u) if $u = v$. Consider the product $\Gamma := T_1 \times T_2 \times \cdots \times T_n$, which is an oriented modular graph. Extend operations $\lfloor (x+y)/2 \rfloor$ and $\lceil (x+y)/2 \rceil$ componentwise. A variation of Theorem 4.5 is the following. Since Γ is not necessarily well-oriented, we consider $K'(\Gamma)$ (instead of $K(\Gamma)$); see Section 2.4 for $K'(\Gamma)$.

Proposition 11. *Let* $T_1, T_2, \ldots, T_n$ *be oriented trees, and let* $\Gamma := T_1 \times T_2 \times \cdots \times T_n$, *For a function* $g : \Gamma \to \overline{\mathbf{R}}$, *the following conditions are equivalent:*

(1) *g is L-convex.*
(2) *Lovász extension* $\overline{g} : K'(\Gamma) \to \overline{\mathbf{R}}$ *is convex.*
(3) $g(x) + g(y) \geq g(\lfloor (x+y)/2 \rfloor) + g(\lceil (x+y)/2 \rceil)$ *holds for any* $x, y \in \Gamma$.

Example 14 (L$^\natural$-convex function)**.** A linearly-oriented grid graph $\vec{\mathbf{Z}}^n$ is regarded as the product of directed paths (of infinite length). In this case, operations $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ on $\mathbf{Z}^n$ coincide with the rounding up and down operators in (1.1), respectively. Thus L$^\natural$-convex functions of [46] coincide with L-convex functions on $\vec{\mathbf{Z}}^n$.

Example 15 (Strongly tree-submodular function)**.** Suppose that $T_1, T_2, \ldots, T_n$ are rooted trees, oriented from roots. In this case, operations $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ coincides with $\sqcup$ and $\sqcap$ in the sense of Kolmogorov [40]. He introduced a *strongly-tree submodular function* as a function g on $\Gamma := T_1 \times T_2 \times \cdots \times T_n$ satisfying $g(x) + g(y) \geq g(x \sqcup y) + g(x \sqcap y)$ for $x, y \in \Gamma$. Thus strongly tree-submodular functions coincide with L-convex functions on Γ with respect to the rooted orientation.

Remark 4. In the case where Γ is the Hasse diagram of a Euclidean building or the product of oriented trees, thanks to the discrete midpoint convexity, Theorem 4.2 holds without the discreteness assumption. The proof is standard; see the proof of [28, Theorem 2.5].

5. Applications

In this section, we present applications of the results in previous sections to combinatorial optimization problems, such as multiflows, multiway cut, and related labeling problems. We show that dual objective functions of several well-behaved multiflow problems [23, 24, 26, 27, 29] can be viewed as submodular/L-convex functions in suitable sense (Proposition 12). This is a far-reaching generalization of a common knowledge in combinatorial optimization: *the cut function, which is the dual objective of the max-flow problem, is submodular*. We present an occurrence of L-extendable functions from the node-multiway cut problem (Example 18). Finally we apply the established iteration bound of SDA to obtain strong polynomial time solvability of the 0-extension problems on orientable modular graphs (Theorem 5.3), whereas the previous work [30] showed only weak polynomiality.

5.1. Multiflows

An *undirected network* $\mathcal{N} = (V, E, c, S)$ consists of undirected graph (V, E), an edge-capacity $c : E \to \mathbf{R}_+$, and a specified set $S \subseteq V$ of nodes, called *terminals*. Suppose that $V = \{1, 2, \ldots, n\}$. An *S-path* is a path connecting distinct terminals in S. A *multiflow* is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of S-paths and a nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint: $f(e) := \sum \{\lambda(P) \mid P \in \mathcal{P} : P \text{ contains } e\} \leq c(e)$ for $e \in E$.

We first consider multiflow maximization problems, where the value of a multiflow is specified by a terminal weight. Let μ be a nonnegative rational-valued function defined on the set of all distinct unordered pairs of terminals in S. The μ-*value* $\mu(f)$ of a multiflow f is defined by $\mu(f) := \sum \mu(s, t)\lambda(P)$, where the sum is taken over all distinct $s, t \in S$ and all (s, t)-paths P. Namely $\mu(s, t)$ is interpreted as the value of a unit (s, t)-flow. The μ-weighted maximum multiflow problem asks to find a multiflow of the maximum μ-value. The weight μ defines the class of problems. For example, if $S = \{s, t\}$ and $\mu(s, t) = 1$, then the problem is the maximum flow problem. There are several combinatorial min-max relations for special weights μ, generalizing Ford-Fulkerson's max-flow min-cut theorem. Examples include Hu's max-biflow min-cut theorem for the maximum 2-commodity flow problem ($\mu(s, t) = \mu(s', t') = 1$ and zero for other terminal pairs) and the Lovász-Cherkassky theorem for the maximum free multiflow problem ($\mu(s, t) = 1$ for $s, t \in S$). See e.g., [37] and [51, Section 73.3b] for further generalizations. Continuing a pioneering work [38] by Karzanov, with inspired by the paper [11] by Chepoi, the author [23, 24, 27] has developed a unified theory for these combinatorial dualities, which we describe below.

A *frame* [38] is an orientable modular graph without any isometric cycle of length greater than 4. An *oriented frame* is a frame endowed with an admissible orientation. An oriented frame is exactly an oriented modular graph Γ such that each pair x, y of vertices with $x \sqsubseteq y$ has distance at most 2, or equivalently, such that the corresponding orthoscheme complex $K'(\Gamma)$ is 2-dimensional. Such an orthoscheme complex is known as a *folder complex* [12, 24]. Consider an oriented frame Γ. A triple (p, q, r) of distinct vertices is called a *triangle* if $p \leftarrow q \leftarrow r$ and $p \sqsubseteq r$ or $r \leftarrow q \leftarrow p$ and $r \sqsubseteq p$. In an oriented frame Γ, a vertex subset X is said to be *normal* [24] if it satisfies:

N1: X is Δ'-connected.

N2: For each triangle (p, q, r), $\{p, q\} \subseteq X$ if and only if $\{q, r\} \subseteq X$.

N3: For any distinct triangles (p, q, r), (p, q, r') sharing an edge pq, if $\{p, r, r'\} \subseteq X$ then $q \in X$.

An *embedding* $\mathcal{E}$ of a terminal weight μ for $\mathcal{N} = (V, E, c, S)$ is a pair $(\Gamma, \{F_s\}_{s \in S})$ of an

oriented frame Γ and a family $\{F_s\}_{s \in S}$ of normal sets F_s indexed by $s \in S$ such that

$$\mu(s,t) = \min_{x \in F_s, y \in F_t} d_\Gamma(x,y) \quad (s,t \in S, s \neq t).$$

Define $\omega_{\mathcal{N},\mathcal{E}} : \Gamma^n \to \overline{\mathbf{R}}$ by

$$x \mapsto \sum_{s \in S} [F_s](x_s) + \sum_{ij \in E} c(ij) d_\Gamma(x_i, x_j),$$

where $[F_s]$ is the indicator function of F_s.

Theorem 5.1 ([24]). *Suppose that μ has an embedding $\mathcal{E} = (\Gamma, \{F_s\}_{s \in S})$. Then the maximum μ-value of a multiflow in $\mathcal{N}$ is equal to the minimum of $\omega_{\mathcal{N},\mathcal{E}}(x)$ over all $x \in \Gamma^n$.*

We verify that the dual objective functions are actually L-convex.

Proposition 12. *Suppose that μ has an embedding $\mathcal{E} = (\Gamma, \{F_s\}_{s \in S})$. Then $\omega_{\mathcal{N},\mathcal{E}}$ is L-convex on Γ^n.*

Proof. By Lemma 6 and Proposition 4.1, it suffices to show that the indicator function $[F]$ of any normal set F is L-convex. Consider Γ^*. Observe that $\mathrm{dom}[F]^*$ is normal in well-oriented Γ^*. Thus we can assume that Γ is well-oriented. Take an arbitrary vertex x in F. By Proposition 8 and N1, it suffices to show that $[F]$ is submodular on $\mathcal{F}_x$ (and $\mathcal{I}_x$). Take $p, q \in \mathcal{F}_x \cap F$. We prove the claim by showing $p \wedge q \in F$ and $\mathcal{E}(p,q) \subseteq F$. We can assume that p and q are incomparable. Notice that the rank of $\mathcal{F}_x$ is at most 2. Suppose that both p and q have rank 1. Then $p \wedge q = x \in F$. If $p \vee q$ exists (i.e., $\mathcal{E}(p,q) = \{p \vee q\}$), then $(x, p, p \vee q)$ is a triangle, and by N2 with $x, p \in F$, we have $p \vee q \in F$. If $p \vee q$ does not exist, then $\mathcal{E}(p,q) = \{p,q\} \subseteq F$. Suppose that p has rank 2 and q has rank 1. Then $p \wedge q = x \in F$, and $u \in \mathcal{E}(p,q) \setminus \{p,q\}$ (if exists) is the join of q and the rank 1-element $p' \in [x,p]$; apply N2 to triangle (x,q,u) to obtain $u \in F$. Suppose that both p and q have rank 2. Then $\mathcal{E}(p,q) = \{p,q\} \subseteq F$. Suppose that $p \wedge q \neq x (\in F)$. Then $(x, p \wedge q, p)$ and $(x, p \wedge q, q)$ are triangles with $x, p, q \in F$. By N3, we obtain $p \wedge q \in F$, as required. $\square$

Example 16. An embedding for the maximum flow case ($S = \{s, t\}$ and $\mu(s,t) = 1$) is easily obtained from $\Gamma = K_2$. Theorem 5.1 coincides with the max-flow min-cut theorem. Consider the free multiflow case: $\mu(s,t) = 1$ for $s, t \in S$. Let Γ be the star with $|S|$ leaves v_s ($s \in S$) and edge-length $1/2$, and let $F := \{v_s\}$ for $s \in S$. Then $(\Gamma, \{F_s\}_{s \in S})$ is an embedding of μ (under an arbitrary orientation of Γ). Theorem 5.1 coincides with Lovász-Cherkassky theorem. If Γ is oriented so that the center node is the sink, then the dual objective function is viewed as a k-submodular function on $\mathcal{S}_k^n$ for $k := |S|$. Consider the two-commodity flow case $S = \{s, s', t, t'\}$, $\mu(s,t) = \mu(s',t') = 1$ and zero for others. An embedding is obtained as follows. Let $\Gamma := (\mathbf{Z}^*)^2$, i.e., the alternating-grid of 2-dimensional half-integer vectors; see Example 4. Let $F_s := \{x \in (\mathbf{Z}^*)^2 \mid x_1 + x_2 \leq 0\}$, $F_t := \{x \in (\mathbf{Z}^*)^2 \mid x_1 + x_2 \geq 1\}$, $F_{s'} := \{x \in (\mathbf{Z}^*)^2 \mid x_1 - x_2 \leq 0\}$, and $F_{t'} := \{x \in (\mathbf{Z}^*)^2 \mid x_1 - x_2 \geq 1\}$. From Theorem 5.1 for this embedding, one can deduce Hu's max-biflow min-cut theorem. The dual objective function is viewed as an L-convex function on $\check{\mathbf{Z}}^{2n}$.

