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STABILITY IN SUPPLY CHAIN NETWORKS: AN APPROACH BY DISCRETE CONVEX ANALYSIS

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Abstract Ostrovsky generalized the stable marriage model of Gale and Shapley to a model on an acyclic directed graph, and showed the existence of a chain stable allocation under the conditions called same-side substitutability and cross-side complementarity. In this paper, we extend Ostrovsky's model and the concepts of same-side substitutability and cross-side complementarity by using value functions which are defined on integral vectors and allow indifference. We give a characterization of chain stability under the extended versions of same-side substitutability and cross-side complementarity, and develop an algorithm which always finds a chain stable allocation. We also verify that twisted M$^\natural$-concave functions, which are variants of M$^\natural$-concave functions central to discrete convex analysis, satisfy these extended conditions. For twisted M$^\natural$-concave value functions of the agents, we analyze the time-complexity of our algorithm.

Keywords: Economics, stability, acyclic networks, discrete convex analysis

1. Introduction

Since the seminal paper by Gale and Shapley [9] the stable marriage model has been widely studied in such fields as economics, operations research, computer science and mathematics. Applications to various real world problems have been highly successful, e.g. the matching market for American physicians in Roth and Peranson [22], school choice in Abdulkadiroğlu and Sönmez [1], while theoretic aspects, particularly in game theory have also been studied (Roth and Sotomayor [23]). Extensions of this problem have been introduced in various settings. In economic theory, Kelso and Crawford [14] showed the existence of a stable allocation in the many-to-one job matching model, if value functions of firms have the gross substitutes condition. In computer science, Gusfield and Irving [10] developed efficient algorithms for several optimization problems on the stable marriage model. On the other hand, the maximum stable marriage problem with incomplete lists and ties was shown to be NP-hard by Iwama et al. [12]. For the above problem and some variations, approximation algorithms have been developed (see e.g., Iwama et al. [13] and McDermid [15].) Proofs of existence of a stable marriage/allocation for various generalizations have been made using Tarski's fixed point theorem [28] by Adachi [2], Fleiner [5], Hatfield and Milgrom [11]. In operations research, the recent development of frameworks in discrete optimization have led to further models of increasing generality. Fleiner [4] extended the stable marriage model to the framework of matroids, showed existence of a stable allocation, Eguchi, Fujishige and Tamura [3] extended the matroidal model to the framework of discrete convex analysis, a field developed by Murota [17, 18] as a unified framework of discrete optimization. Fujishige and Tamura [7, 8] also gave a common generalization of the stable marriage model and the assignment model of Shapley and Shubik [24], by applying discrete convex analysis.

Remarkable progress was recently made by Ostrovsky [21], who extended the two-sided market model (mathematically modeled on a bipartite graph) to a supply chain network model (mathematically modeled on an acyclic directed graph.) The model is based on a supply chain with various agents (e.g. manufacturers, brokers, consumers), who conduct bilateral transactions of various commodities. The model is described by an acyclic directed graph with parallel edges, where vertices correspond to agents, and edges to the possible transactions of one unit of a certain commodity. Preferences of agents are described by choice functions on subsets of edges. In this setting, stable matchings are generalized to 'chain stable allocations.' Ostrovsky [21] newly defined the concepts of same-side substitutability and cross-side complementarity, and proved that if choice functions of the agents satisfy these two conditions then there always exists a chain stable allocation, (see the next section for details.) Fleiner [6] also imported the concept of stability into the discrete/continuous flow problem, and generalized the lattice structure of stable marriages.

In this paper, we extend Ostrovsky's model and the concepts of same-side substitutability (SSS) and cross-side complementarity (CSC) by using value functions which are defined on integral vectors and allow indifference. We give a characterization of chain stability under the extended SSS and CSC. Our main result is that there always exists a chain stable allocation when value functions satisfy the extended SSS and CSC. We show this by giving an algorithm for finding a chain stable allocation together with certificates of stability. We also illustrate that value functions satisfying SSS and CSC are natural in the discrete convex analysis framework, by showing that twisted M$^\natural$-concave functions, a variant of M$^\natural$-concave functions, satisfy both extended SSS and CSC. We also give analysis of the time-complexities of our algorithms when the value functions of agents satisfy twisted M$^\natural$-concavity. The merit of utilizing discrete convex analysis, is that one can easily image and construct concrete examples of choice functions and value functions satisfying SSS and CSC.

This paper is organized as follows. In Section 2, we describe Ostrovsky's supply chain model and extend SSS and CSC, and in Section 3 we define twisted M$^\natural$-concave function, give some examples, and prove that twisted M$^\natural$-concave functions satisfy the extended SSS and CSC. We modify Ostrovsky's model and chain stability in terms of value functions, and show a characterization of chain stable allocations in Section 4. Finally, in Sections 5 and 6, we develop an algorithm for finding a chain stable allocation, and analyze its time complexity when agents' value functions are twisted M$^\natural$-concave.

2. Same-Side Substitutability and Cross-Side Complementarity

In this section, we briefly describe Ostrovsky's supply chain network model [21], and extend the concepts of same-side substitutability and cross-side complementarity.

The model in Ostrovsky [21] is based on a supply chain with various *agents* (e.g. manufacturers, brokers, consumers), who conduct bilateral transactions of various commodities. Each commodity is traded discretely in units, and the buyer pays the seller some price, which depends on the number of units changing hands. For instance, transaction of only one unit of a certain commodity may cost the buyer 10, while purchase of two units will cost 18, and three units 24. This situation is represented by an acyclic directed graph with parallel edges. Each vertex corresponds to an agent, and each edge to the possible transaction of one unit of a certain commodity. More precisely, a directed edge is associated with a *contract* consisting of the seller (understood to be the tail), the buyer (corresponding to the head), an identifier specifying the commodity together with the 'serial number' of the unit being traded, and the price accompanying the transaction. Thus, if some agent u is willing to sell to another agent v, up to three units of commodity A at the prices mentioned above, and up to two units of commodity B at 15 each, there will be five contracts represented by edges from u to v, the first for the sale of the first unit of A with price 10, the second for the second unit of A with price $18 - 10 = 8$, and so on with the fifth standing for the second unit of B with price 15. An *allocation* of this model is specified by a set of contracts. As is in case of brokers who buy from manufacturers, and sell to consumers, each agent can participate in multiple contracts, some as the buyer, and others as the seller, as long as no agent is involved in two contracts of the same unit (with the same serial number) of commodity which differ only in price.

For an allocation X, let X_u be the set of contracts involving agent u as either seller or buyer. For any given X, each u has a choice function which specifies his/her most desirable subset of X, this is denoted by $Ch_u(X) \subseteq X_u$. In these choice functions, contracts involving differing units of the same commodity are assumed to be preferred in lexicographical order, and indifference is not allowed, so that for any X, $Ch_u(X)$ is uniquely determined.

