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ON UNCROSSING GAMES FOR SKEW-SUPERMODULAR FUNCTIONS

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Abstract In this note, we consider the uncrossing game for a skew-supermodular function f, which is a two-player game with players, Red and Blue, and abstracts the uncrossing procedure in the cut-covering linear program associated with f. Extending the earlier results by Karzanov for $\{0,1\}$-valued skew-supermodular functions, we present an improved polynomial time strategy for Red to win, and give a strongly polynomial time uncrossing procedure for dual solutions of the cut-covering LP as its consequence. We also mention its implication on the optimality of laminar solutions.

Keywords: Combinatorial optimization, uncrossing game, skew-supermodular function, cut-covering LP

1. Introduction

Let V be a finite set, and let $\mathcal{S}(V)$ denote the set of all bi-partitions of V. A member $\{X, V \setminus X\}$ of $\mathcal{S}(V)$ is also denoted by X or $V \setminus X$ (if no confusion occurs). A pair X, Y of members in $\mathcal{S}(V)$ is said be *crossing* if $X \cap Y$, $V \setminus (X \cup Y)$, $X \setminus Y$, and $Y \setminus X$ are all nonempty. A family $\mathcal{F} \subseteq \mathcal{S}(V)$ is called *laminar* if there is no crossing pair in $\mathcal{F}$. A function $f : \mathcal{S}(V) \to \mathbf{R}_+$ is called *skew-supermodular* if it satisfies

$$f(X) + f(Y) \leq \max\{f(X \cap Y) + f(X \cup Y), f(X \setminus Y) + f(Y \setminus X)\}$$

for every pair $X, Y \in \mathcal{S}(V)$. For a skew-supermodular function f, we consider the following game involving two players *Blue* and *Red*, which we call the *uncrossing game*.

Uncrossing game for skew-supermodular function f
Input: A family $\mathcal{F} \subseteq \mathcal{S}(V)$.
Step 1: If $\mathcal{F}$ is laminar, then Red wins; the game terminates.
Step 2: Otherwise, Red chooses a crossing pair (X, Y) in $\mathcal{F}$, chooses $(X', Y') \in \{(X \cap Y, X \cup Y), (X \setminus Y, Y \setminus X)\}$ satisfying

$$f(X) + f(Y) \leq f(X') + f(Y'),$$

and replaces X, Y by X', Y' in $\mathcal{F}$.
Step 3: Blue returns one of X, Y to $\mathcal{F}$. Go to **step 1**.

Here we assume that Red has an evaluation oracle of f.

The uncrossing game abstracts the uncrossing procedure in combinatorial optimization, and was originally introduced by Hurkens, Lovász, Schrijver and Tardos [3]. The original formulation is the following. For an input family $\mathcal{F}$ of subsets of V, if $\mathcal{F}$ is a chain, then Red wins. Otherwise Red chooses an incomparable pair (X, Y) in $\mathcal{F}$ and replaces (X, Y) by $(X \cap Y, X \cup Y)$ in $\mathcal{F}$. Then Blue returns one of X and Y to $\mathcal{F}$. Hurkens, Lovász, Schrijver and Tardos showed that there is a strategy (i.e., a way of choosing (X, Y)) for Red to win after a polynomial number $O(|V||\mathcal{F}|)$ of iterations. Karzanov [5] considered a symmetric generalization on cross-closed families. Here a family $\mathcal{S} \subseteq \mathcal{S}(V)$ is called *cross-closed* if for $X, Y \in \mathcal{S}$, $X \cap Y, X \cup Y$ belong to $\mathcal{S}$ or $X \setminus Y, Y \setminus X$ belong to $\mathcal{S}$. In his uncrossing game, the input is a family $\mathcal{F} \subseteq \mathcal{S}$, and Red chooses a crossing pair (X, Y) in $\mathcal{F}$ and chooses $(X', Y') \in \{(X \cap Y, X \cup Y), (X \setminus Y, Y \setminus X)\}$ with $X', Y' \in \mathcal{S}$. As was noticed by Karzanov, the original uncrossing game can be viewed as a special case of the cross-closed family on $V \cup \{s, t\}$ consisting of X with $|X \cup \{s, t\}| = 1$. He showed that there is a strategy for Red to win after $O(|V|^4|\mathcal{F}|)$ iterations.