Papers [23, 24, 27] contain further examples of multiflow combinatorial dualities and constructions of embeddings. Now they all fall into our theory of L-convexity. Other examples of combinatorial multiflow dualities are given.

Example 17 (Minimum-cost node-demand multiflow problem)**.** Suppose that the network $\mathcal{N}$ has an edge-cost $a : E \to \mathbf{Z}_+$ and a node-demand $r : S \to \mathbf{R}_+$ on terminal set S. A multiflow $f = (\mathcal{P}, \lambda)$ is said to be *feasible* if for each $s \in S$, the sum of $\lambda(P)$ over all paths

$P \in \mathcal{P}$ connecting s is at least $r(s)$. The problem is to find a feasible multiflow f of the minimum cost $\sum_{e \in E} a(e)f(e)$. This problem was introduced by Fukunaga [18] in connection with a class of network design problems.

The dual of this problem can be formulated as an optimization over the product of *subdivided stars*, which is constructed as follows. For $s \in S$ let Γ_s be a path of infinite length and one end vertex v_s. Consider the disjoint union $\bigcup_{s \in S} \Gamma_s$ and identify all v_s into one vertex 0. The resulting graph is denoted by Γ, and the edge-length is defined as $1/2$ uniformly.

Then the minimum value of the problem is equal to the maximum of

$$\sum_{s \in S} r(s)d(x_s, 0) - [\Gamma_s](x_s) - \sum_{ij \in E} c(ij) \max\{d(x_i, x_j) - a(ij), 0\}$$

over all $x = (x_1, x_2, \ldots, x_n) \in \Gamma^n$ [28]. The negative of the dual objective is an L-convex function on Γ^n if Γ is endowed with the zigzag orientation. In [28], SDA is applied to this objective function, where each local problem reduces to minimizing a network-representable k-submodular function, and is efficiently solved by minimum cut computation [36]. Combined with domain scaling technique, we obtain the first combinatorial polynomial time algorithm to find a minimum-cost feasible multiflow; see also [31].

Example 18 (Node-capacitated free multiflow problem). Suppose that the network $\mathcal{N}$ has a node-capacity $b : V \setminus S \to \mathbf{R}_+$ instead of an edge-capacity c, where a multiflow $f = (\mathcal{P}, \lambda)$ should satisfy the node-capacity constraint: $\sum\{\lambda(P) \mid P \in \mathcal{P} : P \text{ contains node } i\} \leq b(i)$ for $i \in V \setminus S$. The node-capacitated free multiflow problem asks to a find a multiflow $f = (\mathcal{P}, \lambda)$ of the maximum total flow-value $\sum_{P \in \mathcal{P}} \lambda(P)$. This problem was considered by Garg, Vazirani, and Yannakakis [19] as the dual of an LP-relaxation of the node-multiway cut problem; see Section 5.2 below. They showed the (dual) half-integrality, and designed a 2-approximation algorithm for node-multiway cut; see [57, Section 19.3].

We here reformulate this dual half-integrality according to [26, 29]. Consider a star Γ with center v_0 and leaf set $\{v_s \mid s \in S\}$ indexed by $S = \{1, 2, \ldots, k\}$, where the edge-length is defined as $1/2$ uniformly. Consider further the subdivision Γ^*, where the midpoint between v_0 and v_s is denoted by $\bar{v}_s$. Consider points in $\Gamma^* \times \{0, 1/4, 1/2\}$:

$$(v_0, 0), (v_0, 1/2), (v_1, 0), (v_2, 0), \ldots, (v_k, 0), (\bar{v}_1, 1/4), (\bar{v}_2, 1/4), \ldots, (\bar{v}_k, 1/4).$$

A partial order on these points is given by $(v_s, 0) \to (\bar{v}_s, 1/4) \leftarrow (v_0, 1/2)$ and $(\bar{v}_s, 1/4) \to (v_0, 0)$ for $s \in S$. Let $\mathcal{S}_{2,k}^+$ denote the resulting modular semilattice, which is a subsemilattice of the polar space $\mathcal{S}_{2,k}$ (Example 2); see Figure 5.

The maximum flow-value of a multiflow is equal to the minimum of $\sum_{i \in V \setminus S} 2b(i)r_i$ over all $(p_i, r_i) \in \mathcal{S}_{2,k}^+$ for $i \in V$ satisfying

$$\begin{aligned} r_i + r_j &\geq d_{\Gamma^*}(p_i, p_j) \quad (ij \in E), \\ (p_s, r_s) &= (v_s, 0) \quad (s \in S). \end{aligned} \tag{5.1}$$

This dual objective is viewed as a $(2, k)$-submodular function (Remark 2). An equivalent statement of this fact was presented by Yoichi Iwata at the SOTA seminar in November 9, 2013.

In [29], we showed that this dual objective can be further perturbed into an L-convex function on a Euclidean building so to as each local problem reduces to an easy submodular flow problem. Then our SDA yields the first combinatorial strongly polynomial time algorithm for the maximum node-capacitated free multiflow problem, which in turn implies the
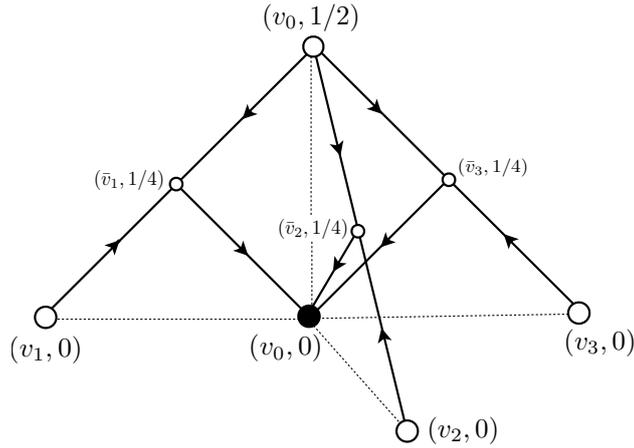
Figure 5: $\mathcal{S}_{2,3}^+$

first combinatorial strongly polynomial time implementation of Garg-Vazirani-Yannakakis algorithm for the node-multiway cut problem; see also [31].

5.2. Multiway cut and 0-extension

For an edge-capacitated network $\mathcal{N} = (V, E, c, S)$, an *(edge-)multiway cut* is a set F of edges such that every S-path meets F. The multiway cut problem is to find a multiway cut F of the minimum capacity $\sum_{e \in F} c(e)$. As mentioned in [21, 36], the multiway cut problem has a natural k-submodular relaxation (see Example 11) that is the dual to the maximum free multiflow problem; see [28, Example 2.17].

An analogous relation holds in the node-capacitated setting. Suppose that b is a node-capacity function on $V \setminus S$. A *node-multiway cut* is a subset C of nodes such that every S-path meets C. The node-multiway cut problem is to find a node-multiway cut C of the minimum capacity $b(C) := \sum_{i \in V \setminus S} b(i)$. This problem can be viewed as an L-extendable function minimization on $\mathcal{S}_{2,k}{}^n$. Recall the notation in Example 18. For a node-multiway cut C, define $(p, r) = ((p_i, r_i) : i \in V) \in (\mathcal{S}_{2,k}^+)^n$ by $(p_i, r_i) := (v_0, 1/2)$ if $i \in C$, $(p_i, r_i) := (v_s, 0)$ if i and terminal s belong to the same component in the network obtained by deleting C, and $(p_i, r_i) := (v_t, 0)$ for an arbitrary fixed $t \in S$ otherwise. Then (p, r) satisfies (5.1) and $\sum_i 2b(i)r_i = b(C)$, and is maximal in $\mathcal{S}_{2,k}{}^n$. Conversely, from a maximal $(p, r) \in (\mathcal{S}_{2,k}^+)^n$ satisfying (5.1), we obtain a node-multiway cut $C := \{i \mid (p_i, r_i) = (v_0, 1/2)\}$ with $\sum_i 2b(i)r_i = b(C)$. Thus the objective function of the node-multiway cut problem is the restriction of the L-convex function on $(\mathcal{S}_{2,k})^n$ in Example 18 to $(K_{2,k})^n$.

We next discuss a generalization of the edge-multiway cut problem. The *minimum 0-extension problem* is: Given an input I consisting of a number n of variables, undirected graph Γ, nonnegative weights b_{iv} ($i \in \{1, 2, \ldots, n\}, v \in \Gamma$) and c_{ij} ($1 \leq i < j \leq n$), find $x = (x_1, x_2, \ldots, x_n) \in \Gamma^n$ that minimizes the function $D_I : \Gamma^n \to \mathbf{R}_+$ defined by

$$x \mapsto \sum_{i=1}^{n} \sum_{v \in \Gamma} b_{iv} d_\Gamma(x_i, v) + \sum_{1 \leq i < j \leq n} c_{ij} d_\Gamma(x_i, x_j).$$

Observe that the multiway cut problem corresponds to $\Gamma = K_k$. The following complexity dichotomy theorem was the starting point of our theory.

Theorem 5.2. (1) *If Γ is orientable modular, then minimum 0-extension problem can be solved in polynomial time* [30].

(2) *If Γ is not orientable modular, then minimum 0-extension problem is NP-hard* [38].

The polynomial solvability is based on the VCSP-tractability of submodular functions and SDA of L-convex functions, applied to the following fact:

Proposition 13 ([30]). *If Γ is oriented modular, then D_I is L-convex on Γ^n.*

The algorithm in [30] is based on SDA with capacity scaling, and is weakly polynomial. By using the l_∞-bound (Theorem 4.3), we show the strongly polynomial time solvability.

Theorem 5.3. *The minimum 0-extension problem on orientable modular graph Γ can be solved in strongly polynomial time.*

Proof. Consider the barycentric subdivision Γ^*, and the minimum 0-extension problem for instance $I^* = (n, \Gamma^*, \{b_{iv}\}, \{c_{ij}\})$, where $b_{iv} := 0$ for $v \in \Gamma^* \setminus \Gamma$. Notice that Γ^* has $O(|\Gamma|^2)$ vertices by Proposition 3, and can be constructed in time polynomial in $|\Gamma|$; see [9, Lemma 3.7]. By Proposition 13, the objective function D_{I^*} is L-convex on $(\Gamma^*)^n$. By Theorem 2.4, it holds $D_{I^*}(u) = (D_I(x) + D_I(y))/2$, where $u = ([x_1, y_1], [x_2, y_2], \ldots, [x_n, y_n])$, $x = (x_1, x_2, \ldots, x_n)$, and $y = (y_1, y_2, \ldots, y_n)$. This means that D_{I^*} is an exact L-convex relaxation of D_I. Therefore, from an optimal solution of the relaxation, we can obtain an optimal solution of the original problem.

