

# EQUIVALENCE OF PARTICLE SURVIVAL MODEL AND RECORD VALUE PROCESS, AND ITS NEW APPLICATION TO LIMIT ORDER BOOKS

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*Abstract* The particle survival model, which was originally proposed to analyze the dynamics of species' coexistence, has surprisingly found to be related to a non-homogeneous Poisson process. It is also well known that successive record values of independent and identically distributed sequences have the spatial distribution of such processes. In this paper, we show that the particle survival model and the record value process are indeed equivalent. Further, we study their application to determine the optimal strategy for placing selling orders on stock exchange limit order books. Our approach considers the limit orders as particles, and assumes that the other traders have zero intelligence.

**Keywords:** Applied probability, particle survival model, record value, non-homogeneous Poisson process, zero-intelligence, limit order book, last arrival problem

## 1. Introduction

The particle survival model was introduced by [18] to model coexistence phenomena in a sequence of emerging species. Accordingly, [11] proposed a mathematical model for the particle survival process, in which successive particles representing species arrive with a random label indicating their fitness. Meanwhile, the existing particles will be killed with a certain probability if their label values are less than those of the arriving particle. This model was fully-analyzed by [7], who found that the steady state counting measure of the surviving particle strengths is a non-homogeneous Poisson process. Inspired by this result, [9] proved that the same result also holds even when the killing probability depends on the label of the particle being killed.

The record value process of an independent and identically distributed (i.i.d.) sequence is known to have the spatial distribution of a non-homogeneous Poisson process whose intensity is equal to the hazard rate function of the underlying random variable (see [14] and [12] for examples).

Although their similarity can be naturally apprehended, but no known relationship has been found between the particle survival model and record value process. In this paper, we show that the order-reversed process of the particle survival model is indeed an extension of the record value process. In addition, we present a simplified proof that their counting measure in the label space is a Poisson process.

Further, we apply the particle survival model and the last arrival problem to the stock market under the assumption that the traders have zero intelligence regarding the limit order book, which is a scenario discussed by [10, 5, 1, 16], and we determine the optimal strategy for placing orders on the stock market.

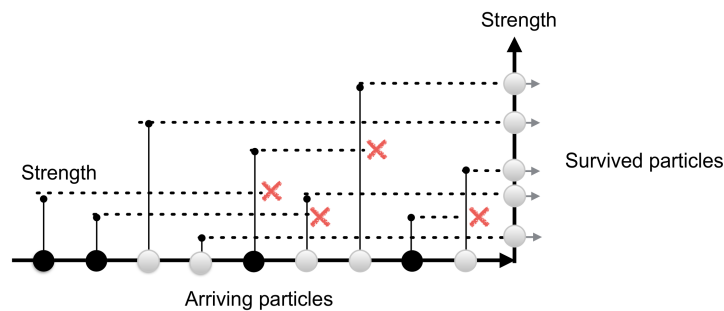
## 2. Equivalence of Particle Survival Model and Record Value Process

Let  $X_1, X_2, \dots$  be a sequence of random variables that are i.i.d. as a distribution function  $F(x)$  that has the density function  $f(x) = dF(x)/dx$ . Particles arrive in succession, and the random variable  $X_i$  represents the strength of the  $i$ -th particle. Particles compete in a special manner: an arriving particle of strength  $y$  challenges existing particles, and challenged particles of strength  $x$  are eliminated with probability  $\alpha(x)$  if  $x < y$  (see Figure 1(a)). Let  $N_n$  be the spatial counting measure of surviving particles in the strength space after  $n$  particles have arrived. This is formally defined as

