Journal of the Operations Research Society of Japan
Vol. 60, No. 2, April 2017, pp. 201–214

EXISTENCE OF A PURE STRATEGY EQUILIBRIUM IN MARKOV GAMES WITH STRATEGIC COMPLEMENTARITIES FOR FINITE ACTIONS AND FINITE STATES

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(Received December 29, 2015; Revised January 18, 2017)

Abstract We provide a sufficient condition for the existence of a Markov perfect equilibrium for pure strategies in a class of Markov games where each stage has strategic complementarities. We assume that both the sets of actions for all players and the set of states are finite and that the horizon is also finite, while the past studies examined Markov games with infinite horizons where the sets of actions and states are assumed to be infinite. We give an elementary proof of the existence and apply the result to a game of Bertrand oligopoly with investment.

Keywords: Game theory, Markov process, Markov perfect equilibrium, supermodular game, pure strategy equilibrium

1. Introduction

A stochastic game is a collection of strategic form games indexed by a state variable. The players choose their actions according to a state, and this state changes from one period to the next according to a stochastic process that depends on the actions of all of the players. Stochastic games were introduced in [16] and Markov games, which provide the most typical and practical models in stochastic games, have been developed in the field of economics and operations research. A number of models of Markov games have been applied to interesting problems such as an oligopolistic industry with investment [9], political economics [1], and competition with inventory control or supply chain management [11].

The existence of a Markov Perfect Equilibrium (MPE) in Markov games with *infinite horizons* is complex, as indicated by several studies [10, 15, 16]. In contrast, the existence of an MPE in Markov games with finite horizons and finite states is easily proved if we consider the finite set of actions allowing mixed strategies or if we assume that the sets of actions are compact in the metric space and that the payoff functions are concave and continuous. However, in terms of *pure strategies* for the finite set of actions, the existence problem remains to be solved.

Even a one-shot game such as rock–paper–scissors may not have a pure strategy equilibrium. Additional conditions are required to ensure the existence of equilibria in pure strategies for one-shot games. Games with strategic complementarities [12, 13, 17–20] are included in the class that guarantees the existence of pure strategy equilibria in strategic form games*.

However, even if each stage of a game has a strategic complementarity for a given Markov game, it is known that the Markov game itself may not preserve these strategic

*Comprehensive surveys have been given by [3] and [22].

complementarities, and so may not have an MPE for pure strategies [21]. Hence, more assumptions, which may appear to be strong, are necessary for the existence of an MPE for pure strategies, even if the Markov game has a finite horizon, finite sets of actions of all players, and a finite set of states†. Curtat [5] and Amir [2] presented sufficient conditions for the existence of an MPE in pure strategies in Markov games with infinite horizons where the sets of actions and states are compact in the metric space. Curtat [5] gave a sufficient condition for the existence of a stationary MPE in pure strategies, whereas Amir [2] presented another proof for the results of [5] and showed that an MPE exists in pure strategies if a Markov game with a finite horizon satisfies the condition stated by Curtat [5].

The aim of this paper is to provide a sufficient condition for the existence of an MPE in pure strategies in Markov games. Our sufficient condition is different from those of [5] and [2] in the point that both the sets of actions and the set of states are finite and that the horizon is finite. Curtat [5] and Amir [2] made assumptions, such as *dominant diagonal conditions* for the payoffs and the transition probability. We successfully exclude these dominant diagonal conditions, which imply the uniqueness of the equilibrium, i.e., the existence of an MPE holds only if the MPE is unique. In contrast to their results, the equilibria are not necessarily unique under our condition.

We also show that an action of any player in the greatest equilibrium of the stage game for each state at any period is monotonically increasing in the state. Similarly, Curtat [5] and Amir [2] showed that if the equilibrium is unique, then an action of any player in the equilibrium in the stage game for each state at any period is monotonically increasing in the state. We extend this result to the greatest equilibrium, even if the uniqueness of the equilibrium may not hold.

Note that Amir [4] independently obtained similar results to those in this paper for Markov games in which the sets of actions are compact metric spaces. Our results are established on multi-dimensional states and actions, whereas Amir [4] only considered one-dimensional states and actions. Moreover, we should emphasize that our proof is simple and elementary because of the restriction to finite sets of actions and states and the finite horizon, whereas [4] used several theorems from functional analysis, such as the compactness and convergence of the set of continuous functions, the Schauder fixed point theorem, and the Arzera–Ascoli theorem.

The remainder of this paper is organized as follows. In Section 2, Markov games and equilibrium concepts are defined. In Section 3, we show the sufficient conditions for the existence and monotonicity of an equilibrium. In Section 4, we apply the conditions to a model of a Bertrand oligopoly with investment. The definitions and properties of lattice theory are summarized in the appendix.

2. Model

2.1. Markov games with finite actions and finite states

We consider a stochastic game in discrete time with a finite horizon indexed by parameter $t = 0, \dots, T$. The set of *players* is denoted by $N = \{1, \dots, n\}$. In the following, for any n-dimensional vectors $x = (x_1, \dots, x_n)$, we denote $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ by x_{-i}, which is the usual notation in game theory. The set of *actions* for player i, denoted by A_i, is

†It has been reported [7] that the extension of strategic complementarities to dynamic games in extensive form, which include Markov games, is very restrictive.

assumed to be an integer interval in $\mathbb{R}^m$:

$$A_i = \{a_i \in \mathbb{Z}^m | \underline{a_i} \leq a_i \leq \overline{a_i}\},$$

for some $\underline{a_i}, \overline{a_i} \in \mathbb{Z}^m$. The n-tuple of actions for all players $a = (a_1, \ldots, a_n)$ is called an *action profile*. The set of action profiles is given by A, i.e., $A = A_1 \times \cdots \times A_n$.

The set of *states*, denoted by $S \subset \mathbb{Z}^k$, is also an integer interval in $\mathbb{R}^k$:

$$S = \{s \in \mathbb{Z}^k | \underline{s} \leq s \leq \overline{s}\},$$

for some $\underline{s}, \overline{s} \in \mathbb{Z}^k$.

A *transition probability* from state $s \in S$ to $s' \in S$ for an action profile $a \in A$ is denoted by $f(s'|s, a)$. The state at time t is denoted by s^t, $\delta \in [0, 1]$ is a *discount factor*, and the (single-period) *payoff function* of player i is denoted by $u_i : S \times A \to \mathbb{R}$.