The key concepts in Ostrovsky [21], are same-side substitutability and cross-side complementarity. For an agent u, denote by X_u^- the contracts of X in which u is a buyer, and X_u^+ the contracts of X for which u is a seller. The preferences of agent u are *same-side substitutable* if

- for any allocations X and Y, with $X_u^+ = Y_u^+$ and $X_u^- \subseteq Y_u^-$,

$$X_u^- \setminus (Ch_u(X))_u^- \subseteq Y_u^- \setminus (Ch_u(Y))_u^-, \tag{2.1}$$

- for any allocations X and Y, with $X_u^+ \subseteq Y_u^+$ and $X_u^- = Y_u^-$,

$$X_u^+ \setminus (Ch_u(X))_u^+ \subseteq Y_u^+ \setminus (Ch_u(Y))_u^+. \tag{2.2}$$

It is easy to show that (2.1) and (2.2) are equivalent to $X_u^- \cap (Ch_u(Y))_u^- \subseteq (Ch_u(X))_u^-$ and $X_u^+ \cap (Ch_u(Y))_u^+ \subseteq (Ch_u(X))_u^+$, respectively. The preferences of agent u are *cross-side complementary* if

- for any allocations X and Y, with $X_u^+ = Y_u^+$ and $X_u^- \subseteq Y_u^-$,

$$(Ch_u(X))_u^+ \subseteq (Ch_u(Y))_u^+, \tag{2.3}$$

- for any allocations X and Y, with $X_u^+ \subseteq Y_u^+$ and $X_u^- = Y_u^-$,

$$(Ch_u(X))_u^- \subseteq (Ch_u(Y))_u^-. \tag{2.4}$$

Presented with a larger set of contracts on one side, same-side substitutability says that the agent will not accept any contract on that side that was rejected in the smaller set, and cross-side complementarity that no contract on the opposite side that was previously accepted will be rejected.

We now extend same-side substitutability (SSS) and cross-side complementarity (CSC) of Ostrovsky [21] to the case where value functions representing preferences of agents are defined on integral vectors and allow indifference.

Let E be a nonempty finite set, and $\mathbb{Z}$ and $\mathbb{R}$ be the sets of integers and reals, respectively. We denote by $\mathbb{Z}^E$ the set of integral vectors $x = (x(e) : e \in E)$ indexed by E, where $x(e)$ denotes the e-th component of vector x. Here E and $x \in \mathbb{Z}^E$ might denote the set of indivisible commodities and the numbers $x(e)$ of commodities e sold or bought by an agent.

The value function of each agent is defined as a function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$. The *effective domain* of f is defined by

$$\mathrm{dom}\, f = \{y \in \mathbb{Z}^E \mid f(y) > -\infty\},$$

and the set of maximizers of f on $U \subseteq \mathbb{Z}^E$ by

$$\text{argmax}\{f(y) \mid y \in U\} = \{x \in U \mid f(x) \geq f(y) \ (\forall y \in U)\}.$$

We assume that each value function f satisfies the following assumption:

$$\text{(A)} \qquad \text{dom}\, f \text{ is bounded and has } \mathbf{0} \text{ as the minimum point.}$$

The boundedness of effective domains implies that a budget constraint is implicitly imposed on each agent's value function. We note that under the assumption, $\text{argmax}\{f(y) \mid y \in U\}$ is well-defined if $U \cap \text{dom}\, f \neq \emptyset$. We can define a choice correspondence C_f by using the value function f as:

$$C_f(q) = \text{argmax}\{f(y) \mid y \leq q\} \quad (q \in \mathbb{Z}^E),$$

where q denotes possible quotas of commodities.

For any $x, y \in \mathbb{Z}^E$, the vectors $x \wedge y$ and $x \vee y$ in $\mathbb{Z}^E$ are defined by

$$(x \wedge y)(e) = \min\{x(e), y(e)\}, \quad (x \vee y)(e) = \max\{x(e), y(e)\} \quad (e \in E).$$

In order to define (SSS) and (CSC) for value functions, we assume that E is partitioned into E^+ and E^-, i.e., $E = E^+ \cup E^-$ and $E^+ \cap E^- = \emptyset$. For any vector $x \in \mathbb{Z}^E$, we respectively denote by x^+ and x^- the subvectors of x on E^+ and E^-. The following property is a natural extension of (SSS) and (CSC). We say that a value function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ satisfies *SSS-CSC property* on (E^+, E^-) if the following conditions hold for any vectors $z_1, z_2 \in \mathbb{Z}^E$ with $z_1 \geq z_2 \geq \mathbf{0}$,

(a) if $z_1^+ \geq z_2^+$ and $z_1^- = z_2^-$ then
- for any $x_1 \in \text{argmax}\{f(y) \mid y \leq z_1\}$, there exists $x_2 \in \text{argmax}\{f(y) \mid y \leq z_2\}$ such that
$$z_2^+ \wedge x_1^+ \leq x_2^+ \quad \text{and} \quad x_2^- \leq x_1^-, \tag{2.5}$$
- for any $x_2 \in \text{argmax}\{f(y) \mid y \leq z_2\}$, there exists $x_1 \in \text{argmax}\{f(y) \mid y \leq z_1\}$ with (2.5),

(b) if $z_1^+ = z_2^+$ and $z_1^- \geq z_2^-$ then
- for any $x_1 \in \text{argmax}\{f(y) \mid y \leq z_1\}$, there exists $x_2 \in \text{argmax}\{f(y) \mid y \leq z_2\}$ such that
$$z_2^- \wedge x_1^- \leq x_2^- \quad \text{and} \quad x_2^+ \leq x_1^+, \tag{2.6}$$
- for any $x_2 \in \text{argmax}\{f(y) \mid y \leq z_2\}$, there exists $x_1 \in \text{argmax}\{f(y) \mid y \leq z_1\}$ with (2.6).

We will give a class of functions with SSS-CSC property in the next section.

A value function with SSS-CSC property has the following properties, which are used in Sections 4 and 5.

Lemma 2.1. *Let $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ be a function with SSS-CSC property on (E^+, E^-) and $x, z_1, z_2 \in \mathbb{Z}^E$ be such that $x \in \text{argmax}\{f(y) \mid y \leq z_1\}$, and $x \in \text{argmax}\{f(y) \mid y \leq z_2\}$. If $\text{argmax}\{f(y) \mid y \leq z_1 \vee z_2\} \neq \emptyset$ and either $z_1^+ = z_2^+$ or $z_1^- = z_2^-$ then we have $x \in \text{argmax}\{f(y) \mid y \leq z_1 \vee z_2\}$.*

Proof. We will prove the case where $z_1^+ = z_2^+$ (we can prove the case $z_1^- = z_2^-$ similarly.) Let $z_3 = z_1 \vee z_2$. Since $z_3^+ = z_2^+$ and $z_3^- \geq z_2^-$, by (b) in SSS-CSC property, there exists $x_3 \in \text{argmax}\{f(y) \mid y \leq z_3\}$ such that $z_2^- \wedge x_3^- \leq x^-$ and $x^+ \leq x_3^+$. For each $e \in E^-$, the first inequality means $x_3^-(e) \leq x^-(e)$ if $x^-(e) < z_2^-(e)$; otherwise $x_3^-(e) \leq z_1^-(e)$ because

$x^-(e) = z_2^-(e)$, $x \le z_1 \wedge z_2$ and $x_3 \le z_1 \vee z_2$. Thus, we have $x_3^- \le z_1^-$. Since $x_3^+ \le z_3^+ = z_1^+$ obviously holds, we have $x_3 \in \mathrm{argmax}\{f(y) \mid y \le z_1\}$, which implies $f(x) = f(x_3)$ and also $x \in \mathrm{argmax}\{f(y) \mid y \le z_3\}$. □