Apply SDA to solve I^*, where each local problem is a submodular VCSP, and can be solved in strongly polynomial time by Theorem 3.1. Now Γ^* is well-oriented. By Lemma 1, two vertices $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are adjacent in $((\Gamma^*)^n)^\Delta$ if and only if x_i and y_i are equal or adjacent in $(\Gamma^*)^\Delta$ for each $i = 1, 2, \ldots, n$. This implies that the diameter of $((\Gamma^*)^n)^\Delta$ is bounded by the diameter of $(\Gamma^*)^\Delta (\leq |\Gamma|^2)$. Thus the whole time complexity is polynomial in n and $|\Gamma|$. $\square$

Remark 5. The minimum 0-extension problem on an swm-graph G is viewed as an L-extendable function minimization on G^n. Indeed, an L-convex relaxation is obtained by relaxing G to G^* (Proposition 10). The relaxed problem is polynomially solvable, and is a kind of a half-integral relaxation. In [9, Section 6.9], we designed a 2-approximation rounding scheme based on this relaxation. This generalizes the classical 2-approximation algorithm [57, Algorithm 4.3] for edge-multiway cut.

6. Proofs

We first note structural properties of intervals $I(p, q)$ of a modular semilattice $\mathcal{L}$, and behavior of submodular functions on $I(p, q)$.

Lemma 8 ([30, Lemmas 3.11 and 3.12]). *Let $u, u' \in \mathcal{E}(p, q)$.*

(1) *If $r(u \wedge p) \geq r(u' \wedge p)$, then $u \wedge p \succeq u' \wedge p$ and $u \wedge u' = (u' \wedge p) \vee (u \wedge q)$.*

(2) *$p \vee_L u = p \vee_L q$.*

See also Figure 4. The following is a slightly sharper version of [30, Lemma 4.2].

Lemma 9. *Let $f : \mathcal{L} \to \overline{\mathbf{R}}$ be a submodular function. If $f(p) > f(q)$, then there exists a sequence $(p = u_0, u_1, \ldots, u_k = q)$ in $I(p, q)$ such that $f(p) > f(u_i)$ for $i > 0$ and u_i and u_{i+1} are comparable for $i \geq 0$. In addition, if $f(p \wedge q) > f(q)$, then $u_1 \succeq p \wedge (p \vee_R q)$.*

Sketch of proof. Suppose $f(p) > f(q)$. By submodular inequality (3.3), there is $u \in \{p \wedge q\} \cup (\mathcal{E}(p, q) \setminus \{p, q\})$ with $f(u) < f(p)$. Apply an inductive argument to p, u and to u, q, to obtain such a sequence, as in the proof of [30, Lemma 4.2]. Suppose further that $f(p \wedge q) > f(q)$. We can assume that $p \wedge (p \vee_R q) \succ p \wedge q$. Then $p \vee_R q \succ q$, and $[C(q; p, q)] = 0$. By submodular inequality (3.3) with $f(q) < f(p \wedge q)$, it holds $f(p) > \sum_{u \in \mathcal{E}(p, q)} [C(u; p, q)] f(u)$. Thus there is $u \in \mathcal{E}(p, q) \setminus \{p, q\}$ with $f(u) < f(p)$. By Lemma 8 (1), $p \wedge u \succeq p \wedge (p \vee_R q)$. By applying the inductive argument to p, u and to u, q as above, we have the latter part. $\square$

6.1. Section 3

6.1.1. Proof of Theorem 3.2

We first show the claim for the case where $\mathcal{L} = \mathcal{S}_2{}^n = \{-1, 0, 1\}^n$. In this case, $p \vee_L q$ is obtained from p by replacing p_i by q_i for each i with $p_i = 0 \neq q_i$ (Example 10). Therefore $p \vee_L q$ and $p \vee_R q$ have the same rank, which is equal to the number N of indices i with $p_i \neq 0$ or $q_i \neq 0$. Each $u \in I(p, q)$ must satisfy $u_i = 0$ for each index i with $p_i = q_i = 0$, and hence $r(u; p, q)$ belongs to $\{(x, y) \in \mathbf{R}_+^2 \mid x + y \leq N\}$. Consequently, it must hold that $\mathrm{Conv}\, I(p, q) = \{(x, y) \in \mathbf{R}_+^2 \mid x + y \leq N, x \leq r(p), y \leq r(q)\}$, $\mathcal{E}(p, q) = \{p, q, p \vee_L q, p \wedge_R q\}$, and the fractional join of p, q is equal to $(1/2)p \vee_L q + (1/2)p \vee_R q$.

Next we consider the general case. Let $p, q \in \mathcal{L}$. Consider a polar frame $\mathcal{F}$ containing chains $p \wedge q, p$ and $p \wedge q, q$. We show that $\mathrm{Conv}\, I(p, q)$ is equal to that considered in polar frame $\mathcal{F}$. For $u = p' \vee q' \in I(p, q)$ with $p' \in [p \wedge q, p]$ and $q' \in [p \wedge q, q]$, consider a polar frame $\mathcal{F}'$ containing chains $p \wedge q, p', p$ and $p \wedge q, q', q$. Then $u = p' \vee q'$ must belong to $\mathcal{F}'$. Indeed, consider a polar frame $\mathcal{F}''$ containing $p \wedge q, p', u$ and $p \wedge q, q', u$, and consider an isomorphism $\mathcal{F}'' \to \mathcal{F}'$ fixing $p \wedge q, p', q'$. The image of u must be the join of p', q' and equal u. Now consider an isomorphism $\phi : \mathcal{F}' \to \mathcal{F}$ fixing $p \wedge q, p, q$, and consider images $\phi(p'), \phi(q')$ and $\phi(u) = \phi(p') \vee \phi(q')$. Then $r(\phi(p')) = r(p')$ and $r(\phi(q')) = r(q')$ must hold. Thus the point $r(u; p, q)$ belongs to $\mathrm{Conv}\, I(p, q)$ considered in $\mathcal{F}$. Necessarily the left and right joins are equal to the left and right joins in $\mathcal{F}$, respectively. Thus the fractional join is equal to $(1/2)p \vee_L q + (1/2)p \vee_R q$. We remark that this argument implies the following rank equality for polar space.

$$r(p \vee_L q) = r(p \vee_R q) \quad (p, q \in \mathcal{L}). \tag{6.1}$$

6.1.2. Proof of Theorem 3.3

For three binary operations $\circ, \circ', \circ''$ on $\mathcal{L}$, let $\circ' \circ \circ''$ denote the operation defined by $(p, q) \mapsto (p \circ' q) \circ (p \circ'' q)$. Define the projection operations L by $(p, q) \mapsto p$ and R by $(p, q) \mapsto q$. Define operations $\wedge_L := L \wedge \vee_R$ and $\wedge_R := R \wedge \vee_L$.

Lemma 10. (1) $p \sqcup q = p \vee q = p \vee_L q = p \vee_R q$ *if $p \vee q$ exists.*

(2) $\vee_L \vee_L \vee_R = \vee_L$ *and* $\vee_L \vee_R \vee_R = \vee_R$.

(3) $\wedge_L \sqcup \wedge_R = \wedge_L \vee \wedge_R = \sqcup$.

(4) $\wedge_L \wedge \wedge_R = \wedge$.

(5) $L \sqcup \sqcup = L \vee \sqcup = \vee_L$ *and* $R \sqcup \sqcup = R \vee \sqcup = \vee_R$.

(6) $L \wedge \sqcup = \wedge_L$ *and* $R \wedge \sqcup = \wedge_R$.

Proof. (1) is obvious from the definition. (2) follows from Lemma 8 (2) (with $p \vee_L q = u_0$ or u_1). (3) follows from (1) and Lemma 8 (2). (4) and (6) follow from the definition of $\wedge_L$ and $\wedge_R$. (5) follows from $p \sqcup (p \sqcup q) = p \vee (p \sqcup q) = p \vee (p \wedge_L q) \vee (p \wedge_R q) = p \vee (p \wedge_R q) = ((p \vee_L q) \wedge p) \vee ((p \vee_L q) \wedge q) = p \vee_L q$, where we use (1) for the first equality, (3) for the second, and the unique representation of elements in $I(p, q)$ for the third. $\qquad\square$

We are ready to prove Theorem 3.3. Fix arbitrary $p, q \in \mathcal{L}$. For an operation $\circ$, $f(p \circ q)$ is simply denoted by $f(\circ)$.

(1) $\Rightarrow$ (2). Suppose that f is submodular. By applying Lemma 10 (2) to (3.6) for $(p \vee_L q, p \vee_R q)$, we obtain $f(\vee_L) + f(\vee_R) \geq f(\sqcup) + \{f(\vee_L) + f(\vee_R)\}/2$. In particular we have $f(\vee_L) + f(\vee_R) \geq 2f(\sqcup)$. Thus we have $f(L) + f(R) \geq f(\wedge) + \{f(\vee_L) + f(\vee_R)\}/2 \geq f(\wedge) + f(\sqcup)$. Hence f satisfies the inequality in (2).

(2) $\Rightarrow$ (1). Suppose that f satisfies the inequality in (2). By applying Lemma 10 to this inequality, we have $f(\wedge_L) + f(\wedge_R) \geq f(\wedge) + f(\sqcup)$, $f(L) + f(\sqcup) \geq f(\wedge_L) + f(\vee_L)$,

and $f(R) + f(\sqcup) \geq f(\wedge_R) + f(\vee_R)$. Adding them, we have $f(L) + f(R) + f(\sqcup) \geq f(\wedge) + f(\vee_L) + f(\vee_R)$. By applying Lemma 10 (3) to the inequality for $(p \vee_L q, p \vee_R q)$, we have $f(\vee_L) + f(\vee_R) \geq f(\sqcup) + f(\sqcup)$. Adding them (multiplying the second by $1/2$), we obtain (3.6) as required.

(3) $\Leftrightarrow$ (2). As seen in the proof of Theorem 3.2, for each polar frame $\mathcal{F}$, the left and right joins in $\mathcal{F}$ are equal to those in $\mathcal{L}$. Consequently, the pseudo join in $\mathcal{F}$ is equal to that in $\mathcal{L}$. Now the inequality in (2) is nothing but the bisubmodularity inequality under $\mathcal{S}_2{}^n \simeq \{-1, 0, 1\}^n$. From this, we see the equivalence (3) $\Leftrightarrow$ (2).