In stochastic games with observable actions, the action of each player at time t generally depends on time t, state s^t, and both the history of action profiles and the history of states until time $t-1$. In this paper, we restrict the strategies of any player to *Markovian strategies*, in which the action of each player at time t depends only on time t and the state at time t. $\sigma_i^t : S \to A_i$ is said to be a *strategy of player i at time t*, where $\sigma_i^t(s)$ specifies the action of player i at time t and state s. The sequence of strategies of player i from time 0 to time T, denoted by $\boldsymbol{\sigma}_i = (\sigma_i^0, \ldots, \sigma_i^T)$, is simply said to be a *strategy* of player i. $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n)$ is said to be a *strategy profile*.

For a strategy profile $\boldsymbol{\sigma}$, $\sigma^t = (\sigma_1^t, \ldots, \sigma_n^t)$ denotes the strategies of all players at time t. An induced action profile at state $s \in S$ from σ^t is denoted by $\sigma^t(s) = (\sigma_1^t(s), \ldots, \sigma_n^t(s))$.

For a strategy profile $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^{\geq t}$ and $\boldsymbol{\sigma}_i^{\geq t}$ are the subsequences from t to T of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_i$, respectively, i.e., $\boldsymbol{\sigma}^{\geq t} = (\sigma^t, \sigma^{t+1}, \ldots, \sigma^T)$ and $\boldsymbol{\sigma}_i^{\geq t} = (\sigma_i^t, \sigma_i^{t+1}, \ldots, \sigma_i^T)$.

2.2. Definitions of payoffs and equilibria

For a Markov game with finite horizons and any strategy profile $\boldsymbol{\sigma}$, $U_i^t(\boldsymbol{\sigma}^{\geq t})(s)$ denotes the expected payoff of player i at time t with the realized state $s^t = s$, given by

$$U_i^t(\boldsymbol{\sigma}^{\geq t})(s) = E[\sum_{\tau=t}^{T} \delta^{\tau-t} u_i(s^\tau, \sigma^\tau(s^\tau)) | s^t = s].$$

Thus, $U_i^t(\boldsymbol{\sigma}^{\geq t})$ can be regarded as a function from S to $\mathbb{R}$, where $U_i^t(\boldsymbol{\sigma}^{\geq t})(s)$ determines the expected payoff of player i at time t and state s. This function $U_i^t(\boldsymbol{\sigma}^{\geq t})$ is called the *continuation value function* of player i at time t for a given strategy profile $\boldsymbol{\sigma}$.

For any strategy profile $\boldsymbol{\sigma}$, $U_i^t(\boldsymbol{\sigma}^{\geq t})(s)$ can be calculated recursively as follows:

$$U_i^t(\boldsymbol{\sigma}^{\geq t})(s) = \begin{cases} u_i(s, \sigma^t(s)) & t = T, \\ u_i(s, a) + \delta \sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a) & 0 \leq t \leq T-1. \end{cases}$$

Now we define an MPE.

Definition 2.1 (MPE). *A strategy profile $\boldsymbol{\sigma}^*$ is said to be a Markov Perfect Equilibrium if, for any $i \in N$, $t \leq T$, and $s \in S$,*

$$U_i^t(\boldsymbol{\sigma}^{*\geq t})(s) \geq U_i^t(\hat{\boldsymbol{\sigma}}_i^{\geq t}, \boldsymbol{\sigma}_{-i}^{*\geq t})(s)$$

for any strategy of player i $\hat{\boldsymbol{\sigma}}_i$.

Under this definition, we find that the strategy profile $\boldsymbol{\sigma}^*$ is an MPE if and only if, for any $i \in N$, $t \leq T$, and $s \in S$,

$$U_i^T(\boldsymbol{\sigma}^{*\geq t})(s) = \begin{cases} \max_{a_i \in A_i} \ u_i(s, a_i, \sigma_{-i}^{*T}(s)) & t = T, \\ \max_{a_i \in A_i} \ u_i(s, a_i, \sigma_{-i}^{t}(s)) & \\ \qquad + \delta \sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a_i, \sigma_{-i}^t(s)) & 0 \leq t \leq T-1. \end{cases}$$

3. Results

To describe a sufficient condition for the existence of an MPE, we add some notation and definitions.

For any $x', x \in \mathbb{R}^m$, $x' \vee x$ and $x' \wedge x$ are defined by

$$x' \vee x = (\max\{x'_1, x_1\}, \dots, \max\{x'_m, x_m\}) \quad \text{and} \quad x' \wedge x = (\min\{x'_1, x_1\}, \dots, \min\{x'_m, x_m\}). \tag{1}$$

For a fixed transition probability f, let $P_f(\hat{S}|s,a)$ be the probability of the set $\hat{S} \subseteq S$ occurring with respect to $f(s'|s,a)$, i.e.,

$$P_f(\hat{S}|s,a) = \sum_{z \in \hat{S}} f(z|s,a).$$

$\hat{S} \subseteq S$ is said to be *an increasing set* if $s' \in \hat{S}$ and $s'' \geq s'$ imply $s'' \in \hat{S}$.

The sufficient condition for the existence of equilibria provided by our result is described by the following eight assumptions. The first four conditions are assumptions about the payoff functions, whereas the latter four are assumptions about the transition probability.

(U1) For any $i \in N$, $s \in S$ and $a_{-i} \in A_{-i}$, $u_i(s, a_i, a_{-i})$ is supermodular in a_i:

$$\forall a'_i, a_i \in A_i, \quad u_i(s, a'_i \vee a_i, a_{-i}) + u_i(s, a'_i \wedge a_i, a_{-i}) \geq u_i(s, a'_i, a_{-i}) + u_i(s, a_i, a_{-i}).$$

(U2) For any $i \in N$ and $s \in S$, $u_i(s, a_i, a_{-i})$ has increasing differences in (a_i, a_{-i}):

$$\forall a'_i \geq a_i, \quad \forall a'_{-i} \geq a_{-i}, \quad u_i(s, a'_i, a'_{-i}) - u_i(s, a_i, a'_{-i}) \geq u_i(s, a'_i, a_{-i}) - u_i(s, a_i, a_{-i}).$$

(U3) For any $i \in N$ and $a_{-i} \in A_{-i}$, $u_i(s, a_i, a_{-i})$ has increasing differences in (a_i, s):

$$\forall a'_i \geq a_i, \quad \forall s' \geq s, \quad u_i(s', a'_i, a_{-i}) - u_i(s', a_i, a_{-i}) \geq u_i(s, a'_i, a_{-i}) - u_i(s, a_i, a_{-i}).$$