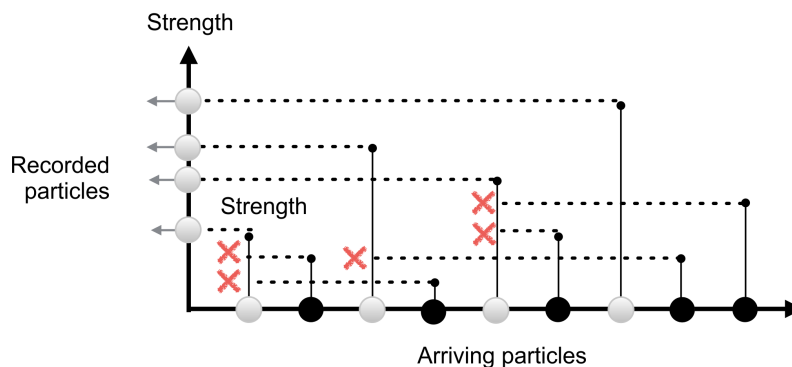
$$N_n(B) = \sum_{i=1}^n 1_{\{U_i=1, X_i \in B\}}, \tag{2.1}$$

where  $B$  is a Borel set in  $\mathbb{R}$  and  $U_i$  indicates whether the  $i$ -th particle is alive ( $U_i = 1$ ) or not ( $U_i = 0$ ). Our aim is to evaluate the stochastic features of the limit counting measure  $N = \lim_{n \rightarrow \infty} N_n$ .

We can also define the record value process for the same sequence of particles (see Figure 1(b)). If a new particle has a larger strength value than all other particles at its



(a) Particle survival model: The first particle has survived a challenge by the third particle, but is eliminated by the fifth particle



(b) Extended record value process: The fifth particle is not an actual record, but is allowed to be recorded by the third particle, which is larger than the fifth particle

Figure 1: Particle survival model and extended record value process

arrival, its strength is registered as a new record. We extend this process to record the strengths of new particles that are not necessarily the strongest ever. That is, when a new particle of strength  $x$  arrives, each particle that has already arrived and is stronger than  $x$  independently determines whether the new particle can register as an extended record with probability  $1 - \alpha(x)$ . Thus, if there exist  $m$  stronger particles (including non-recorded particles) at the time of arrival, the new particle has a probability of  $\{1 - \alpha(x)\}^m$  of being registered as a record. Obviously, when  $\alpha(x) \equiv 1$ , the process is an ordinary record value process. Let  $M_n$  be the spatial counting measure of particles recorded in the strength space after  $n$  particles have arrived. Similarly to (2.1), this is formally defined by

$$M_n(B) = \sum_{i=1}^n 1_{\{U_i=1, X_i \in B\}}. \tag{2.2}$$

Here, we use the same notation  $U_i$  to indicate whether the particle is recorded or not. We set the limit counting measure  $M = \lim_{n \rightarrow \infty} M_n$ .

Consider the order-reversed version of the particle survival model (Figure 2). Particles arrive in reverse order, i.e.,  $\dots, X_n, X_{n-1}, \dots, X_2, X_1$ , starting from the infinite past. We use the same notation for the counting process  $N_n$  (for the last  $n$  particles) and its limit  $N$  in the reverse particle survival model. We show the equivalence of the extended record value process and the order-reversed particle survival model.