6.1.3. Proof of Theorem 3.5

Let $\mathcal{L}$ be a modular lattice of rank n. The proof uses the following facts:

(1) For two maximal chains there is a distributive sublattice of $\mathcal{L}$ containing them.
(2) For a distributive sublattice $\mathcal{D}$ of rank n, the orthoscheme subcomplex $K(\mathcal{D})$ is convex in $K(\mathcal{L})$.
(3) $f : \mathcal{L} \to \overline{\mathbf{R}}$ is submodular if and only if f is submodular on every distributive sublattice of $\mathcal{L}$.

(1) follows from [20, Theorem 363]. (2) follows from [9, Lemma 7.13 (4)]. The only-if part of (3) is obvious. The if-part of (3) follows from the fact that for $p, q \in \mathcal{L}$ there is a distributive sublattice containing $p, p \wedge q, q, p \vee q$ (by (1)).

Suppose that the Lovász extension $\overline{f} : K(\mathcal{L}) \to \overline{\mathbf{R}}$ is convex. For every distributive sublattice $\mathcal{D}$ (of rank n) the restriction of $\overline{f}$ to $K(\mathcal{D}) \subseteq K(\mathcal{L})$ is also convex by (2). By Theorem 3.4, f is submodular on $\mathcal{D}$. By (3), f is submodular on $\mathcal{L}$. Suppose that $f : \mathcal{L} \to \overline{\mathbf{R}}$ is submodular. Take arbitrary two points x, y in $K(\mathcal{L})$. Then x and y are represented as formal convex combinations of two maximal chains C and C', respectively. By (1), we can take a (maximal) distributive sublattice $\mathcal{D}$ containing C and C'. The orthoscheme subcomplex $K(\mathcal{D})$ contains x, y, and a geodesic $[x, y]$ by (2). By Theorem 3.4, the Lovász extension $\overline{f}$ is convex on $K(\mathcal{D})$. Therefore $\overline{f}$ satisfies the convexity inequality (2.1) on $[x, y]$. Consequently $\overline{f}$ is convex on $K(\mathcal{L})$.

6.1.4. Proof of Theorem 3.7

Let $\mathcal{L}$ be a polar space of rank n. Then the following hold.

(0) For a polar frame $\mathcal{F}$, the left and right joins in $\mathcal{F}$ are equal to those in $\mathcal{L}$.
(1) For two maximal chains in $\mathcal{L}$ there is a polar frame $\mathcal{F}$ containing them.
(2) For a polar frame $\mathcal{F}$, the orthoscheme subcomplex $K(\mathcal{F})$ is convex in $K(\mathcal{L})$.
(3) $f : \mathcal{L} \to \overline{\mathbf{R}}$ is submodular if and only if f is submodular on every polar frame.

We saw (0) in the proof of Theorem 3.2. (1) is axiom (P1). (2) follows from the argument of the proof of [9, Proposition 7.4] (the existence of nonexpansive retraction from $K(\mathcal{L})$ to $K(\mathcal{F})$). (3) follows from the combination of (0) and (1). Now Theorem 3.7 is proved in precisely the same way as Theorem 3.5 above; replace (maximal) distributive sublattices by polar frames, and Theorem 3.4 by Theorem 3.6. Notice that submodular functions on a polar frame $\mathcal{S}_2{}^n$ are exactly bisubmodular functions.

6.1.5. Proof of Proposition 6

We start with some notation. For $0 \leq a \leq b \leq \infty$, let $\mathrm{Cone}(a, b)$ be the convex cone in $\mathbf{R}_+^2$ defined by $\mathrm{Cone}(a, b) := \{(x, y) \in \mathbf{R}_+^2 \mid ax \leq y, x \geq y/b\}$, where we let $1/\infty := 0$. Then $[\mathrm{Cone}(a, b)] = b/(1 + b) - a/(1 + a) = 1/(1 + a) - 1/(1 + b)$.

We next determine the fractional join operation on $\mathcal{S}_2 = \{0, +, -\}$ with respect to valuation v_i. By $I(+, -) = \{0, +, -\}$, $\mathrm{Conv}\, I(+, -)$ is the triangle with vertices $v_i(0; +, -) = (0, 0), v_i(+; +, -) = (1, 0), v_i(-; +, -) = (0, \alpha_i)$, and $\mathrm{Conv}\, I(-, +)$ is also the triangle

with vertices $v_i(0; +, -) = (0, 0), v_i(-; -, +) = (\alpha_i, 0), v_i(+; -, +) = (0, 1)$. In particular, $\mathcal{E}(+, -) = \mathcal{E}(-, +) = \{-, +\}$, $C(+; +, -) = \mathrm{Cone}(0, 1/\alpha_i)$, $C(-; +, -) = \mathrm{Cone}(1/\alpha_i, \infty)$, $C(-; -, +) = \mathrm{Cone}(0, \alpha_i)$, and $C(+; -, +) = \mathrm{Cone}(\alpha_i, \infty)$. If the join $x \vee y$ exists, then $C(x \vee y; x, y) = \mathbf{R}_+^2$, and any operation θ in the fractional join operation satisfies $\theta(x, y) = x \vee y$. Hence $C(\theta) = C(\theta(+, -); +, -) \cap C(\theta(-, +); -, +)$. The operation θ assigning $(-, +)$ to $-$ and $(+, -)$ to $-$ does not appear in the fractional join operation, since $C(-; +, -) \cap C(-; -, +) = \mathrm{Cone}(1/\alpha_i, \infty) \cap \mathrm{Cone}(0, \alpha_i) = \{0\}$. The other operations are the left join $\vee_L$, the right join $\vee_R$, and $\sqcup_+$, where the corresponding cones are given by

$$C(\vee_L) = \mathrm{Cone}(0, \alpha_i), \; C(\vee_R) = \mathrm{Cone}(1/\alpha_i, \infty), \; C(\sqcup_+) = \mathrm{Cone}(\alpha_i, 1/\alpha_i).$$

To see this, for example, $C(\sqcup_+) = C(+; +, -) \cap C(+; -, +) = \mathrm{Cone}(0, 1/\alpha_i) \cap \mathrm{Cone}(\alpha_i, \infty) = \mathrm{Cone}(\alpha_i, 1/\alpha_i)$.

By Proposition 5, the fractional join operation on $\mathcal{S}_2^n$ relative to v is equal to

$$\sum_{\theta_1, \theta_2, \ldots, \theta_n \in \{\vee_L, \vee_R, \sqcup_+\}} [C(\theta_1) \cap C(\theta_2) \cap \cdots \cap C(\theta_n)](\theta_1, \theta_2, \ldots, \theta_n),$$

where $C(\theta_i)$ is considered under valuation v_i for $i = 1, 2, \ldots, n$. If $\theta_i \in \{\vee_L, \vee_R\}$, $\theta_j = \sqcup_+$, and $i < j$, then $C(\theta_i) \cap C(\theta_j)$ has no interior point. If $\theta_i = \vee_L$ and $\theta_j = \vee_R$, then $C(\theta_i) \cap C(\theta_j)$ has no interior point. Thus the fractional join operation equals

$$\sum_{i=0}^{n-1} [\mathrm{Cone}(\alpha_i, 1/\alpha_i) \cap \mathrm{Cone}(0, \alpha_{i+1})](\underbrace{\sqcup_+, \ldots \sqcup_+}_{i}, \vee_L, \ldots, \vee_L)$$

$$+ \sum_{i=0}^{n-1} [\mathrm{Cone}(\alpha_i, 1/\alpha_i) \cap \mathrm{Cone}(1/\alpha_{i+1}, \infty)](\underbrace{\sqcup_+, \ldots \sqcup_+}_{i}, \vee_R, \ldots, \vee_R)$$

$$+ [\mathrm{Cone}(\alpha_n, 1/\alpha_n)](\sqcup_+, \sqcup_+, \ldots, \sqcup_+)$$

$$= \sum_{i=0}^{n-1} [\mathrm{Cone}(\alpha_i, \alpha_{i+1})] \vee_L^i + \sum_{i=0}^{n-1} [\mathrm{Cone}(1/\alpha_{i+1}, 1/\alpha_i)] \vee_R^i + [\mathrm{Cone}(\alpha_n, 1/\alpha_n)] \sqcup_+ .$$

From this, we obtain the desired formula (3.10).

6.1.6. Proof of Theorem 3.9

Lemma 11. *For $1 \le i, j \le n$, it holds $\vee_L^i \wedge \vee_R^i = \sqcup^i$, $\vee_L^i \vee_L^j \vee_R^i = \vee_L^{\max\{i,j\}}$, $\vee_L^i \vee_R^j \vee_R^i = \vee_R^{\max\{i,j\}}$, and $\vee_L^i \sqcup^j \vee_R^i = \sqcup^{\max\{i,j\}}$.*

Proof. We can verify these equations by applying $\vee_L \wedge \vee_R = \sqcup$, $\vee_L \vee_L \vee_R = \vee_L$, $\vee_L \vee_R \vee_R = \vee_R$, $\vee_L \sqcup_+ \vee_R = \sqcup_+ \vee_L \sqcup_+ = \sqcup_+ \vee_R \sqcup_+ = \sqcup \sqcup_+ \sqcup = \sqcup_+$ to each component. $\square$

Let f be a submodular function on $\mathcal{S}_2^n$ (in our sense). Fix an arbitrary pair (p, q) of elements in $\mathcal{S}_2^n$. We use notation $f(\circ) = f(p \circ q)$. By Proposition 6 we have

$$f(L) + f(R) \ge f(\wedge) + \sum_{i=0}^{n-1} \left(\frac{1}{1 + \alpha_i} - \frac{1}{1 + \alpha_{i+1}} \right) \{f(\vee_L^i) + f(\vee_R^i)\} + \frac{1 - \alpha_n}{1 + \alpha_n} f(\sqcup^n).$$

For $k = 0, 1, 2, \ldots, n$, let B_k be defined by

$$B_k := \sum_{i=k}^{n-1} \left(\frac{1}{1 + \alpha_i} - \frac{1}{1 + \alpha_{i+1}} \right) \{f(\vee_L^i) + f(\vee_R^i)\} + \frac{1 - \alpha_n}{1 + \alpha_n} f(\sqcup^n). \tag{6.2}$$

For $k = 0, 1, 2, \ldots, n$, we are going to show, by induction, that

$$f(L) + f(R) \geq f(\wedge) + \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) f(\sqcup^i) + (1 + \alpha_k) B_k. \tag{6.3}$$

This inequality (6.3) coincides with the submodularity inequality if $k = 0$, and coincides with the desired inequality if $k = n$.