(U4) For any $i \in N$ and $a_i \in A_i$, $u_i(s, a)$ is increasing in (s, a_{-i}):

$$\forall s' \geq s, \quad \forall a'_{-i} \geq a_{-i}, \quad u_i(s', a_i, a'_{-i}) \geq u_i(s, a_i, a_{-i}).$$

(T1) For any $s \in S$ and $a_{-i} \in A_{-i}$, $f(\cdot|s,a)$ is stochastically supermodular in a_i:

$$\begin{aligned} \forall a'_i, a_i \in A_i, \quad & P_f(\hat{S}|s, a'_i \vee a_i, a_{-i}) + P_f(\hat{S}|s, a'_i \wedge a_i, a_{-i}) \\ & \geq P_f(\hat{S}|s, a'_i, a_{-i}) + P_f(\hat{S}|s, a_i, a_{-i}) \end{aligned}$$

for any increasing set $\hat{S} \subseteq S$.

(T2) For any $s \in S$, $f(\cdot|s,a)$ has stochastically increasing differences in (a_i, a_{-i}):

$$\begin{aligned} \forall a'_i \geq a_i, \quad \forall a'_{-i} \geq a_{-i}, \quad & P_f(\hat{S}|s, a'_i, a'_{-i}) - P_f(\hat{S}|s, a_i, a'_{-i}) \\ & \geq P_f(\hat{S}|s, a'_i, a_{-i}) - P_f(\hat{S}|s, a_i, a_{-i}) \end{aligned}$$

for any increasing set $\hat{S} \subseteq S$.

(T3) For any $a_{-i} \in A_{-i}$, $f(\cdot|s,a)$ has stochastically increasing differences in (a_i, s):

$$\begin{aligned} \forall a'_i \geq a_i, \quad \forall s' \geq s, \quad & P_f(\hat{S}|s', a'_i, a_{-i}) - P_f(\hat{S}|s', a_i, a_{-i}) \\ & \geq P_f(\hat{S}|s, a'_i, a_{-i}) - P_f(\hat{S}|s, a_i, a_{-i}) \end{aligned}$$

for any increasing set $\hat{S} \subseteq S$.

(T4) For any $a_{-i} \in A_{-i}$, $f(\cdot|s,a)$ is stochastically increasing in (s, a_{-i}):

$$\forall s' \geq s, \quad \forall a'_{-i} \geq a_{-i}, \quad P_f(\hat{S}|s', a_i, a'_{-i}) \geq P_f(\hat{S}|s, a_i, a_{-i})$$

for any increasing set $\hat{S} \subseteq S$.

Our main result is stated as follows.

Theorem 3.1. *If a Markov game with a finite horizon satisfies (U1)–(U4) and (T1)–(T4), then*

(i) *the game has an MPE* $\boldsymbol{\sigma}^*$,

(ii) $\sigma^t(s)$ *is increasing in state* s *for any* $0 \leq t \leq T$, *and*

(iii) $U_i^t(\boldsymbol{\sigma}^{*\geq t})(s)$ *is also increasing in state* s *for any* $i \in N$ *and for any* $0 \leq t \leq T$.

To prove Theorem 3.1, we state two lemmas. In these lemmas, we fix an arbitrary strategy profile $\boldsymbol{\sigma}$ and any time $t \geq 1$. For any $s \in S$, let $\phi_i(s)$ be the function from A to $\mathbb{R}$ defined as follows:

$$\phi_i(s)(a) = u_i(s,a) + \delta \sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s,a).$$

Note that $\phi_i(s)$ is regarded as a payoff function of player i parameterized by s, and we can define a (one-shot) n-person game $\Gamma(s)$ given a three-tuple $\Gamma(s) = (N, (A_i)_{i=1}^n, (\phi_i(s))_{i=1}^n)$. The following lemma states that this game $\Gamma(s)$ has the greatest equilibrium, and the greatest equilibrium is increasing in s if (i) (U1)–(U3) and (T1)–(T3) are satisfied and (ii) $U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(s)$ is increasing in s. To prove the lemmas and the theorem, we use some properties and definitions of supermodular games on lattice theory [12, 13, 19]. These definitions and properties are summarized in the appendix.

Lemma 3.2. *Suppose that (i)* u_i *satisfies (U1)–(U3) and* f *satisfies (T1)–(T3) for any* $i \in N$, *and (ii)* $U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(s)$ *is increasing in* s.

Then, the game $\Gamma(s)$ *has the greatest equilibrium* $a^*(s)$, *and* $a^*(s)$ *is increasing in* s.

Proof. As f satisfies (T1) and $U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(s)$ is increasing in s, (i) in Proposition A.3 implies the following (C1) by replacing x to $\hat{s}$, y to a_i and z to (s, a_{-i}):

(C1) $\sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s,a)$ is supermodular in a_i for any $s \in S$ and $a_{-i} \in A_{-i}$:

$$\begin{aligned}&\textstyle\sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a'_i \vee a_i, a_{-i}) + \sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a'_i \wedge a_i, a_{-i}) \\ &\geq \textstyle\sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a'_i, a_{-i}) + \sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a_i, a_{-i})\end{aligned}$$

for any $a'_i, a_i \in A_i$.

(C1) and (U1) imply that $\phi_i(s)$ is supermodular in s for any $s \in S$ and $a_{-i} \in A_{-i}$:

$$\begin{aligned}\forall a'_i, a_i \in A_i, \quad &\phi_i(s)(s, a'_i \vee a_i, a_{-i}) + \phi_i(s)(s, a'_i \wedge a_i, a_{-i}) \\ &\geq \phi_i(s)(s, a'_i, a_{-i}) + \phi_i(s)(s, a_i, a_{-i}).\end{aligned}$$

Similarly, as f satisfies (T2) and $U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(s)$ is increasing in s, (ii) in Proposition A.3 implies the following (C2) by replacing x to $\hat{s}$ and (y,z) to (a_i, a_{-i}):

(C2) $\sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s,a)$ has increasing differences in (a_i, a_{-i}) for any $s \in S$:

$$\begin{aligned}&\textstyle\sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a'_i, a'_{-i}) - \sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a_i, a'_{-i}) \\ &\geq \textstyle\sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a'_i, a_{-i}) - \sum_{\hat{s} \in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a_i, a_{-i})\end{aligned}$$

for any $a'_i \geq a_i$ and $a'_{-i} \geq a_{-i}$.

