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A SEARCH ALLOCATION GAME WITH PRIVATE INFORMATION OF INITIAL TARGET POSITION

Ryusuke Hohzaki
National Defense Academy of Japan

Kwanshik Joo
Ministry of National Defense of Republic of Korea

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Abstract This paper deals with a two-person zero-sum search game, in which a searcher distributes search resource to detect a target and the target moves to evade the searcher. The game includes private information of an initial position of the target and a detection probability of the target as payoff. The searcher estimates the initial position with a probability distribution. We model the problem as an incomplete-information game, and propose a convex programming formulation and a linear programming one to derive an optimal distribution of search resource of the searcher and an optimal target strategy of selecting paths. However, the number of paths exponentially increases as the number of time points becomes larger. To cope with the combinatorial explosion, we propose a new approach using Markov movement strategy of the target. By some numerical examples, we analyze players' optimal strategies and evaluate the value of information of the target initial position.

Keywords: Search, game theory, private information, linear programming, convex programming, dynamic programming

1. Introduction

A search game usually has two players, a searcher and a target. In the so-called search allocation game (SAG), the target takes a strategy of movement in a search space and the searcher distributes a limited amount of search resource in the space to detect the target. This paper deals with the SAG, in which the target knows his initial position at the beginning of the search but the searcher does not, i.e. the initial position is the private information of the target. In actual search operations, the target's initial position has been said to be the most important information and it considerably affects the results of the operations. We model our SAG as an incomplete-information game and derive its equilibrium point, by which we can evaluate the value of the private information.

Search theory has been a major theme in Operations Research. In the well-known book *Methods of Operations Research* [35], Morse and Kimball have already considered the interception problem of the passage of a submarine in straits by an ASW-airplane as a search game. Now let us survey past research on one-sided search problems. Koopman [34] first discussed the search problem to find an optimal distribution of search resource when knowing the distribution probability of a stationary target in a search space. From Koopman's work, the so-called resource allocation problem [26] has started as a fruitful research field. De Guenin [6] and Kadane [30] extended Koopman's model to optimal distribution problems of continuous or discrete search resource just for stationary targets. Their research has been extended further to moving-target problems, in which a target moves as time goes by.

Pollock [38], Dobbie [7], Hellman [12], Iida [27] and Kan [31] are early analytical works on an optimal search for a moving target. Stone [39] contributed to elaborate mathematical formulations for many versions of search problems of stationary and moving targets. Brown [3] and Washburn [42] developed numerical algorithms to derive an optimal distribution of search resource to maximize the detection probability of a target. Stromquist and Stone [40] generalized Brown's and Washburn's works. The one-sided search problems mentioned above were numerically solved by making use of convex programming or nonlinear programming methods in order to obtain an optimal distribution of search resource from the probabilistic law on the distribution of the target. Eagle and Yee [9] and Hohzaki and Iida [21] studied NP-complete problems for an optimal strategy of looking in cells when the searcher has some restrictions on his movement or his resource distribution. The one-sided problem was naturally extended to two-sided problems or search games including two players of a searcher and a target.

Almost all search games are categorized into two models: search allocation game (SAG) and search-and-evasion game (SAEG), depending on the type of searcher's strategy [10, 20]. In the SAEG, the searcher takes a moving strategy as the target does. We can find much research on the SAEG, such as Danskin [5], Nakai [36] and Kikuta [33]. Washburn [41] dealt with a multi-stage game with the traveling time of the searcher as payoff, in which each player decides his next position after being informed of his enemy's current position. Eagle and Washburn [8] considered a one-stage game with the total reward given by sequential positions of the searcher as payoff. Alpern [1] and Baston and Kikuta [2] considered specific SAEG models, in which heuristics were proposed for their solutions.

Many researchers have also studied the SAG. Nakai [37] and Iida et al. [29] considered SAGs with stationary targets. Iida et al. [28] and Hohzaki and Iida [22, 23] considered SAG models with moving targets. Some researchers introduced practical constraints on target motion to make their models more realistic. Wasburn and Hohzaki [25, 43] introduced energy constraints on target motion. Hohzaki et al. [24] and Hohzaki [13] generalized the SAG model with energy constraints and elucidated a relation between two types of SAGs defined in a continuous space and in a discrete one. The basic model of the SAG was extended in various ways. Practical entity of search resource, such as search time, search operation forces and others, would have a variety of individual characteristics. Dambreville and Le Cadre [4] and Hohzaki [16, 18] took account of some attributes, such as rechargeability, long-distance effectiveness and temporal durability, in their SAG models. Hohzaki [14] and Kekka and Hohzaki [32] considered disturbance effects in the SAG as often seen in practical search spaces, in which some false detection signals randomly occur. Some studied several versions of the SAG model: a multi-stage game version [15], a cooperative game version [17] and a nonzero-sum game version [19].

As is true in search-and-rescue and military operations, information about an initial distribution of targets plays an important role in search operations. In the past research on search games surveyed above, they basically considered just models with complete information of the initial target position. Therefore, players know the rules of the game on available search resource of the searcher and initial positions of the target as common knowledge, as well as time and geographic spaces. If the search starts after some time delay from an initial time, a target distribution could spread in some areas and the target is allowed to choose his initial position within the areas at the beginning of the search. Such an intentional choice of initial position is available to the target in this situation. However, in the ordinary case of the search operation, the target can notice the search only after the searcher takes action to start the search. In such a situation, an initial position is seemingly given to the target as a random event by nature and the target cannot choose his initial position at first hand. The searcher would decide to start the search, motivated by some uncertain information on target positions. The uncertainty depends on sensing technology, situations of the search space and others. We assume that the uncertainty is given by a probability distribution in our model. The target would reasonably know the probability distribution if the target observes the situation of the search space and the sensing technology is in circulation in the world. To handle the above situation in a theoretical way, we give up the past models of complete-information game and adopt a new game model with incomplete information. By our model, we quantitatively evaluate the effects of the incomplete information on optimal strategies of players and the results of the game. This is a purpose of this study.

In the next section, we model our SAG as a two-person zero-sum game having private information of initial target position, define players' strategies and formulate the payoff of the game. In Section 3, we derive an equilibrium point for the target strategy of selecting paths and the searcher's strategy of distributing search resource. We can imagine that as a time horizon becomes larger, the number of target paths exponentially increases and our proposed solution would be practically difficult to be applied to large size of problems. To cope with this, we discuss a new approach using a Markov movement strategy of the target for an equilibrium in Section 4. In Section 5, we analyze some examples to clarify the characteristics of players' optimal strategies and evaluate the value of information of the initial position of the target. In Section 6, we conclude the paper with some remarks.

2. A Search Game under the Uncertainty of Initial Target Position

In a search game in which a searcher and a target compete with each other, the searcher starts his search operation for the target after getting uncertain information of target position. The beginning of the search has some randomness or unpredictability for the target and so his current position is thought to be a random number decided by nature. A search sensor could not be so precise to give the searcher a certain position of the target but would teach him some information about the target position, i.e. a probabilistic law of the target position. Here we make a model of a one-shot search game with uncertain information of the target's initial position.

(A1) A search space consists of a discrete cell space and a discrete time space. The cell space is denoted by $\boldsymbol{K} = \{1, \cdots, K\}$ and the time space by $\boldsymbol{T} = \{1, \cdots, T\}$. A target and a searcher play a search game.

(A2) Nature decides a cell k as an initial target position within a region $I_0 \subseteq \boldsymbol{K}$, which is given as a random number with a probability distribution $\{f(k), k \in I_0\}$. The probability distribution satisfies $\sum_{k \in I_0} f(k) = 1$ and is known to the searcher and the target in advance of the game.

(A3) The target moves from his initial position k, which is private information of the target, but his movement is restricted as follows. At time t, he can move from cell i to some cell $j \in N(i,t) \subseteq \boldsymbol{K}$. It takes some energy $\mu(i,j)$ for the target to move from cell i to j. He has initial moving energy e_0 but its exhaustion prohibits the target any more moving except staying at the current cell.
By P_k, we denote the set of all paths starting from initial target position k, which satisfy the constraints of movement above. The target chooses a path among P_k at the beginning of the game and moves along the path. The position of the target taking path $\omega \in P_k$ is $\omega(t) \in \boldsymbol{K}$ at time $t \in \boldsymbol{T}$.

(A4) The searcher distributes his search resource in the search space to detect the target. He can start the search from time τ. $\Phi(t)$ search resources are available to the searcher at each time t of a time period $\widehat{\boldsymbol{T}} \equiv \{\tau, \tau+1, \cdots, T\}$. The resource is continuously divisible. A searcher's strategy of distributing the search resource is denoted by $\varphi = \{\varphi(i,t),\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}$, where $\varphi(i,t)$ is the amount of the resource scattered into cell i at time t.