A cross-closed family $\mathcal{S}$ is naturally identified with a $\{0, 1\}$-valued skew-supermodular function f defined by $f(X) := 1$ for $X \in \mathcal{S}$ and $f(X) := 0$ otherwise. By this identification, the uncrossing game on the cross-closed family $\mathcal{S}$ reduces to our setting. We do not know whether the converse reduction is possible. Also Karzanov's strategy [5] seems not to be applied to our generalization. Indeed, his strategy includes selection rules of type "if $X \notin \mathcal{S}$, then Red takes..."[5, p. 222]. The main aim of this note is to present an improved Red-win strategy for our generalization.

Theorem 1. *For every skew-supermodular function f and every input $\mathcal{F}$, there exists a strategy for Red to win after $O(|V|^3|\mathcal{F}|)$ iterations*

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1. Our strategy basically follows Karzanov's one in high level, and incorporates nontrivial modifications in an essential part. In Section 3, we explain implications of this result on skew-supermodular cut-covering linear programs, which constitute an important subclass of linear programs appearing from a wide variety of network design problems in combinatorial optimization. As was noted by [5] (for $\{0, 1\}$-skew-supermodular cases), the uncrossing game naturally arises from the uncrossing procedure of dual solutions, and a Red-win strategy gives rise to a strongly polynomial time uncrossing algorithm. Our strategy in Theorem 1 is applicable to general skew-supermodular cut-covering linear programs beyond that treated in [5]. We also present an unexpected consequence on the optimality property of laminar dual solutions.

2. Proof

Let $\mathcal{F}$ be a set of bi-partitions on V. A member $X \in \mathcal{F}$ is said to be *trivial* (in $\mathcal{F}$) if no member of $\mathcal{F}$ is crossing with X. If Z is not crossing with X and Y, then it is not crossing with $X \cap Y$, $X \cup Y$, $X \setminus Y$, and $Y \setminus X$. In particular, a trivial member remains trivial after uncrossings. Therefore we can assume that a trivial member is removed from $\mathcal{F}$ whenever it appears. Define an equivalence relation $\sim_{\mathcal{F}}$ on V by $i \sim_{\mathcal{F}} j$ if there is no $X \in \mathcal{F}$ with $|\{i, j\} \cap X| = 1$. An equivalence class of this relation is called an *atom*. If $i \not\sim_{\mathcal{F}} j$, then i and j are said to be *separated*. Obviously each $X \in \mathcal{F}$ is a disjoint union of atoms. Therefore we can identify the ground set V with the set of all atoms. Notice that $i \sim_{\{X \cap Y, X \cup Y\}} j$ implies $i \sim_{\{X,Y\}} j$, and $i \sim_{\{X \setminus Y, Y \setminus X\}} j$ implies $i \sim_{\{X,Y\}} j$. Therefore, if $\mathcal{F}'$ is obtained from $\mathcal{F}$ through iterations of the game, then $\sim_{\mathcal{F}'}$ coarsens $\sim_{\mathcal{F}}$, i.e., $i \sim_{\mathcal{F}} j$ implies $i \sim_{\mathcal{F}'} j$. By this fact together with the removal of trivial members, it happens that several atoms in $\mathcal{F}$ are joined to a single atom in $\mathcal{F}'$. In this case, the cardinality of the ground set decreases.

We first consider the essential case of input $\mathcal{F}$ to which the argument in [5] is not directly applicable. Let $V = \{1, 2, \ldots, n\}$ $(n \geq 4)$. Suppose that $\mathcal{F}$ satisfies:

(A) $\mathcal{F}$ is the disjoint union of two subsets $\mathcal{A}$ and $\mathcal{B}$ such that every member in $\mathcal{A}$ takes form $\{1, 2, \ldots, i\}$ for some $2 \leq i \leq n-2$, and every member of $\mathcal{B}$ takes form $\{2, 3, \ldots, j\}$ for some $3 \leq j \leq n-1$.

In the following, we denote $\{i, i+1, i+2, \dots, j\}$ by $[i, j]$. We may assume that $\mathcal{B} \neq \emptyset$. Let $d(\geq 3)$ be the minimum number for which $[2, d]$ belongs to $\mathcal{B}$. We can assume that $[1, i]$ belongs to $\mathcal{A}$ for $i = 2, 3, \dots, d-1$; otherwise some i and $i+1$ are not separated, and are joined to a single element. We give a strategy for Red to keep the game of form (A) and to decrease $n + |\mathcal{B}|$ after $O(d)$ iterations.