(C2) and (U2) imply that $\phi_i(s)$ has increasing differences in (a_i, a_{-i}) for any $s \in S$:

$$\begin{aligned}\forall a_i' \geq a_i, \quad \forall a_{-i}' \geq a_{-i}, \quad & \phi_i(s)(s, a_i', a_{-i}') - \phi_i(s)(s, a_i, a_{-i}') \\ & \geq \phi_i(s)(s, a_i', a_{-i}) - \phi_i(s)(s, a_i, a_{-i}).\end{aligned}$$

Since A_i is an integer interval in $\mathbb{Z}^m$, A_i is a finite lattice. Then, $\{\Gamma(s)\}_{s\in S} = \{(N, (A_i)_{i=1}^n, (\phi_i(s))_{i=1}^n)\}_{s\in S}$ is a family of supermodular games parameterized by $s \in S$.

As f satisfies (T3) for any $a_{-i} \in A_{-i}$ and $U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(s)$ is increasing in s, (ii) in Proposition A.3 implies the following (C3) by replacing x to $\hat{s}$ and (y, z) to (s, a_i):

(C3) $\sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a)$ has increasing differences in (s, a_i) for any $a_{-i} \in A_{-i}$:

$$\begin{aligned}&\textstyle\sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s', a_i', a_{-i}) - \sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s', a_i, a_{-i}) \\ &\geq \textstyle\sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a_i', a_{-i}) - \sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a_i, a_{-i})\end{aligned}$$

for any $s' \geq s$ and $a_i' \geq a_i$.

(C3) and (U3) imply that $\phi_i(s)$ has increasing differences in (s, a_i) for any $a_{-i} \in A_{-i}$. By Theorem A.1, (i) $\Gamma(s)$ has the greatest equilibrium $a^*(s)$ and (ii) $a^*(s)$ is increasing in s. □

Lemma 3.3 asserts that, by adding (U4) and (T4) to (U1)–(U3) and (T1)–(T3), the continuation value function at time t $U_i^t(\boldsymbol{\sigma}^{\geq t})$ is increasing in s if the continuation value function at time $t+1$ $U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})$ is increasing in s.

Lemma 3.3. *Suppose that u_i satisfies (U1)–(U4) and f satisfies (T1)–(T4) for any $i \in N$. If $U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})$ is increasing in s, then $\phi_i(s)(a^*(s))$ is increasing in s.*

Proof. By Lemma 3.2, the game $\Gamma(s)$ has the greatest equilibrium $a^*(s)$, which is increasing in s. For any $s \in S$, let $a_i^*(s)$ be a strategy of player i in equilibrium $a^*(s)$ and $a_{-i}^*(s)$ be strategies of other players in equilibrium $a^*(s)$, respectively.

Consider two states $s', s \in S$ such that $s' \geq s$. Then, $a^*(s') \geq a^*(s)$, i.e., $a_i^*(s') \geq a_i^*(s)$ and $a_{-i}^*(s') \geq a_{-i}^*(s)$.

As $a_i^*(s')$ is an equilibrium strategy of player i for game $\Gamma(s')$, the payoff of player i is non-increasing as the strategy changes from $a_i^*(s')$ to $a_i^*(s)$:

$$\begin{aligned}\phi_i(s')(a^*(s')) &= u_i(s', a_i^*(s'), a_{-i}^*(s')) + \delta \textstyle\sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s', a_i^*(s'), a_{-i}^*(s')) \\ &\geq u_i(s', a_i^*(s), a_{-i}^*(s')) + \delta \textstyle\sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s', a_i^*(s), a_{-i}^*(s')).\end{aligned} \tag{2}$$

$s' \geq s$, $a_{-i}^*(s') \geq a_{-i}^*(s)$ and (U4) imply that

$$u_i(s', a_i^*(s), a_{-i}^*(s')) \geq u_i(s, a_i^*(s), a_{-i}^*(s)). \tag{3}$$

Similarly, $s' \geq s$, $a_{-i}^*(s') \geq a_{-i}^*(s)$ and (T4) implies

$$P_f(\hat{S}|s', a_i^*(s), a_{-i}^*(s')) \geq P_f(\hat{S}|s, a_i^*(s), a_{-i}^*(s))$$

for any increasing set $\hat{S} \subseteq S$. Then, Proposition A.2 implies

$$\sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s', a_i^*(s), a_{-i}^*(s')) \geq \sum_{\hat{s}\in S} U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(\hat{s}) f(\hat{s}|s, a_i^*(s), a_{-i}^*(s)) \tag{4}$$

because $U_i^{t+1}(\boldsymbol{\sigma}^{\geq t+1})(s)$ is increasing in s.

By (2), (3) and (4), we conclude that $\phi_i(s')(a^*(s')) \geq \phi_i(s)(a^*(s))$. □

Proof of Theorem 3.1. We consider a Markov game with the terminal period T. The proof is given by induction based on T.

Suppose $T = 0$ and fix a state s. Then, the game is a one-shot n-person game. (U1)–(U3) imply that the game has the greatest equilibrium $\sigma^{*0}(s) = a^*(s)$, which is increasing in s by Theorem A.1. (U4) implies that the payoff of the greatest equilibrium is increasing in s. Hence, (i)–(iii) hold for a Markov game with the terminal period $T = 0$.

Next, suppose that $T \geq 1$ and (i)–(iii) hold for a Markov game with terminal period $T-1$. As a subgame following $t = 1$ is a Markov game with $T-1$ periods, there exists an MPE, according to the inductive hypothesis of (i). Hence, we denote the MPE of the subgame following $t = 1$ by $\boldsymbol{\sigma}^{*\geq 1} = (\sigma^{1*}, \sigma^{2*}, \ldots, \sigma^{T*})$. The inductive hypotheses of (ii) and (iii) also imply that $\boldsymbol{\sigma}^{*\geq 1}$ and $U_i^1(\boldsymbol{\sigma}^{*\geq 1})(s)$ are increasing in s. We define $\phi_i(s)(a)$ by

$$\phi_i(s)(a) = u_i(s,a) + \delta \sum_{\hat{s} \in S} U_i^1(\boldsymbol{\sigma}^{\geq 1})(\hat{s}) f(\hat{s}|s,a)$$

for any $i \in N$, and consider a strategic form game $\Gamma(s) = (N, (A_i)_{i=1}^n, (\phi_i(s))_{i=1}^n)$. Lemma 3.3 implies that

(r1) $\Gamma(s)$ has the greatest equilibrium $a^*(s)$,
(r2) $a^*(s)$ is increasing in state s, and
(r3) $\phi_i(s)(a^*(s))$ is increasing in state s for any $i \in N$.