By applying Lemma 11 to the submodular inequality for $\vee_L^k, \vee_R^k$, we obtain

$$f(\vee_L^k) + f(\vee_R^k) \geq f(\sqcup^k) + \left(1 - \frac{1}{1 + \alpha_{k+1}}\right) \left\{f(\vee_R^k) + f(\vee_L^k)\right\} + B_{k+1}.$$

Hence we obtain $f(\vee_L^k) + f(\vee_R^k) \geq (1 + \alpha_{k+1}) \left\{f(\sqcup^k) + B_{k+1}\right\}$, and

$$
\begin{aligned}
(1 + \alpha_k) B_k &= (1 + \alpha_k) \left[\left(\frac{1}{1 + \alpha_k} - \frac{1}{1 + \alpha_{k+1}}\right) \left\{f(\vee_L^k) + f(\vee_R^k)\right\} + B_{k+1} \right] \\
&\geq (1 + \alpha_k) \left[\left(\frac{1}{1 + \alpha_k} - \frac{1}{1 + \alpha_{k+1}}\right) (1 + \alpha_{k+1}) \left\{f(\sqcup^k) + B_{k+1}\right\} + B_{k+1} \right] \\
&= (\alpha_{k+1} - \alpha_k) f(\sqcup^k) + (1 + \alpha_{k+1}) B_{k+1}.
\end{aligned}
$$

Substituting this to (6.3) at k, we obtain (6.3) at $k + 1$. Thus f is $\boldsymbol{\alpha}$-bisubmodular.

Next we consider the converse direction. Define $\sqcup_L := \sqcup_+ \wedge \vee_L$ and $\sqcup_R := \sqcup_+ \wedge \vee_R$. Then we observe $\sqcup = \sqcup_L \wedge \sqcup_R$ and $\sqcup_+ = \sqcup_L \sqcup \sqcup_R = \sqcup_L \sqcup_+ \sqcup_R$. For $i = 0, 1, 2, \ldots, n-1$, define operations $\square_L^i$ and $\bigcirc_L^i$ on $\mathcal{S}_2{}^n$ by

$$\square_L^i := (\sqcup_+, \ldots, \sqcup_+, \underbrace{\sqcup_L}_{i}, \sqcup, \ldots, \sqcup), \quad \bigcirc_L^i := (\sqcup_+, \ldots, \sqcup_+, \underbrace{\sqcup_L}_{i}, \vee_L, \ldots, \vee_L).$$

Operations $\square_R^i$ and $\bigcirc_R^i$ are defined by replacing L by R. By using $\sqcup \sqcup \vee_L = \sqcup \sqcup_+ \vee_L = \vee_L$, $\sqcup_+ \sqcup \sqcup_L = \sqcup_+ \sqcup_+ \sqcup_L = \sqcup_+$, $\sqcup_+ \sqcup \vee_L = \sqcup_L$, and $\sqcup_+ \sqcup_+ \vee_L = \sqcup_+$ componentwise, we have

$$
\begin{aligned}
\square_L^i \wedge \square_R^i = \sqcup^{i-1}, \quad \square_L^i \sqcup^j \square_R^i = \sqcup^i, \quad \sqcup^i \wedge \bigcirc_L^i = \square_L^i, \quad \sqcup^i \sqcup^j \bigcirc_L^i = \vee_L^i, \\
\vee_L^{i-1} \wedge \vee_L^i = \bigcirc_L^i, \quad \vee_L^{i-1} \sqcup^j \vee_L^i = \begin{cases} \vee_L^i & \text{if } j \geq i, \\ \bigcirc_L^i & \text{otherwise.} \end{cases}
\end{aligned}
$$

The relations replacing L by R also hold. Applying these relations to the $\boldsymbol{\alpha}$-bisubmodularity inequality (3.9), we obtain

$$
\begin{aligned}
f(\square_L^i) + f(\square_R^i) &\geq f(\sqcup^{i-1}) + f(\sqcup^i), \\
f(\sqcup^i) + f(\bigcirc_L^i) &\geq f(\square_L^i) + f(\vee_L^i), \\
f(\sqcup^i) + f(\bigcirc_R^i) &\geq f(\square_R^i) + f(\vee_R^i), \\
f(\vee_L^{i-1}) + f(\vee_L^i) &\geq (1 + \alpha_i) f(\bigcirc_L^i) + (1 - \alpha_i) f(\vee_L^i), \\
f(\vee_R^{i-1}) + f(\vee_R^i) &\geq (1 + \alpha_i) f(\bigcirc_R^i) + (1 - \alpha_i) f(\vee_R^i).
\end{aligned}
$$

Adding them with the fourth and the fifth divided by $1 + \alpha_i$, we obtain

$$\frac{1}{1 + \alpha_i} \left\{f(\vee_L^{i-1}) + f(\vee_R^{i-1})\right\} + f(\sqcup^i) \geq f(\sqcup^{i-1}) + \frac{1}{1 + \alpha_i} \left\{f(\vee_L^i) + f(\vee_R^i)\right\}. \tag{6.4}$$

We are going to show, by reverse induction on $i = n - 1, n - 2, \ldots, 0$,

$$f(\vee_L^i) + f(\vee_R^i) \geq f(\sqcup^i) + \left(1 - \frac{1}{1 + \alpha_{i+1}} \right) \left\{ f(\vee_L^i) + f(\vee_R^i) \right\} + B_{i+1}. \tag{6.5}$$

Recall (6.2) for B_i. By $\sqcup^n = \vee_L^n = \vee_R^n$, the inequality (6.4) with $i = n$ gives the base case. Suppose that (6.5) is true for $i > 0$. Adding (6.4) to (6.5), we obtain (6.5) for $i - 1$:

$$f(\vee_L^{i-1}) + f(\vee_R^{i-1}) \geq f(\sqcup^{i-1}) + \left(1 - \frac{1}{1 + \alpha_i} \right) \left\{ f(\vee_L^{i-1}) + f(\vee_R^{i-1}) \right\} + B_i.$$

Thus we have

$$f(\vee_L) + f(\vee_R) \geq f(\sqcup) + B_0. \tag{6.6}$$

From Lemma 10 and the fact that $p \sqcup q = p \sqcup^+ q$ if $p \vee q$ exists, we see $\sqcup \sqcup^j L = \vee_L$, $\sqcup \sqcup^j R = \vee_R$, $\wedge_L \sqcup^j \wedge_R = \sqcup$, and obtain $f(\sqcup) + f(L) \geq f(\wedge_L) + f(\vee_L)$, $f(\sqcup) + f(R) \geq f(\wedge_R) + f(\vee_R)$, and $f(\wedge_L) + f(\wedge_R) \geq f(\wedge) + f(\sqcup)$. Adding them, we obtain

$$f(L) + f(R) + f(\sqcup) \geq f(\vee_L) + f(\vee_R) + f(\wedge). \tag{6.7}$$

Adding (6.6) and (6.7), we obtain the submodularity inequality in our sense.

6.2. Section 4

6.2.1. Proof of Lemma 7

Let X be a d_Γ-convex set in Γ. Let X^* denote the set of vertices $[p, q]$ in Γ^* with $p, q \in X$. Then $[X]^* = [X^*]$ holds since $[X]^*([p, q]) = ([X](p) + [X](q))/2 = [X^*]([p, q])$. We first show that X^* is d_{Γ^*}-convex in Γ^*. It suffices to show that any common neighbor $[u, v]$ of any distinct $[p, q], [p', q'] \in X^*$ belongs to X^* (by Lemma 2). We can assume $q' \not\preceq q$. Then $p \leftarrow p' = u, v = q \leftarrow q'$ or $u = p \leftarrow p', q \leftarrow q' = v$ or $p = p' = u$ with v being a common neighbor of q, q' or $q = q' = v$ with u being a common neighbor of p, p'. Then $u, v \in X$ and $[u, v] \in X^*$, where the last two cases follow from the d_Γ-convexity of X. Next we prove the L-convexity of $[X]$. The Δ'-connectivity of $[X]$ follows from the connectivity of the subgraph induced by X. It suffices to show that $[X^*]$ is submodular on each $\mathcal{I}_p^*$ that is d_{Γ^*}-convex (Proposition 2). The intersection $[X^*] \cap \mathcal{I}_p^*$ is d-convex in the Hasse diagram of $\mathcal{I}_p^*$ with path-metric d. Thus $I([p, q], [p, q'])$ of any $[p, q], [p, q'] \in [X^*] \cap \mathcal{I}_p^*$ is contained in $[X^*] \cap \mathcal{I}_p^*$. Thus $\{[p, q] \wedge [p', q']\} \cup \mathcal{E}([p, q], [p', q']) \subseteq [X^*] \cap \mathcal{I}_p^*$, and $[X^*]$ is submodular on $\mathcal{I}_p^*$.

6.2.2. Proof of Proposition 8

It suffices to show the if-part. By well-orientedness, Γ^* is the poset of all intervals of Γ with reverse inclusion order (Lemma 3). In particular, $\mathcal{I}_p^*$ is the poset of all intervals containing p, and is isomorphic to $\mathcal{I}_p \times \mathcal{F}_p$ by $[q, q'] \mapsto (q, q')$. By this isomorphism, g^* can be regarded as a function on the product $\mathcal{I}_p \times \mathcal{F}_p$ of two modular semilattices $\mathcal{I}_p$ and $\mathcal{F}_p$, defined by $g^*(q, q') := (g(q) + g(q'))/2$. Since g is submodular on $\mathcal{I}_p$ and on $\mathcal{F}_p$, the direct sum g^* is also submodular. This means that g^* is submodular on every neighborhood semilattice, and hence g is L-convex.

6.2.3. Proof of Proposition 9

We start with a preliminary argument. Let $\mathcal{L}$ be a complemented modular lattice with Hasse diagram Γ. Now Γ is thick (since every interval of $\mathcal{L}$ is a complemented modular lattice). By Theorem 2.3, Γ is a dual polar space. Consider the barycentric subdivision Γ^* of Γ, which is the poset of all intervals $[p, q]$ of $\mathcal{L}$ with respect to the reverse inclusion order. Then Γ^* is equal to the polar space corresponding to Γ.

