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ON POLYHEDRAL APPROXIMATION OF L-CONVEX AND M-CONVEX FUNCTIONS

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Abstract In discrete convex analysis, L-convexity and M-convexity are defined for functions in both discrete and continuous variables. Polyhedral L-/M-convex functions connect discrete and continuous versions. Specifically, polyhedral L-/M-convex functions with certain integrality can be identified with discrete versions. Here we show another role of polyhedral L-/M-convex functions: every closed L-/M-convex function in continuous variables can be approximated by polyhedral L-/M-convex functions, uniformly on every compact set. The proof relies on L-M conjugacy under the Legendre-Fenchel transformation.

Keywords: Discrete optimization, discrete convex analysis, L-convex function, M-convex function, polyhedral approximation

1. Introduction

In discrete convex analysis [4, 9, 10, 12], "convexity" concepts are defined for functions in both discrete and continuous variables. Specifically, three types of functions:

$$f : \mathbb{Z}^n \to \mathbb{Z}, \qquad f : \mathbb{Z}^n \to \mathbb{R}, \qquad f : \mathbb{R}^n \to \mathbb{R}$$

are considered in discussing "convexity." Furthermore, polyhedral and non-polyhedral (typically smooth) functions are distinguished for functions of type $\mathbb{R}^n \to \mathbb{R}$. Set functions form a remarkable subclass of functions of type $\mathbb{Z}^n \to \mathbb{Z}$ or $\mathbb{Z}^n \to \mathbb{R}$.

L-convexity and M-convexity in discrete convex analysis are convexity concepts of combinatorial nature, defined for each of these classes of functions. $\text{L}^\natural$-convexity and $\text{M}^\natural$-convexity are variants of L-convexity and M-convexity, respectively. Submodular set functions are captured as $\text{L}^\natural$-convex functions of type $\mathbb{Z}^n \to \mathbb{R}$, and matroids (basis families) are captured as M-convex functions of type $\mathbb{Z}^n \to \mathbb{Z}$. L-convex functions of type $\mathbb{Z}^n \to \mathbb{R}$ or $\mathbb{R}^n \to \mathbb{R}$ find applications in operations research, queueing and inventory in particular (e.g., [1, 8, 20, 21]), through the equivalence between L-convexity and multimodularity [11]. M-convex functions play substantial roles in economics and game theory (e.g., [3, 5, 6, 17]) due to the equivalence between M-convexity and gross substitutes property.

Polyhedral L-/M-convex functions connect discrete and continuous versions in two directions: (i) convex extensions of L-/M-convex functions in discrete variables are (locally) polyhedral L-/M-convex functions in continuous variables, and (ii) discretization (or restriction to integer vectors) of polyhedral L-/M-convex functions with a certain integrality property results in L-/M-convex functions in discrete variables. Although polyhedral L-/M-convex functions are continuous functions of type $\mathbb{R}^n \to \mathbb{R}$, they are endowed with combinatorial properties, sometimes called "discreteness in direction" [10].

In this paper we demonstrate another role of polyhedral L-/M-convex functions by establishing theorems stating that every closed L-/M-convex function in continuous variables can be approximated by polyhedral L-/M-convex functions, uniformly on every compact set. These theorems will serve to reinforce the connection between discrete and continuous versions of L-/M-convex functions.

As a motivation of the present work, a subtle technical aspect in polyhedral (or piecewise-linear) approximation of L-/M-convex functions is explained here. A standard technique of constructing a piecewise-linear convex approximation of a given function $f : \mathbb{R}^n \to \mathbb{R}$ is to evaluate $f(x)$ at finitely many sample points, say, $x = x_1, \ldots, x_N$, and then take the convex lower envelope of the points $(x_1, f(x_1)), \ldots, (x_N, f(x_N))$ in $\mathbb{R}^{n+1}$. A natural choice of the sample points for an L-/M-convex function $f : \mathbb{R}^n \to \mathbb{R}$ is those points of $(\frac{1}{k}\mathbb{Z})^n$ contained in a finite interval, where k is an integer. It can be shown that this standard technique basically works for L- or L$^\natural$-convex functions. However, it does not work for M- or M$^\natural$-convex functions. To be specific, a quadratic function $f(x) = \frac{1}{2}x^\top Ax$ in $x \in \mathbb{R}^3$ with

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

is an example of an M$^\natural$-convex function for which the standard procedure results in a piecewise-linear function that is not M$^\natural$-convex. We overcome this difficulty via conjugacy under the Legendre–Fenchel transformation. Given f, we first consider its Legendre–Fenchel transform, say, g. We apply the above-mentioned standard technique to g to obtain a piecewise-linear approximation, say, g_k to g. We define f_k to be the Legendre–Fenchel transform of g_k, and adopt f_k as a piecewise-linear approximation to f. It can be shown that this method of construction works for M- or M$^\natural$-convex functions.

The rest of the paper is organized as follows. Section 2 offers preliminaries from discrete convex analysis, Section 3 presents the theorems (Theorems 3.1, 3.2 and 3.3) for L-convex functions, and Section 4 gives the corresponding results (Theorems 4.1 and 4.2) for M-convex functions.

2. Preliminaries

2.1. Convex functions

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$, the effective domain and the epigraph are defined as

$$\mathrm{dom}\, f = \{x \in \mathbb{R}^n \mid -\infty < f(x) < +\infty\}, \tag{2.1}$$
$$\mathrm{epi}\, f = \{(x, y) \in \mathbb{R}^{n+1} \mid y \geq f(x)\}. \tag{2.2}$$

The interior and the relative interior of the effective domain of f are denoted as $\mathrm{int}\,(\mathrm{dom}\, f)$ and $\mathrm{ri}\,(\mathrm{dom}\, f)$, respectively.

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *convex* if it satisfies the following inequality:

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y) \qquad (0 \leq \lambda \leq 1). \tag{2.3}$$

Definition 2.2. A convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *proper* if $\mathrm{dom}\, f$ is nonempty, and *closed* if $\mathrm{epi}\, f$ is a closed subset of $\mathbb{R}^{n+1}$.

Definition 2.3. A function defined on $\mathbb{R}^n$ is said to be *polyhedral convex* if its epigraph is a convex polyhedron in $\mathbb{R}^{n+1}$. A polyhedral convex function is exactly such a function that can be represented as the maximum of a finite number of affine functions on a polyhedral effective domain.