Let $\sigma^{0*}(s) = a^*(s)$. Then, (r1) and inductive hypothesis (i) imply that $\boldsymbol{\sigma}^* = (\sigma^{0*}, \sigma^{1*}, \sigma^{2*}, \ldots, \sigma^{T*})$ is an MPE. As $\sigma^{0*}(s)$ is increasing in s by (r2), inductive hypothesis (ii) implies that $\boldsymbol{\sigma}^*$ is increasing in s. Finally, $U_i^0(\boldsymbol{\sigma}^{*\geq 0})(s) = \phi_i(s)(a^*(s))$ is increasing in s by (r3). Inductive hypothesis (iii) implies that $U_i^t(\boldsymbol{\sigma}^{*\geq t})(s)$ is also increasing in state s for any $i \in N$ and for any $0 \leq t \leq T$. Hence, (i)–(iii) hold for T. This concludes the proof. □

4. An Application: Bertrand Oligopoly with Investment

One of the applications satisfying (U1)–(U4) and (T1)–(T4) is a Bertrand oligopoly game with investments, which is stated as follows. Firm i ($i = 1, \ldots, n$) sells product i with price $p_i \in \{p \in \mathbb{Z} | 0 \leq p \leq \bar{p}\}$. The demand for product i depends on both the prices of all types of products $(p_1, \ldots, p_n)$ and the appeal of product i, s_i. The accumulation of investment increases the appeal of product i, thereby increasing the demand for the product. At time t, firm i decides the price of product i, p_i, and the amount of the investment $I_i \in \{I \in \mathbb{Z} | 0 \leq I \leq \bar{I}\}$. The appeal of the product of firm i at time t is denoted by s_i^t and is given by $s_i^{t+1} = s_i^t + h_i(s_i^t, I_i)$, where $h_i(s_i^t, I_i)$ is an increment in the appeal of the product. Thus, we assume that the increment in the appeal of product i depends only on the amount of investment by firm i, I_i, and the current appeal of the product, s_i^t, because we ignore the spillover effects of investment by other firms. We also assume that $h_i(s_i, I_i)$ is a random variable according to some distribution function F_i defined by

$$F_i(x|s_i, I_i) = Prob[h_i(s_i, I_i) \leq x].$$

The payoff of firm i is defined by

$$u_i(s,a) = (p_i - c_i) D_i(s_i, p_1, \ldots, p_n) - k_i I_i,$$

where $c_i \geq 0$ is the marginal cost of product i and $k_i \geq 0$ is the marginal cost of investment of firm i. $D_i(s_i, p_1, \ldots, p_n)$ is the demand for product i when its appeal is s_i and for prices $(p_1, \ldots, p_n)$.

This Bertrand oligopoly game with investments can be modeled as a Markov game with a finite horizon T, finite actions, and a finite state. The set of states $S \subset \mathbb{Z}^n$ is the set of n-tuples of the appeal of the products of all firms, given by

$$S = \{(s_1, \dots, s_n) \in \mathbb{Z}^n \mid 0 \le s_i \le \bar{s}_i,\ i = 1, \dots, n\},$$

where $\bar{s}_i$ gives the upper bound of the appeal of product i. The set of actions of firm i is given by

$$A_i = \{(p_i, I_i) \in \mathbb{Z}^2 \mid 0 \le p_i \le \bar{p},\ 0 \le I_i \le \bar{I},\ i = 1, \dots, n\ \}.$$

To apply Theorem 3.1, we assume that the demand function D_i is a linear function given by

$$\textbf{(D1)}\quad \begin{array}{lll} D_i(s_i, p_1, \dots, p_n) = \alpha(s_i) - \beta_{ii} p_i + \sum_{j \ne i} \beta_{ij} p_j & & \\ s'_i \ge s_i \quad \to & \alpha(s'_i) \ge \alpha(s_i), & \\ \beta_{ii} \ge 0,\ \ \beta_{ij} \ge 0,\ \ \alpha(s_i) > \bar{p}\beta_{ii} & & \text{for any } s_i, \end{array}$$

and that the probability distribution of the increment in the investment F_i of each firm $i \in N$ satisfies

(F1) $F_i(x - s'_i | s'_i, I'_i) \le F_i(x - s_i | s_i, I_i)$ for any $s'_i \ge s_i$ and $I'_i \ge I_i$, and for any x, and

(F2) $F_i(x - s'_i | s'_i, I'_i) - F_i(x - s_i | s_i, I'_i) \le F_i(x - s'_i | s'_i, I_i) - F_i(x - s_i | s_i, I_i)$ for any $s'_i \ge s_i$ and $I'_i \ge I_i$, and for any x.

Then, we can establish the following result.

Proposition 4.1. *If D_i is given by (D1) and F_i satisfies (F1) and (F2) for any $i \in N$, then*

(i) *the game has an MPE,*

(ii) *the price and the amount of investment is increasing in state s, for any t, and*

(iii) *$U_i^t(\boldsymbol{\sigma}^{*\ge t})(s)$ is also increasing in state s for any $i \in N$ and for any $0 \le t \le T$.*

Before proving Proposition 4.1, we present an example of F_i satisfying (F1) and (F2):

$$F_i(y | s_i, I_i) = \begin{cases} \left(1 - \epsilon(\bar{I} - I_i)\right)(y/\bar{h}) + \epsilon(\bar{I} - I_i) & y < \bar{h} \\ 1 & y \ge \bar{h} \end{cases}$$

where $\bar{h} > 0$ is the upper bound of the increment in the appeal of the product satisfying $\bar{h} < \bar{s}_i - s_i$, and ϵ is a constant satisfying $\epsilon \le 1/\bar{I}$. Note that $F_i(0 | s_i, I_i) > 0$ for $I_i > 0$, i.e., there is a positive probability that the increment in the appeal of the product is zero, even if the amount of investment I_i is positive. We find that $F_i(\bar{h} | s_i, I_i) = 1$ and that F_i is increasing in y, and hence F_i is a probability distribution function.