We use the following explicit formulas of $\wedge$, $\vee_L$, $\vee_R$, and $\sqcup$ in Γ^*:

$$[p,q] \wedge [p',q'] = [p \wedge p', q \vee q'], \tag{6.8}$$

$$[p,q] \vee_L [p',q'] = [p \vee (q \wedge p'), q \wedge (p \vee q')] = [p',q'] \vee_R [p,q], \tag{6.9}$$

$$[p,q] \sqcup [p',q'] = [(p \vee p') \wedge (q \wedge q'), (p \vee p') \vee (q \wedge q')]. \tag{6.10}$$

Notice that $p \vee (q \wedge p') = q \wedge (p \vee p') \preceq q \wedge (p \vee q')$ holds by modularity, and hence the left and right joins are well-defined intervals. Also the join $[p,q] \vee [p',q']$ is equal to nonempty intersection $[p,q] \cap [p',q']$; thus the join exists if and only if $p \vee p' \preceq q \wedge q'$.

It is easy to see (6.8). To see (6.9), consider a minimal interval $[s,t]$ with $[p',q'] \subseteq [s,t] \subseteq [p \wedge p', q \vee q']$ and $[s,t] \cap [p,q] \neq \emptyset$. Then $t \succeq q \wedge t \succeq p \vee s \succeq p$ necessarily holds. This implies $t \succeq p \vee q'$. Similarly $s \preceq q \wedge p'$. On the other hand, $[q \wedge p', p \vee q'] \cap [p,q] \neq \emptyset$ since $(q \wedge p') \vee p = (p \vee p') \wedge q \preceq (p \vee q') \wedge q$. By minimality we have $t = p \vee q'$ and $s = q \wedge p'$. From this, we obtain (6.9). The equality (6.10) is obtained from definition (3.4) of $\sqcup$ by using the modular equality $x \vee (y \wedge z) = (x \vee y) \wedge z$ for $x \succeq z$.

Let us start the proof of Proposition 9. It suffices to show that g^* is an L-convex relaxation. We have seen in the proof of Proposition 8 that g^* is submodular on every neighborhood semilattice, and hence on every principal ideal. By Lemma 2.4, Γ^* is well-oriented. Therefore it suffices to show that g^* is submodular on every principal filter of Γ^*. Take an arbitrary vertex X of Γ^* that is represented as $X = [u,v]$ for $u \sqsubseteq v$. The principal filter of $[u,v]$ is the semilattice of all subintervals of $[u,v]$, and is equal to the interval poset of the complemented modular lattice $[u,v]$. Thus the principal filter is a polar space, and it suffices to show the inequality in Theorem 3.3 (2). By (6.8), (6.10), and submodularity of g on $[u,v]$, we have

$$
\begin{aligned}
&g^*([p,q] \wedge [p',q']) + g^*([p,q] \sqcup [p',q']) \\
&= g^*([p \wedge p', q \vee q']) + g^*([(p \vee p') \wedge (q \wedge q'), (p \vee p') \vee (q \wedge q')]) \\
&= \{g(p \wedge p') + g(q \vee q'))\}/2 + \{g((p \vee p') \wedge (q \wedge q')) + g((p \vee p') \vee (q \wedge q'))\}/2 \\
&\leq \{g(p \wedge p') + g(q \vee q'))\}/2 + \{g(p \vee p') + g(q \wedge q')\}/2 \\
&\leq \{g(p) + g(p') + g(q) + g(q')\}/2 = g^*([p,q]) + g^*([p',q']).
\end{aligned}
$$

Thus g^* is submodular on the principal filter of every interval, and hence g^* is L-convex. The exactness is immediate from the definition of g^*.

6.2.4. Proof of Lemma 6

Let H be an swm-graph. For vertices x,y, let $\langle\langle x,y \rangle\rangle$ denote the minimum d_H-gated set containing x,y. For vertices x,y, a Δ-path $(x = x_0, x_1, \ldots, x_m = y)$ is called a *normal Δ-path* (*normal Boolean-gated path*) from x to y [9, Section 6.6] if for every index i with $0 < i < m$ and every Boolean-gated set B containing $\langle\langle x_{i-1}, x_i \rangle\rangle$ it holds $B \cap \langle\langle x_i, x_{i+1} \rangle\rangle = \{x_i\}$.

Theorem 6.1 ([9, Theorem 6.20]). *For vertices x,y, there exists a unique normal Δ-path from x to y.*

We prove a global convexity property of the domain of L-extendable functions.

Proposition 14. *Let h be an L-extendable function on an swm-graph H. For any $x,y \in \mathrm{dom}\, h$, the normal Δ-path from x to y is contained in $\mathrm{dom}\, h$.*

Proof. The proof is based on the idea of Abram and Ghrist [2] to find normal cube paths in CAT(0) cube complex. Let $g : H^* \to \overline{\mathbf{R}}$ be an L-convex relaxation of h. For $x,y \in \mathrm{dom}\, h$, take a Δ-path $P = (x = x_0, x_1, \ldots, x_m = y)$ in $\mathrm{dom}\, h$ such that $I_P := \sum_{i=1}^{m} i \cdot d_H(x_{i-1}, x_i)$ is minimum. We remark that $\langle\langle x_i, x_{i-1} \rangle\rangle$ is also Boolean-gated ([9, Lemma

6.8]), and $\{x_i\} \wedge \{x_{i-1}\} = \langle\langle x_i, x_{i-1}\rangle\rangle$ in $\mathcal{F}_{\langle\langle x_i, x_{i-1}\rangle\rangle}$. By submodularity of g on $\mathcal{F}_{\langle\langle x_i, x_{i-1}\rangle\rangle}$ with $x_i, x_{i-1} \in \mathrm{dom}\, g$, it holds $\langle\langle x_i, x_{i-1}\rangle\rangle \in \mathrm{dom}\, g$. By using $\wedge_R$, the definition of a normal path is rephrased as: P is normal if and only if $\langle\langle x_{i-1}, x_i\rangle\rangle \wedge_R \langle\langle x_i, x_{i+1}\rangle\rangle = \{x_i\}$ in $\mathcal{I}_{\{x_i\}}$ for $i = 1, 2, \ldots, m - 1$.

We show that P is normal. Suppose not. There is an index i $(0 < i < m)$ such that $U := \langle\langle x_{i-1}, x_i\rangle\rangle \wedge_R \langle\langle x_i, x_{i+1}\rangle\rangle \supset \{x_i\}$ in $\mathcal{I}_{\{x_i\}}$. Then U is contained in $\mathrm{dom}\, g$ since U is the meet of $\langle\langle x_i, x_{i+1}\rangle\rangle \in \mathrm{dom}\, g$ and $\langle\langle x_{i-1}, x_i\rangle\rangle \vee_L \langle\langle x_i, x_{i+1}\rangle\rangle \in \mathrm{dom}\, g$. Now $\langle\langle x_i, x_{i+1}\rangle\rangle \supseteq U \supset \{x_i\}$. Consider $U \vee_L \{x_{i+1}\}$ in the polar space $\mathcal{F}_{\langle\langle x_i, x_{i+1}\rangle\rangle}$, which consists of a single vertex x_i' by (6.1). Also x_i' belongs to $\mathrm{dom}\, g$ and to $\mathrm{dom}\, h$, and is different from x_i (by $\{x_i'\} \wedge \{x_{i+1}\} \subset \langle\langle x_i, x_{i+1}\rangle\rangle = \{x_i\} \wedge \{x_{i+1}\}$). Then x_{i+1} and x_i' are Δ-adjacent by $x_{i+1}, x_i' \in \langle\langle x_i, x_{i+1}\rangle\rangle$. Also x_{i-1} and x_i' are Δ-adjacent since $x_i' \in U \subseteq U \vee \langle\langle x_i, x_{i-1}\rangle\rangle \supseteq \langle\langle x_i, x_{i-1}\rangle\rangle \ni x_{i-1}$. Replacing x_i by x_i' in P leads again to a Δ-path in $\mathrm{dom}\, h$. We finally show that I_P strictly decreases in this modification. Since $x_i' \in I_{H^*}(x_{i+1}, x_i)$, it holds $d_{H^*}(x_{i+1}, x_i) = d_{H^*}(x_{i+1}, x_i') + d_{H^*}(x_i', x_i)$, and $d(x_{i+1}, x_i) = d(x_{i+1}, x_i') + d(x_i', x_i)$ (by Proposition 2.4). Also $d(x_i', x_{i-1}) \leq d(x_i', x_i) + d(x_i, x_{i-1})$. Therefore $id(x_{i-1}, x_i) + (i+1)d(x_i, x_{i+1}) \geq id(x_{i-1}, x_i') + (i+1)d(x_i', x_{i+1}) + d(x_i', x_i) > id(x_{i-1}, x_i') + (i+1)d(x_i', x_{i+1})$. Thus I_P strictly deceases. This is a contradiction to the minimality of I_P. $\square$

We are ready to prove Lemma 6. Let g and g' be L-convex functions on an oriented modular graph Γ. It suffices to show that $\mathrm{dom}\, g \cap \mathrm{dom}\, g'$ is Δ'-connected. Now g and g' are also regarded as L-extendable functions on the swm-graph Γ (Proposition 9). For $x, y \in \mathrm{dom}\, g \cap \mathrm{dom}\, g'$, the normal Δ-path $(x = x_0, x_1, \ldots, x_m = y)$ from x to y is contained in $\mathrm{dom}\, g \cap \mathrm{dom}\, g'$ (Proposition 14). By submodularity on each interval, the Δ'-path $(x = x_0, x_0 \wedge x_1, x_1, x_1 \wedge x_2, \ldots, x_m = y)$ is also contained in $\mathrm{dom}\, g \cap \mathrm{dom}\, g'$. Thus $\mathrm{dom}\, g \cap \mathrm{dom}\, g'$ is Δ'-connected.

6.2.5. Proof of Theorem 4.2

We can assume that Γ is well-oriented. Otherwise, consider the subdivision Γ^* and exact L-convex relaxation g^* on Γ^*. If $g(p) = g^*([p, p]) > g^*([p', q'])$ and $p' \sqsubseteq p \sqsubseteq q'$, then $g(p) > g(p')$ or $g(p) > g(q')$. By the Δ'-connectivity of $\mathrm{dom}\, g$ and nonoptimality of g, there is a Δ'-path $(p = p_0, p_1, \ldots, p_m)$ such that $g(p) > g(p_m)$. Consider all such paths minimizing $\max_i g(p_i)$; their existence is guaranteed by the discreteness of the image of g and the inequality $\max_i g(p_i) \geq g(p)$. Among these paths, choose a path with the index set $I := \{i \mid g(p_i) = \max_j g(p_j)\}$ minimal. We show that $I = \{0\}$. We can choose $i \in I$ such that $g(p_{i-1}) \leq g(p_i) > g(p_{i+1})$. By the well-orientedness and the minimality, it holds $p_{i-1} \prec p_i \succ p_{i+1}$ or $p_{i-1} \succ p_i \prec p_{i+1}$. We may assume that $p_{i-1}, p_{i+1} \in \mathcal{F}_{p_i}$, and p_{i-1} and p_{i+1} are incomparable. By Lemma 9, there are $p_{i-1} = q_0, q_1, \ldots, q_k = p_{i+1} \in \mathcal{F}_{p_i}$ such that $g(p_i) > g(q_j)$ for $j = 1, 2, \ldots, k$ and $q_i \prec q_{i+1}$ or $q_i \succ q_{i+1}$. Replacing p_i by $q_1, q_2, \ldots, q_k$ in the path, the resulting Δ'-path either decreases $\max_i g(p_i) \geq g(p)$ or the size of I. This is a contradiction to the minimality choice.