Definition 2.4. A function is said to be *locally polyhedral convex* if it is a polyhedral convex function on any finite closed interval $[a, b]$ with $a \leq b$.

See [7, 18] for more about convex functions.

2.2. L-convex functions

L-convex and L$^\natural$-convex functions are defined as follows.

Definition 2.5. A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *L-convex* if it is a convex function that satisfies the following two conditions:

• [Submodularity]:

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (p, q \in \mathbb{R}^n), \tag{2.4}$$

where $p \vee q$ and $p \wedge q$ are, respectively, the componentwise maximum and minimum of p and q.

• [Linearity in direction $\mathbf{1}$]: There exists a real number r such that

$$g(p + \alpha\mathbf{1}) = g(p) + \alpha r \qquad (\alpha \in \mathbb{R}, p \in \mathbb{R}^n), \tag{2.5}$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

Definition 2.6. A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *L$^\natural$-convex* if it is a convex function that satisfies the following inequality:

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})) \qquad (0 \leq \alpha \in \mathbb{R},\ p, q \in \mathbb{R}^n). \tag{2.6}$$

The property expressed by (2.6) is referred to as *translation-submodularity.*

Proposition 2.1 ([15, Proposition 3.10])**.** *A function g is L-convex if and only if it is a convex function that satisfies*

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})) \qquad (\alpha \in \mathbb{R},\ p, q \in \mathbb{R}^n). \tag{2.7}$$

Proof. [*] If g is an L-convex function, then

$$\begin{aligned} g(p) + g(q) &= g(p) + g(q + \alpha\mathbf{1}) - \alpha r \\ &\geq g(p \vee (q + \alpha\mathbf{1})) + g(p \wedge (q + \alpha\mathbf{1})) - \alpha r \\ &= g((p \vee (q + \alpha\mathbf{1})) - \alpha\mathbf{1}) + g(p \wedge (q + \alpha\mathbf{1})) \\ &= g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})). \end{aligned}$$

Conversely, suppose that g satisfies the inequality (2.7). Submodularity (2.4) follows as a special case of (2.7) with $\alpha = 0$. Linearity in direction $\mathbf{1}$ in (2.5) can be derived as follows. The inequality (2.7) with $p = q = s$, $\alpha = -\beta \leq 0$ yields $2g(s) \geq g(s + \beta\mathbf{1}) + g(s - \beta\mathbf{1})$, whereas (2.7) with $p = s + \beta\mathbf{1}$, $q = s - \beta\mathbf{1}$, $\alpha = \beta$ yields $g(s + \beta\mathbf{1}) + g(s - \beta\mathbf{1}) \geq 2g(s)$. Therefore,

$$g(s + \beta\mathbf{1}) + g(s - \beta\mathbf{1}) = 2g(s) \quad (0 \leq \beta \in \mathbb{R},\ s \in \mathbb{R}^n).$$

Since g is a convex function, this implies (2.5). □

The inequality (2.7) is the same as (2.6) in form, but different in the range of α. Since α is nonnegative in (2.6), whereas it can be both negative and positive in (2.7), L-convex functions form a subclass of L$^\natural$-convex functions. Nevertheless, L-convex functions and L$^\natural$-convex functions are essentially the same, in the sense that L$^\natural$-convex functions in n variables

[*] The proof is given here as it is omitted in [15].

can be identified, up to the constant r in (2.5), with L-convex functions in $n+1$ variables [10].

L$^\natural$-convex functions in discrete variables are defined in terms of a discrete version of translation-submodularity.

Definition 2.7. A function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called *L$^\natural$-convex* if it satisfies

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})) \qquad (0 \leq \alpha \in \mathbb{Z},\ p, q \in \mathbb{Z}^n). \tag{2.8}$$

2.3. M-convex functions

M-convex and M$^\natural$-convex functions are defined as follows. We denote by χ_i the i-th unit vector, i.e., $\chi_i = (0, \ldots, 0, \overset{\overset{i}{\vee}}{1}, 0, \ldots, 0)$ for $1 \leq i \leq n$, and the zero vector for $i = 0$, i.e., $\chi_0 = \mathbf{0}$. The positive and negative supports of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are denoted as

$$\mathrm{supp}^+(x) = \{i \mid x_i > 0,\ 1 \leq i \leq n\}, \qquad \mathrm{supp}^-(x) = \{i \mid x_i < 0,\ 1 \leq i \leq n\}. \tag{2.9}$$

Definition 2.8. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *M-convex* if it is a convex function that satisfies the following exchange axiom:

(M-EXC) For any $x, y \in \mathbb{R}^n$ and any $i \in \mathrm{supp}^+(x-y)$, there exists $j \in \mathrm{supp}^-(x-y)$ and a positive real number α_0 such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 \leq \alpha \leq \alpha_0).$$

Definition 2.9. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *M$^\natural$-convex* if it is a convex function that satisfies the following exchange axiom:

(M$^\natural$-EXC) For any $x, y \in \mathbb{R}^n$ and any $i \in \mathrm{supp}^+(x - y)$, there exists $j \in \mathrm{supp}^-(x - y) \cup \{0\}$ and a positive real number α_0 such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 \leq \alpha \leq \alpha_0).$$

Since $j = 0$ is allowed in (M$^\natural$-EXC) and not in (M-EXC), M-convex functions form a subclass of M$^\natural$-convex functions. Nevertheless, M-convex functions and M$^\natural$-convex functions are essentially the same, in the sense that M$^\natural$-convex functions in n variables can be obtained as projections of M-convex functions in $n+1$ variables [10].

2.4. Conjugacy

Conjugacy between L-convex functions and M-convex functions plays an important role in this paper. For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, the conjugate of f is a function $f^\bullet : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{R}^n\} \qquad (p \in \mathbb{R}^n), \tag{2.10}$$

where $\langle p, x\rangle$ denotes the standard inner product of two vectors p and x. The function $f^\bullet$ is also called the Legendre–Fenchel transform of f, and the mapping $f \mapsto f^\bullet$ is referred to as the Legendre–Fenchel transformation.

Theorem 2.2 ([14, Theorem 1.1])**.**
(1) *The classes of closed proper M-convex functions and closed proper L-convex functions are in one-to-one correspondence under the Legendre–Fenchel transformation* (2.10)*. That is, if f is a closed proper M-convex function and g is a closed proper L-convex function, then $f^\bullet$ is a closed proper L-convex function, $g^\bullet$ is a closed proper M-convex function, $(f^\bullet)^\bullet = f$, and $(g^\bullet)^\bullet = g$.*
(2) *The classes of closed proper M$^\natural$-convex functions and closed proper L$^\natural$-convex functions are in one-to-one correspondence under the Legendre–Fenchel transformation* (2.10).