Lemma 2.2. *Let $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ be a function with SSS-CSC property on (E^+, E^-) and $x, z_1, z_2 \in \mathbb{Z}^E$ be such that $z_1 \ge z_2$, $\mathrm{argmax}\{f(y) \mid y \le z_1\} \ne \emptyset$, and $x \in \mathrm{argmax}\{f(y) \mid y \le z_2\}$. If for each $e \in E$, $x(e) < z_2(e)$ whenever $z_2(e) < z_1(e)$ then we have $x \in \mathrm{argmax}\{f(y) \mid y \le z_1\}$.*

Proof. Let z_3 be the vector defined by $z_3^+ = z_2^+$ and $z_3^- = z_1^-$. By (b) in SSS-CSC property for z_3, z_2 and x, there exists $x_3 \in \mathrm{argmax}\{f(y) \mid y \le z_3\}$ such that $z_2^- \wedge x_3^- \le x^-$ and $x^+ \le x_3^+$. For each $e \in E^-$ with $z_2^-(e) < z_1^-(e)$, we have $x(e) < z_2(e)$ from the assumption, and hence, $x_3^-(e) \le x^-(e)$ from the first inequality $z_2^- \wedge x_3^- \le x^-$. For each $e \in E^-$ with $z_2^-(e) = z_1^-(e)$, the first inequality together with $x_3^-(e) \le z_3^-(e) = z_1^-(e) = z_2^-(e)$ means $x_3^-(e) \le x^-(e)$. Thus, we have $x_3^- \le z_2^-$. Since $x_3^+ \le z_3^+ = z_2^+$ obviously holds, we have $x_3 \in \mathrm{argmax}\{f(y) \mid y \le z_2\}$, which implies $f(x) = f(x_3)$ and also $x \in \mathrm{argmax}\{f(y) \mid y \le z_3\}$.

Since $z_1^+ \ge z_3^+$ and $z_1^- = z_3^-$ hold, by using (a) in SSS-CSC property for z_1, z_3, and x, we can show $x \in \mathrm{argmax}\{f(y) \mid y \le z_1\}$ in precisely the same way as above. □

3. M$^\natural$-Concave Functions and Twisted M$^\natural$-Concave Functions

In this section, we give a useful class of functions with SSS-CSC property.

Let E be a nonempty finite set. Given a vector $x \in \mathbb{Z}^E$, its positive support and negative support are defined by

$$\mathrm{supp}^+(x) = \{e \in E \mid x(e) > 0\}, \quad \mathrm{supp}^-(x) = \{e \in E \mid x(e) < 0\}.$$

For each $S \subseteq E$, we denote by χ_S the characteristic vector of S defined by $\chi_S(e) = 1$ if $e \in S$; otherwise 0, and simply write χ_e instead of $\chi_{\{e\}}$ for each $e \in E$.

An *M-concave function* defined by Murota [17, 18] is a function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \ne \emptyset$ satisfying

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall e \in \mathrm{supp}^+(x - y)$, $\exists e' \in \mathrm{supp}^-(x - y)$:

$$f(x) + f(y) \le f(x - \chi_e + \chi_{e'}) + f(y + \chi_e - \chi_{e'}).$$

From (M-EXC), the effective domain of an M-concave function lies on a hyperplane $\{y \in \mathbb{R}^E \mid y(E) = \text{constant}\}$, where $y(E) = \sum_{e \in E} y(e)$.

The concept of M$^\natural$-concavity which is a variant of M-concavity was proposed by Murota and Shioura [20]. Let 0 denote a new element not in E and define $\widetilde{E} = \{0\} \cup E$. A function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \ne \emptyset$ is called *M$^\natural$-concave* if it is expressed in terms of an M-concave function $\widetilde{f} : \mathbb{Z}^{\widetilde{E}} \to \mathbb{R} \cup \{-\infty\}$ as: for all $x \in \mathbb{Z}^E$

$$f(x) = \widetilde{f}(x_0, x) \quad \text{with } x_0 = -x(E).$$

Namely, an M$^\natural$-concave function is a function obtained as the projection of an M-concave function. Conversely, an M$^\natural$-concave function f determines the corresponding M-concave function $\widetilde{f}$ by

$$\widetilde{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(E)) \\ -\infty & (\text{otherwise}) \end{cases} \qquad ((x_0, x) \in \mathbb{Z}^{\widetilde{E}}).$$

An M$^\natural$-concave function can also be defined by using the following exchange property.

Theorem 3.1 (Murota and Shioura [20])**.** *A function* $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$ *is* $M^\natural$*-concave if and only if it satisfies*

($M^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall e \in \mathrm{supp}^+(x-y)$, $\exists e' \in \mathrm{supp}^-(x-y) \cup \{0\}$:

$$f(x) + f(y) \leq f(x - \chi_e + \chi_{e'}) + f(y + \chi_e - \chi_{e'}),$$

where we assume χ_0 *is the zero vector on* E*.*

An $M^\natural$-concave function has nice features as a value function, e.g., submodularity, natural generalizations of gross substitutability, single improvement property and substitutability (see Murota [19], Fujishige and Tamura [8] for details.)

As in Section 2, let E^+ and E^- be a partition of E (i.e. $E = E^+ \cup E^-$ and $E^+ \cap E^- = \emptyset$) and for $x = (x^+, x^-) \in \mathbb{Z}^E$, define $\mathrm{tw}(x) \in \mathbb{Z}^E$ by $(x^+, -x^-)$. Let $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ be a function defined by

$$f(x) = \widehat{f}(\mathrm{tw}(x)) \quad (x \in \mathbb{Z}^E) \tag{3.1}$$

for some $M^\natural$-concave function $\widehat{f}$. We call a function defined as above a *twisted* $M^\natural$*-concave function* on (E^+, E^-). Note that if either $E^+ = \emptyset$ or $E^- = \emptyset$ then twisted $M^\natural$-concave functions are also $M^\natural$-concave. Twisted $M^\natural$-concave functions are generalizations of the class of functions called GM-concave functions in Sun and Yang [26]. We give some simple examples of $M^\natural$-concave and twisted $M^\natural$-concave functions.