(A5) If the target exists in cell i at time t, the searcher detects the target with detection probability

$$1 - \exp(-\alpha_i \varphi(i,t)) \tag{2.1}$$

by the distribution of $\varphi(i,t)$ resources there. A coefficient α_i indicates the effectiveness of detection by unit resource at cell i.

Only if the searcher detects the target, the searcher gets reward 1 but the target loses the same.

(A6) The probability distribution $f(k)$ is common knowledge to both players. The searcher can certainly know the target's initial energy e_0 and then he can estimate the path set P_k. Other assumptions or rules of the game stated above are also known to both players.

Assumption (A6) implies that the sensor technology, by which the initial target position is estimated, universally spreads to the world and the target can also estimate its sensing capability. The target mobility is under the same situation as for the sensor so that the searcher certainly estimates the initial target energy e_0. From Assumption (A5), the problem is a two-person zero-sum (TPZS) game with the detection probability of the target as payoff.

Before a main discussion, let us explicitly express the constraints on a target path $\omega \in P_k$ starting from an initial position $k \in I_0$. Using a symbol $e(t)$ as the remaining energy of the target at time t, we can construct the constraints on any target path $\omega \in P_k$ from Assumption (A3), as follows:

(i) Initial position: $\omega(1) = k$

(ii) Geographic restriction on movement: $\omega(t+1) \in N(\omega(t), t),\ t = 1, \cdots, T-1$

(iii) Initial energy: $e(1) = e_0$

(iv) Conservation of energy: $e(t+1) = e(t) - \mu(\omega(t), \omega(t+1)),\ t = 1, \cdots, T-1$

(v) Energetic restriction on movement: $\mu(\omega(t), \omega(t+1)) \le e(t),\ t = 1, \cdots, T-1$

By a pair of a target position and its remaining energy, $(\omega(t), e(t))$, at each time $t \in \boldsymbol{T}$, satisfying all constraints above, we can enumerate a set of paths P_k starting from cell k.

From Assumption (A4), we have a feasible region of the searcher's strategy φ as

$$\Psi \equiv \left\{ \varphi \;\middle|\; \sum_{i \in \boldsymbol{K}} \varphi(i,t) \le \Phi(t),\ \varphi(i,t) \ge 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \right\}. \tag{2.2}$$

We call the target moving from its initial position k the k-type target. First, we express the payoff function by a searcher's strategy φ and a strategy $\omega \in P_k$ of the k-type target. At time t, the k-type target is at cell $\omega(t)$ and the searcher distributes $\varphi(\omega(t), t)$ resources there. Therefore, from expression (2.1), the detection probability of the target is given by

$$R_k(\varphi, \omega) = 1 - \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right),$$

which is the payoff for pure strategies φ and ω. Now we assume that the k-type target takes a mixed strategy $\pi_k \equiv \{\pi_k(\omega),\ \omega \in P_k\}$, where $\pi_k(\omega)$ is the probability of choosing path ω. The mixed strategy π_k has the following feasible region:

$$\Pi_k \equiv \left\{ \{\pi_k(\omega)\} \;\middle|\; \sum_{\omega \in P_k} \pi_k(\omega) = 1, \quad \pi_k(\omega) \ge 0, \omega \in P_k \right\}.$$

For a searcher's pure strategy φ and a target mixed strategy π_k, the expected payoff is given by

$$\begin{aligned} R_k(\varphi,\pi_k) &= \sum_{\omega\in P_k} \pi_k(\omega) R_k(\varphi,\omega) \\ &= \sum_{\omega\in P_k} \pi_k(\omega) \left\{ 1-\exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \right) \right\} \\ &= 1-\sum_{\omega\in P_k} \pi_k(\omega) \exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \right). \end{aligned} \tag{2.3}$$

The target desires to minimize the expected payoff. On the other hand, the searcher has a probability distribution $\{f(k), k\in I_0\}$ as an estimation on the type of target or the target initial position, and so the searcher wants to maximize the following expected payoff

$$\begin{aligned} R(\varphi,\pi) &= \sum_{k\in I_0} f(k) R_k(\varphi,\pi_k) \\ &= \sum_{k\in I_0} f(k) \sum_{\omega\in P_k} \pi_k(\omega) \left\{ 1-\exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \right) \right\} \\ &= 1-\sum_{k\in I_0} f(k) \sum_{\omega\in P_k} \pi_k(\omega) \exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \right), \end{aligned} \tag{2.4}$$

taking account of all types of target strategies $\pi\equiv\{\pi_k, k\in I_0\}$.

In the game, a specific type of target and the searcher seem to have different criteria, shown by Equation (2.3) and (2.4). In the next section, we derive an equilibrium point of the game.

3. An Equilibrium Point of the Game

The target has to prepare a moving strategy π consisting of π_k for every k-type target. As seen from Equation (2.3) and (2.4), an optimal strategy π_k^* of the k-type target, which minimizes the expected payoff $R_k(\varphi,\pi_k)$, also minimizes the expected payoff $R(\varphi,\pi)$ in the aggregate for all types $k\in I_0$. The searcher desires to maximize $R(\varphi,\pi)$. Therefore, we can regard our SAG as a TPZS game with the payoff $R(\varphi,\pi)$.

An optimal searcher's strategy is derived as a maximin strategy of $R(\varphi,\pi)$. We can transform the minimization of $R(\varphi,\pi)$ with respect to π by making $\pi_k(\omega)=0$ for $\omega\notin\Omega^{+k}\equiv\{\omega\in P_k | R_k(\varphi,\omega)=\min_{p\in P_k} R_k(\varphi,p)\}$, as follows:

$$\begin{aligned} \min_{\pi} R(\varphi,\pi) &= \sum_{k\in I_0} f(k) \sum_{\omega\in P_k} \pi_k(\omega) R_k(\varphi,\omega) = \sum_{k\in I_0} f(k) \min_{\omega\in P_k} R_k(\varphi,\omega) \\ &= \sum_{k\in I_0} f(k) \min_{\omega\in P_k} \left\{ 1-\exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \right) \right\}. \end{aligned} \tag{3.1}$$

As a maximization problem of the minimum value above, we have the following formulation.

$$
\begin{aligned}
(P_S^0) \qquad & D^* = \max_{\varphi,\{\nu_k\}} \sum_{k\in I_0} f(k)\nu_k \\
s.t. \qquad & 1-\exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t)\right) \ge \nu_k,\ \omega\in P_k,\ k\in I_0, \\
& \sum_{i\in\boldsymbol{K}} \varphi(i,t)=\Phi(t),\ t\in\widehat{\boldsymbol{T}}, \\
& \varphi(i,t)\ge 0,\ i\in\boldsymbol{K},\ t\in\widehat{\boldsymbol{T}}.
\end{aligned}
$$

If we substitute $\eta_k \equiv -\log(1-\nu_k)$ for ν_k, i.e. $\nu_k \equiv 1-\exp(-\eta_k)$, we can transform the problem (P_S^0) to a final formulation, using $\sum_k f(k)=1$.

$$
\begin{aligned}
(P_S) \qquad & ND^* = \min_{\varphi,\{\eta_k\}} \sum_{k\in I_0} f(k)\exp(-\eta_k) \\
s.t. \qquad & \sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \ge \eta_k,\ \omega\in P_k,\ k\in I_0, \qquad (3.2) \\
& \sum_{i\in\boldsymbol{K}} \varphi(i,t)=\Phi(t),\ t\in\widehat{\boldsymbol{T}}, \\
& \varphi(i,t)\ge 0,\ i\in\boldsymbol{K},\ t\in\widehat{\boldsymbol{T}}.
\end{aligned}
$$

We have $D^* = 1-ND^*$ as a relation between optimal values of problems (P_S^0) and (P_S). Problem (P_S) is a convex programming problem and is easy to be solved for an optimal searcher's strategy φ^* by some commercial solvers. In problem (P_S), the searcher anticipates an optimal strategy of every k-type target who can take an optimal response to the strategy φ and then makes the minimum detection probability $\sum_k f(k)\{1-\exp(-\eta_k)\}$ as large as possible to obtain an optimal strategy φ^*. Therefore, the searcher could have larger detection probability than the value of the game, D^*, for some type of target and also have smaller one for another type of target.

Here we derive an optimal strategy of the k-type target. The k-type target wants an optimal strategy π_k to minimize $R_k(\varphi^*,\pi_k)$ after the evaluation of an optimal searcher's strategy φ^* by solving problem (P_S). The minimization is done as follows:

$$
\begin{aligned}
\min_{\pi_k} R_k(\varphi^*,\pi_k) &= \min_{\omega\in P_k} R_k(\varphi^*,\omega) = \min_{\omega\in P_k}\left\{1-\exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}}\alpha_{\omega(t)}\varphi^*(\omega(t),t)\right)\right\} \\
&= 1-\exp\left(-\min_{\omega\in P_k}\sum_{t\in\widehat{\boldsymbol{T}}}\alpha_{\omega(t)}\varphi^*(\omega(t),t)\right). \qquad (3.3)
\end{aligned}
$$

From Equation (3.2), we can see that the minimum value in the parentheses of Equation (3.3) is given by optimal η_k^*,

$$
\eta_k^* = \min_{\omega\in P_k}\sum_{t\in\widehat{\boldsymbol{T}}}\alpha_{\omega(t)}\varphi^*(\omega(t),t),
$$

and $1 - \exp(-\eta_k^*)$ is the minimum detection probability of the k-type target. Therefore we can redefine Ω^{+k} by $\Omega^{+k} \equiv \{\omega \in P_k | \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi^*(\omega(t), t) = \eta_k^*\}$.