Polyhedral M-convex and L-convex functions are conjugate to each other.

Theorem 2.3 ([13, Theorem 5.1],[10, Theorem 8.4])**.**
(1) *The classes of polyhedral M-convex functions and polyhedral L-convex functions are in one-to-one correspondence under the Legendre–Fenchel transformation* (2.10)*.*
(2) *The classes of polyhedral M$^\natural$-convex functions and polyhedral L$^\natural$-convex functions are in one-to-one correspondence under the Legendre–Fenchel transformation* (2.10)*.*

3. Approximation of L-convex Functions

3.1. Theorems

Theorem 3.1.
(1) *If a sequence of L$^\natural$-convex functions $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ converges to a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at every point of $\mathbb{R}^n$, then g is an L$^\natural$-convex function†.*
(2) *The same statement with "L$^\natural$-convex" replaced by "L-convex" also holds.*

Proof. The proof is given in Section 3.2.1. □

Theorem 3.2.
(1) *For any closed proper L$^\natural$-convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, there exists a nonincreasing sequence $\{g_k\}$ of polyhedral L$^\natural$-convex functions $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ that converges to g uniformly on every compact subset of* $\mathrm{ri}\,(\mathrm{dom}\, g)$ *(the relative interior of the effective domain of g). In particular, for each $p \in \mathrm{ri}\,(\mathrm{dom}\, g)$, we have $g(p) = \lim_{k\to\infty} g_k(p)$.*
(2) *The same statement with "L$^\natural$-convex" replaced by "L-convex" also holds.*

Proof. The proof is given in Section 3.2.2. □

Example 3.1. The function g defined by

$$g(p) = \begin{cases} \frac{1}{p+1} & (p > -1) \\ +\infty & (\text{otherwise}) \end{cases}$$

is a closed proper L$^\natural$-convex function ($n = 1$) with $\mathrm{dom}\, g = (-1, +\infty)$. This function can be represented as the limit of a sequence of polyhedral L$^\natural$-convex functions that converges to g uniformly on every compact subset of the interval $(-1, +\infty) = \mathrm{ri}\,(\mathrm{dom}\, g)$. This fact follows from Theorem 3.2.

Example 3.2. The function g defined by

$$g(p) = \begin{cases} p \log p & (p > 0) \\ 0 & (p = 0) \\ +\infty & (p < 0) \end{cases}$$

is a closed proper L$^\natural$-convex function ($n = 1$) with $\mathrm{dom}\, g = [0, +\infty)$. At the end point $p = 0$ of $\mathrm{dom}\, g$, it has no subgradients. This function can be represented as the limit of a sequence of polyhedral L$^\natural$-convex functions that converges to g uniformly on every compact subset of $\mathrm{dom}\, g = [0, +\infty)$. To see this, consider the piecewise-linear function that interpolates g at $\frac{1}{k}\mathbb{Z}$ and let g_k be its restriction to the interval $[0, k]$. Then each g_k is a polyhedral L$^\natural$-convex function and the sequence $\{g_k\}$ converges to g uniformly on every compact subset S of $\mathrm{dom}\, g = [0, +\infty)$. In particular, the sequence converges to g uniformly on $S = [0, 1]$, say. But this fact does not follow from Theorem 3.2, since $S = [0, 1]$ is not contained in $\mathrm{ri}\,(\mathrm{dom}\, g)$.

†The assumption means that for each $p \in \mathbb{R}^n$, the limit $\lim_{k\to\infty} g_k(p)$ exists in $\mathbb{R} \cup \{+\infty\}$ and $g(p) = \lim_{k\to\infty} g_k(p)$. In particular, the possibility of $g_k(p) \to -\infty$ is excluded.

In Theorem 3.2 above the convergence is established in $\mathrm{ri}\,(\mathrm{dom}\,g)$, whereas in the next theorem (Theorem 3.3) we extend this to $\mathrm{dom}\,g$ under the assumption of compactness of $\mathrm{dom}\,g$.

Theorem 3.3.

(1) *Let* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a closed proper* $\mathrm{L}^\natural$*-convex function with compact effective domain* $\mathrm{dom}\,g$*. Then there exists a sequence*[‡] $\{g_k\}$ *of polyhedral* $\mathrm{L}^\natural$*-convex functions* $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ *that converges to* g *uniformly on* $\mathrm{dom}\,g$*, i.e.,*

$$\lim_{k \to \infty} \sup_{p \in \mathrm{dom}\,g} |g_k(p) - g(p)| = 0. \tag{3.1}$$

(2) *The same statement with "*$\mathrm{L}^\natural$*-convex" replaced by "L-convex" also holds.*

Proof. The proof relies on Theorem 3.2. See Section 3.2.3. □

Example 3.3. The function g defined by

$$g(p) = \begin{cases} p^2 & (|p| < 1) \\ 2 & (|p| = 1) \\ +\infty & (|p| > 1) \end{cases}$$

is a (non-closed) $\mathrm{L}^\natural$-convex function $(n = 1)$ with $\mathrm{dom}\,g = [-1, 1]$. This function cannot be equal to the uniform limit of a sequence of polyhedral $\mathrm{L}^\natural$-convex functions. This example shows the necessity of the closedness assumption on g in Theorem 3.3. We add that a pointwise convergent sequence of polyhedral $\mathrm{L}^\natural$-convex functions does exist. For example, let g_k be the piecewise-linear function that interpolates g at $\frac{1}{k}\mathbb{Z}$; we have $g_k(1) = g_k(-1) = 2$ and $g_k(i/k) = g_k(-i/k) = (i/k)^2$ for $i = 0, 1, \ldots, k-1$. Then $\lim_{k \to \infty} g_k(p) = g(p)$ for each $p \in [-1, 1]$.

Remark 3.1. Here are two remarks about Theorems 3.2 and 3.3. First, in Theorem 3.2 we have a nonincreasing sequence $\{g_k\}$, but this may not be the case in Theorem 3.3. Second, it seems difficult to derive Theorem 3.2 from Theorem 3.3.

3.2. Proofs

We first recall a fundamental fact.