Example 3.1. *We call a nonempty family* $\mathcal{T}$ *of subsets of* E *a* laminar family *if* $X \cap Y = \emptyset$, $X \subseteq Y$ *or* $Y \subseteq X$ *holds for every* $X, Y \in \mathcal{T}$*. For a laminar family* $\mathcal{T}$ *and a family of univariate concave functions* $f_Y : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *indexed by* $Y \in \mathcal{T}$*, the function* $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ *defined by*

$$f(x) = \sum_{Y \in \mathcal{T}} f_Y(x(Y)) \quad (x \in \mathbb{Z}^E)$$

is called a laminar concave function*. It is known that a laminar concave function is* $M^\natural$*-concave if* $\mathrm{dom}\, f \neq \emptyset$ *(see Murota [19].) For a partition* (E^+, E^-) *of* E*, the function defined by*

$$g(x) = \begin{cases} 0 & (x \geq \mathbf{0},\ x(E^-) = x(E^+)) \\ -\infty & (\textit{otherwise}) \end{cases} \quad (x \in \mathbb{Z}^E)$$

is twisted $M^\natural$*-concave on* (E^+, E^-) *because* g *corresponds to the laminar concave function with the laminar family* $E \cup \{E\}$ *and the following univariate concave functions: for each* $e \in E^+$*,*

$$\widehat{g}_e(y) = \begin{cases} 0 & (y \geq 0) \\ -\infty & (\textit{otherwise}) \end{cases} \quad (y \in \mathbb{R}),$$

for each $e \in E^-$*,*

$$\widehat{g}_e(y) = \begin{cases} 0 & (y \leq 0) \\ -\infty & (\textit{otherwise}) \end{cases} \quad (y \in \mathbb{R}),$$

and for E*,*

$$\widehat{g}_E(y) = \begin{cases} 0 & (y = 0) \\ -\infty & (\textit{otherwise}) \end{cases} \quad (y \in \mathbb{R}).$$

Similarly, the function defined by

$$h(x) = \begin{cases} x(E^-) - x(E^+) & (x \geq \mathbf{0},\ x(E^-) \geq x(E^+)) \\ -\infty & (\textit{otherwise}) \end{cases} \quad (x \in \mathbb{Z}^E)$$

is also twisted M$^\natural$-concave on (E^+, E^-). *If* E^- *and* E^+ *respectively denote the input commodities and output commodities, function* g *can be used to enforce the conservation between input and output commodities, and* h *can be used to forbid the total of output commodities to exceed that of the input commodities. //*

Example 3.2. *Let* f *be a twisted M$^\natural$-concave function on* (E^+, E^-), *and* $\widehat{f}$ *its corresponding M$^\natural$-concave function. For any two vectors* $a, b \in \mathbb{Z}^E$ *with* $a \leq b$, *the function defined by*

$$\widehat{f}_{[a,b]}(x) = \begin{cases} \widehat{f}(x) & (a \leq x \leq b) \\ -\infty & (otherwise) \end{cases} \qquad (x \in \mathbb{Z}^E)$$

is also M$^\natural$-concave if $\mathrm{dom}\, \widehat{f}_{[a,b]} \neq \emptyset$ *(see Murota [19].) Hence the function defined by*

$$f_{[a,b]}(x) = \begin{cases} f(x) & (a \leq x \leq b) \\ -\infty & (otherwise) \end{cases} \qquad (x \in \mathbb{Z}^E)$$

is also twisted M$^\natural$-concave if $\mathrm{dom}\, f_{[a,b]} \neq \emptyset$. *//*

Example 3.3. *Let* f *be a twisted M$^\natural$-concave function on* (E^+, E^-), *and* $\widehat{f}$ *its corresponding M$^\natural$-concave function. Given univariate concave functions* $f_e : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *indexed by* $e \in E$, *it is known that*

$$\widehat{f}(x) + \sum_{e \in E} f_e(x(e)) \quad (x \in \mathbb{Z}^E)$$

is also M$^\natural$-concave if its effective domain is nonempty (see Murota [19].) This says that the sum of an M$^\natural$-concave function and a separable concave function is also M$^\natural$-concave. Hence the sum of a twisted M$^\natural$-concave function and a separable concave function, i.e.,

$$f(x) + \sum_{e \in E} f_e(x(e)) \quad (x \in \mathbb{Z}^E)$$

is also twisted M$^\natural$-concave if its effective domain is nonempty. //

The next two lemmas, namely, Lemma 3.1 and Lemma 3.2, say that twisted M$^\natural$-concave functions satisfy SSS-CSC property on (E^+, E^-).

Lemma 3.1. *Let* $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ *be a twisted M$^\natural$-concave function on* (E^+, E^-) *and* $z_1, z_2 \in \mathbb{Z}^E$ *be such that* $z_1^+ \geq z_2^+$, $z_1^- = z_2^-$, $\mathrm{argmax}\{f(y) \mid y \leq z_1\} \neq \emptyset$, *and* $\mathrm{argmax}\{f(y) \mid y \leq z_2\} \neq \emptyset$.

(a) *For any* $x_1 \in \mathrm{argmax}\{f(y) \mid y \leq z_1\}$, *there exists* $x_2 \in \mathrm{argmax}\{f(y) \mid y \leq z_2\}$ *such that*

$$z_2^+ \wedge x_1^+ \leq x_2^+ \quad \text{and} \quad x_2^- \leq x_1^-. \tag{3.2}$$

(b) *For any* $x_2 \in \mathrm{argmax}\{f(y) \mid y \leq z_2\}$, *there exists* $x_1 \in \mathrm{argmax}\{f(y) \mid y \leq z_1\}$ *with (3.2).*

Proof. Let $\widehat{f}$ be an M$^\natural$-concave function satisfying (3.1).

We begin by making some observations on a pair x_1, x_2 with $x_1 \in \mathrm{argmax}\{f(y) \mid y \leq z_1\}$ and $x_2 \in \mathrm{argmax}\{f(y) \mid y \leq z_2\}$ not satisfying (3.2). Since (3.2) does not hold, there must exist $e \in E$ such that either $\min\{z_2^+(e), x_1^+(e)\} > x_2^+(e)$ or $x_2^-(e) > x_1^-(e)$. In the former case we have $e \in E^+$, $z_2(e) > x_2(e)$ and $x_1(e) > x_2(e)$, and in the latter case we have $e \in E^-$

and $x_2(e) > x_1(e)$. In both cases, $e \in \mathrm{supp}^+(\mathrm{tw}(x_1) - \mathrm{tw}(x_2))$ must hold. By (M$^\natural$-EXC) for $\widehat{f}$, there exists $e' \in \mathrm{supp}^-(\mathrm{tw}(x_1) - \mathrm{tw}(x_2)) \cup \{0\}$ such that

$$\widehat{f}(\mathrm{tw}(x_1)) + \widehat{f}(\mathrm{tw}(x_2)) \leq \widehat{f}(\mathrm{tw}(x_1) - \chi_e + \chi_{e'}) + \widehat{f}(\mathrm{tw}(x_2) + \chi_e - \chi_{e'}). \tag{3.3}$$

Let $\widetilde{x}_1 = \mathrm{tw}(\mathrm{tw}(x_1) - \chi_e + \chi_{e'})$ and $\widetilde{x}_2 = \mathrm{tw}(\mathrm{tw}(x_2) + \chi_e - \chi_{e'})$. If $e \in E^+$ then $\widetilde{x}_1(e) = x_1(e) - 1 \leq z_1(e)$ and $\widetilde{x}_2(e) = x_2(e) + 1 \leq z_2(e)$ hold; otherwise, $\widetilde{x}_1(e) = x_1(e) + 1 \leq x_2(e) \leq z_2(e) = z_1(e)$ and $\widetilde{x}_2(e) = x_2(e) - 1 \leq z_2(e)$. If $e' \in E^+$ then we have $\widetilde{x}_1(e') = x_1(e') + 1 \leq x_2(e') \leq z_2(e') \leq z_1(e')$ and $\widetilde{x}_2(e') = x_2(e') - 1 \leq z_2(e')$. If $e' \in E^-$ then we have $\widetilde{x}_1(e') = x_1(e') - 1 \leq z_1(e')$ and $\widetilde{x}_2(e') = x_2(e') + 1 \leq x_1(e') \leq z_1(e') = z_2(e')$. Hence $\widetilde{x}_1 \leq z_1$ and $\widetilde{x}_2 \leq z_2$ hold whichever e' belongs to. Recalling that $x_1 \in \mathrm{argmax}\{f(y) \mid y \leq z_1\}$ we have $f(x_1) \geq f(\widetilde{x}_1)$, which implies $\widehat{f}(\mathrm{tw}(x_1)) \geq \widehat{f}(\mathrm{tw}(x_1) - \chi_e + \chi_{e'})$, which considered together with (3.3), in turn yields $\widehat{f}(\mathrm{tw}(x_2)) \leq \widehat{f}(\mathrm{tw}(x_2) + \chi_e - \chi_{e'})$, i.e., $f(x_2) \leq f(\widetilde{x}_2)$ and hence $\widetilde{x}_2 \in \mathrm{argmax}\{f(y) \mid y \leq z_2\}$. Applying the same argument to x_2 and $\widetilde{x}_1$, we obtain that $\widetilde{x}_1 \in \mathrm{argmax}\{f(y) \mid y \leq z_1\}$.