Choose arbitrary x, $s'_i \ge s_i$, and $I'_i \ge I_i$. Suppose that $x < \bar{s}$. Since $x - s'_i < \bar{s} - s'_i$ and $x - s_i < \bar{s} - s_i$, we have

$$\begin{aligned} & F_i(x - s'_i | s'_i, I'_i) - F_i(x - s_i | s_i, I_i) \\ & = \{\left(1 - \epsilon(\bar{I} - I'_i)\right)(x - s'_i/\bar{h}) + \epsilon(\bar{I} - I'_i)\} - \{\left(1 - \epsilon(\bar{I} - I_i)\right)(x - s_i/\bar{h}) + \epsilon(\bar{I} - I_i)\} \\ & \le \{\left(1 - \epsilon(\bar{I} - I'_i)\right)(x - s_i/\bar{h}) + \epsilon(\bar{I} - I'_i)\} - \{\left(1 - \epsilon(\bar{I} - I_i)\right)(x - s_i/\bar{h}) + \epsilon(\bar{I} - I_i)\} \\ & = -\epsilon \left(1 - \frac{x - s_i}{\bar{h}}\right)(I'_i - I_i) \le 0, \end{aligned}$$

Hence, (F1) holds. Similarly, (F2) holds by

$$\begin{aligned} & \{F_i(x - s'_i | s'_i, I'_i) - F_i(x - s_i | s_i, I'_i)\} - \{F_i(x - s'_i | s'_i, I_i) - F_i(x - s_i | s_i, I_i)\} \\ & \quad = \{\left(1 - \epsilon(\bar{I} - I'_i)\right)(x - s'_i/\bar{h}) + \epsilon(\bar{I} - I'_i)\} - \{\left(1 - \epsilon(\bar{I} - I'_i)\right)(x - s_i/\bar{h}) + \epsilon(\bar{I} - I'_i)\} \\ & \quad - \{\left(1 - \epsilon(\bar{I} - I_i)\right)(x - s'_i/\bar{h}) + \epsilon(\bar{I} - I_i)\} + \{\left(1 - \epsilon(\bar{I} - I_i)\right)(x - s_i/\bar{h}) + \epsilon(\bar{I} - I_i)\} \\ & \quad = -(\epsilon/\bar{h})(I'_i - I_i)(s'_i - s_i) \le 0. \end{aligned}$$

Suppose that $x \geq \bar{s}$. Then, $x - s_i' \geq \bar{s} - s_i'$ and $x - s_i \geq \bar{s} - s_i$. This implies $F_i(x - s_i'|s_i', I_i') = F_i(x - s_i|s_i, I_i') = F_i(x - s_i'|s_i', I_i) = F_i(x - s_i|s_i, I_i) = 1$, and (F1) and (F2) holds as an equality. Hence, we conclude that F_i satisfies (F1) and (F2).

Proof of Proposition 4.1. Fix any firm $i \in N$. In the first part of the proof, we will assert that $u_i(s, a)$ satisfies (U1)–(U4). For any $s \in S$ and $a_{-i} \in A_{-i}$, we obtain

$$u_i(s, a_i' \vee a_i, a_{-i}) + u_i(s, a_i' \wedge a_i, a_{-i}) = u_i(s, a_i', a_{-i}) + u_i(s, a_i, a_{-i})$$

for any $a_i', a_i \in A_i$. Hence, (U1) holds as an equality. For any $s \in S$, $a_i' \geq a_i$, and $a_{-i}' \geq a_{-i}$,

$$\begin{aligned}&\left(u_i(s, a_i', a_{-i}') - u_i(s, a_i, a_{-i}')\right) - \left(u_i(s, a_i', a_{-i}) - u_i(s, a_i, a_{-i})\right)\\ &= (p_i' - p_i)\left\{\sum_{j \neq i} \beta_{ij}(p_j' - p_j)\right\} \geq 0.\end{aligned}$$

Then, (U2) holds. We find that (U3) holds by a calculation such as

$$\begin{aligned}&(u_i(s', a_i', a_{-i}) - u_i(s', a_i, a_{-i})) - (u_i(s, a_i', a_{-i}) - u_i(s, a_i, a_{-i}))\\ &= (p_i' - p_i)(\alpha(s_i') - \alpha(s_i)) \geq 0\end{aligned}$$

for any $a_i' \geq a_i$, $s' \geq s$, and $a_{-i} \in A_{-i}$. Finally, (U4) also holds by a calculation such as

$$\begin{aligned} u_i(s', a_i, a_{-i}') &= \left(\alpha(s_i') - \beta_{ii}p_i + \sum_{j \neq i} \beta_{ij} p_j'\right) p_i - k_i I_i \\ &\geq \left(\alpha(s_i) - \beta_{ii}p_i + \sum_{j \neq i} \beta_{ij} p_j\right) p_i - k_i I_i \\ &= u_i(s, a_i, a_{-i}) \end{aligned}$$

for any $a_i \in A_i$, $s' \geq s$, and $a_{-i}' \geq a_{-i}$.

In the rest of the proof, we will show that the transition probability satisfies (T1)–(T4) if F_i satisfies (F1)–(F2). If $\hat{S} \subseteq S$ is an increasing set, then $\hat{S}$ must be a rectangle in S of which the maximum point is $(\overline{s}_1, \ldots, \overline{s}_n)$. Let $x(\hat{S}) = (\hat{s}_1(\hat{S}) \ldots \hat{s}_n(\hat{S}))$ be the minimum point of $\hat{S}$. Thus, for any increasing set $\hat{S} \subseteq S$, $\hat{S}$ is denoted by $\times_{i=1}^{n}\{s_i | \hat{s}_i(\hat{S}) \leq s_i \leq \overline{s}_i\}$.

Let $\bar{F}_j(x|s_j, I_j) = 1 - F_j(x|s_j, I_j)$. Then, for any increasing set $\hat{S}$, we have

$$P_f(\hat{S}|s, a) = \Pi_{j \in N} \bar{F}_j(\hat{s}_j(\hat{S}) - s_j | s_j, I_j).$$

Fix arbitrary increasing set $\hat{S}$, $j \in N$ and $a_j' \geq a_j$. $a_j' \geq a_j$ yields $I_j' \geq I_j$. (F1) implies that

$$\bar{F}_j(\hat{s}_j(\hat{S}) - s_j' | s_j', I_j') \geq \bar{F}_j(\hat{s}_j(\hat{S}) - s_j | s_j, I_j), \tag{5}$$

and (F2) implies that

$$\bar{F}_j(\hat{s}_j(\hat{S}) - s_j'|s_j', I_j') - \bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j') \geq \bar{F}_j(\hat{s}_j(\hat{S}) - s_j'|s_j', I_j) - \bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j), \tag{6}$$

for any $s_j' \geq s_j$.