Lemma 3.4. *The pointwise limit of convex functions is a convex function.*

Proof. The proof is given for completeness. Assume that a sequence of convex functions $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ converges pointwise, and denote by $g(p)$ the limit of $g_k(p)$ for each p, i.e., $g(p) = \lim_{k \to \infty} g_k(p)$. It may be that $g(p) = -\infty$ for some p or $g(p) \equiv +\infty$. In the inequality

$$\lambda g_k(p) + (1 - \lambda) g_k(q) \geq g_k(\lambda p + (1 - \lambda) q) \qquad (0 \leq \lambda \leq 1)$$

for the convexity of g_k, we let $k \to \infty$ with λ fixed, to obtain

$$\lambda g(p) + (1 - \lambda) g(q) \geq g(\lambda p + (1 - \lambda) q) \qquad (0 \leq \lambda \leq 1).$$

Hence g is convex. □

[‡]Unlike in Theorem 3.2, this sequence g_k is not necessarily nonincreasing.

3.2.1. Proof of Theorem 3.1

Convexity of the limit function follows from Lemma 3.4 above. In addition, L$^\natural$-convexity and L-convexity of the limit function can be proved as follows.

(1) Each g_k, being L$^\natural$-convex, has translation-submodularity in (2.6), i.e.,

$$g_k(p) + g_k(q) \geq g_k((p - \alpha\mathbf{1}) \vee q) + g_k(p \wedge (q + \alpha\mathbf{1})) \qquad (0 \leq \alpha \in \mathbb{R},\ p, q \in \mathbb{R}^n).$$

By letting $k \to \infty$, we obtain translation-submodularity (2.6) for g.

(2) By a similar argument with the use of (2.7) in place of (2.6).

3.2.2. Proof of Theorem 3.2

We make use of the following general convergence theorem.

Lemma 3.5 ([18, Th.10.8]). *Let C be a relatively open convex set, and let $f_1, f_2, \ldots$ be a sequence of finite convex functions on C. Suppose that the sequence converges pointwise on a dense subset of C, i.e., that there exists a subset C' of C such that $\mathrm{cl}\, C' \supseteq C$ and, for each $x \in C'$, the limit of $f_1(x), f_2(x), \ldots$ exists and is finite. The limit then exists for every $x \in C$, and the function f, where*

$$f(x) = \lim_{k\to\infty} f_k(x),$$

is finite and convex on C. Moreover the sequence of $f_1, f_2, \ldots$ converges to f uniformly on each closed bounded subset of C.

Lemma 3.6. *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an L$^\natural$-convex function, and $p_0 \in \mathrm{dom}\, g$.*

(1) [*Discretization with $1/2^{k-1}$ mesh*] *For $k = 1, 2, \ldots,$ define $h_k : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ by*

$$h_k(q) = g(p_0 + \frac{q}{2^{k-1}}) \qquad (q \in \mathbb{Z}^n).$$

Then h_k is an L$^\natural$-convex function in discrete variables.

(2) *Let $\hat{h}_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the convex extension (convex closure) of h_k, and define $\hat{g}_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by*

$$\hat{g}_k(p) = \hat{h}_k(2^{k-1}(p - p_0)), \qquad i.e., \qquad \hat{g}_k(p_0 + \frac{q}{2^{k-1}}) = \hat{h}_k(q).$$

Then each $\hat{g}_k$ is a locally polyhedral L$^\natural$-convex function that satisfies $\hat{g}_k \geq g$ on $\mathbb{R}^n$. Moreover, the sequence $(\hat{g}_k \mid k = 1, 2, \ldots)$ is monotone nonincreasing.

(3) *Let $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the restriction of $\hat{g}_k$ onto $D_k = \{p \in \mathbb{R}^n \mid |p(i) - p_0(i)| \leq k\ (i = 1, 2, \ldots, n)\}$. Each g_k is a polyhedral L$^\natural$-convex function that satisfies $g_k \geq g$ on $\mathbb{R}^n$. Moreover, the sequence $(g_k \mid k = 1, 2, \ldots)$ is monotone nonincreasing.*

(4) $(g_k \mid k = 1, 2, \ldots)$ *converges to g uniformly on every compact subset of* $\mathrm{ri}\,(\mathrm{dom}\, g)$.

Proof. (1) Obviously, h_k is endowed with the discrete translation-submodularity (2.8).

(2) It is known [10] that an L$^\natural$-convex function in discrete variables is convex-extensible, and its convex closure is a locally polyhedral L$^\natural$-convex function. Therefore, $\hat{g}_k$ is a locally polyhedral L$^\natural$-convex function. The monotonicity is obvious.

(3) D_k is a bounded L$^\natural$-convex set, and an L$^\natural$-convex function remains to be L$^\natural$-convex when it is restricted to an L$^\natural$-convex set. Therefore, g_k is a polyhedral L$^\natural$-convex function. The monotonicity of $\{g_k\}$ follows from the monotonicity of $\{\hat{g}_k\}$ and the inclusion $D_k \subseteq D_{k+1}$.

(4) Take any compact set S contained in $\mathrm{ri}\,(\mathrm{dom}\, g)$. There exists a bounded convex set C that is open relative to the affine hull of $\mathrm{dom}\, g$ and[§]

$$S \subset C \subset \mathrm{cl}\, C \subset \mathrm{ri}\,(\mathrm{dom}\, g).$$

By the construction of g_k, there exists an integer $k(C)$ such that $\mathrm{dom}\, g_k \supseteq C$ for all $k \geq k(C)$. For $k \geq k(C)$, let g_k^C denote the restriction of g_k to C. Then $(g_k^C \mid k \geq k(C))$ is a sequence of finite convex functions on C, to which we apply Lemma 3.5 with

$$C' = \{p \in C \mid 2^{k-1}p \in \mathbb{Z}^n \text{ for some } k \geq k(C),\ k \in \mathbb{Z}\}.$$

Note that C' is a dense subset of C, i.e., $\mathrm{cl}\, C' \supseteq C$.