To prove (a), we consider $x_1 \in \mathrm{argmax}\{f(y) \mid y \leq z_1\}$ to be fixed, and choose x_2 to be an element in $\mathrm{argmax}\{f(y) \mid y \leq z_2\}$ that minimizes

$$\sum\{x_1(e) - x_2(e) \mid e \in \mathrm{supp}^+((z_2^+ \wedge x_1^+) - x_2^+)\} + \sum\{x_2(e) - x_1(e) \mid e \in \mathrm{supp}^+(x_2^- - x_1^-)\}. \tag{3.4}$$

For such x_2, (3.2) must hold, since otherwise, for $\widetilde{x}_2$ defined as above, $\widetilde{x}_2 \in \mathrm{argmax}\{f(y) \mid y \leq z_2\}$, and the value of (3.4) for x_1 and $\widetilde{x}_2$ is smaller by exactly one than that for x_1 and x_2, which is a contradiction.

Similarly, for fixed $x_2 \in \mathrm{argmax}\{f(y) \mid y \leq z_2\}$, any x_1 in $\mathrm{argmax}\{f(y) \mid y \leq z_1\}$ minimizing (3.4), satisfies (3.2), proving (b). □

We can show the following lemma in the same way as Lemma 3.1.

Lemma 3.2. *Let $f : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ be a twisted M$^\natural$-concave function on (E^+, E^-) and $z_1, z_2 \in \mathbb{Z}^E$ be such that $z_1^+ = z_2^+$, $z_1^- \geq z_2^-$, $\mathrm{argmax}\{f(y) \mid y \leq z_1\} \neq \emptyset$, and $\mathrm{argmax}\{f(y) \mid y \leq z_2\} \neq \emptyset$.*

(a) *For any $x_1 \in \mathrm{argmax}\{f(y) \mid y \leq z_1\}$, there exists $x_2 \in \mathrm{argmax}\{f(y) \mid y \leq z_2\}$ such that*

$$z_2^- \wedge x_1^- \leq x_2^- \quad \textit{and} \quad x_2^+ \leq x_1^+. \tag{3.5}$$

(b) *For any $x_2 \in \mathrm{argmax}\{f(y) \mid y \leq z_2\}$, there exists $x_1 \in \mathrm{argmax}\{f(y) \mid y \leq z_1\}$ with (3.5).*

4. Our Model

Let $G = (V, E)$ be an acyclic directed graph. For each $v \in V$, let $\delta(v)$, $\delta^+(v)$ and $\delta^-(v)$ respectively denote the set of all edges incident to v, the set of outgoing edges from v and the set of incoming edges to v. As in Ostrovsky [21], the vertex set corresponds to the agents (e.g. manufacturers, brokers, consumers) involved in a supply chain, and the edge set represents possible transactions of commodities. However, instead of creating multiple edges to deal with different units of the same commodity, we use integral vectors defined on the edge set. Each agent $v \in V$ has a value function f_v on $\delta(v)$, i.e., $f_v : \mathbb{Z}^{\delta(v)} \to \mathbb{R} \cup \{-\infty\}$. We recall that each value function f_v satisfies Assumption (A). We call $x \in \mathbb{Z}^E$ an *allocation*. For each allocation x and agent $v \in V$, $x_{\delta(v)}$ denotes the subvector $(x(e) : e \in \delta(v))$ of x. We say that an allocation $x \in \mathbb{Z}^E$ is *feasible* if $x_{\delta(v)} \in \mathrm{dom}\, f_v$ for all $v \in V$, and that a feasible allocation x is *individually rational* if $f_v(x_{\delta(v)}) = \max\{f_v(y) \mid y \le x_{\delta(v)}\}$ for all $v \in V$. Individual rationality means that no agent would like to unilaterally decrease the number of units in any transaction he/she participates in. For a feasible allocation $x \in \mathbb{Z}^E$, a *blocking path* for x is a directed path $P = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$ such that

- $f_{v_0}(x_{\delta(v_0)}) < \max\{f_{v_0}(y) \mid y \le (x + \chi_{e_1})_{\delta(v_0)}\}$,
- $f_{v_i}(x_{\delta(v_i)}) < \max\{f_{v_i}(y) \mid y \le (x+\chi_{e_i}+\chi_{e_{i+1}})_{\delta(v_i)},\ y(e_i) = x(e_i)+1,\ y(e_{i+1}) = x(e_{i+1})+1\}$ for $i = 1, \ldots, k-1$,
- $f_{v_k}(x_{\delta(v_k)}) < \max\{f_{v_k}(y) \mid y \le (x + \chi_{e_k})_{\delta(v_k)}\}$.

We say that a feasible allocation x^* is *chain stable* or simply *stable* if it is individually rational and has no blocking path. Blocking paths are generalizations of blocking pairs in stable matchings, and represent a sequence of agents, each one buying from the previous and selling to the next, who would like to increase their transactions by one unit each (and possibly decrease other deals.)

We give a sufficient condition for a given allocation $x^* \in \mathbb{Z}^E$ to be chain stable.

Lemma 4.1. *For a feasible allocation $x^* \in \mathbb{Z}^E$, if there exist $\overline{z}, \underline{z} \in (\mathbb{Z} \cup \{+\infty\})^E$ such that*

$$x^*_{\delta(v)} \in \operatorname{argmax}\{f_v(y) \mid y_{\delta^+(v)} \le \overline{z}_{\delta^+(v)},\ y_{\delta^-(v)} \le \underline{z}_{\delta^-(v)}\} \quad (v \in V), \tag{4.1}$$
$$\overline{z}(e) = +\infty \quad \text{or} \quad \underline{z}(e) = +\infty \quad (e \in E), \tag{4.2}$$

then x^* is chain stable.