To obtain an optimal target strategy, let us discuss the minimax optimization of $R(\varphi, \pi)$. In this optimization, the searcher can optimally respond to any target strategy π after knowing π. Therefore, we have to make an optimal target strategy π such that φ^*, already obtained from problem (P_S), becomes an optimal response to π. To derive an optimal target strategy corresponding to φ^*, we minimize $R(\varphi^*, \pi)$ with respect to variable π. This can be done by making $\pi_k(\omega) = 0$ for every $\omega \notin \Omega^{+k}$, as seen from the transformation in Equation (3.1).

A solution φ of problem $\max_\varphi R(\varphi, \pi)$ $s.t.$ $\varphi \in \Psi$ for given π becomes optimal if and only if it satisfies the so-called Karush-Kuhn-Tucker (KKT) conditions. Using Lagrangian multipliers $\{\lambda(t),\ t \in \widehat{\boldsymbol{T}}\}$, $\{\eta(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}\}$ and a Lagrange function defined by

$$L(\varphi; \lambda, \eta) \equiv R(\varphi, \pi) + \sum_{t \in \widehat{\boldsymbol{T}}} \lambda(t) \left(\Phi(t) - \sum_{i \in K} \varphi(i,t) \right) + \sum_{(i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}} \eta(i,t) \varphi(i,t),$$

we have the following KKT conditions for an optimal φ.

$$\begin{aligned} \frac{\partial L}{\partial \varphi(i,t)} &= \frac{\partial R(\varphi, \pi)}{\partial \varphi(i,t)} - \lambda(t) + \eta(i,t) \\ &= \alpha_i \sum_{k \in I_0} f(k) \sum_{\omega \in \Omega_{it}^k} \pi_k(\omega) \exp\left(- \sum_{t' \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t')} \varphi(\omega(t'), t') \right) \\ &\qquad -\lambda(t) + \eta(i,t) = 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}, \end{aligned} \tag{3.4}$$

$$\varphi(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}},$$

$$\sum_{i \in \boldsymbol{K}} \varphi(i,t) = \Phi(t),\ t \in \widehat{\boldsymbol{T}},$$

$$\eta(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}, \tag{3.5}$$

$$\eta(i,t)\varphi(i,t) = 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}, \tag{3.6}$$

where $\Omega_{it}^k \equiv \{\omega \in P_k | \omega(t) = i\}$. Here we again confirm that optimal target strategy π_k makes $\pi_k(\omega) = 0$ for any $\omega \notin \Omega^{+k}$ and $\sum_{t'} \alpha_{\omega(t')} \varphi^*(\omega(t'), t') = \eta_k^*$ for any path $\omega \in \Omega^{+k}$. Therefore we can simplify the right-hand side of Equation (3.4) to

$$\alpha_i \sum_{k \in I_0} f(k) \exp(-\eta_k^*) \sum_{\omega \in \Omega_{it}^{+k}} \pi_k(\omega) - \lambda(t) + \eta(i,t) = 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}, \tag{3.7}$$

where Ω_{it}^{+k} is defined by

$$\Omega_{it}^{+k} \equiv \{\omega \in P_k |\ \omega(t) = i,\ \sum_{t' \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t')} \varphi^*(\omega(t'), t') = \eta_k^*\}$$

as a product set of Ω_{it}^k and Ω^{+k}.

We sort the KKT conditions (3.4), (3.5) and (3.6) out in the following way. In the case of $\varphi^*(i,t) > 0$, we have $\eta(i,t) = 0$ from Equation (3.6) and then

$$\alpha_i \sum_{k \in I_0} f(k) \exp(-\eta_k^*) \sum_{\omega \in \Omega_{it}^{+k}} \pi_k(\omega) = \lambda(t) \tag{3.8}$$

from Equation (3.7). In the other case of $\varphi^*(i,t)=0$, we have

$$\alpha_i \sum_{k\in I_0} f(k)\exp(-\eta_k^*) \sum_{\omega\in\Omega_{it}^{+k}} \pi_k(\omega) \le \lambda(t) \tag{3.9}$$

from Equation (3.5). Conversely, we easily generate optimal multipliers $\{\lambda^*(t)\}$ and $\{\eta^*(i,t)\}$ of the KKT conditions from two conditions above. We regard the two conditions as necessary conditions of π such that the already-obtained φ^* becomes optimal to π.

So far we have discussed the optimality conditions of the target strategy $\pi=\{\pi_k, k\in I_0\}$. If φ satisfies these conditions, the optimal solution φ^* of problem (P_S) becomes optimal to π and simultaneously π becomes optimal to φ^*.

Finally, we can construct a linear programming problem, from which we derive an optimal target strategy, as follows:

$$(P_T) \quad \min_{\pi,\lambda} \sum_{k\in I_0} f(k) \sum_{\omega\in P_k} \pi_k(\omega) \left\{ 1-\exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi^*(\omega(t),t) \right) \right\}$$

$$s.t. \quad \alpha_i \sum_{k\in I_0} f(k)\exp(-\eta_k^*) \sum_{\omega\in\Omega_{it}^{+k}} \pi_k(\omega) = \lambda(t),\ for\ (i,t)\in \boldsymbol{K}\times\widehat{\boldsymbol{T}}\ of\ \varphi^*(i,t)>0, \tag{3.10}$$

$$\alpha_i \sum_{k\in I_0} f(k)\exp(-\eta_k^*) \sum_{\omega\in\Omega_{it}^{+k}} \pi_k(\omega) \le \lambda(t),\ for\ (i,t)\in \boldsymbol{K}\times\widehat{\boldsymbol{T}}\ of\ \varphi^*(i,t)=0,$$

$$\sum_{\omega\in P_k} \pi_k(\omega)=1,\ k\in I_0, \tag{3.11}$$

$$\pi_k(\omega)\ge 0,\ \omega\in P_k,\ k\in I_0.$$

The total number of variables π and λ in problem (P_T) is $\sum_{k\in I_0}|P_k|+|\widehat{\boldsymbol{T}}|$. The largest total number of equations (3.10) and (3.11) is $|\boldsymbol{K}||\widehat{\boldsymbol{T}}|+|I_0|$. If there is no restriction on the target motion, the number of all paths $|P_k|$ could be $|\boldsymbol{K}|^{(|\widehat{\boldsymbol{T}}|-1)}$ at largest. If the total energy e_0, the energy consumption $\mu(i,j)$ and the neighboring cell restriction $N(i,t)$ are not so tight, the number of paths would be so large that problem (P_T) has some feasible solutions. However, we are concerned about the non-existence of feasible solution for (P_T) in the case that the number of paths is small and the number of equations such as (3.10) would overcount the number of variables. We leave the proof of the existence of a feasible solution to Appendix section.

4. Markov Strategy of Target Movement

We have derived an equilibrium point of the game in Section 3 by enumerating all paths $\{P_k, k\in I_0\}$ from restrictions on the target movement stated in Assumption (A3) in Section 2.

Here we handle not the target paths but the target movement between cells and its energy as another type of target strategy. We define a Markov strategy for the target movement. By the Markov strategy, the target moves depending on not the past movement but just his current state in a probabilistic manner. Any target state is expressed by a quadruple (k,i,t,e) of the target's type k, cell i, remaining energy e and time t, or a triplet (i,t,e) without posting the self-evident target type k. From the concept of Markov strategy, the target uses the same probabilistic rule for his movement from the same state. We denote a

Markov strategy by $v_k(i,j,t,e)$, which is the probability that the k-type target moves from the present state (k,i,t,e) to cell $j \in \boldsymbol{K}$ at the next time $t+1$. Let us derive an equilibrium point for the Markov strategy of the target. To this end, we have to discriminate the beginning point of time t from the ending point even at the same time t.