For each $p \in C'$ there exists $k = k(p)$ such that $2^{k-1}p \in \mathbb{Z}^n$, where we may assume $k(p) \geq k(C)$. Since $g_k^C(p) = g_{k(p)}^C(p) = g(p)$ for all $k \geq k(p)$, the sequence $(g_k^C \mid k \geq k(C))$ converges pointwise on C'. The first half of Lemma 3.5 shows that for each $p \in C$, the limit $g^C(p) = \lim_{k\to\infty} g_k^C(p) = \lim_{k\to\infty} g_k(p)$ exists, and the function g^C is a convex function, which is finite-valued on C. By the latter half of Lemma 3.5, the sequence $(g_k^C \mid k \geq k(C))$ converges to g^C uniformly on each compact subset of C. Obviously, we have $g^C(p) = g(p)$ for $p \in C'$, and hence $g^C(p) = g(p)$ for $p \in C$, since a convex function is continuous in the relative interior of the effective domain. Therefore, $(g_k^C \mid k \geq k(C))$ converges to g uniformly on every compact subset of C, and, in particular, on S. Thus we conclude that $(g_k \mid k = 1, 2, \ldots)$ converges to g uniformly on S. □

Theorem 3.2 follows from Lemma 3.6 above.

Example 3.4. The function g defined by

$$g(p) = \begin{cases} -\sqrt{2-p^2} & (|p| \leq \sqrt{2}) \\ +\infty & (|p| > \sqrt{2}) \end{cases}$$

is a closed proper L$^\natural$-convex function with $\mathrm{dom}\, g = [-\sqrt{2}, \sqrt{2}]$. In the construction in Lemma 3.6 we may choose $p_0 = 0$ to obtain polyhedral L$^\natural$-convex functions g_k. Since $\sqrt{2} \notin \mathrm{dom}\, g_k$ and $g_k(\sqrt{2}) = +\infty$ for every k, the sequence $g_k(p)$ does not converge to $g(p)$ at $p = \sqrt{2} \in \mathrm{dom}\, g$. Thus $\{g_k\}$ does not converge to g on $\mathrm{dom}\, g$, although it certainly does on $\mathrm{ri}\,(\mathrm{dom}\, g) = (-\sqrt{2}, \sqrt{2})$.

3.2.3. Proof of Theorem 3.3

We first recall two fundamental facts that we use.

Lemma 3.7 ([16, Theorem 1.2])**.** *A closed proper L$^\natural$-convex function is continuous on its effective domain.*

Lemma 3.8 (Dini's theorem, e.g., [2, Theorem 8.2.6], [19, Theorem 7.1.2])**.** *If a monotone sequence of continuous functions on a compact set converges pointwise to a continuous function, then the convergence is uniform on the compact set.*

In proving Theorem 3.3 we may assume that $\mathrm{dom}\, g$ is full-dimensional, since otherwise, we may project it onto an appropriate coordinate plane while preserving L$^\natural$-convexity. For any positive number $a > 0$, define

$$g^a(p) = \min\{g(q) \mid \|p - q\|_\infty \leq a\}. \tag{3.2}$$

We consider a sequence $\{g^a\}$ by fixing a (strictly) decreasing sequence of a's converging to zero; e.g., $a = 1/2, 1/2^2, 1/2^3, \ldots$. We shall first apply Theorem 3.2 to g^a to obtain a

[§]We may assume that $\mathrm{cl}\, C$ is a bounded L$^\natural$-convex set.

sequence of polyhedral L$^\natural$-convex functions g_k^a $(k = 1, 2, \ldots)$, and then extract a sequence $\tilde{g}_m$ $(m = 1, 2, \ldots)$ from $\{g_k^a\}$ by choosing appropriate pairs (a_m, k_m). Our construction is summarized as: $g \to g^a \to g_k^a \to \tilde{g}_m$.

The functions g^a have the following properties.

1. Each g^a is an L$^\natural$-convex function.
(Proof) Let δ_S denote the indicator function of $S = \{p \in \mathbb{R}^n \mid \|p\|_\infty \leq a\}$. Then δ_S is a separable convex function, and g^a is equal to the infimum convolution of g and δ_S. The infimum convolution of an L$^\natural$-convex function and a separable convex function is known to be L$^\natural$-convex.
2. $\mathrm{dom}\, g^a = \mathrm{dom}\, g + [-a\mathbf{1}, a\mathbf{1}]$ (Minkowski sum). In particular, $\mathrm{int}\,(\mathrm{dom}\, g^a) \supsetneqq \mathrm{dom}\, g$.
3. The sequence $\{g^a\}$ is nondecreasing as $a \downarrow 0$. That is, $g^a(p) \leq g^b(p)$ if $a > b > 0$.
4. For each $p \in \mathrm{dom}\, g$, the sequence $\{g^a(p)\}$ converges to $g(p)$ as $a \downarrow 0$, i.e.,
$$\lim_{a \downarrow 0} g^a(p) = g(p) \qquad (p \in \mathrm{dom}\, g). \tag{3.3}$$
(Proof) By Lemma 3.7, g is continuous on $\mathrm{dom}\, g$. Then (3.3) follows from the definition (3.2).
5. As $a \downarrow 0$, the sequence $\{g^a\}$ converges to g uniformly on $\mathrm{dom}\, g$, i.e.,
$$\lim_{a \downarrow 0} \sup_{p \in \mathrm{dom}\, g} |g^a(p) - g(p)| = 0. \tag{3.4}$$
(Proof) The effective domain $\mathrm{dom}\, g$ is a compact set by the assumption, and g^a and g are continuous on $\mathrm{dom}\, g$ by Lemma 3.7. Moreover, as $a \downarrow 0$, the sequence $\{g^a\}$ is nondecreasing and converges pointwise to g, as shown above. Therefore, the convergence is uniform by Dini's theorem (Lemma 3.8).