Initially, Red evaluates $f([1, d-1]) + f([2, d])$, $f(\{1\}) + f(\{d\})$, and $f([2, d-1]) + f([1, d])$. Suppose that $f([1, d-1]) + f([2, d]) \leq f(\{1\}) + f(\{d\})$ holds. Red chooses $X = [1, d-1], Y = [2, d]$, chooses $X' = X \setminus Y = \{1\}$, and $Y' = Y \setminus X = \{d\}$, and replaces X, Y by X', Y'; both X', Y' are singletons (trivial) and vanish. If Blue returns $X = [1, d-1]$, then $|\mathcal{B}|$ decreases. If Blue returns $Y = [2, d]$, then $|\mathcal{B}|$ does not change, d and $d-1$ are not separated, and hence n decreases.

Suppose that $f([1, d-1]) + f([2, d]) > f(\{1\}) + f(\{d\})$ holds. By skew-supermodularity, $f([1, d-1]) + f([2, d]) \leq f([2, d-1]) + f([1, d])$ must hold. Suppose that $d = 3$. Red chooses $X = [1, 2]$, $Y = [2, 3]$, $X' = X \cup Y = [1, 3]$, and $Y' = X \cap Y = \{2\}$. Then $\{2\}$ vanishes; in particular $|\mathcal{B}|$ does not increase. If Blue returns $X = [1, 2]$, then $|\mathcal{B}|$ decreases. If Blue returns $Y = [2, 3]$, then 2 and 3 are not separated, and n decreases.

Suppose that $d > 3$. Red computes the smallest $k \in [2, d-1]$ such that

$$f([1, l]) + f([2, l+1]) \leq f([2, l]) + f([1, l+1]) \quad (l = k, k+1, \dots, d-1). \tag{2.1}$$

Such an index k actually exists since $f([1, d-1]) + f([2, d]) \leq f([2, d-1]) + f([1, d])$. If $k > 2$, then it holds $f([1, k-1]) + f([2, k]) > f([2, k-1]) + f([1, k])$, and by skew-supermodularity, it holds

$$f([1, k-1]) + f([2, k]) \leq f(\{1\}) + f(\{k\}). \tag{2.2}$$

Also, in the case of $k = 2$, (2.2) holds since $f(\{1\}) + f(\{2\}) \leq f(\{1\}) + f(\{2\})$. Adding inequalities (2.1), we obtain

$$f([1, k]) + f([2, d]) \leq f([1, d]) + f([2, k]).$$

Red chooses $X = [1, k]$, $Y = [2, d]$, $X' = X \cup Y = [1, d]$, and $Y' = X \cap Y = [2, k]$, and replaces X, Y by X', Y'. If Blue returns $X = [1, k]$, then $|\mathcal{B}|$ does not change, d decreases, and Red goes to the initial stage above. Suppose that Blue returns $Y = [2, d]$. If $k = 2$, then Y' is a singleton and vanishes, and 2 and 3 are not separated; hence $n + |\mathcal{B}|$ decreases. Suppose that $k > 2$. In the next iteration, $|\mathcal{B}|$ increases by one, and d becomes k. By (2.2), Red chooses $X = [1, k-1]$, $Y = [2, k]$, $X' = X \setminus Y = \{1\}$, and $Y' = Y \setminus X = \{k\}$. Both X' and Y' are singletons and vanish in the next iteration. If Blue returns $[1, k-1]$, then $\mathcal{B}$ decreases by one, k and $k+1$ are not separated (since now $[1, k]$ does not exist), and $n + |\mathcal{B}|$ is smaller than that in two iterations before. If Blue returns $[2, k]$, then $k-1$ and k are not separated, $n + |\mathcal{B}|$ is equal to that in two iterations before, $d = k$ is smaller than before, and Red goes to the initial stage above.

Summarizing, by using the above strategy, Red can keep the game of form (A) and decrease $n + |\mathcal{B}|$ after $O(d)$ iterations. Thus Red wins after $O(n^2)$ iterations. We remark that the above strategy is easily adapted for the case where Blue is allowed to return none of X, Y.

The rest of arguments is exactly the same as that given in [5], and is sketched as follows. A subset $X \subseteq V$ is said to be 2-*partitioned* for $\mathcal{F}$ if X intersects at most two atoms with respect to $\mathcal{F}$. Suppose that

(B) the input $\mathcal{F}$ is the disjoint union of two laminar families $\mathcal{C}$ and $\mathcal{D}$ such that for each $X \in \mathcal{C}$, X or $V \setminus X$ is 2-partitioned for $\mathcal{D}$.