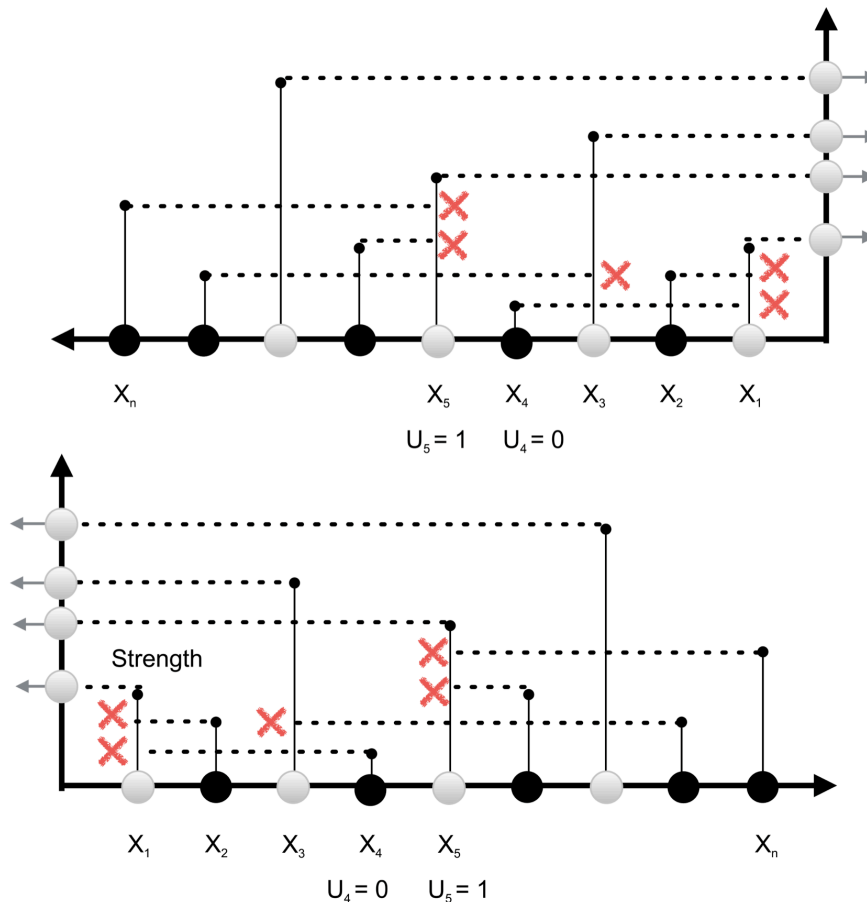


Figure 2: Order-reversed particle survival model (top) and extended record value process (bottom)

**Theorem 2.1.** *Coupling the Bernoulli killings in the order-reversed particle survival model and the extended recordings, the surviving particles are indeed the recorded particles in the extended record value process, that is,  $U_i$ , which is the indicator of the  $i$ -th particle, has the same value in both models.*

*Further, in the limit of  $n \rightarrow \infty$ , the counting measure  $N$  of the (original as well as order-reversed) particle survival model and  $M$  of the extended record value process are the same stochastic process.*

*Proof.* The first particle is always the first record, and this is the last particle to have arrived in the order-reversed particle survival model. This particle always survives (see Figure 2). In the order-reversed particle survival model, particle  $i$  with strength  $X_i = x$  survives when the particles indexed  $1, 2, \dots, i-1$ , which arrive later, are either weaker than  $x$ , or stronger than  $x$  but fail the Bernoulli killing with probability  $1 - \alpha(x)$ . In the record value process, the  $i$ -th particle is registered as the record either when it is the real record (i.e., stronger than the particles indexed  $1, 2, \dots, i-1$ ) or the extended record, which occurs with probability  $\{1 - \alpha(x)\}^m$ , where  $m$  is the number of previously arrived particles that are stronger than  $x$ . Thus, the conditions for a particle to survive and be recorded are equivalent, and  $U_i$  of both the order-reversed particle survival model and the extended record value process is the same.

Since  $X_1, X_2, \dots$  is an i.i.d. sequence and  $N_n$  is monotonically increasing with respect to  $n$  in the reversed system, the spatial counting process of the order-reversed particle survival model converges to the limit  $N$  ([7]), which is stochastically equivalent to the original process.  $\square$

Let us say that the particle survival model is simple, when  $\alpha(x) = 1$ . The simple particle survival model corresponds to the ordinary record value process whose counting measure  $M$  is known to be inhomogeneous Poisson process (see [14] and [12] for example). Thus, by using Theorem 2.1, the following corollary is readily apparent - it is an alternate proof of [7] and [9], for the limited case when  $\alpha(x) \equiv 1$ .

**Corollary 2.1.** *The spatial counting measure  $N$  of the simple particle survival model is an inhomogeneous Poisson process.*

This may be an answer to the surprize raised by [7] as in the title “a surprising poisson process arising from a species competition model.”

For the case of the constant killing probability ( $\alpha(x) \equiv \alpha$ ), [7] used the argument of splitting and merging of Poisson processes. The general case (general function  $\alpha(x)$ ) was later proved by [9] by the argument of the limit of sums of dependent point processes. Employing these results for the survival particle models and Theorem 2.1, we can deduce the property for the extended record value process.

**Corollary 2.2.** *The spatial counting measure  $M$  of the extended record value process is an inhomogeneous Poisson process.*

### 3. Alternate Proof of [7] and [9]

Now, we give an alternate proof of [7] and [9], by applying Watanabe’s characterization to the extended record value process.