Example 3.5. For the function
$$g(p) = \begin{cases} -\sqrt{2 - p^2} & (|p| \leq \sqrt{2}), \\ +\infty & (|p| > \sqrt{2}) \end{cases}$$
treated in Example 3.4, we have
$$g^a(p) = \begin{cases} -\sqrt{2} & (|p| \leq a), \\ -\sqrt{2 - (|p| - a)^2} & (a \leq |p| \leq \sqrt{2} + a), \\ +\infty & (|p| > \sqrt{2} + a), \end{cases}$$
and hence
$$\sup_{p \in \mathrm{dom}\, g} |g^a(p) - g(p)| = |g^a(\sqrt{2}) - g(\sqrt{2})| = \sqrt{2\sqrt{2}a - a^2} \quad \to 0 \qquad (a \downarrow 0).$$

For each $a > 0$ we apply Theorem 3.2 to g^a to obtain a sequence of polyhedral L$^\natural$-convex functions g_k^a $(k = 1, 2, \ldots)$ that converges to g^a on every compact set contained in $\mathrm{ri}\,(\mathrm{dom}\, g^a) = \mathrm{int}\,(\mathrm{dom}\, g^a)$. Since $\mathrm{dom}\, g$ is a compact set contained in $\mathrm{int}\,(\mathrm{dom}\, g^a)$, we have
$$\lim_{k \to \infty} \sup_{p \in \mathrm{dom}\, g} |g_k^a(p) - g^a(p)| = 0. \tag{3.5}$$

By (3.4), on the other hand, $\{g^a\}$ converges to g uniformly on $\mathrm{dom}\, g$ as $a \downarrow 0$, which implies that for any $\varepsilon > 0$, there exists $\hat{a} = \hat{a}(\varepsilon) > 0$ such that
$$\sup_{p \in \mathrm{dom}\, g} |g^{\hat{a}}(p) - g(p)| < \varepsilon. \tag{3.6}$$

By (3.5) for $\hat{a} = \hat{a}(\varepsilon)$, there exists $\hat{k} = \hat{k}(\varepsilon)$ such that

$$\sup_{p \in \mathrm{dom}\, g} |g_k^{\hat{a}}(p) - g^{\hat{a}}(p)| < \varepsilon \tag{3.7}$$

for all $k \geq \hat{k}$. In particular, with $k = \hat{k}$, we obtain

$$\sup_{p \in \mathrm{dom}\, g} |g_{\hat{k}}^{\hat{a}}(p) - g^{\hat{a}}(p)| < \varepsilon. \tag{3.8}$$

A combination of (3.6) and (3.8) yields

$$\sup_{p \in \mathrm{dom}\, g} |g_{\hat{k}}^{\hat{a}}(p) - g(p)| \leq \sup_{p \in \mathrm{dom}\, g} |g_{\hat{k}}^{\hat{a}}(p) - g^{\hat{a}}(p)| + \sup_{p \in \mathrm{dom}\, g} |g^{\hat{a}}(p) - g(p)| < 2\varepsilon. \tag{3.9}$$

By choosing ε as $\varepsilon = 1/m$ for $m = 1, 2, \ldots$, we construct a sequence $\{\tilde{g}_m\}$ as

$$\tilde{g}_m = g_{\hat{k}(1/m)}^{\hat{a}(1/m)} \qquad (m = 1, 2, \ldots). \tag{3.10}$$

Then we have the following.

1. $\mathrm{dom}\, \tilde{g}_m = \mathrm{dom}\, g_{\hat{k}(1/m)}^{\hat{a}(1/m)} \supsetneqq \mathrm{dom}\, g$.
2. Each $\tilde{g}_m$ is a polyhedral L$^\natural$-convex function.
3. $\{\tilde{g}_m\}$ converges to g uniformly on $\mathrm{dom}\, g$.
   (Proof) By (3.9) with $\varepsilon = 1/m$ we have

$$\sup_{p \in \mathrm{dom}\, g} |\tilde{g}_m(p) - g(p)| < 2/m. \tag{3.11}$$

   Therefore,

$$\lim_{m \to \infty} \sup_{p \in \mathrm{dom}\, g} |\tilde{g}_m(p) - g(p)| = 0. \tag{3.12}$$

The proof of Theorem 3.3 is completed.

4. Approximation of M-convex Functions

4.1. Theorems

Theorem 4.1.
(1) *If a sequence of closed proper M$^\natural$-convex functions $f_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ converges to a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at every point of $\mathbb{R}^n$, then f is an M$^\natural$-convex function (not necessarily closed)*[¶].
(2) *The same statement with "M$^\natural$-convex" replaced by "M-convex" also holds.*

Proof. The proof is based on Theorem 3.2 and the conjugacy theorems (Theorems 2.2 and 2.3). See Section 4.2.1. □

Example 4.1. Consider functions $f_k(x) = \max(1 - kx, 0)$ with $\mathrm{dom}\, f_k = [0, 1]$. Each f_k is a closed proper M$^\natural$-convex function, and the limit

$$\lim_{k \to \infty} f_k(x) = \begin{cases} 1 & (x = 0), \\ 0 & (0 < x \leq 1), \\ +\infty & (x \notin [0, 1]) \end{cases}$$

is an M$^\natural$-convex function, which is not closed.

[¶] The assumption means that for each $x \in \mathbb{R}^n$, the limit $\lim_{k \to \infty} f_k(x)$ exists in $\mathbb{R} \cup \{+\infty\}$ and $f(x) = \lim_{k \to \infty} f_k(x)$. In particular, the possibility of $f_k(x) \to -\infty$ is excluded.

Theorem 4.2.
(1) *For any closed proper* $\mathrm{M}^\natural$*-convex function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *there exists a nondecreasing sequence* $\{f_k\}$ *of polyhedral* $\mathrm{M}^\natural$*-convex functions* $f_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ *that converges to* f *uniformly on every compact subset of* $\mathrm{dom}\, f$*. In particular, for each* $x \in \mathrm{dom}\, f$*, we have* $f(x) = \lim_{k\to\infty} f_k(x)$*.*
(2) *The same statement with "*$\mathrm{M}^\natural$*-convex" replaced by "M-convex" also holds.*

Proof. The proof is given in Section 4.2.2. □

Remark 4.1. Note that Theorem 4.2 asserts uniform convergence on every compact subset of $\mathrm{dom}\, f$ (that may not be a subset of $\mathrm{ri}\,(\mathrm{dom}\, f)$). Also note that no compactness assumption is imposed on $\mathrm{dom}\, f$.

Remark 4.2. In applications, $\mathrm{M}^\natural$-convex functions often appear as laminar convex functions, for which a polyhedral approximation can be constructed easily. By a *laminar family* we mean a nonempty family $\mathcal{T}$ of subsets of $\{1, \ldots, n\}$ such that $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$ for any $A, B \in \mathcal{T}$. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *laminar convex* if it can be represented as

$$f(x) = \sum_{A\in\mathcal{T}} \varphi^A(x(A)) \qquad (x \in \mathbb{R}^n)$$

for a laminar family $\mathcal{T}$ and a family of univariate convex functions $\varphi^A : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ indexed by $A \in \mathcal{T}$, where $x(A) = \sum_{i\in A} x_i$ for $x = (x_1, \ldots, x_n)$. A laminar convex function is $\mathrm{M}^\natural$-convex.