In this case, for any $X \in \mathcal{C}$ the family $\mathcal{D} \cup \{X\}$ satisfies the condition (A) after removing trivial members in $\mathcal{D} \cup \{X\}$. Indeed, $\mathcal{D} \cup \{X\}$ consists of X and members $Y_1, Y_2, \ldots, Y_m$ in $\mathcal{D}$ crossing with X. We can assume that X is 2-partitioned for $\mathcal{D}$. By this condition, X is the disjoint union of Z_1 and Z_2 such that $X \cap Y_i \in \{Z_1, Z_2\}$ for $i = 1, 2, \ldots, m$. We can assume that both Z_1 and Z_2 are nonempty, and $Z_2 = X \cap Y_i$ for each i (by $Y_i \leftrightarrow V \setminus Y_i$). By the laminarity of $\mathcal{D}$ together with $\emptyset \neq Z_1 \subseteq V \setminus (Y_i \cup Y_j)$ and $\emptyset \neq Z_2 \subseteq Y_i \cap Y_j$, we have $Y_i \subset Y_j$ or $Y_j \subset Y_i$ for $i \neq j$. By rearranging them, we have $Z_2 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_m$. Atoms are Z_1, Z_2, $Y_1 \setminus Z_2$, $V \setminus (Y_m \cup Z_1)$, and $Y_{i+1} \setminus Y_i$ for $i = 1, 2, \ldots, m-1$. Thus the situation reduces to (A) in the setting of $\mathcal{A} = \{[1, 2]\}$, $\mathcal{B} = \{[2, j] \mid 3 \leq j \leq m+2\}$, and $n = m + 3$.

The strategy for Red is the following. Red chooses *maximal* $\{X, V \setminus X\} \in \mathcal{C}$ in the sense that X is 2-partitioned for $\mathcal{D}$ and there is no $\{Y, V \setminus Y\} \in \mathcal{C}$ such that $X \subset Y$ (proper inclusion) and Y is 2-partitioned for $\mathcal{D}$. As above, the family $\mathcal{D} \cup \{X\}$ satisfies the condition (A) (after removing trivial members). Therefore Red plays the game within $\mathcal{D} \cup \{X\}$ and obtains a laminar family $\mathcal{D}'$ by the above strategy after $O(|V|^2)$ iterations. Since $\mathcal{D}'$ is obtained from $\mathcal{D} \cup \{X\}$ by uncrossings, $\sim_{\mathcal{D}'}$ coarsens $\sim_{\mathcal{D} \cup \{X\}}$ (see the beginning of Section 2). By this fact together with the maximality of X and the laminarity of $\mathcal{C}$, the union of $\mathcal{C}' := \mathcal{C} \setminus \{X\}$ and $\mathcal{D}'$ satisfies the condition (B). Let $\mathcal{C} \leftarrow \mathcal{C}'$ and $\mathcal{D} \leftarrow \mathcal{D}'$. Red repeats the same procedure for $\mathcal{C}$ and $\mathcal{D}$, and wins after $|\mathcal{C}|$ steps. The total number of iterations is $O(|\mathcal{C}||V|^2) = O(|V|^3)$; recall that the size of any laminar family on V is $O(|V|)$.

Suppose finally that $\mathcal{F}$ is arbitrary. Red chooses a (maximal) laminar subset $\mathcal{C}$ of $\mathcal{F}$, and chooses an arbitrary $X \in \mathcal{B} := \mathcal{F} \setminus \mathcal{C}$. Then the union of $\mathcal{C}$ and $\mathcal{D} = \{X\}$ obviously satisfies (B). Red applies the above strategy for $\mathcal{C} \cup \mathcal{D}$, and obtains a laminar family $\mathcal{C}'$ after $O(|V|^3)$ iterations. Let $\mathcal{C} \leftarrow \mathcal{C}'$ and $\mathcal{B} \leftarrow \mathcal{B} \setminus \{X\}$. Red repeats the same procedure for $\mathcal{C}$ and $\mathcal{B}$ (until $\mathcal{B} = \emptyset$), and wins after $O(|\mathcal{F}|)$ steps. The total number of iterations is $O(|V|^3|\mathcal{F}|)$, as required.