First we denote a set of energy levels by $\boldsymbol{E} \equiv \{0,1,\cdots,e_0\}$ and define an optimized value $z_k(i,t,e)$ as the maximized non-detection probability after the beginning point of time t by an optimal motion of the target from the present state (k,i,t,e), in order to formulate our problem by dynamic programming. In this section, we use the non-detection probability of the target as a criterion of the game, which is simply given by one minus the detection probability. The target acts to maximize the non-detection probability but the searcher behaves to minimize it. For the convenience of formulation, we define $N(i,t,e) \equiv \{j \in \boldsymbol{K} | j \in N(i,t),\ \mu(i,j) \leq e\}$ as the set of cells to which any type of target can move from state (i,t,e) and $N^*(i,t,e) \equiv \{j \in \boldsymbol{K} | i \in N(j,t-1),\ e+\mu(j,i) \leq e_0\}$ as a set of cells from which any type of target can moves to state (i,t,e). The feasible region of the Markov strategy is given by

$$V_k \equiv \left\{ \{v_k(i,j,t,e), i,j \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{T\}, e \in \boldsymbol{E}\} \;\middle|\; \sum_{j \in N(i,t,e)} v_k(i,j,t,e) = 1, \right.$$
$$\left. v_k(i,j,t,e) = 0 (j \notin N(i,t,e)),\ v_k(i,j,t,e) \geq 0 \right\}.$$

Here we prove that the previous presentation $\{\pi_k(\omega)\}$ and the Markov strategy $\{v_k(i,j,t,e)\}$ are equivalent to each other. We just need to show that both representations are transformable to each other using a notation $e(\omega,n) \equiv e_0 - \sum_{t=1}^{n-1} \mu(\omega(t),\omega(t+1))$, as follows:

$$\begin{aligned} \pi_k(\omega) &= \prod_{t=1}^{T-1} v_k(\omega(t),\omega(t+1),t,e(\omega,t)) \ \mathit{for}\ \omega \in P_k, \\ v_k(i,j,t,e) &= \frac{\sum_{\{\omega \in \Omega_{it}^k | e(\omega,t)=e, \omega(t+1)=j\}} \pi_k(\omega)}{\sum_{\{\omega \in \Omega_{it}^k | e(\omega,t)=e\}} \pi_k(\omega)} \ \mathit{for}\ i,j \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{T\}, e \in \boldsymbol{E}. \end{aligned}$$

If $\sum_{\{\omega \in \Omega_{it}^k | e(\omega,t)=e\}} \pi_k(\omega) = 0$, namely, the state (k,i,t,e) is never reachable by the strategy of path selection, any Markov strategy $\{v_k(i,j,t,e)\}$ is acceptable in the state.

Anyway, we still use $\varphi(i,t)$ as a searcher's pure strategy. Considering the survival of the target moving from state (i,t,e) to cell j with no detection at time $t \in \widehat{\boldsymbol{T}}$, we have the following recursive equation in terms of $z_k(i,t,e)$.

$$z_k(i,t,e) = \max_{v_k(i,\cdot,t,e)} e^{-\alpha_i \varphi(i,t)} \sum_{j \in N(i,t,e)} v_k(i,j,t,e) z_k(j,t+1,e-\mu(i,j)).$$

At the final time T, $z_k(i,T,e) = \exp(-\alpha_i \varphi(i,T))$ is self-evident. Taking account of the feasible region V_k, we can generate an inequality condition from the recursive equation, as follows:

$$z_k(i,t,e) = \max_{j \in N(i,t,e)} e^{-\alpha_i \varphi(i,t)} z_k(j,t+1,e-\mu(i,j)) \geq e^{-\alpha_i \varphi(i,t)} z_k(j,t+1,e-\mu(i,j)).$$

Similarly, we have the other recursive equation and an inequality condition at time $t \in \boldsymbol{T} \backslash \widehat{\boldsymbol{T}}$, when the searcher does not do any search operation.

$$\begin{aligned} z_k(i,t,e) &= \max_{v_k(i,\cdot,t,e)} \sum_{j \in N(i,t,e)} v_k(i,j,t,e) z_k(j,t+1,e-\mu(i,j)) \\ &= \max_{j \in N(i,t,e)} z_k(j,t+1,e-\mu(i,j)) \geq z_k(j,t+1,e-\mu(i,j)). \end{aligned}$$

Because the maximum non-detection probability of the k-type target is $z_k(k,1,e_0)$ at an initial state $(k,1,e_0)$, the searcher has to minimize the non-detection probability expected over all types of target, $\sum_k f(k) z_k(k,1,e_0)$. The minimization gives us the minimax value of the non-detection probability or the maximin value for the original payoff of the detection probability. We have formulated the minimax optimization into the following convex programming problem.

$$\begin{aligned} (P_S^{M0}) \quad & \min_{\varphi,z} \sum_{k \in I_0} f(k) z_k(k,1,e_0) \\ s.t. \quad & z_k(i,t,e) \geq z_k(j,t+1,e-\mu(i,j)),\ j \in N(i,t,e), i \in \boldsymbol{K}, t \in \boldsymbol{T} \backslash \widehat{\boldsymbol{T}}, e \in \boldsymbol{E}, k \in I_0, \\ & z_k(i,t,e) \geq e^{-\alpha_i \varphi(i,t)} z_k(j,t+1,e-\mu(i,j)), \\ & \qquad\qquad j \in N(i,t,e), i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}} \backslash \{T\}, e \in \boldsymbol{E}, k \in I_0, \\ & z_k(i,T,e) = e^{-\alpha_i \varphi(i,T)},\ i \in \boldsymbol{K}, e \in \boldsymbol{E}, k \in I_0, \\ & \sum_{i \in \boldsymbol{K}} \varphi(i,t) = \Phi(t),\ t \in \widehat{\boldsymbol{T}}, \\ & \varphi(i,t) \geq 0,\ i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}. \end{aligned}$$

Substituting $w_k(i,t,e) \equiv -\log z_k(i,t,e)$ for $z_k(i,t,e)$, we have the following problem with a convex objective function and linear conditions. Since $z_k(i,t,e)$ lies between 0 and 1, $w_k(i,t,e)$ is nonnegative.

$$\begin{aligned} (P_S^{M}) \quad & \min_{\varphi,w} \sum_{k \in I_0} f(k) \exp(-w_k(k,1,e_0)) \\ s.t. \quad & w_k(i,t,e) \leq w_k(j,t+1,e-\mu(i,j)),\ j \in N(i,t,e), i \in \boldsymbol{K}, t \in \boldsymbol{T} \backslash \widehat{\boldsymbol{T}}, e \in \boldsymbol{E}, k \in I_0, \\ & w_k(i,t,e) \leq \alpha_i \varphi(i,t) + w_k(j,t+1,e-\mu(i,j)), \\ & \qquad\qquad j \in N(i,t,e), i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}} \backslash \{T\}, e \in \boldsymbol{E}, k \in I_0, \\ & w_k(i,T,e) = \alpha_i \varphi(i,T),\ i \in \boldsymbol{K}, e \in \boldsymbol{E}, k \in I_0, \\ & \sum_{i \in \boldsymbol{K}} \varphi(i,t) = \Phi(t),\ t \in \widehat{\boldsymbol{T}}, \\ & \varphi(i,t) \geq 0,\ i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}. \end{aligned}$$

Anyway, variables $\{w_k(i,1,e), i \neq k, e \neq e_0\}$ never have any effect on the optimal value of the problem (P_S^M). We only need to set such variables zeros. The setting is equivalent to making corresponding z_ks ones in problem (P_S^{M0}).

Next let us derive an optimal Markov strategy of the target, $\{v_k(i,j,t,e)\}$, using optimal solutions φ^* and w^* of (P_S^M). First, we calculate variable z_k in (P_S^{M0}) by $z_k^*(i,t,e) = \exp(-w_k^*(i,t,e))$ from w^*. From the definition of $z_k^*(i,t,e)$, the expression $\widehat{z}_k^*(i,t,e) \equiv z_k^*(i,t,e) \exp(\alpha_i \varphi^*(i,t))$ represents the maximal non-detection probability accomplished by an optimal target movement after a state (i,t,e) given that the k-type target is in the state (i,t,e) at the ending point of time t. For the convenience of formulation, we define a new Markov target strategy and denote it by $\{\widehat{v}_k(i,j,t,e),\ i,j\in\boldsymbol{K},t\in\boldsymbol{T}\backslash\{T\},e\in\boldsymbol{E}\}$ other than the original Markov strategy v_k. $\widehat{v}_k(i,j,t,e)$ is the probability that the k-type target reaches state (i,t,e) surviving at the ending point of t and moves to cell j at the next time $t+1$ while the previous notation $v_k(i,j,t,e)$ represents the conditional transition probability given that the target has survived until the ending point of t. Related to the strategy, we use other variables representing the existence probability of surviving target. $q_k(i,t,e)$ is the probability that the k-type target reaches state (i,t,e) surviving at the beginning point of t. $q'_k(i,t,e)$ is the probability that the k-type target reaches state (i,t,e) surviving and then is undetected until the ending point of t. We illustrate probabilities z^*_k, $\widehat{z}^*_k$, q_k and q'_k in Figure 1. The time period during which each probability is evaluated is depicted by an arrow. If possible, a search operation by search resource $\varphi(i,t)$ is carried out between the beginning point and the ending point of the time t for each state (i,t,e).