6.2.6. Proof of Theorem 4.3

Let Γ be an oriented modular graph. In this case, $\langle\langle x, y\rangle\rangle$ (the smallest d-gated set containing x, y) is the smallest d_Γ-convex set containing x, y (by Lemma 2). Consider the thickening Γ^Δ. If vertices x and y are adjacent in Γ^Δ, then x and y are said to be Δ-*adjacent*, and y is called a Δ-*neighbor* of x.

Lemma 12. (1) *vertices* x, y *are* Δ-*adjacent if and only if both* $x \vee y$ *and* $x \wedge y$ *exist and* $x \wedge y \sqsubseteq x \vee y$.

(2) *If* $x \preceq y$, *then* $I(x, y) = [x, y] = \langle\langle x, y\rangle\rangle$.

(3) *For $x, y, z, v \in \Gamma$ with $x \preceq z \preceq y$, if $d^\Delta(v, x) = d^\Delta(v, y) = k$, then $d^\Delta(v, z) \leq k$.*

Proof. (1) follows from Lemma 3. (2) follows from [30, Lemmas 4.13 and 4.14]. (3). $d^\Delta(v, x) = d^\Delta(v, y) = k$ implies $x, y \in B_k^\Delta(v)$. By (2) and Lemma 5, we have $B_k^\Delta(v) \supseteq I(x, y) = [x, y] \ni z$. This means $d^\Delta(v, z) \leq k$. $\qquad\square$

Proposition 15 ([9, Lemma 6.17, Proposition 6.18])**.** *For distinct vertices x, y, there exists a unique Δ-neighbor u of x having the following properties:*

(1) $d^\Delta(x, y) = 1 + d^\Delta(u, x)$.

(2) *For a Δ-neighbor v of x, if $d^\Delta(x, y) = 1 + d^\Delta(v, x)$, then $u \in \langle\langle x, v \rangle\rangle$.*

(3) *For a Δ-neighbor v of x, $d^\Delta(v, y) = 1 + d^\Delta(x, y)$ if and only if v is not Δ-adjacent to u.*

This vertex u is called the Δ-*gate* of y at x. To obtain an intuition of the Δ-gate, consider the case of $\Gamma = \mathbf{Z}^n$. For distinct $x, y \in \mathbf{Z}^n$ with $x \leq y$, the Δ-gate of y at x is equal to $x + \sum\{e_i \mid y_i - x_i = \|x - y\|_\infty\}$.

Let us start the proof of Theorem 4.3. Suppose that Γ is well-oriented. Let $x = x^0, x^1, \ldots, x^N$ be a sequence generated by SDA applied to an L-convex function g and an initial vertex x.

Lemma 13. *Suppose that $g(x) = \min\{g(y) \mid y \in \mathcal{F}_x\}$ or $g(x) = \min\{g(y) \mid y \in \mathcal{I}_x\}$. For $z \in \mathcal{F}_{x^k} \cup \mathcal{I}_{x^k}$, if $g(x^k) > g(z)$, then $d^\Delta(x, z) = k + 1$.*

Proof. By the well-orientedness of Γ and the definition of SDA (with reversed orientation if necessarily) we have $x = x^0 \succ x^1 \prec x^2 \succ x^3 \prec \cdots$. Then it hold $g(x^i) = \min\{g(y) \mid y \in \mathcal{I}_{x^i}\}$ if i is odd, and $g(x^i) = \min\{g(y) \mid y \in \mathcal{F}_{x^i}\}$ if i is even. We use the induction on k. Suppose that k is odd. Then x^{k-1} and z belong to $\mathcal{F}_{x^k}$. By induction, we have $d^\Delta(x, x^{k-1}) = k - 1$ and $d^\Delta(x, x^k) = k$. By $g(x^{k-1}) > g(z) < g(x^k) \leq g(x^{k-1} \wedge z)$ and Lemma 9, there is $y \in I(x^{k-1}, z) \subseteq \mathcal{F}_{x^k}$ with $g(y) < g(x^{k-1})$ such that $y \in [x^{k-1} \wedge (z \vee_L x^{k-1}), x^{k-1}]$ or $y \succ x^{k-1}$. Then $y \succ x^{k-1}$ is impossible by $g(x^{k-1}) = \min\{g(y) \mid y \in \mathcal{F}_{x^{k-1}}\}$. Thus $y \in [x^{k-1} \wedge z, x^{k-1}]$ holds. By induction, $d^\Delta(x, y) = k$ holds. Let h be the Δ-gate of x at x^k. Then $d^\Delta(x, h) = k - 1$ (Proposition 15 (1)). By Proposition 15 (2) and Lemma 12 (2), we have $\langle\langle h, x^k \rangle\rangle \subseteq \langle\langle x^{k-1}, x^k \rangle\rangle = [x^{k-1}, x^k]$. In particular, h belongs to $[x^{k-1}, x^k]$. Then $h \not\preceq y$ must hold. Otherwise, by Lemma 12 (3), $d^\Delta(x, x^{k-1}) = d^\Delta(x, h) = k - 1$ and $y \in [h, x^{k-1}]$ imply $d^\Delta(x, y) \leq k - 1$, contradicting $d^\Delta(x, y) = k$. Consider $y' := x^{k-1} \wedge (z \vee_L x^{k-1})$. Now $y' \preceq y$. Thus $y' \vee h$ is strictly greater than y' (by $h \not\preceq y$). Consequently h and z cannot have the join. Otherwise, since $y' \vee h$ and $y' \vee z$ exist, by definition of modular semilattice, $y' \vee z \vee h$ exists, and is strictly greater than $z \vee_L x^{k-1}$, which contradicts the definition of $\vee_L$. By Lemma 12, z and h are not Δ-adjacent. By Proposition 15 (3), it holds $d^\Delta(x, z) = k+1$, as required. The case of k even is similar. $\qquad\square$

Consider a function $g' := g + [B_r^\Delta(x)]$ with $r := d^\Delta(x, \mathrm{opt}(g))$, which is L-convex by Lemmas 5, 6, and 7. Any optimal solution of g' is also optimal to g. If $N \leq r$, a sequence $x = x^0, x^1, \ldots x^N$ generated by SDA applied to g' can be a sequence generated by SDA for g, and vice versa. Suppose that x satisfies the condition $g(x) = \min\{g(y) \mid y \in \mathcal{F}_x\}$ or $g(x) = \min\{g(y) \mid y \in \mathcal{I}_x\}$. By the above lemma, the optimal solution x^N satisfies $d^\Delta(x, x^N) = N \leq r$, and the number of iteration N must be equal to $r = d^\Delta(x, \mathrm{opt}(g))$. Observe that $g(x^1) = \min\{g(y) \mid y \in \mathcal{F}_{x^1}\}$ or $g(x^1) = \min\{g(y) \mid y \in \mathcal{I}_{x^1}\}$ always holds. Therefore, apply the above argument with regarding x^1 as the initial vertex. Then it holds $N - 1 = d^\Delta(x^1, \mathrm{opt}(g))) \leq 1 + d^\Delta(x, \mathrm{opt}(g)))$. Thus $N \leq 2 + d^\Delta(x^1, \mathrm{opt}(g)))$ always holds, as required.

6.2.7. Proof of Theorem 4.4

Choose a minimizer y of h over H with minimum $d^\Delta(x^*, y)$. For small $\epsilon > 0$, define $g' : H^* \to \overline{\mathbf{R}}$ by $x \mapsto g(x) + \epsilon d(x^*, x)$. Then g' is L-convex by Lemma 6 and Theorem 4.1, and x^* is a unique minimizer of g'. Let $y = y^0, y^1, \ldots$ be a sequence generated by SDA applied to g' and initial vertex y. We show $y^1 = x^*$, which implies $y \in \mathcal{F}_{x^*}$ as required. Suppose to the contrary that $y^1 \neq x^*$. Then $g'(y^2) < g'(y^1) \leq g'(y \wedge y^2)$. Now $\mathcal{F}_{y \wedge y^2}$ is a polar space (Lemma 4). By Theorem 3.2 and submodularity of g' on $\mathcal{F}_{y \wedge y^2}$, we have

$$g'(y) + g'(y^2) \geq g'(y \wedge y^2) + \frac{1}{2} g'(y \vee_L y^2) + \frac{1}{2} g'(y \vee_R y^2).$$

By $g'(y^2) < g'(y \wedge y^2)$ and $y \vee_L y^2 = y$ (since $y \in H$ is maximal), we have $g'(y) > g'(y \vee_R y^2)$, which means $g(y) > g(y \vee_R y^2) + \epsilon(d(x^*, y \vee_R y^2) - d(x^*, y))$. Since $\epsilon > 0$ is arbitrary, it must hold $g(y) \geq g(y \vee_R y^2)$. On the other hand, by (6.1), elements y and $y \vee_R y^2$ have the same rank in $\mathcal{F}_{y \wedge y^2}$. This means that $y \vee_R y^2$ is also maximal, and belongs to H. Consequently, $y \vee_R y^2$ is also a minimizer of g over H. However, by Lemma 13, it must hold $d^\Delta(x^*, y \vee_R y^2) \leq d^\Delta(x^*, y^2) + 1 = d^\Delta(x^*, y) - 2 + 1 < d^\Delta(x^*, y)$. This is a contradiction to the minimality of $d^\Delta(x^*, y)$.

6.2.8. Proof of Theorem 4.5

$(2) \Rightarrow (3)$. Suppose that $\overline{g}$ is convex on $K(\Gamma)$. Take vertices $x, y \in \Gamma$. Consider the midpoint $(x + y)/2$ of the geodesic between x and y. By convexity of $\overline{g}$ with $\overline{g}(x) = g(x)$ and $\overline{g}(y) = g(y)$, we have $g(x) + g(y) \geq 2\overline{g}((x + y)/2)$. Now $(x + y)/2$ is the midpoint of the edge between $\lceil (x + y)/2 \rceil$ and $\lfloor (x + y)/2 \rfloor$. Here $\lfloor (x + y)/2 \rfloor \sqsubseteq \lceil (x + y)/2 \rceil$. Therefore we have $\overline{g}((x + y)/2) = g(\lceil (x + y)/2 \rceil)/2 + g(\lfloor (x + y)/2 \rfloor)/2$. Thus we obtain (3).