Proof. Condition (4.1) guarantees that x^* is individually rational. Suppose, to the contrary, that there is a blocking path $P = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$. By the condition for v_0 in the definition of a blocking path together with (4.1), $\overline{z}(e_1) = x^*(e_1)$ must hold, which implies $\underline{z}(e_1) = +\infty$ by (4.2). For agent v_1, we have $e_1 \in \delta^-(v_1)$, $e_2 \in \delta^+(v_1)$, $x^*_{\delta^+(v_1)} \le \overline{z}_{\delta^+(v_1)}$, $(x^* + \chi_{e_1})_{\delta^-(v_1)} \le \underline{z}_{\delta^-(v_1)}$ and

$$f_{v_1}(x^*_{\delta(v_1)}) < \max\{f_{v_1}(y) \mid y \le (x^*+\chi_{e_1}+\chi_{e_2})_{\delta(v_1)},\ y(e_1) = x^*(e_1)+1,\ y(e_2) = x^*(e_2)+1\},$$
$$f_{v_1}(x^*_{\delta(v_1)}) = \max\{f_{v_1}(y) \mid y_{\delta^+(v_1)} \le \overline{z}_{\delta^+(v_1)},\ y_{\delta^-(v_1)} \le \underline{z}_{\delta^-(v_1)}\}.$$

Thus $\overline{z}(e_2) = x^*(e_2)$ must hold, and hence, $\underline{z}(e_2) = +\infty$ by (4.2). In the same way as above, we obtain $\overline{z}(e_i) = x^*(e_i)$ and $\underline{z}(e_i) = +\infty$ for $i = 1, 2, \ldots, k$. Since $e_k \in \delta^-(v_k)$ and $\underline{z}(e_k) = +\infty$, the condition for v_k in the definition of a blocking path, i.e., $f_{v_k}(x^*_{\delta(v_k)}) < \max\{f_{v_k}(y) \mid y \le (x^* + \chi_{e_k})_{\delta(v_k)}\}$ contradicts (4.1) for v_k. Hence there is no blocking path for x^*, and hence, x^* must be chain stable. □

Example 4.1. *We consider the supply chain network in Figure 1 in which agents 1 and 2 are producers, agents 3 and 4 brokers, and agents 5 and 6 consumers. Note that transactions can only occur between producers and brokers, and brokers and consumers. We assume that*

- *producers 1 and 2 respectively supply 5 and 7 homogeneous commodities per week,*
- *each producer wants to sell (to brokers) primarily as many commodities as possible, and secondly the same amounts as possible,*
- *each broker must sell all commodities he/she bought, within this condition each wants to primarily buy and sell as many commodities as possible, and secondly to buy from producers and sell to consumers the same amounts as possible,*
- *each consumer can buy up to 5 commodities; within this limit each wants to buy primarily as many commodities as possible, and secondly the same amounts as possible.*

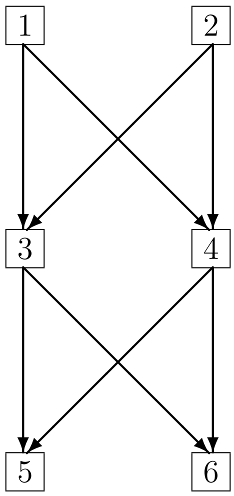

Figure 1: An example of supply chain network: 1 and 2 are producers, 3 and 4 brokers, and 5 and 6 consumers.

Let x_{ab} *be a variable representing the number of commodities traded between agents a and b. In order to describe preferences of the producers and consumers, we adopt the following* $M^\natural$*-concave function* $g^\alpha : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$ *defined for integer* α*:*

$$g^\alpha(y_1, y_2) = \begin{cases} 0 & (y_1, y_2 \geq 0,\ y_1 + y_2 \leq \alpha) \\ -\infty & (\textit{otherwise}) \end{cases} \qquad (y_1, y_2 \in \mathbb{Z}).$$

By using g^α *and a sufficiently small positive value* ε*, we can describe preferences of agents* 1*,* 2*,* 5 *and* 6 *by using the following functions:*

$$\begin{aligned} f_1(x_{13}, x_{14}) &= g^5(x_{13}, x_{14}) + (x_{13} + x_{14}) - \varepsilon(x_{13}^2 + x_{14}^2) \quad (x_{13}, x_{14} \in \mathbb{Z}), \\ f_2(x_{23}, x_{24}) &= g^7(x_{23}, x_{24}) + (x_{23} + x_{24}) - \varepsilon(x_{23}^2 + x_{24}^2) \quad (x_{23}, x_{24} \in \mathbb{Z}), \\ f_5(x_{35}, x_{45}) &= g^5(x_{35}, x_{45}) + (x_{35} + x_{45}) - \varepsilon(x_{35}^2 + x_{45}^2) \quad (x_{35}, x_{45} \in \mathbb{Z}), \\ f_6(x_{36}, x_{46}) &= g^5(x_{36}, x_{46}) + (x_{36} + x_{46}) - \varepsilon(x_{36}^2 + x_{46}^2) \quad (x_{36}, x_{46} \in \mathbb{Z}), \end{aligned}$$

in which the second terms show preference to trade as many commodities as possible, and the third terms that they trade with the two brokers as the same amounts as possible. Functions f_1*,* f_2*,* f_5 *and* f_6 *are* $M^\natural$*-concave (i.e., twisted* $M^\natural$*-concave) because these functions are the sums of* $M^\natural$*-concave function* g^α *and separable concave functions (see Example 3.3.) In order to describe preferences of the brokers, we use the following function* $h : \mathbb{Z}^4 \to \mathbb{R} \cup \{-\infty\}$*:*

$$h(y_1^-, y_2^-, y_5^+, y_6^+) = \begin{cases} 0 & (y_1^-, y_2^-, y_5^+, y_6^+ \geq 0,\ y_1^- + y_2^- = y_5^+ + y_6^+) \\ -\infty & (\textit{otherwise}) \end{cases} \qquad (y_1^-, y_2^-, y_5^+, y_6^+ \in \mathbb{Z}),$$

where h is the twisted $M^\natural$*-concave shown in Example 3.1. By using h, we can describe preferences of agents* 3 *and* 4 *by using the following functions:*

$$\begin{aligned} f_3(x_{13}, x_{23}, x_{35}, x_{36}) &= h(x_{13}, x_{23}, x_{35}, x_{36}) + (x_{13} + x_{23} - \varepsilon(x_{13}^2 + x_{23}^2)) + (x_{35} + x_{36} - \varepsilon(x_{35}^2 + x_{36}^2)) \\ &\qquad (x_{13}, x_{23}, x_{35}, x_{36} \in \mathbb{Z}), \\ f_4(x_{14}, x_{24}, x_{45}, x_{46}) &= h(x_{14}, x_{24}, x_{45}, x_{46}) + (x_{14} + x_{24} - \varepsilon(x_{14}^2 + x_{24}^2)) + (x_{45} + x_{46} - \varepsilon(x_{45}^2 + x_{46}^2)) \\ &\qquad (x_{14}, x_{24}, x_{45}, x_{46} \in \mathbb{Z}). \end{aligned}$$

In these functions, h enforces the conservation between input and output commodities, and the second and third terms describe the brokers' preference. By Examples 3.1 and 3.3, f_3 and f_4 are twisted $M^\natural$-concave.

In the instance, let us consider the following allocations x and x^*.