Let  $X_{(1)} > \dots > X_{(n)}$  be the decreasingly-ordered statistics of the first  $n$  particles. We use  $(k)$  to indicate the  $k$ -th particle in this decreasing order, and let  $U_{(k)}$  be the record indicator for particle  $(k)$ . The actual arrival order of these particles is randomly assigned. We extend a well-known fact about the independence of record times ([13]) to our extended

record value process. Note that this claim for the particle survival model is embedded in the proof of [9].

**Lemma 3.1.** *By conditioning on  $X_{(k)} = x$ ,  $U_{(k)}$  and  $(U_{(k+1)}, \dots, U_{(n)}, X_{(k+1)}, \dots, X_{(n)})$  are independent. In particular,*

$$\begin{aligned} P(U_{(k)} = 1 | X_{(k)} = x, (U_{(k+1)}, \dots, U_{(n)}, X_{(k+1)}, \dots, X_{(n)})) &= P(U_{(k)} = 1 | X_{(k)} = x) \\ &= \frac{1 - \{1 - \alpha(x)\}^k}{k\alpha(x)}. \end{aligned} \quad (3.1)$$

*Proof.* Given its strength  $X_{(k)} = x$ , the number of challenges to  $(k)$  depends only on the time at which  $(k)$  arrived among the  $k - 1$  stronger particles, which happens at random with uniform probability  $1/k$ . Thus, conditioning on the number of stronger particles that arrived before  $(k)$  among  $k - 1$  stronger particles, which is denoted by  $l$ , we have for  $k < i$  and  $x > y$ ,

$$\begin{aligned} &P(U_{(k)} = 1, U_{(i)} = 1, X_{(i)} \in (y, y + dy) | X_{(k)} = x) \\ &= \sum_{l=0}^{k-1} \frac{1}{k} P(U_{(k)} = 1, U_{(i)} = 1, X_{(i)} \in (y, y + dy) \\ &\quad | X_{(k)} = x, l \text{ stronger particles arrived before } (k)) \\ &= \frac{1}{k} \sum_{l=0}^{k-1} (1 - \alpha(x))^l P(U_{(i)} = 1, X_{(i)} \in (y, y + dy) | X_{(k)} = x) \\ &= \frac{1 - \{1 - \alpha(x)\}^k}{k\alpha(x)} P(U_{(i)} = 1, X_{(i)} \in (y, y + dy) | X_{(k)} = x). \end{aligned}$$

The second equality holds because  $\{U_{(i)} = 1\}$  depends on the position of particle  $(i)$  in the  $(i - 1)$  stronger particles, and it does not depend on the position of  $(k)$  among  $(1), \dots, (k)$ . We can generalize the above argument to show that  $(U_{(k+1)}, \dots, U_{(n)}, X_{(k+1)}, \dots, X_{(n)})$  and  $U_{(k)}$  are independent by conditioning on  $X_{(k)} = x$ .  $\square$

**Theorem 3.1** (Extension of [7, 9]). *When  $0 < \alpha(x) \leq 1$ , the spatial counting measure  $N$  of the particle survival model and its equivalent  $M$  in the extended record value process are made up of an inhomogeneous Poisson process with the intensity measure:*

$$\lambda(x)dx = \frac{h(x)dx}{\alpha(x)} = \frac{f(x)dx}{\alpha(x)\{1 - F(x)\}}, \quad (3.2)$$

where  $h(x)$  is the hazard rate function of the particle strength  $X$ .

*Proof.* By Theorem 2.1, it is sufficient to prove the result for the extended record value process. First, we show that the limit of the intensity is obtained for each  $x \in \mathbb{R}$  as

$$E[M_n(x, x + dx)] \rightarrow E[M(x, x + dx)] = \lambda(x)dx, \quad (3.3)$$

as  $n \rightarrow \infty$ . Since we can ignore the case when two or more particles have the same strength, we have

$$E[M_n(x, x + dx)] = P(M_n(x, x + dx) = 1) = \sum_{k=1}^n P(X_{(k)} \in (x, x + dx), U_{(k)} = 1). \quad (3.4)$$