3. Implications

As mentioned in Introduction, the uncrossing game abstracts the uncrossing procedure arising from a class of cut-covering linear programs. Let $G = (V, E)$ be an undirected graph with edge-cost $a : E \to \mathbf{R}_+$, and let $f : \mathcal{S}(V) \to \mathbf{R}_+$ be a skew-supermodular function. The *skew-supermodular cut-covering LP* is the following linear program: Minimize the cost $\sum_{e \in E} a(e)x(e)$ over all edge-weights $x : E \to \mathbf{R}_+$ satisfying the covering constraint:

$$\sum_{e \in \delta X} x(e) \geq f(X) \quad (X \in \mathcal{S}(V)),$$

where δX denotes the set of edges $e = ij \in E$ with $i \in X$ and $j \notin X$. This class of LP and its variation capture a wide variety of network design problems and their fractional relaxations. Examples include matching, T-join, network synthesis, survivable network, traveling salesman, Steiner tree/forest, connectivity augmentation and so on; see, e.g., [4, 6]. An important feature of this LP is that its dual always admits a *laminar* optimal solution. The dual LP is given as: Maximize $\sum_{X \in \mathcal{S}(V)} \lambda(X) f(X)$ over all $\lambda : \mathcal{S}(V) \to \mathbf{R}_+$ satisfying

$$\sum_{X \in \mathcal{S}(V): e \in \delta X} \lambda(X) \leq a(e) \quad (e \in E).$$

A feasible solution λ is called *laminar* if its nonzero support $\mathcal{F}(\lambda) := \{X \in \mathcal{S}(V) \mid \lambda(X) \neq 0\}$ is laminar. Then there always exists a laminar optimal solution. This useful property has played key roles in algorithm design and analysis: Edmonds' blossom algorithm [1] for weighted matching works with a laminar dual solution, which enables us to avoid keeping exponential number of inequalities/variables for matching polytope. Jain's iterative rounding algorithm [4] for survivable network was obtained by analyzing the larminarity property of skew-supermodular covering LPs.

The existence of a laminar dual solution can be seen from the following standard uncrossing argument. Let λ be an arbitrary feasible solution. Suppose that $\mathcal{F}(\lambda)$ is not laminar. Choose a crossing pair (X, Y) in $\mathcal{F}(\lambda)$, and choose $(X', Y') \in \{(X \cap Y, X \cup Y), (X \setminus Y, Y \setminus X)\}$ with $f(X) + f(Y) \leq f(X') + f(Y')$. Decrease λ by $\alpha := \min\{\lambda(X), \lambda(Y)\}$ on X and Y, and increase λ by α on X' and Y'. The objective value does not decrease (by skew-supermodularity) and the feasibility is also preserved. This operation is called an *uncrossing.* For rational λ, we obtain a laminar solution after a finite number of uncrossings, where λ is said to be *uncrossed.*

The uncrossing process gives rise to the uncrossing game with input $\mathcal{F} = \mathcal{F}(\lambda)$, as mentioned in [5] (under the setting of a cross-closed family). Red chooses a crossing pair X, Y in $\mathcal{F}$, and replaces X, Y by X', Y' in $\mathcal{F}$. Blue returns one $\tilde{X}$ of X, Y for $\lambda(\tilde{X}) > \min\{\lambda(X), \lambda(Y)\}$; Blue returns none of them if $\alpha = \lambda(X) = \lambda(Y)$. Then $\mathcal{F} = \mathcal{F}(\lambda)$ holds in the next iteration. Suppose that λ is integer-valued. Observe that $\sum_{X \in \mathcal{S}(V)} |X||V \setminus X|\lambda(X)$ strictly decreases in one uncrossing. Therefore, in any choices of uncrossing pairs, this process terminates after $O(|V|^2|\mathcal{F}(\lambda)|\|\lambda\|)$ iterations, where $\|\lambda\| := \max_{X \in \mathcal{S}(V)} \lambda(X)$. By using the Red-win strategy in Theorem 1, we can conduct the uncrossing procedure in time polynomial in $|V|$ and $|\mathcal{F}(\lambda)|$, not depending on the bit length of λ. Thus we have:

Theorem 2. *Any $\lambda : \mathcal{S}(V) \to \mathbf{R}_+$ can be uncrossed in time polynomial of $|V|$ and $|\mathcal{F}(\lambda)|$.*

This is a natural extension of [5, Theorem 2] to general skew-supermodular functions. In [2, Section 3.3], we found that the Red-win strategy of the uncrossing game also brings an interesting optimality property of laminar solutions, where we proved this property for $\{0, 1\}$-valued skew-supermodular functions (as a consequence of Karzanov's uncrossing algorithm). Now we can state and prove this for general skew-supermodular functions.