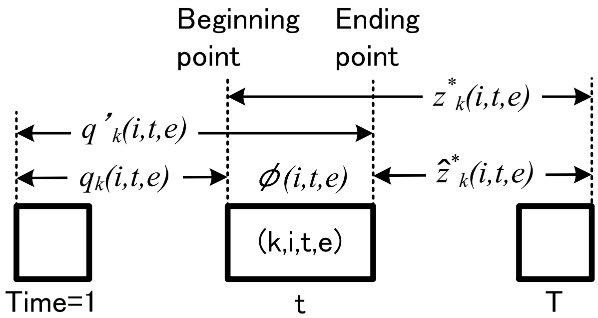

Figure 1: Definition of probabilities

Considering the transition of the states of the k-type target, we can construct the following equations.

$$\begin{aligned}
q_k(k,1,e_0) &= 1, k\in I_0,\\
q_k(i,1,e) &= 0, k\neq i\in\boldsymbol{K}, e_0\neq e\in\boldsymbol{E}, k\in I_0,\\
q'_k(i,t,e) &= q_k(i,t,e), i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\widehat{\boldsymbol{T}}, e\in\boldsymbol{E}, k\in I_0,\\
q'_k(i,t,e) &= q_k(i,t,e)\exp(-\alpha_i\varphi^*(i,t)), i\in\boldsymbol{K}, t\in\widehat{\boldsymbol{T}}, e\in\boldsymbol{E}, k\in I_0,\\
q_k(i,t,e) &= \sum_{j\in N^*(i,t,e)}\widehat{v}_k(j,i,t-1,e+\mu(j,i)), i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{1\}, e\in\boldsymbol{E}, k\in I_0,\\
q'_k(i,t,e) &= \sum_{j\in N(i,t,e)}\widehat{v}_k(i,j,t,e), i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{T\}, e\in\boldsymbol{E}, k\in I_0.
\end{aligned}$$

Focusing on the distribution of search resource at a specific time $t\in\widehat{\boldsymbol{T}}$, $\{\varphi(i,t), i\in\boldsymbol{K}\}$, we can express the total non-detection probability all through the search by

$$g_t(\varphi)\equiv\sum_{k\in I_0}\sum_{i\in\boldsymbol{K}}\sum_{e\in\boldsymbol{E}}f(k)q_k(i,t,e)\exp(-\alpha_i\varphi(i,t))\widehat{z}^*_k(i,t,e)$$

The already-obtained optimal distribution $\{\varphi^*(i,t), i\in\boldsymbol{K}\}$ has to minimize the above function under the constraints of $\sum_i\varphi(i,t)=\Phi(t)$ and $\varphi(i,t)\geq 0$ $(i\in\boldsymbol{K})$. By Lagrangian multipliers $\lambda(t)$, $\mu(i,t)$ and the definition of a Lagrange function

$$L(\varphi;\lambda,\mu) \equiv \sum_{k\in I_0}\sum_{i\in\boldsymbol{K}}\sum_{e\in\boldsymbol{E}} f(k)q_k(i,t,e)\exp(-\alpha_i\varphi(i,t))\widehat{z}_k^*(i,t,e) \\ +\lambda(t)\left(\sum_{i\in\boldsymbol{K}}\varphi(i,t)-\Phi(t)\right)-\sum_{i\in\boldsymbol{K}}\mu(i,t)\varphi(i,t),$$

we have the following KKT conditions for an optimal solution $\{\varphi(i,t), i\in\boldsymbol{K}\}$.

$$\frac{\partial L}{\partial\varphi(i,t)} = -\alpha_i\exp(-\alpha_i\varphi(i,t))\sum_{k\in I_0}\sum_{e\in\boldsymbol{E}} f(k)q_k(i,t,e)\widehat{z}_k^*(i,t,e) \\ +\lambda(t)-\mu(i,t)=0,\ i\in\boldsymbol{K}, \tag{4.1}$$

$$\mu(i,t)\geq 0,\ i\in\boldsymbol{K}, \tag{4.2}$$

$$\mu(i,t)\varphi(i,t)=0,\ i\in\boldsymbol{K}, \tag{4.3}$$

$$\sum_{i\in\boldsymbol{K}}\varphi(i,t)=\Phi(t), \tag{4.4}$$

$$\varphi(i,t)\geq 0,\ i\in\boldsymbol{K}. \tag{4.5}$$

We unify conditions (4.1)–(4.3) into an equivalent system of conditions.

(i) If $\varphi(i,t)>0$,

$$\alpha_i\exp(-\alpha_i\varphi(i,t))\sum_{k\in I_0}\sum_{e\in\boldsymbol{E}} f(k)q_k(i,t,e)\widehat{z}_k^*(i,t,e)=\lambda(t). \tag{4.6}$$

(ii) If $\varphi(i,t)=0$,

$$\alpha_i\sum_{k\in I_0}\sum_{e\in\boldsymbol{E}} f(k)q_k(i,t,e)\widehat{z}_k^*(i,t,e)\leq\lambda(t). \tag{4.7}$$

We regard the system as the necessary conditions of the target strategy to which the optimal searcher's strategy φ^* already calculated must be an optimal response. Please note the case that the search is executed at initial time $t=1$, i.e. $t\in\widehat{\boldsymbol{T}}$. In this case, $q_k(k,1,e_0)=1$ and $q_k(i,1,e)=0$ $(i\neq k\ or\ e\neq e_0)$ should be applied to Equation (4.6) and (4.7). Anyway, the already-obtained optimal searcher's strategy φ^* satisfies conditions (4.4) and (4.5).

The total non-detection probability is expressed by $\sum_{k,i,e} f(k)q'_k(i,T,e)$. An optimal Markov strategy of the target, $\widehat{v}^*$, must maximize the non-detection probability as an optimal response to φ^*. If the conditions (4.6) and (4.7) holds for any $i\in\boldsymbol{K}$ and $t\in\widehat{\boldsymbol{T}}$, φ^* is optimal to $\widehat{v}$. From the discussion so far, we have a final problem to derive an optimal Markov strategy of the target $\widehat{v}^*$. The problem is a linear programming problem including variables $\widehat{v}$, q, q' and λ, embedded with the optimal searcher's strategy φ^* and $\widehat{z}^*$ already given from (P_S^{M0}).

$$
\begin{aligned}
(P_T^M) \quad & \max_{\widehat{v},q,q',\lambda} \sum_{k\in I_0}\sum_{i\in \boldsymbol{K}}\sum_{e\in \boldsymbol{E}} f(k)q'_k(i,T,e) \\
s.t. \quad & q_k(k,1,e_0)=1, k\in I_0, \\
& q_k(i,1,e)=0, k\neq i\in \boldsymbol{K}, e_0\neq e\in \boldsymbol{E}, k\in I_0, \\
& q'_k(i,t,e)=q_k(i,t,e), i\in \boldsymbol{K}, t\in \boldsymbol{T}\backslash\widehat{\boldsymbol{T}}, e\in \boldsymbol{E}, k\in I_0, \\
& q'_k(i,t,e)=q_k(i,t,e)\exp(-\alpha_i\varphi^*(i,t)), i\in \boldsymbol{K}, t\in \widehat{\boldsymbol{T}}, e\in \boldsymbol{E}, k\in I_0, \\
& q_k(i,t,e)=\sum_{j\in N^*(i,t,e)} \widehat{v}_k(j,i,t-1,e+\mu(j,i)), i\in \boldsymbol{K}, t\in \boldsymbol{T}\backslash\{1\}, e\in \boldsymbol{E}, k\in I_0, \\
& q'_k(i,t,e)=\sum_{j\in N(i,t,e)} \widehat{v}_k(i,j,t,e), i\in \boldsymbol{K}, t\in \boldsymbol{T}\backslash\{T\}, e\in \boldsymbol{E}, k\in I_0, \\
& \alpha_i\exp(-\alpha_i\varphi^*(i,t))\sum_{k\in I_0}\sum_{e\in \boldsymbol{E}} f(k)q_k(i,t,e)\widehat{z}_k^*(i,t,e)=\lambda(t), \\
& \qquad\qquad for\ (i,t)\in \boldsymbol{K}\times\widehat{\boldsymbol{T}}\ of\ \varphi^*(i,t)>0, \\
& \alpha_i\sum_{k\in I_0}\sum_{e\in \boldsymbol{E}} f(k)q_k(i,t,e)\widehat{z}_k^*(i,t,e)\leq\lambda(t), \\
& \qquad\qquad for\ (i,t)\in \boldsymbol{K}\times\widehat{\boldsymbol{T}}\ of\ \varphi^*(i,t)=0, \\
& \widehat{v}_k(i,j,t,e)\geq 0,\ i,j\in \boldsymbol{K}, t\in \boldsymbol{T}\backslash\{T\}, e\in \boldsymbol{E}, k\in I_0, \\
& \widehat{v}_k(i,j,t,e)=0,\ j\notin N(i,t,e), i\in \boldsymbol{K}, t\in \boldsymbol{T}\backslash\{T\}, e\in \boldsymbol{E}, k\in I_0.
\end{aligned}
$$

Using an optimal solution $\widehat{v}_k^*(i,j,t,e)$, we can calculate an optimal target strategy as the original Markov strategy, as follows:

$$
v_k^*(i,j,t,e)=\frac{\widehat{v}_k^*(i,j,t,e)}{\sum_{j\in N(i,t,e)}\widehat{v}_k^*(i,j,t,e)}. \tag{4.8}
$$

In the special case that the target does not reach state (k,i,t,e) by its optimal movement, the denominator of Equation (4.8) becomes zero. In the special case, any strategy $v_k(i,j,t,e)$ does not matter because the target is never in the state.