To construct a polyhedral approximation of f, let $\hat{\varphi}_k^A$ be the piecewise-linear function that interpolates φ^A at $\frac{1}{k}\mathbb{Z}$, and let φ_k^A denote its restriction to the interval $[-k, k]$. Then the function f_k defined by

$$f_k(x) = \sum_{A\in\mathcal{T}} \varphi_k^A(x(A)) \qquad (x \in \mathbb{R}^n)$$

is a polyhedral $\mathrm{M}^\natural$-convex function, and the sequence $\{f_k\}$ converges (pointwise) to f. It is noted, however, that, unlike in Theorem 4.2, the sequence $\{f_k\}$ is nonincreasing and the convergence is not necessarily uniform on every compact subset of $\mathrm{dom}\, f$.

4.2. Proofs

4.2.1. Proof of Theorem 4.1

It suffices to consider the case of M-convex functions. First recall from Lemma 3.4 that the limit of convex functions is a convex function.

To show (M-EXC) for f, take distinct $x, y \in \mathrm{dom}\, f$ and $i \in \mathrm{supp}^+(x - y)$. Since f_k converges to f pointwise, we have $x, y \in \mathrm{dom}\, f_k$ for all sufficiently large k. Each f_k is an M-convex function, and, by Lemma 4.3 below, there exists $j_k \in \mathrm{supp}^-(x - y)$ such that

$$f_k(x) + f_k(y) \geq f_k(x - \alpha(\chi_i - \chi_{j_k})) + f_k(y + \alpha(\chi_i - \chi_{j_k})) \qquad (0 \leq \alpha \leq \alpha_0),$$

where

$$\alpha_0 = \frac{x(i) - y(i)}{2|\mathrm{supp}^-(x - y)|} > 0.$$

Since $\mathrm{supp}^-(x - y)$ is a finite set, there exists some $j \in \mathrm{supp}^-(x - y)$ such that j_k equals j for infinitely many k. Fix such j and take a subsequence $k_1 < k_2 < \cdots < k_l < \cdots$ with $j = j_{k_l}$ $(l = 1, 2 \ldots)$. Then we have

$$f_{k_l}(x) + f_{k_l}(y) \geq f_{k_l}(x - \alpha(\chi_i - \chi_j)) + f_{k_l}(y + \alpha(\chi_i - \chi_j)) \qquad (0 \leq \alpha \leq \alpha_0),$$

where α_0 is independent of l. Letting $l \to \infty$ we obtain

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 \leq \alpha \leq \alpha_0),$$

which shows (M-EXC) for f.

Lemma 4.3 ([14, Theorem 3.11]). *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function. Then, f satisfies* (M-EXC) *if and only if it satisfies*

(M-EXC$_\mathbf{s}$) *For any $x, y \in \mathrm{dom}\, f$ and any $i \in \mathrm{supp}^+(x - y)$, there exists $j \in \mathrm{supp}^-(x - y)$ such that*

$$f(x)+f(y) \geq f(x-\alpha(\chi_i-\chi_j))+f(y+\alpha(\chi_i-\chi_j)) \qquad \left(0 \leq \alpha \leq \frac{x(i) - y(i)}{2|\mathrm{supp}^-(x - y)|}\right).$$

4.2.2. Proof of Theorem 4.2

Recall the notation (2.10) for the conjugate function:

$$g^\bullet(x) = \sup\{\langle p, x\rangle - g(p) \mid p \in \mathbb{R}^n\} \qquad (x \in \mathbb{R}^n). \tag{4.1}$$

Our proof uses the following general facts about conjugate functions.

Lemma 4.4 ([18, Corollary 12.2.2]). *For any convex function $g : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, we have*

$$g^\bullet(x) = \sup\{\langle p, x\rangle - g(p) \mid p \in \mathrm{ri}\,(\mathrm{dom}\, g)\} \qquad (x \in \mathbb{R}^n). \tag{4.2}$$

Lemma 4.5. *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ be convex functions with $\mathrm{dom}\, g \neq \emptyset$ and $\mathrm{dom}\, g_k \neq \emptyset$ $(k = 1, 2, \ldots)$. Assume that for each $p \in \mathbb{R}^n$, the sequence $\{g_k(p)\}$ is nonincreasing, bounded from below by $g(p)$, i.e.,*

$$g_1(p) \geq g_2(p) \geq \cdots \geq g_k(p) \geq g_{k+1}(p) \geq \cdots \geq g(p) \qquad (p \in \mathbb{R}^n), \tag{4.3}$$

and that $\{g_k\}$ converges to g pointwise on $\mathrm{ri}\,(\mathrm{dom}\, g)$, i.e.,

$$\lim_{k\to\infty} g_k(p) = \inf_k g_k(p) = g(p) \qquad (p \in \mathrm{ri}\,(\mathrm{dom}\, g)). \tag{4.4}$$

Also assume that $g^\bullet$ is continuous on $\mathrm{dom}\, g^\bullet$. Then the following hold.

(1) The sequence $\{g_k^\bullet\}$ is nondecreasing and converges to $g^\bullet$ pointwise on $\mathrm{dom}\, g^\bullet$. That is, for each $x \in \mathrm{dom}\, g^\bullet$, we have $g_k^\bullet(x) \leq g_{k+1}^\bullet(x)$ and $\lim_{k\to\infty} g_k^\bullet(x) = g^\bullet(x)$.

(2) The sequence $\{g_k^\bullet\}$ converges to $g^\bullet$ uniformly on every compact subset of $\mathrm{dom}\, g^\bullet$.

Proof. (1) It follows from the monotonicity (4.3) of g_k and

$$g_k^\bullet(x) = \sup\{\langle p, x\rangle - g_k(p) \mid p \in \mathbb{R}^n\} \qquad (x \in \mathbb{R}^n) \tag{4.5}$$

that $g_k^\bullet(x) \leq g_{k+1}^\bullet(x) \leq \cdots \leq g^\bullet(x)$. Define

$$h(x) = \sup_k g_k^\bullet(x) = \lim_{k\to\infty} g_k^\bullet(x) \qquad (x \in \mathbb{R}^n),$$

where $h(x) \in \mathbb{R} \cup \{+\infty\}$.