	(1,3)	(1,4)	(2,3)	(2,4)	(3,5)	(3,6)	(4,5)	(4,6)
x	2	2	2	3	2	2	2	3
x^*	2	3	2	3	2	2	3	3
$\overline{z}$	$+\infty$	$+\infty$	2	3	2	2	3	3
$\underline{z}$	$+\infty$	3	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

Allocation x^* is chain stable because x^*, $\overline{z}$ and $\underline{z}$ defined as above satisfy (4.1) and (4.2) in Lemma 4.1. On the other hand, x is not chain stable, because $P = (1,(1,3),3,(3,5),5)$ is a blocking path for x. //

In general, conditions (4.1) and (4.2) might not be a necessary condition for which a feasible allocation is chain stable. In the sequel, we deal with the case where each value function f_v satisfies SSS-CSC property on $(\delta^+(v), \delta^-(v))$. In this case, (4.1) and (4.2) are a necessary condition for the chain stability of a feasible allocation x^*. To show this, we will first show that the following procedure decides whether a given individually rational allocation x^* is chain stable or not.

STABILITY_CHECK

input:

an individually rational allocation $x^* \in \mathbb{Z}^E$, i.e. $x^*_{\delta(v)} \in \arg\max\{f_v(y) \mid y \le x^*_{\delta(v)}\}$ for $v \in V$;

output:

$\overline{z}, \underline{z} \in (\mathbb{Z} \cup \{+\infty\})^E$ satisfying (4.1) and (4.2) if x^* is stable; otherwise a blocking path;

Step 0.

$O := E$, $I^0 := I^1 := \emptyset$ and $k := 1$;

For $v \in V$ and $e = vw \in O \cap \delta^+(v)$ with $f_v(x^*_{\delta(v)}) < \max\{f_v(y) \mid y \le (x^* + \chi_e)_{\delta(v)}\}$,

- $I^k := I^k \cup \{e\}$, $O := O \setminus \{e\}$;
- $p(e) = nil$,
- if $f_w(x^*_{\delta(w)}) < \max\{f_w(y) \mid y \le (x^* + \chi_e)_{\delta(w)}\}$ then output blocking path $P = (v, e, w)$ for x^* and stop,

$k := 2$, $I^k := I^{k-1}$;

Step 1.

For $v \in V$ and $e = vw \in O \cap \delta^+(v)$ with $f_v(x^*_{\delta(v)}) < \max\{f_v(y) \mid y \le (x^* + \chi_{I^{k-1} \cap \delta^-(v)} + \chi_e)_{\delta(v)}\}$,

- $I^k := I^k \cup \{e\}$, $O := O \setminus \{e\}$;
- select $e' \in (I^{k-1} \setminus I^{k-2}) \cap \delta^-(v)$ with
  $f_v(x^*_{\delta(v)}) < \max\{f_v(y) \mid y \le (x^* + \chi_{e'} + \chi_e)_{\delta(v)},\ y(e') = x^*(e')+1,\ y(e) = x^*(e)+1\}$,
  (we will show the existence of e' later) and set $p(e) := e'$,
- if $f_w(x^*_{\delta(w)}) < \max\{f_w(y) \mid y \le (x^* + \chi_e)_{\delta(w)}\}$ then construct and output a path P of length k by backtracking $e, p(e), p(p(e)), \cdots$, and stop,

If $I^k = I^{k-1}$ then go to Step 2; otherwise $k := k+1$, $I^k := I^{k-1}$ and continue Step 1;

Step 2.

Output $\overline{z}$ and $\underline{z}$ defined by

$$\overline{z}(e) := \begin{cases} x^*(e) & (e \in I^k) \\ +\infty & (e \in O), \end{cases} \qquad \underline{z}(e) := \begin{cases} +\infty & (e \in I^k) \\ x^*(e) & (e \in O), \end{cases}$$

and stop.

Recall that each value function f_v for $v \in V$ satisfies SSS-CSC property on $(\delta^+(v), \delta^-(v))$. Since set O is reduced by one element in Steps 0 and 1, Stability_Check terminates in finite time. At the end of each iteration in Steps 0 and 1 of Stability_Check, O and I^k form a partition of E, i.e., $E = O \cup I^k$ and $O \cap I^k = \emptyset$, which imply that $\overline{z}$ and $\underline{z}$ in Step 2 are well-defined and satisfy (4.2).

If Stability_Check stops in Step 0 then $P = (v, e, w)$ is obviously a blocking path for x^*. Hereafter, we consider the case where Step 1 is executed. Just after Step 0, for each $w \in V$, we have

$$x^*_{\delta(w)} \in \max\{f_w(y) \mid y \leq (x^* + \chi_e)_{\delta(w)}\}$$

for all $e \in I^1 \cap \delta^-(w)$. By repeatedly using Lemma 2.1, we obtain that

$$x^*_{\delta(w)} \in \mathrm{argmax}\{f_w(y) \mid y \leq (x^* + \chi_{I^1 \cap \delta^-(w)})_{\delta(w)}\} \quad (w \in V).$$

For $k \geq 2$, just before the iteration for k in Step 1, we assume that

$$x^*_{\delta(v)} \in \mathrm{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^{k-1} \cap \delta^-(v)})_{\delta(v)}\} \quad (v \in V) \tag{4.3}$$

and consider the iteration for k in Step 1. We may abbreviate $(\chi_e)_{\delta(v)}$ as χ_e if $e \in \delta(v)$, for example, for $y \in \mathbb{Z}^{\delta(v)}$, we denote $y + (\chi_e)_{\delta(v)}$ by $y + \chi_e$. Let v and e be a vertex and an edge such that $e = vw \in O \cap \delta^+(v)$ and

$$f_v(x^*_{\delta(v)}) < \max\{f_v(y) \mid y \leq (x^* + \chi_{I^{k-1} \cap \delta^-(v)} + \chi_e)_{\delta(v)}\}. \tag{4.4}$$

We claim that there is $e' \in (I^{k-1} \setminus I^{k-2}) \cap \delta^-(v)$ with

$$f_v(x^*_{\delta(v)}) < \max\{f_v(y) \mid y \leq (x^* + \chi_{e'} + \chi_e)_{\delta(v)},\ y(e') = x^*(e') + 1,\ y(e) = x^*(e) + 1\}. \tag{4.5}$$

By the description of Stability_Check,

$$x^*_{\delta(v)} \in \mathrm{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^{k-2} \cap \delta^-(v)} + \chi_e)_{\delta(v)}\} \tag{4.6}$$

must hold. From (4.6) and (b) in SSS-CSC property for

$$z_1 = (x^* + \chi_{I^{k-1} \cap \delta^-(v)} + \chi_e)_{\delta(v)}, \quad z_2 = (x^* + \chi_{I^{k-2} \cap \delta^-(v)} + \chi_e)_{\delta(v)}, \quad x_2 = x^*_{\delta(v)},$$

there exists $y^* \in \mathrm{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^{k-1} \cap \delta^-(v)} + \chi_e)_{\delta(v)}\}$ such that

$$(x^* + \chi_{I^{k-2} \cap \delta^-(v)} + \chi_e)_{\delta^-(v)} \wedge y^*_{\delta^-(v)} \leq x^*_{\delta^-(v)},$$

which together with (4.4) implies

$$y^*_{I^{k-2} \cap \delta^-(v)} \leq x^*_{I^{k-2} \cap \delta^-(v)}, \quad y^*_{I^{k-1} \cap \delta^-(v)} \not\leq x^*_{I^{k-1} \cap \delta^-(v)}.$$