First, we show that (T1) holds in the equality. For any $a'_i, a_i \in A_i$, $s \in S$, and increasing set $\hat{S}$, we can assume $I'_I \geq I_i$ without loss of generality, and

$$
\begin{aligned}
&P_f(\hat{S}|s, a'_i \vee a_i, a_{-i}) + P_f(\hat{S}|s, a'_i \wedge a_i, a_{-i}) \\
&= \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, \max\{I'_i, I_i\})\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) \\
&+\bar{F}_j(\hat{s}_i(\hat{S}) - s_i|s_i, \min\{I'_i, I_i\})\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) \\
&= \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|I'_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) + \bar{F}_j(\hat{s}_i(\hat{S}) - s_i|I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) \\
&= P_f(\hat{S}|s, a'_i, a_{-i}) + P_f(\hat{S}|s, a_i, a_{-i}).
\end{aligned}
$$

Second, we will show that (T2) is satisfied. For any s, $a'_i \geq a_i$, $a'_{-i} \geq a_{-i}$, and any increasing set $\hat{S}$,

$$
\begin{aligned}
&P_f(\hat{S}|s, a'_i, a'_{-i}) - P_f(\hat{S}|s, a_i, a'_{-i}) \\
&= \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I'_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I'_j) - \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I'_j) \\
&= \left(\bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I'_i) - \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I_i)\right)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I'_j) \\
&\geq \left(\bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I'_i) - \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I_i)\right)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) \\
&= \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I'_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) - \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) \\
&= P_f(\hat{S}|s, a'_i, a_{-i}) - P_f(\hat{S}|s, a_i, a_{-i}),
\end{aligned}
$$

where the inequality is obtained by (5) setting $s'_j = s_j$.

Third, we will show that (T3) holds. For any $s' \geq s$, $a'_i \geq a_i$, a_{-i}, and any increasing set $\hat{S}$,

$$
\begin{aligned}
&P_f(\hat{S}|s', a'_i, a_{-i}) - P_f(\hat{S}|s', a_i, a_{-i}) \\
&= \bar{F}_i(\hat{s}_i(\hat{S}) - s'_i|s'_i, I'_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s'_j|s'_j, I_j) - \bar{F}_i(\hat{s}_i(\hat{S}) - s'_i|s'_i, I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s'_j|s'_j, I_j) \\
&= \left(\bar{F}_i(\hat{s}_i(\hat{S}) - s'_i|s'_i, I'_i) - \bar{F}_i(\hat{s}_i(\hat{S}) - s'_i|s'_i, I_i)\right)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s'_j|s'_j, I_j) \\
&\geq \left(\bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I'_i) - \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I_i)\right)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s'_j|s'_j, I_j) \\
&\geq \left(\bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I'_i) - \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I_i)\right)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) \\
&= P_f(\hat{S}|s, a'_i, a_{-i}) - P_f(\hat{S}|s, a_i, a_{-i}),
\end{aligned}
$$

where the first inequality is implied by (6) and the second inequality is implied by (5) setting $I'_j = I_j$.

Finally, we will show that (T4) is true. For any a_i,$s' \geq s$, $a'_{-i} \geq a_{-i}$, and any increasing set $\hat{S}$,

$$
\begin{aligned}
P_f(\hat{S}|s', a_i, a'_{-i}) &= \bar{F}_i(\hat{s}_i(\hat{S}) - s'_i|s'_i, I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s'_j|s'_j, I'_j) \\
&\geq \bar{F}_i(\hat{s}_i(\hat{S}) - s_i|s_i, I_i)\Pi_{j\neq i}\bar{F}_j(\hat{s}_j(\hat{S}) - s_j|s_j, I_j) \\
&= P_f(\hat{S}|s, a_i, a_{-i}),
\end{aligned}
$$

where the inequality is implied by (5). □

5. Conclusions

The present paper established sufficient conditions for the existence of an MPE in pure strategies for a class of stochastic games with a finite horizon and finite states, in which any stage game has strategic complementarities for finite actions. The results for finite states and actions have advantages for numerical computation. Indeed, the computation of equilibria of Markov games has attracted a great deal of attention in the modeling of industrial dynamics [6, 9]. It would be useful to extend the results of this study to these models.

Appendix: Summary of Lattice Theory: Definitions and Results

In this appendix, we summarize some properties of lattice theory on a finite set that were used herein.

A.1. A partially ordered set and a lattice

Let X be a nonempty finite set and $\succeq$ be a partial order on X, i.e., $\succeq$ is a binary relation that is reflexive, transitive, and anti-symmetric.

For a subset Y of X, $\hat{y} \in X$ is referred to as an upper (lower) bound on Y if $\hat{y} \succeq y$ ($\hat{y} \preceq y$) for all $y \in Y$. A supremum (infimum) of $Y \subseteq X$ is a least upper bound (greatest lower bound), and is denoted by $\sup_X Y$ ($\inf_X Y$). If a supremum (infimum) of Y belongs to Y, then it is said to be the greatest (least) element of Y.

For any two elements $x', x \in X$, $x' \vee x$ and $x' \wedge x$ are defined by $\sup_X\{x', x\}$ and $\inf_X\{x', x\}$, respectively.

For a subset of $X \subseteq \mathbb{R}^m$, $\succeq$ becomes a partial order if it is defined by the usual order in $\mathbb{R}^m$, i.e.,

$$x' \succeq x \iff x'_i \geq x_i \quad \text{for any } i.$$

In this order, $x' \vee x$ and $x' \wedge x$ are equivalent to

$$x' \vee x = (\max\{x'_1, x_1\}, \ldots, \max\{x'_m, x_m\}), \quad x' \wedge x = (\min\{x'_1, x_1\}, \ldots, \min\{x'_m, x_m\})$$

defined as (1).

For a finite set X with a partial order $\succeq$ and a subset Y of X, Y is said to be a *lattice* if, for any $x', x \in Y$, both $x' \vee x$ and $x' \wedge x$ also belong to Y.