5. Numerical Examples

Here we apply our methodology, proposed in previous sections, to some numerical examples to analyze the characteristics of optimal players' strategies.

We consider time points $\boldsymbol{T}=\{1,\cdots,5\}$ and a search space $\boldsymbol{K}$ having 19 cells, shown in Figure 2. Cells 9 and 11 are obstacles to which a target is prohibited to move. Coefficients of cells α_i are categorized into three groups. α_i is set to be 0.7 for cells $i\in\{1,2,3,17,18,19\}$, 0.5 for $i\in\{4,5,6,10,15\}$ and 0.3 for $i\in\{7,12,16,8,13,14\}$ which lie behind the obstacles and are most inconvenient for search operations. The target can move from his current cell to neighboring or 2nd-neighboring cells, which locate at one or two-cell distance from the current position. The movement constraint $N(i,t)$ is independent of time t. He spends energy $\mu(i,j)=1$ to move to a neighboring cell and $\mu(i,j)=4$ to move to a 2nd-neighboring cell while staying at the current cell needs no energy, i.e. $\mu(i,i)=0$. In this example, we perform sensitivity analysis by changing initial energy of the target, e_0. A searcher can start a search operation from time $\tau=2$. $\Phi(t)=1$ search resource is available to him at each time $t \in \widehat{\boldsymbol{T}} = \{2, \cdots, 5\}$. The searcher has information of the target's initial position, given by a probability distribution of $f(4)$ and $f(2)$ ($f(2) = 1 - f(4)$) for two cells $I_0 = \{4, 2\}$.

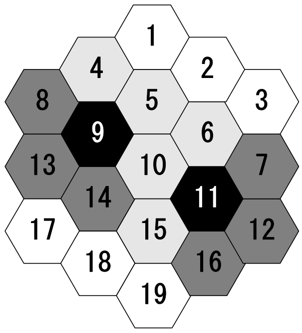

Figure 2: A search space

We change probability $f(4)$ from 0 through 1 to get the values of the game, i.e. the detection probability $R(\varphi^*, \pi^*)$ calculated by players' optimal strategies φ^* and π^*. In Figure 3–6, diamonds show the detection probabilities for $e_0 = 2, \ldots, 5$, respectively. For the convenience of comparison, by squares we show the detection probabilities calculated by the model in which the target's initial position is public information and then the searcher knows the position (refer to [13]).

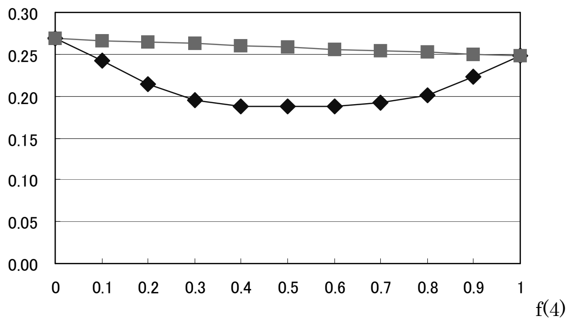

Figure 3: Detection probability for $e_0 = 2$

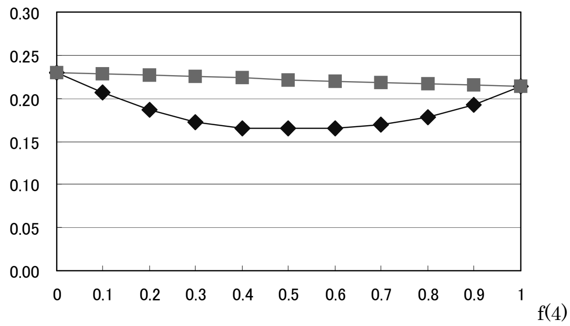

Figure 4: Detection probability for $e_0 = 3$

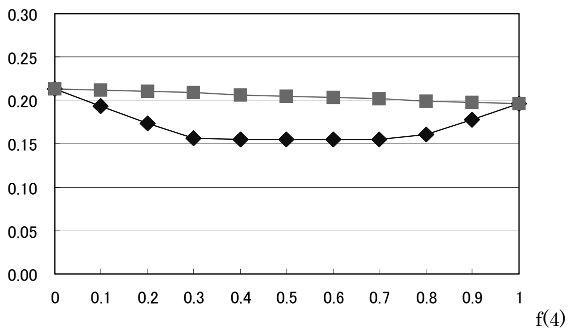

Figure 5: Detection probability for $e_0 = 4$

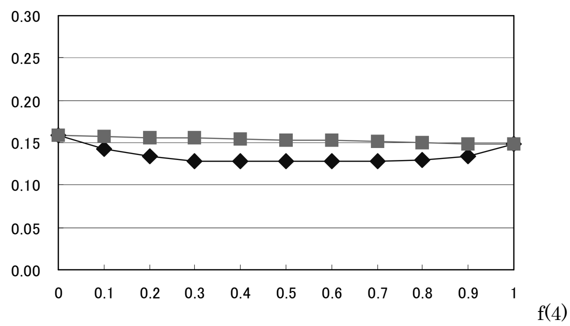

Figure 6: Detection probability for $e_0 = 5$

We can evaluate the value of the information of the initial position by a difference between the two curves. The results of our model always stay below the public-information model because of the searcher's ignorance of the target's initial position. From these figures, we summarize some characteristics of equilibrium points in terms of the value of the game.

(1) As e_0 increases, both curves change their positions lower and the game becomes more disadvantageous to the searcher's side. At two end points of the figures, $f(4) = 0$ or $f(4) = 1$ indicates that the searcher certainly knows the target's initial position 2 or 4, respectively. That is why two figures cross there. The value at $f(4) = 1$ is a little smaller than that at $f(4) = 0$ because cell 4 is located nearer to ineffective cells with lower coefficients $\alpha_i = 0.5$ and 0.3, and then the target starting from cell 4 is a little easier to survive.

(2) When the information of the initial position is vague around $f(4) = f(2) = 0.5$, the detection probability becomes the lowest and the value of the information becomes the largest. This means that the information profits the searcher more when he is more uncertain of the initial position.
The tendency becomes more remarkable and the detection probability $R(\varphi^*, \pi^*)$ has a deeper bathtub curve as e_0 gets smaller. The target possessing small energy cannot go farther from his initial position and the information of his initial position is more credible even if time passes by, that is, the credibility of the information is kept longer.

To check a diffusive motion of the target, we compare two cases of $e_0 = 2$ and $e_0 = 5$ in terms of the survival probability of the target estimated by $q'(i,t) \equiv \sum_k \sum_e q'_k(i,t,e)$ in Table 1 and 2. The estimation of the initial target position is set to be $f(4) = f(2) = 0.5$.

Table 1: $q'(i,t)$ for smaller energy $e_0 = 2$

cell\t	1	2	3	4	5
1	0	.0975	.0566	.0541	.0521
2	.5	.0975	.0566	.0542	.0521
3	0	.0975	.0566	.0541	.0521
4	.5	.1365	.0792	.0758	.0729
5	0	.1364	.0792	.0758	.0729
6	0	.1364	.0792	.0758	.0729
7	0	0	.1349	.1288	.1215
8	0	.2275	.1320	.1264	.1215
9	0	0	0	0	0
10	0	0	.0808	.0771	.0729
11	0	0	0	0	0
12	0	0	0	0	0
13	0	0	.1335	.1273	.1215
14	0	0	0	0	0
15	0	0	0	0	0
16	0	0	0	0	0
17	0	0	0	0	0
18	0	0	0	0	0
19	0	0	0	0	0
Total	1	.9292	.8885	.8495	.8121