[Proof of $h(x) \leq g^\bullet(x)$] By (4.5) and (4.3) we have

$$g_k^\bullet(x) = \sup_{p\in\mathbb{R}^n}\{\langle p, x\rangle - g_k(p)\} \leq \sup_{p\in\mathbb{R}^n}\{\langle p, x\rangle - g(p)\} = g^\bullet(x) \tag{4.6}$$

for any $x \in \mathbb{R}^n$. Taking the supremum over k and using the definition of $h(x)$, we obtain $h(x) \leq g^\bullet(x)$. This implies, in particular, that $\{g_k^\bullet(x)\}$ has a finite limit for $x \in \operatorname{dom} g^\bullet$.

[Proof of $h(x) \geq g^\bullet(x)$] For $x \in \operatorname{dom} g^\bullet$ we have

$$\begin{aligned} h(x) &= \sup_k g_k^\bullet(x) = \sup_k \left(\sup_{p \in \mathbb{R}^n} \{\langle p, x\rangle - g_k(p)\} \right) = \sup_{p \in \mathbb{R}^n} \left(\sup_k \{\langle p, x\rangle - g_k(p)\} \right) \\ &= \sup_{p \in \mathbb{R}^n} \{\langle p, x\rangle - \inf_k g_k(p)\} \geq \sup_{p \in \mathrm{ri}\,(\operatorname{dom} g)} \{\langle p, x\rangle - \inf_k g_k(p)\} \\ &= \sup_{p \in \mathrm{ri}\,(\operatorname{dom} g)} \{\langle p, x\rangle - g(p)\} = g^\bullet(x), \end{aligned}$$

where the last equality is due to (4.2) in Lemma 4.4.

(2) Let $S \subseteq \operatorname{dom} g^\bullet$ be a compact set. The sequence $\{g_k^\bullet\}$ is nondecreasing and converges to $g^\bullet$ pointwise on S, where $g^\bullet$ is continuous by the assumption. Then, by Dini's theorem (Lemma 3.8), $\{g_k^\bullet\}$ converges to $g^\bullet$ uniformly on S. □

The following two lemmas show properties specific to M$^\natural$-convex and L$^\natural$-convex functions.

Lemma 4.6 ([16, Theorem 1.1])**.** *A closed proper M$^\natural$-convex function is continuous on its effective domain.*

Lemma 4.7. *For a closed proper L$^\natural$-convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, define polyhedral L$^\natural$-convex functions $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, as in Lemma 3.6.*

(1) $(g_k^\bullet \mid k = 1, 2, \ldots)$ *is nondecreasing and converges to $g^\bullet$ pointwise on* $\operatorname{dom} g^\bullet$*. That is, for each $x \in \operatorname{dom} g^\bullet$, we have $g_k^\bullet(x) \leq g_{k+1}^\bullet(x)$ and $\lim_{k\to\infty} g_k^\bullet(x) = g^\bullet(x)$.*

(2) $(g_k^\bullet \mid k = 1, 2, \ldots)$ *converges to $g^\bullet$ uniformly on every compact subset of* $\operatorname{dom} g^\bullet$.

(3) *Each $g_k^\bullet$ is a polyhedral M$^\natural$-convex function.*

Proof. (1) & (2) We have $g_1 \geq g_2 \geq \cdots \geq g$ on $\mathbb{R}^n$ by Lemma 3.6(3), and the sequence $\{g_k\}$ converges to g pointwise on $\mathrm{ri}\,(\operatorname{dom} g)$ by Lemma 3.6(4). The conjugate function $g^\bullet$ is a closed proper M$^\natural$-convex function by Theorem 2.2, and is continuous on $\operatorname{dom} g^\bullet$ by Lemma 4.6. Hence Lemma 4.5 applies.

(3) $g_k^\bullet$ is a polyhedral M$^\natural$-convex function by the polyhedral version of M-L conjugacy theorem (Theorem 2.3). □

We now begin the proof of Theorem 4.2. For a closed proper M$^\natural$-convex function f, its conjugate $g = f^\bullet$ is a closed proper L$^\natural$-convex function and $f = g^\bullet$ by Theorem 2.2. From this g construct g_k as in Lemma 3.6, and then define $f_k = g_k^\bullet$. Then Lemma 4.7 shows that, f_k is a polyhedral M$^\natural$-convex function, and f_k converges to f uniformly on every compact subset of $\operatorname{dom} f$. Our construction is summarized as follows:

			$(\operatorname{dom} \hat{g}_k \subseteq \operatorname{dom} g)$		$(\operatorname{dom} g_k : \text{bounded})$
L :	g	$\rightarrow$	$\hat{g}_k$	$\rightarrow$	g_k
	$\uparrow$				$\downarrow$
M :	f				f_k
					$(\operatorname{dom} f_k = \mathbb{R}^n)$

Remark 4.3. Here is an alternative proof, due to Shinji Ito, of the pointwise convergence in Lemma 4.5(1). Since $g_k \geq g$ we have $\operatorname{dom}\, g_k \subseteq \operatorname{dom}\, g$. By the assumption (4.4), there exists some k' such that $\operatorname{aff}(\operatorname{dom}\, g_k) = \operatorname{aff}(\operatorname{dom}\, g)$ and $\mathrm{ri}(\operatorname{dom}\, g_k) \subseteq \mathrm{ri}(\operatorname{dom}\, g)$ for all $k \geq k'$, where $\operatorname{aff}(\cdot)$ means the affine hull. Then it follows from Lemma 4.4 that

$$g^\bullet(x) = \sup\{\langle p, x\rangle - g(p) \mid p \in \mathrm{ri}(\operatorname{dom}\, g)\}, \qquad g_k^\bullet(x) = \sup\{\langle p, x\rangle - g_k(p) \mid p \in \mathrm{ri}(\operatorname{dom}\, g)\}.$$

Therefore,

$$\lim_{k\to\infty} g_k^\bullet(x) = \sup_{k\geq k'} g_k^\bullet(x) = \sup_{k\geq k'} \left(\sup_{p\in \mathrm{ri}(\mathrm{dom}\ g)} \{\langle p, x\rangle - g_k(p)\} \right)$$
$$= \sup_{p\in \mathrm{ri}(\mathrm{dom}\ g)} \left(\sup_{k\geq k'} \{\langle p, x\rangle - g_k(p)\} \right)$$
$$= \sup_{p\in \mathrm{ri}(\mathrm{dom}\ g)} \{\langle p, x\rangle - g(p)\} = g^\bullet(x).$$

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