A.2. Supermodularity and increasing differences

A function $g : X \to \mathbb{R}$ on a lattice X endowed with a partial order $\succeq$ is said to be *supermodular* if, for any $x, x' \in X$, $g(x' \vee x) + g(x' \wedge x) \succeq g(x') + g(x)$.

Let X be a lattice with a partial order $\succeq_X$, and let Y be a finite set with a partial order $\succeq_Y$. The function $g : X \times Y \to \mathbb{R}$ has *increasing differences* in (x, y) if $g(x', y) - g(x, y)$ is increasing in y for $x' \succeq_X x$, namely,

$$g(x', y') - g(x, y') \geq g(x', y) - g(x, y) \quad \forall x' \succeq_X x \quad \text{and} \quad \forall y' \succeq_Y y. \tag{7}$$

A.3. Supermodular games

A (finite) *n-person game* Γ is a three tuple $\Gamma = (N, (A_i)_{i=1}^n, (\gamma_i)_{i=1}^n)$ where

- $N = \{1, \cdots, n\}$ is the set of players,
- A_i is the finite set of actions for player i, and
- $\gamma_i : A \to R$ is the payoff function of player i, where $A = A_1 \times \cdots \times A_n$.

An n-person game is said to be supermodular if (i) the set of actions A_i for each player i is a lattice with some partial order $\succeq_i$, (ii) the payoff function $\gamma_i(a_i, a_{-i})$ is supermodular in $a_i \in A_i$ for fixed $a_{-i} \in A_{-i}$, where $A_{-i} = A_1 \times \cdots \times A_{i-1} \times A_{i+1} \cdots \times A_n$, and (iii) the payoff function $\gamma_i(a_i, a_{-i})$ has increasing differences in (a_i, a_{-i}), where the partial order $\succeq_{-i}$ on A_{-i} is defined as $a'_{-i} \succeq_{-i} a_{-i}$ iff $a'_j \succeq_j a_j$ for any $j \neq i$.

The following result was shown by [8, 12, 19].

Theorem A.1. *Let Y be a partially ordered set, and let $\{(N, (A_i)_{i=1}^n, (\gamma_i(y))_{i=1}^n))\}_{y \in Y}$ be a family of supermodular games in which the payoff function of player i, $\gamma_i(y)$, is parameterized by $y \in Y$.*

Then, there exists the greatest equilibrium a^* in the n-person game $\Gamma(y) = (N, (A_i)_{i=1}^n, (\gamma_i(y))_{i=1}^n)$ for any $y \in Y$, that is, $\Gamma(y)$ has an equilibrium $a^*(y)$ such that $a^*(y) \succeq a$ for any equilibrium a in $\Gamma(y)$.

Moreover, suppose that $\gamma_i(y)(a_i, a_{-i})$ has increasing differences in (y, a_i) for any $i \in N$ and any $a_{-i} \in A_{-i}$. Then, $a^*(y)$ is increasing in y.

A.4. Monotonicity of probability functions

Let $X = \{x \in \mathbb{Z}^k | \underline{x} \leq x \leq \overline{x}\}$ be a k-dimensional integer interval, and let Y be a finite set endowed with a partial order $\succeq$. Suppose that $f(x, y)$ is a probability function on X parameterized by $y \in Y$: f satisfies that for any $y \in Y$, (i) $f(x, y) \geq 0$ for any $x \in X$, and (ii) $\sum_{x \in X} f(x, y) = 1$.

For a fixed $y \in Y$, let $P_f(X', y)$ be a probability of the set $X' \subseteq X$ occurring with respect to $f(x, y)$, which is defined by

$$P_f(X', y) = \sum_{z \in X'} f(z, y).$$

$X' \subseteq X$ is said to be an increasing set, if $x \in X'$ and $x' \geq x$ imply $x' \in X'$.

$f(x, y)$ is said to be *stochastically increasing* in y if, for any $y' \succeq y$, $P_f(X', y') \geq P_f(X', y)$ for any increasing set X'.

The following proposition is known as the theory of multivariate stochastic orders [14, 19].

Proposition A.2 (Multivariate Stochastic Dominance)**.** *Let $v(x)$ be an increasing function in x, and suppose that $f(x, y)$ is stochastically increasing in y. Then, the expectation $\sum_{x \in X} v(x) f(x, y)$ is increasing in y.*

Topkis [19] showed that, for an increasing function, Proposition A.2 implies that the supermodularity and increasing differences of the expectation of a function for specific parameter values are derived from the submodularity and decreasing differences of a distribution function for those parameters.

$f(x, y)$ is said to be *stochastically supermodular* in y if, for any increasing set $X' \subseteq X$ and for any $y', y \in Y$,

$$P_f(X', y' \vee y) + P_f(X', y' \wedge y) \geq P_f(X', y') + P_f(X', y).$$

Consider a probability function $f(x, y, z)$ on X parameterized by $(y, z) \in Y \times Z$, where y and Z are finite sets endowed with partial orders $\succeq_Y$ and $\succeq_Z$, respectively. Here, $f(x, y, z)$ is said to represent *stochastically increasing differences* in (y, z) if, for any increasing set $X' \subseteq X$, $y' \succeq_Y y$, and $z' \succeq_Z z$,

$$P_f(X', y', z') - P_f(X', y, z') \geq P_f(X', y', z) - P_f(X', y, z).$$

Proposition A.3 ([19])**.** *Let $v : X \to R$ be an increasing function of $x \in X$, and let $f(x, y, z)$ be a probability function on X parameterized by $(y, z) \in Y \times Z$. Then, the following two properties hold:*

(i) *If f is stochastically supermodular in $y \in Y$ for any $(x, z) \in X \times Z$, then the expectation $\sum_{x \in X} v(x) f(x, y, z)$ is supermodular in $y \in Y$, and*

(ii) *If f has stochastically increasing differences in (y, z) for any $x \in X$, then the expectation $\sum_{x \in X} v(x) f(x, y, z)$ has increasing differences in (y, z).*

Acknowledgments

We thank two anonymous referees and the editors of this journal for their helpful comments. We are also grateful to the participants of the 2008 Spring Meeting of the Operations Research Society of Japan, SING5, and SWET09. We especially thank Chiaki Hara, Atsushi Kajii, Tadashi Sekiguchi, and Takashi Ui for their helpful comments. This work was supported by JSPS KAKENHI Grant Numbers JP21530169 and JP16K03553.

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