Table 2: $q'(i,t)$ for larger energy $e_0 = 5$

cell\t	1	2	3	4	5
1	0	.0613	.0405	.0397	.0343
2	.5	.0613	.0405	.0397	.0343
3	0	.0613	.0405	.0397	.0343
4	.5	.0858	.0567	.0556	.0480
5	0	.0858	.0567	.0556	.0480
6	0	.0858	.0567	.0556	.0480
7	0	.1430	.0945	.0927	.0800
8	0	.1430	.0945	.0927	.0800
9	0	0	0	0	0
10	0	.0858	.0567	.0556	.0480
11	0	0	0	0	0
12	0	0	.0977	.0926	.0800
13	0	.1430	.0945	.0927	.0800
14	0	0	.0978	.0921	.0800
15	0	0	.0584	.0550	.0480
16	0	0	0	0	.0800
17	0	0	.0417	.0395	.0345
18	0	0	0	0	.0232
19	0	0	0	0	.0062
Total	1	.9561	.9272	.8989	.8870

(3) The target distribution in the case of $e_0 = 5$ expands more quickly and more widely than that in the case of $e_0 = 2$. We can see the first characteristic of the optimal target movement in the "expansion" or "diffusion". Comparing two cases of $e_0 = 2$ and $e_0 = 5$ in terms of $q'(i,t)$ in the same state (i,t), the latter case looks to have smaller probability because the total probability spreads over wider areas. The total survival probability of

the target is larger in the latter case, as you can see in the last rows of the tables. In the case of $e_0 = 2$, the total probability is 1, 0.93, 0.89, 0.85 and 0.81 for $t = 1, \cdots, 5$, respectively. In the case of $e_0 = 5$, it is 1, 0.96, 0.93, 0.90 and 0.88. In this way, we can quantitatively evaluate how easy the mobile target is to evade from the searcher and survive.

(4) At each time t, the distribution of $q'(i,t)$ looks almost uniform over each area of three groups of cells categorized by $\alpha_i = 0.7,\ 0.5$ and 0.3. $q'(i,t)$ in ineffective cells with lower α_i is larger than in the effective cells with larger α_i. It is natural that the target more likely moves to ineffective cells for more survivability. We refer to the way to move as "the preference for ineffective cells". The property of uniform distribution, referred to as the "uniformity", is almost kept even when the distribution is spread at later time points.

Table 3 and 4 show optimal distribution plans of search resource in the cases of $e_0 = 2$ and $e_0 = 5$ of the preceding examples, respectively. The distribution starts from $\tau = 2$ and $\Phi(t) = 1$ resource is available to the searcher.

Table 3: $\varphi(i,t)$ for smaller energy $e_0 = 2$

cell \t	1	2	3	4	5
1		.1047	.0744	.0627	.0559
2		.1051	.0741	.0620	.0567
3		.1052	.0735	.0624	.0560
4		.1464	.1039	.0872	.0790
5		.1473	.1038	.0876	.0784
6		.1473	.1029	.0874	.0785
7		0	.0979	.1541	.1956
8		.2440	.1722	.1453	.1313
9		0	0	0	0
10		0	.0634	.0930	.1121
11		0	0	0	0
12		0	0	0	0
13		0	.1339	.1583	.1565
14		0	0	0	0
15		0	0	0	0
16		0	0	0	0
17		0	0	0	0
18		0	0	0	0
19		0	0	0	0
Total	0	1	1	1	1

Table 4: $\varphi(i,t)$ for larger energy $e_0 = 5$

cell \t	1	2	3	4	5
1		.0641	.0588	.0306	.0430
2		.0641	.0588	.0306	.0430
3		.0641	.0588	.0306	.0430
4		.0897	.0823	.0429	.0602
5		.0897	.0823	.0429	.0602
6		.0897	.0823	.0429	.0602
7		.1496	.1371	.0715	.1003
8		.1496	.1371	.0715	.1003
9		0	0	0	0
10		.0897	.0823	.0429	.0602
11		0	0	0	0
12		0	.0252	.1722	.1114
13		.1496	.1371	.0715	.1002
14		0	.0207	.1734	.1147
15		0	.0217	.1031	.0604
16		0	0	0	.0001
17		0	.0158	.0736	.0430
18		0	0	0	.0001
19		0	0	0	.0001
Total	0	1	1	1	1

(5) The distribution areas of resource coincide with the areas which have positive probabilities of target distribution, as seen in Table 1 and 2. The amount of distributed resource is larger in the area with larger probability of target distribution and is almost the same within the cells categorized by the same coefficient α_i. Thus the searcher's strategy corresponds to the target strategy in a rational way. However, against the uniformity, there are some disturbance of the distribution in the frontier cells where the target is forced to stay by the energy exhaustion driven by his expansion strategy. There would also be some disturbance of the probability that the target reaches the

frontier cells. Through an interaction between the target distribution and the resource distribution with some disturbance, the survival probability of the target, $q'(i,t)$, after the search by $\varphi(i,t)$ at time t is constructed to be uniform in such cells, as shown in Table 1 and 2.

We can see the properties explained above, (i) expansion, (ii) uniformity, and (iii) preference for ineffective cells, in the optimal Markov movement in general. Let us check these properties from the optimal Markov movement of the target. Table 5 shows $v_k(i,j,t,e)$ at time $t=1,2$ in the case of $e_0=2$.

Table 5: Optimal Markov movement of the target $v_k(i,j,t,e)$ ($e_0=2$)

k	(i,t,e)	j				
		j=4	j=5	j=8		
4	(i=4, t=1, e=2)	.2937	.2168	.4895		
		j=5	j=10			
	(i=5, t=2, e=1)	.1723	.8277			
		j=8	j=13			
	(i=8, t=2, e=1)	.3889	.6111			
		j=1	j=2	j=3	j=5	j=6
2	(i=2, t=1, e=2)	.2098	.2098	.2098	.0769	.2937
		j=1	j=2	j=5		
	(i=1, t=2, e=1)	.4256	.0993	.4751		
		j=2	j=5	j=6	j=7	
	(i=6, t=2, e=1)	.0524	.1260	.2720	.5497	

(6) Note that the target can move from his current cell just to neighboring cells in the case of $e_0=2$. At time $t=1$, the target moves from state $(k=4, e=2)$ to cell $j=8$ with about probability 0.5 and we can find 'expansion' and 'preference for ineffective cells' in the target motion. The target also moves to cell 4 and 5 with total probability 0.5. In the movement, we find 'uniformity' and 'expansion'. At $t=2$, the 4-type target moves from cell $i=5$ to 10 with higher probability with the intention of 'expansion'. From cell 8, the target moves to cell 13 with probability 0.6 with the intention of 'expansion' and 'ineffective-cell preference' and stays at the same cell with the residual probability aiming for 'uniformity'.

Let us check the movement strategy of the 2-type target starting from initial cell $k=2$. At time $t=1$, the target moves from cell 2 to 6 with probability 0.3. The movement is adopted for 'ineffective-cell preference' by the target and it is also advantageous to move to ineffective cell 7 at the next time. With the residual probability 0.7, the target movement to cells 1, 2, 3 and 5 is proper for 'uniformity'. From cell 6 at time $t=2$, the target moves to cell 7 with probability 0.5 and to other cells with the residual probability. The movement is contributive to 'uniformity'. We can make sure of three properties in an optimal Markov movement of the target in the case of $e_0=5$, though we skip it.

The Markov movement strategy of the target has a direct link to the survival probability $q'_k(\cdot)$, shown in Table 1 and 2, which would be an optimal response to the searcher's optimal behavior. At the same time, the searcher's strategy is optimal to the Markov movement of the target.

The survival probability $q(i,t)$ is the expectation weighted by the probability distribution $\{f(k), k \in I_0\}$ on the type of target or its initial position. In a real search, however, just one type of target must appear. The detection probability of the true type of target depends on the difference between the true type and the searcher's estimation $\{f(k), k \in I_0\}$. If the true type is k and $f(k)$ is one, there is no difference between them.

We calculate the detection probability $R_k(\varphi^*, \pi_k^*)$ by optimal strategies φ^* and π_k^* for each type of $k = 4$ and 2 in the case of $e_0 = 2$ and show them in Table 6. The aggregated detection probability $R(\varphi^*, \pi^*)$ is also listed in the last column but it is the same as in Figure 3. Table 7 shows the results in the case of $e_0 = 5$ although $R(\varphi^*, \pi^*)$ is the same as Figure 6.