$(3) \Rightarrow (1)$. Suppose that g satisfies (3). We first show the Δ'-connectivity of $\text{dom}\, g$. Take $x, y \in \text{dom}\, g$, and take an apartment Σ containing x, y. Regard Σ as $\check{\mathbf{Z}}^n$. We show by induction on $k := \|x - y\|_\infty$. Suppose that $k \leq 1$. In this case, x and y belong to the principal ideal or filter of some vertex z. The inequality in (3) is equivalent to $g(x) + g(y) \geq g(x \sqcup y) + g(x \wedge y)$. Then $(x, x \wedge y, y)$ is a Δ'-path, as required. Suppose that $k \geq 2$. Then $\lceil (x + y)/2 \rceil$ and $\lfloor (x + y)/2 \rfloor$ belong to $\text{dom}\, g$. Also $\lfloor (x + y)/2 \rfloor \sqsubseteq \lceil (x + y)/2 \rceil$. Both $\|x - \lceil (x + y)/2 \rceil\|_\infty$ and $\|y - \lfloor (x + y)/2 \rfloor\|_\infty$ are at most $\lceil k/2 \rceil$. By induction, pairs $(x, \lceil (x + y)/2 \rceil)$ and $(\lfloor (x + y)/2 \rfloor, y)$ are connected by Δ'-paths, respectively. Hence x and y are connected by a Δ'-path.

Since Γ is well-oriented modular (by Theorem 2.2), it suffices to show that g is submodular on the principal ideal $\mathcal{I}_p$ and filter $\mathcal{F}_p$ of every vertex p (by Proposition 8). Then both $\mathcal{I}_p$ and $\mathcal{F}_p$ are polar spaces; compare definitions of polar space and Euclidean building. Take any x, y in $\mathcal{I}_p$, and take an apartment Σ containing $\{x, p\}$ and $\{y, p\}$. The intersection of $\mathcal{I}_p$ and Σ forms a polar frame in the polar space $\mathcal{I}_p$. Thus Σ is identified with $\mathbf{Z}^n$ and $\mathcal{I}_p \cap \Sigma$ is identified with $\{-1, 0, 1\}^n \subseteq \mathbf{Z}^n$. Under this identification, it is easy to see that $\lfloor (x + y)/2 \rfloor = x \wedge y$ and $\lceil (x + y)/2 \rceil = x \sqcup y$. Therefore (3) coincides with the submodularity inequality in Theorem 3.3.

$(1) \Rightarrow (2)$. Suppose that g is an L-convex function on Γ. By Δ'-connectivity of g, the domain of $\overline{g}$ is connected. By the Tietze-Nakajima theorem for CAT(0) space [50], it suffices to show that $\overline{g}$ is locally convex. Since the geodesic between two points lies on an apartment, it suffices to show that $\overline{g}$ is locally convex on each apartment. Thus we may assume that Γ is a single apartment, i.e., $\Gamma = \check{\mathbf{Z}}^n$.

For a positive integer m, define Γ^{*m} by: $\Gamma^{*m} := (\Gamma^{*(m-1)})^*$ if $m \geq 1$ and $\Gamma^{*0} := \Gamma$. Define $g^{*m} : \Gamma^{*m} \to \overline{\mathbf{R}}$ by: $g^{*m} := (g^{*(m-1)})^*$ if $m \geq 1$ and $g^{*0} := g$. Then Γ^{*m} is regarded as

$(\mathbf{Z}/2^m)^n$, where the orientation of $\mathbf{Z}/2^m$ is given as $x \pm 1/2^m \leftarrow x \in \mathbf{Z}/2^{m-1}$. In particular, Γ^{*m} is isomorphic to $\check{\mathbf{Z}}^n$. Also $K(\Gamma^{*m})$ is a simplicial subdivision of $K(\Gamma)$ and is isometric to $\mathbf{R}^n$; see also [9, Proposition 8.7]. By $\overline{g^{*(m-1)}}(x) = g^{*m}(x)$ for $x \in (\mathbf{Z}/2^m)^n$, we have

$$\overline{g^{*m}} = \overline{g} \quad (m = 0, 1, 2, \ldots) \tag{6.11}$$

Choose an arbitrary point $x \in \mathrm{dom}\,\overline{g}$. For sufficiently large m, there is a vertex p of Γ^{*m} such that the point x belongs to the interior of the subcomplex $K(\mathcal{F}_p)$ of $K(\Gamma^{*m})$ (for $\mathcal{F}_p = \mathcal{F}_p(\Gamma^{*m})$). We show that $\overline{g}$ is convex on $K(\mathcal{F}_p)$. Since $\overline{g}$ is equal to $\overline{g^{*m}}$ and g^{*m} is L-convex on Γ^{*m} (Proposition 9), g^{*m} is (bi)submodular on $\mathcal{F}_p \simeq \mathcal{S}_2{}^n$ (Theorem 3.3). By Theorem 3.6, $\overline{g^{*m}} = \overline{g}$ is convex on $K(\mathcal{F}_p)$. Thus $\overline{g}$ is locally convex, as required.

6.2.9. Proof of Proposition 11

We can assume that each T_i is a tree without leaves.

(1) $\Rightarrow$ (2). Suppose that g is L-convex. Consider the barycentric subdivision $\Gamma^* = T_1^* \times T_2^* \times \cdots \times T_n^*$. Then Γ^* is the Hasse diagram of a Euclidean building, where apartments are $P_1 \times P_2 \times \cdots \times P_n$ for infinite paths $P_i \subseteq T_i^*$ $(i = 1, 2, \ldots, n)$. Therefore, by Proposition 9, g^* is L-convex on Γ^*. By Theorem 4.5, the Lovász extension $\overline{g^*} : K(\Gamma^*) \to \overline{\mathbf{R}}$ is convex. As in the proof of (1) $\Rightarrow$ (2) of Theorem 4.5, $K(\Gamma^*)$ is a simplicial subdivision of $K'(\Gamma)$, and $\overline{g} = \overline{g^*}$ holds as in (6.11). Hence the Lovász extension $\overline{g} : K'(\Gamma) \to \overline{\mathbf{R}}$ is convex.

(2) $\Rightarrow$ (3). Take two vertices $x, y \in \Gamma$. Since $\overline{g}$ is convex, we have $\overline{g}(x) + \overline{g}(y) \geq 2\overline{g}((x+y)/2)$. Since $\overline{g}((x+y)/2) = g^*([\lfloor (x+y)/2 \rfloor, \lceil (x+y)/2 \rceil]) = \{g(\lfloor (x+y)/2 \rfloor) + g(\lceil (x+y)/2 \rceil)\}/2$, we obtain (3), as required.

(3) $\Rightarrow$ (1). We start with a preliminary argument. Let T be an oriented tree. For vertices p, q in T, let $p \circ q$ and $p \bullet q$ denote $\lceil (p+q)/2 \rceil$ and $\lfloor (p+q)/2 \rfloor$, respectively. Let T^* be the subdivision of T. Take an arbitrary vertex p of T. Consider neighborhood semilattice $\mathcal{I}_p^* \subseteq T^*$. Take two $x, y \in \mathcal{I}_p^*$. Suppose that $x = [\underline{x}, \overline{x}]$ and $y = [\underline{y}, \overline{y}]$ for $\underline{x}, \overline{x}, \underline{y}, \overline{y} \in T$. By case-by-case analysis, it holds

$$[\overline{x} \circ \overline{y} \bullet \underline{x} \bullet \underline{y},\ \overline{x} \circ \overline{y} \circ \underline{x} \bullet \underline{y}] = x \sqcup y, \quad [\overline{x} \bullet \overline{y} \bullet \underline{x} \circ \underline{y},\ \overline{x} \bullet \overline{y} \circ \underline{x} \circ \underline{y}] = x \wedge y, \tag{6.12}$$

where $a \diamond b \diamond' c \diamond'' d$ means $(a \diamond b) \diamond' (c \diamond'' d)$ for $\diamond, \diamond', \diamond'' \in \{\circ, \bullet\}$. For example, if $\overline{x} \to \underline{x} = p = \overline{y} \to \underline{y}$, then one can see (6.12) from $x \sqcup y = x \wedge y = [p, p]$, $\overline{x} \circ \overline{y} = \overline{x}$, $\underline{x} \bullet \underline{y} = \underline{y}$, and $\overline{x} \bullet \overline{y} = \underline{x} \circ \underline{y} = \overline{x} \bullet \underline{y} = \overline{x} \circ \underline{y} = p$.

Suppose that g satisfies (3). Again the Δ'-connectivity can be shown in a similar way as in the proof of Theorem 4.5 (3) $\Rightarrow$ (1). Take any vertex $p = (p_1, p_2, \ldots, p_n)$. We show that g^* is submodular on $\mathcal{I}_p^*(\Gamma) = \mathcal{I}_{p_1}^*(T_1) \times \mathcal{I}_{p_2}^*(T_2) \times \cdots \times \mathcal{I}_{p_n}^*(T_n)$. Take any $x, y \in \mathcal{I}_p^*$. Suppose that $x_i = [\underline{x_i}, \overline{x_i}]$ and $y_i = [\underline{y_i}, \overline{y_i}]$ for $i = 1, 2, \ldots, n$. Let $\overline{x} = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$, $\underline{x} = (\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n})$, $\overline{y} = (\overline{y_1}, \overline{y_2}, \ldots, \overline{y_n})$, and $\underline{y} = (\underline{y_1}, \underline{y_2}, \ldots, \underline{y_n})$, Then we have

$$\begin{aligned}
2(g^*(x) + g^*(y)) &= g(\overline{x}) + g(\underline{x}) + g(\overline{y}) + g(\underline{y}) \\
&\geq g(\overline{x} \circ \overline{y}) + g(\overline{x} \bullet \overline{y}) + g(\underline{x} \circ \underline{y}) + g(\underline{x} \bullet \underline{y}) \\
&\geq g(\overline{x} \circ \overline{y} \bullet \underline{x} \bullet \underline{y}) + g(\overline{x} \circ \overline{y} \circ \underline{x} \bullet \underline{y}) + g(\overline{x} \bullet \overline{y} \bullet \underline{x} \circ \underline{y}) + g(\overline{x} \bullet \overline{y} \circ \underline{x} \circ \underline{y}) \\
&= 2(g^*(x \sqcup y) + g^*(x \wedge y)),
\end{aligned}$$

where we apply (6.12) to the last equality (componentwise). Thus g^* is submodular on $\mathcal{I}_p^*$, and g is L-convex on Γ.

Acknowledgments

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