Using the probability density of the ordered statistic  $X_{(k)}$  to uncondition (3.1), we obtain

$$P(X_{(k)} \in (x, x + dx], U_{(k)} = 1) = \frac{1 - \{1 - \alpha(x)\}^k}{k\alpha(x)} n \binom{n-1}{k-1} (1 - F(x))^{k-1} f(x) dx F(x)^{n-k}. \quad (3.5)$$

Substituting this in (3.4) gives

$$\begin{aligned} E[M_n(x, x + dx)] &= \frac{f(x)dx}{\alpha(x)\{1 - F(x)\}} \sum_{k=1}^n \left[ 1 - \{1 - \alpha(x)\}^k \right] \binom{n}{k} (1 - F(x))^k F(x)^{n-k} \\ &= \frac{f(x)dx}{\alpha(x)\{1 - F(x)\}} \left\{ 1 - F(x)^n - \sum_{k=1}^n \binom{n}{k} \{(1 - \alpha(x))(1 - F(x))\}^k F(x)^{n-k} \right\} \\ &= \frac{f(x)dx}{\alpha(x)\{1 - F(x)\}} \left\{ 1 - F(x)^n - \{(1 - \alpha(x))(1 - F(x)) + F(x)\}^n + F(x)^n \right\}. \end{aligned}$$

Since  $0 < \alpha(x) \leq 1$ , taking  $n \rightarrow \infty$ , we obtain (3.3).

We now prove that  $M$  is a non-homogeneous Poisson counting measure by showing that

$$E[M(x, x + dx) | \mathcal{F}_x] = \lambda(x) dx, \quad (3.6)$$

where  $\mathcal{F}_x$  is the  $\sigma$ -algebra generated by  $\{M(-\infty, s], s \leq x\}$ , since then  $M(-\infty, x] - \int_{-\infty}^x \lambda(u) du$  is  $\mathcal{F}_x$ -martingale and we can use Watanabe's theorem for Poisson process characterization (see [3] or [15]).

Let  $\mathcal{F}_x^n$  be the restriction of  $\mathcal{F}_x$  to the first  $n$  particles. Then, Lemma 3.1 implies

$$\begin{aligned} E[M_n(x, x + dx) | \mathcal{F}_x^n] &= \sum_{k=1}^n P(X_{(k)} \in (x, x + dx], U_{(k)} = 1 | \sigma(U_{(k+1)}, \dots, U_{(n)}, X_{(k+1)}, \dots, X_{(n)}) \\ &= \sum_{k=1}^n P(X_{(k)} \in (x, x + dx], U_{(k)} = 1), \end{aligned} \quad (3.7)$$

where  $\sigma(A)$  is the  $\sigma$ -algebra generated by the random variables  $A$ . Letting  $n \rightarrow \infty$ , we obtain (3.6) from (3.3).  $\square$

#### 4. Limit Order Book with Zero-intelligence Traders

On the stock market, transactions are executed as buy and sell orders placed by traders (double auction: see [17, 6]). There are two types of orders: (1) market orders involving immediately buying or selling stocks at the best price, and (2) limit orders, which involve waiting until they can buy or sell at a specific price. Limit orders are stored in a queue called the limit order book. An arriving buy order above the ask price (which is the lowest sell price in the book) will automatically be executed, i.e., matched with the sell order at that ask price, and cleared from the book. If there are too many orders in the book, some limit orders with no hope of execution in the immediate future are canceled after their arrival. Because of the symmetry in the treatment of buy and sell orders, we focus only on buy limit orders.

The zero-intelligence model of the limit order book was first proposed by [6], and has since been studied by [17, 8, 2], and [19]. Because traders are assumed to have zero-intelligence, the orders are randomly placed as a Poisson process with an i.i.d. sequence of order prices  $X_k$  that is independent of the market situation. We use the particle survival