Table 6: The detection probability for each type of target ($e_0 = 2$)

f(4)	$R_4(\varphi^*, \pi_k^*)$	$R_2(\varphi^*, \pi_k^*)$	$R(\varphi^*, \pi^*)$
0	0	.2686	.2686
0.1	0	.2686	.2417
0.2	0	.2686	.2149
0.3	.1259	.2253	.1955
0.4	.1878	.1878	.1878
0.5	.1878	.1878	.1878
0.6	.1878	.1878	.1878
0.7	.2202	.1259	.1919
0.8	.2202	.1259	.2014
0.9	.2482	0	.2234
1	.2482	0	.2482

Table 7: The detection probability for each type of target ($e_0 = 5$)

f(4)	$R_4(\varphi^*, \pi_k^*)$	$R_2(\varphi^*, \pi_k^*)$	$R(\varphi^*, \pi^*)$
0	0	.1586	.1586
0.1	0	.1586	.1427
0.2	.0885	.1450	.1337
0.3	.1284	.1285	.1285
0.4	.1285	.1285	.1285
0.5	.1285	.1285	.1285
0.6	.1285	.1285	.1285
0.7	.1285	.1285	.1285
0.8	.1390	.0885	.1289
0.9	.1390	.0885	.1340
1	.1481	0	.1481

(7) As seen from the tables, the searcher focuses on the target of the type for which his estimation $f(k)$ is large, and the detection probability for the type of the target becomes larger. However we can notice that the probability $R_k(\varphi^*, \pi_k^*)$ varies not in a continuous way but in a discrete way as $f(k)$ changes. For $f(4) = 0.4, \ldots, 0.6$ in Table 6, for example, $R_k(\varphi^*, \pi_k^*)$ of both types of $k = 4$ and 2 is kept constant being 0.1878. For $f(4) = 0, \ldots, 0.2$, $R_4(\varphi^*, \pi_4^*)$ is 0 and $R_2(\varphi^*, \pi_2^*)$ is 0.2686. We can also find the constancy during $f(4) = 0.7, 0.8$ and $f(4) = 0.9, 1.0$. Against the discrete change for each type, the aggregated detection probability $R(\varphi^*, \pi^*) = \sum_k f(k) R_k(\varphi^*, \pi_k^*)$ shows a continuous change according to the change of $f(4)$. We can find more remarkable constancy of the detection probability for each type of target in the case of higher energy $e_0 = 5$ in Table 7. $R_k(\varphi^*, \pi_k^*)$ has a flat form around the center, from $f(4) = 0.3$ through 0.7. For a high energy of target, the searcher can do the search causing the similar detection probability for both types of target because the high mobile target would make its distribution randomized over expanded areas from the properties of 'uniformity' and 'expansion'.
On the other hand, the distribution of a poor mobile target is not randomized enough that the searcher has an advantage, and then the detection probability is more biased depending on his estimation $f(k)$. If there is a bigger gap between the true type of the target and the estimation $f(k)$, the target has an advantage of incurring low detection probability and the true type of target is generally easier to survive from the search. In Table 6, for example, the detection probability of the 4-type target is zero for $f(4) = 0, \ldots, 0.2$ and the target can survive definitely while that of the 2-type target is zero for $f(4) = 0.9$ and 1. Thus the gap between the true target's type and the searcher's

estimation is more disadvantageous to the search for the low-energy target than for the high-energy target.

6. Conclusions

An initial position of a target is so important that it affect the results of search operations. The most past research on the search allocation game (SAG) assumed that the position is decided by the target. However, as explained in Introduction section, the decision is not up to the target but the position must be regarded as a random event if the searcher begins a search for the target. The searcher might have uncertain information of the initial target position when he begins the search. This is a motivation by which we first began to discuss the SAG model in which the initial position of target is private information of the target and the searcher knows just a probability distribution of the position.

We notice that information plays an important role in games, as Harsanyi [11] pointed out. We apply the concept of incomplete information to our SAG and develop a methodology to obtain an equilibrium point of the game and evaluate the value of the information of the target initial position. The first proposed method is designed for a path selection as a target strategy. However the total number of target paths increase in an exponential manner as the size of a search space becomes larger and our proposed method does not work for a large size of problems in practice . To cope with this, we develop the second method to obtain the equilibrium. In both methods, the derivation of an optimal searcher's strategy is formulated into a convex programming problem and an optimal target strategy is given by a linear programming problem. Recalling that the past research on SAGs with complete information were formulated into linear programming problems, the nonlinear programming formulation would be essential to handling incomplete information in the SAG.

In this paper, we investigate the equilibrium by some numerical examples and clarify the properties of optimal movement strategy of the target: expansion, uniformity and preference for ineffective cells, and the rationality in an optimal searcher's strategy of distributing search resource. By comparing the results of our model and the complete-information game model, we evaluate the value of the information of initial target position. As well as the information of the target position, we can evaluate the importance of system parameters involved in the SAG from the point of view of the detection probability, which is the payoff of the game. In the numerical examples, we analyze some cases by changing initial movement energy of the target and then we gain an insight on effects of the target mobility.

As a future study, we want to extend our model to a multi-stage game, by which we can deal with a repeated game of datum search or elementary search operation. We also want to investigate the value of information involved in the SAG other than the initial target position.

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Appendix.

Proof that problem (P_T) has feasible solutions

An optimal searcher's strategy φ^* is an optimal solution of the convex minimization problem (P_S). Setting Lagrangian multipliers $\{\widehat{\pi}_k(\omega), \omega \in P_k, k \in I_0\}$, $\{\lambda(t), t \in \widehat{\boldsymbol{T}}\}$, $\{\xi(i,t), (i,t) \in$

$\boldsymbol{K} \times \widehat{\boldsymbol{T}}\}$ and defining a Lagrange function

$$L(\varphi, \eta; \widehat{\pi}, \lambda, \xi) \equiv \sum_{k \in I_0} f(k) \exp(-\eta_k) + \sum_{k \in I_0} \sum_{\omega \in P_k} \widehat{\pi}_k(\omega) \left\{ \eta_k - \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right\}$$
$$+ \sum_{t \in \widehat{\boldsymbol{T}}} \lambda(t) \left(\sum_{i \in \boldsymbol{K}} \varphi(i,t) - \Phi(t) \right) - \sum_{(i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}} \xi(i,t) \varphi(i,t),$$

we have the following KKT conditions, which are the necessary and sufficient conditions for φ^*.

$$\frac{\partial L}{\partial \varphi(i,t)} = -\alpha_i \sum_{k \in I_0} f(k) \sum_{\omega \in \Omega_{it}^k} \widehat{\pi}_k(\omega) + \lambda(t) - \xi(i,t) = 0, (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}},$$
$$\frac{\partial L}{\partial \eta_k} = -f(k) \exp(-\eta_k) + \sum_{\omega \in P_k} \widehat{\pi}_k(\omega) = 0,\ k \in I_0, \tag{A.1}$$
$$\widehat{\pi}_k(\omega) \geq 0,\ \omega \in P_k,\ k \in I_0, \tag{A.2}$$
$$\widehat{\pi}_k(\omega) \left(\eta_k - \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) = 0,\ \omega \in P_k,\ k \in I_0, \tag{A.3}$$
$$\xi(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}},$$
$$\xi(i,t)\varphi(i,t) = 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}},$$
$$\sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \geq \eta_k,\ \omega \in P_k,\ k \in I_0,$$
$$\sum_{i \in \boldsymbol{K}} \varphi(i,t) = \Phi(t),\ t \in \widehat{\boldsymbol{T}},$$
$$\varphi(i,t) \geq 0,\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}.$$

We replace $\widehat{\pi}_k(\omega)$ by a new multiplier $\pi_k(\omega) \equiv f^{-1}(k) \exp(\eta_k) \widehat{\pi}_k(\omega)$. In a similar way to the derivation of conditions (3.8) and (3.9), we simplify the KKT conditions into an equivalent system of two conditions: $\alpha_i \sum_{k \in I_0} f(k) \exp(-\eta_k) \sum_{\omega \in \Omega_{it}^{+k}} \pi_k(\omega) = \lambda(t)$ if $\varphi(i,t) > 0$ and $\alpha_i \sum_{k \in I_0} f(k) \exp(-\eta_k) \sum_{\omega \in \Omega_{it}^{+k}} \pi_k(\omega) \leq \lambda(t)$ if $\varphi(i,t) = 0$. We can derive $\sum_{\omega \in P_k} \pi_k(\omega) = 1$ from Equation (A.1), $\pi_k(\omega) \geq 0$ from (A.2) and $\pi_k(\omega) = 0$ if $\sum_t \alpha_{\omega(t)} \varphi(\omega(t), t) > \eta_k$ from Equation (A.3). The last condition is equivalent to the condition of $\pi_k(\omega) = 0$ if $\omega \notin \Omega^{+k}$, which is always satisfied when we minimize the function $R(\varphi^*, \pi)$. Conversely, the KKT conditions can be properly constructed from these five conditions. That shows that the KKT conditions are equivalent to the system of the five conditions.

From the discussion so far, when we solve the problem (P_S), we have optimal multipliers π and λ as well as its optimal solution φ^* and η^*. We can see that the problem (P_T) provides the necessary and sufficient conditions of these multipliers. Additionally noting the boundedness of its objective function, the problem (P_T) always has an optimal solution. □

We have proved the existence of an optimal solution for (P_T) by making use of the KKT conditions of (P_S). We can also reach to the same conclusion from the KKT conditions of (P_S^0) through more complicated process.

Ryusuke Hohzaki
Department of Computer Science
National Defense Academy
1-10-20 Hashirimizu
Yokosuka 239-8686, Japan
E-mail: hozaki@cc.nda.ac.jp