Proposition 3. *Let λ be a nonoptimal laminar feasible function. For any $\epsilon > 0$, there exists a laminar feasible solution λ^* such that $\|\lambda - \lambda^*\| \leq \epsilon$ and the objective value of λ^* is greater than that of λ.*

Proof. The following proof method is due to [2]. We can assume that $\lambda \neq 0$. We can choose a positive integer N such that every $\lambda : \mathcal{S}(V) \to \mathbf{R}_+$ can be uncrossed by at most N uncrossings in the strategy of Theorem 1. Choose $\epsilon' > 0$. Since λ is not optimal, there exists a feasible (not necessarily laminar) solution λ' such that $\|\lambda - \lambda'\| \leq \epsilon'$ and the objective value of λ' is greater than that of λ; by convexity, λ' can be taken from the segment between λ and an optimal solution. Apply the uncrossing procedure to λ' according to the strategy of Theorem 1. Then we obtain a laminar feasible solution λ^* with the objective value not less than that of λ'.

We show that $\|\lambda - \lambda^*\| \leq \epsilon$ if ϵ' is sufficiently small. Let λ^k denote λ' after k uncrossings; $\lambda^0 = \lambda'$. Choose ϵ' with $\epsilon' \leq 2^{-N} \min_{X \in \mathcal{F}(\lambda)} \lambda(X)$. Then it holds

$$\min_{Z \in \mathcal{F}(\lambda)} \lambda^k(Z) \geq \min_{Z \in \mathcal{F}(\lambda)} \lambda(Z) - 2^k \epsilon' \geq 2^k \epsilon' \geq \max_{Z \notin \mathcal{F}(\lambda)} \lambda^k(Z) \quad (k = 0, 1, 2, \ldots, N-1). \tag{3.1}$$

We show (3.1) by induction on k. In the case of $k = 0$, this indeed holds (by $\|\lambda - \lambda'\| \leq \epsilon'$). The second inequality is obvious from the definition of ϵ'. Suppose that (3.1) holds for $k < N - 1$. Each uncrossing step chooses X, Y so that at least one of X, Y does not belong to laminar family $\mathcal{F}(\lambda)$. By (3.1), in the $(k+1)$-th uncrossing, $\alpha(= \min\{\lambda(X), \lambda(Y)\})$ is attained at $\mathcal{S}(V) \setminus \mathcal{F}(\lambda)$ and is bound by $2^k\epsilon'$. Thus $\min_{Z\in\mathcal{F}(\lambda)} \lambda^{k+1}(Z) \geq \min_{Z\in\mathcal{F}(\lambda)} \lambda^k(Z) - 2^k\epsilon' \geq 2^{k+1}\epsilon'$, and $\max_{Z\notin\mathcal{F}(\lambda)} \lambda^{k+1}(Z) \leq \max_{Z\notin\mathcal{F}(\lambda)} \lambda^k(Z) + 2^k\epsilon' \leq 2^{k+1}\epsilon'$. Thus (3.1) holds after the $(k+1)$-th uncrossing.

In particular, α is attained at $\mathcal{S}(V) \setminus \mathcal{F}(\lambda)$ in every step $k \leq N$, and $\|\lambda^{k+1} - \lambda^k\| \leq 2^k\epsilon'$. Hence we have $\|\lambda^* - \lambda'\| \leq \sum_{k=1}^N \|\lambda^k - \lambda^{k-1}\| \leq \sum_{k=1}^N 2^{k-1}\epsilon' = (2^N - 1)\epsilon'$, and $\|\lambda - \lambda^*\| \leq \|\lambda - \lambda'\| + \|\lambda^* - \lambda'\| \leq 2^N\epsilon'$. Thus, by choosing ϵ' as $\epsilon' \leq 2^{-N}\min\{\epsilon, \min_{X\in\mathcal{F}(\lambda)} \lambda(X)\}$, we obtain the desired result. $\square$

From the view of convexity, this local-to-global optimality property is obvious without the laminarity requirement. The set of all laminar feasible solutions is not convex in the space of all functions on $\mathcal{S}(V)$. Nevertheless this proposition says that the objective behaves convex or unimodal over the space of laminar feasible solutions. This may suggest a laminarity-preserving primal-dual algorithm, like Edmonds' blossom algorithm, for general skew-supermodular cut-covering LPs.

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