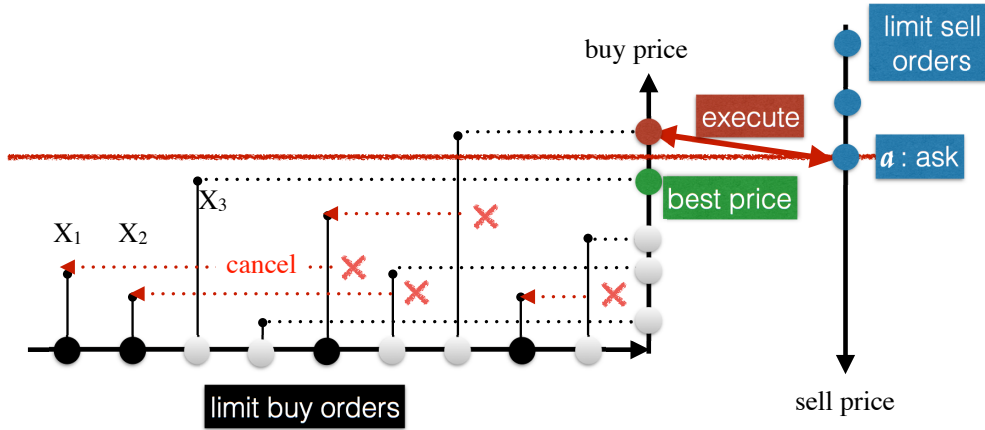


Figure 3: Limit order book as a particle survival model: the best price is the order just below the ask price

model to analyze the zero-intelligence model of the limit order book, by assuming that the  $k$ -th arriving particle is the  $k$ -th limit order to buy a stock at price  $X_k$  (see Figure 4). Note that our model does not require the Poisson arrival of orders. The killing of existing particles can be interpreted as the canceling of limit orders: if someone places a new buy order, a smaller order of price  $x$  that exists in the book is canceled with probability  $\alpha(x)$ . Note that we ignore the possibility that the arrival and staying time of orders affect the cancelling decision.

We assume the transaction speed to be sufficiently high to enable the steady-state condition to hold, i.e., the number of orders that have arrived is infinite ( $n \rightarrow \infty$ ). By applying Theorem 3.1, the spatial counting measure  $N$  of buy limit orders in the steady state is a non-homogeneous Poisson process with intensity  $\lambda(x)dx = f(x)dx / \{\alpha(x)(1 - F(x))\}$ . Most existing models of limit order books assume that the arrivals form a Poisson process. This assumption is mainly because of its analytical tractability, rather than for any specific reason. Our model predicts that the orders in the book form a Poisson process, even when the temporal arrival of the orders itself is not a Poisson process.

### 5. Optimal Selling Strategy by Observing Limit Order Book

We now discuss the optimal selling strategy for a given ask price  $a$ . To beat the current ask price placed by other traders while maximizing our profit by selling a stock, we must determine the highest buying order that is smaller than  $a$  (see Figure 4). This is equivalent to finding some value  $y$  that maximizes the function  $g(y)$  defined by

$$g(y) := P(\text{only one buy limit order in } (y, a)), \tag{5.1}$$

and, once a buy order larger than  $y$  has been found, sell your stock at this buy order price immediately. If  $y$  is set too close to  $a$ , no potential buy order to match your selling may exist in  $(y, a)$ . If  $y$  is too small, the stock may be sold at one (possibly not the best) of many other buy order prices in  $(y, a)$ . Hence, there is an optimal choice of  $y$  that maximizes the success probability  $g(y)$ . This is also known as the last arrival problem ([4]) and is related to the secretary problem.

As discussed in the previous sections,  $N(y, a)$  is a Poisson random variable with mean

$\int_y^a \lambda(x)dx$  and

$$g(y) = \left( \int_y^a \lambda(x)dx \right) e^{-\int_y^a \lambda(x)dx}. \quad (5.2)$$

Differentiating (5.2) and setting it to 0, we find the optimal  $y$  satisfying:

$$\int_y^a \lambda(x)dx = \int_y^a \frac{f(x)dx}{\alpha(x)\{1 - F(x)\}} = 1. \quad (5.3)$$

Since  $\int_0^a \lambda(x)dx \geq -\log(1 - F(a))$ , we can always find  $y$  satisfies (5.3) for any ask price  $a$  with  $1 - e^{-1} \leq F(a) < 1$ .

Substituting this into (5.2), the success probability of our optimal strategy to sell a stock at the best price is always the same as

$$P(\text{only one arrival in } (y, a)) = e^{-1}. \quad (5.4)$$

That is, the well-known  $e^{-1}$  rule from the secretary problem appears again.

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