Let $S = \{e'' \in (I^{k-1} \setminus I^{k-2}) \cap \delta^-(v) \mid y^*(e'') = x^*(e'') + 1\}$. Then, there exists $e' \in S$ such that

$$x^*_{\delta(v)} \notin \mathrm{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^{k-2} \cap \delta^-(v)} + \chi_{e'} + \chi_e)_{\delta(v)}\},$$

since otherwise, Lemma 2.1 implies $x^*_{\delta(v)} \in \mathrm{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^{k-2} \cap \delta^-(v)} + \chi_S + \chi_e)_{\delta(v)}\}$ contradicting (4.4). In the same way as above, (b) in SSS-CSC property for (4.6) implies that there exists y' such that

$$y' \in \mathrm{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^{k-2} \cap \delta^-(v)} + \chi_{e'} + \chi_e)_{\delta(v)}\},$$
$$f_v(y') > f_v(x^*_{\delta(v)}), \quad y'_{I^{k-2} \cap \delta^-(v)} \leq x^*_{I^{k-2} \cap \delta^-(v)}.$$

It follows from (4.3) and (4.6) that $y'(e) = x^*(e) + 1$ and $y'(e') = x^*(e') + 1$, and hence, e' has (4.5).

We assume that there exist $v, w \in V$ and $e = vw \in O \cap \delta^+(v)$ such that $f_v(x^*_{\delta(v)}) < \max\{f_v(y) \mid y \leq (x^* + \chi_{I^{k-1} \cap \delta^-(v)} + \chi_e)_{\delta(v)}\}$ and $f_w(x^*_{\delta(w)}) < \max\{f_w(y) \mid y \leq (x^* + \chi_e)_{\delta(w)}\}$ in Step 1. By backtracking edges from e by using p, we can construct a path $P = (v_0, e_1, v_1, \ldots, e_{k-1} = p(e), v_{k-1} = v, e_k = e, v_k = w)$ such that

$$f_{v_0}(x^*_{\delta(v_0)}) < \max\{f_{v_0}(y) \mid y \leq (x + \chi_{e_1})_{\delta(v_0)}\},$$
$$f_{v_i}(x^*_{\delta(v_i)}) < \max\{f_{v_i}(y) \mid y \leq (x^* + \chi_{e_i} + \chi_{e_{i+1}})_{\delta(v_i)},\ y(e_i) = x^*(e_i) + 1,\ y(e_{i+1}) = x^*(e_{i+1}) + 1\} \quad (i = 1, \ldots, k-1),$$
$$f_{v_k}(x^*_{\delta(v_k)}) < \max\{f_{v_k}(y) \mid y \leq (x^* + \chi_{e_k})_{\delta(v_k)}\}.$$

Hence P is a blocking path for x^*.

We next consider the situation just after the iteration for k (and before updating k) in Step 1. In the same way as the proof for the case just after Step 0, by repeatedly using Lemma 2.1, we can show

$$x^*_{\delta(v)} \in \operatorname{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^k \cap \delta^-(v)})_{\delta(v)}\} \quad (v \in V). \tag{4.7}$$

By induction on k, it is shown that (4.3) holds just before the iteration for k in Step 1 for $k = 2, 3, \ldots$.

We finally consider the case where Step 2 is executed. Just before Step 2, (4.7) and

$$x^*_{\delta(v)} \in \operatorname{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^k \cap \delta^-(v)} + \chi_e)_{\delta(v)}\}$$

hold for $v \in V$ and $e \in O \cap \delta^+(v)$. By repeatedly using Lemma 2.1, we have

$$x^*_{\delta(v)} \in \operatorname{argmax}\{f_v(y) \mid y \leq (x^* + \chi_{I^k \cap \delta^-(v)} + \chi_{O \cap \delta^+(v)})_{\delta(v)}\} \quad (v \in V). \tag{4.8}$$

For each $v \in V$, by setting $z_1 = (\overline{z}_{\delta^+(v)}, \underline{z}_{\delta^-(v)})$, $z_2 = ((x^* + \chi_{O \cap \delta^+(v)})_{\delta^+(v)}, (x^* + \chi_{I^k \cap \delta^-(v)})_{\delta^-(v)})$, condition (4.8) is rewritten by $x^*_{\delta(v)} \in \operatorname{argmax}\{f_v(y) \mid y \leq z_2\}$ and $z_1 \geq z_2$ holds. Furthermore, for each $e \in \delta(v)$, $x^*(e) < z_2(e)$ holds whenever $z_2(e) < z_1(e)$. Thus, Lemma 2.2 guarantees $x^*_{\delta(v)} \in \operatorname{argmax}\{f_v(y) \mid y \leq z_1\}$ for all $v \in V$, and hence, (4.1).

Summing up the above discussion together with Lemma 4.1, we obtain a characterization of the chain stability in our model.

Theorem 4.1. *Assume that for each v, a value function $f_v : \mathbb{Z}^{\delta(v)} \to \mathbb{R} \cup \{-\infty\}$ satisfies Assumption (A) and SSS-CSC property on $(\delta^+(v), \delta^-(v))$. An individually rational allocation $x^* \in \mathbb{Z}^E$ is chain stable if and only if there exist $\overline{z}, \underline{z} \in (\mathbb{Z} \cup \{+\infty\})^E$ satisfying (4.1) and (4.2), that is,*

$$x^*_{\delta(v)} \in \operatorname{argmax}\{f_v(y) \mid y_{\delta^+(v)} \leq \overline{z}_{\delta^+(v)},\ y_{\delta^-(v)} \leq \underline{z}_{\delta^-(v)}\} \quad (v \in V),$$
$$\overline{z}(e) = +\infty \quad \text{or} \quad \underline{z}(e) = +\infty \quad (e \in E).$$

In the next section, we give an algorithm which always finds x^*, $\overline{z}$ and $\underline{z}$ satisfying (4.1) and (4.2). Hence we obtain our main theorem.

Theorem 4.2. *Let $G = (V, E)$ be an acyclic directed graph and $f_v : \mathbb{Z}^{\delta(v)} \to \mathbb{R} \cup \{-\infty\}$ be a value function with (A) and SSS-CSC property on $(\delta^+(v), \delta^-(v))$ for $v \in V$. Then there always exists a chain stable allocation x^*.*

5. An Algorithm Finding A Chain Stable Allocation

Let $G = (V, E)$ be an acyclic directed graph. We assume that the elements of V are renamed as $V = \{1, 2, \ldots, n\}$ in topological order, i.e., so that $i < j$ for all $(i, j) \in E$. We deal with the case where each value function f_v for $v \in V$ satisfies SSS-CSC property on $(\delta^+(v), \delta^-(v))$, and also assume that the effective domain $\mathrm{dom}\, f_v$ is contained in the box $\{y \in \mathbb{R}^{\delta(v)} \mid \mathbf{0} \leq y \leq (L{-}1)\mathbf{1}\}$ for some integer L. This section gives an algorithm finding a chain stable allocation $x^* \in \mathbb{Z}^E$ together with $\overline{z}, \underline{z} \in (\mathbb{Z} \cup \{+\infty\})^E$ satisfying (4.1) and (4.2). The algorithm described below manipulates two allocations x^s and x^b with $x^b \leq x^s$. For each $e = (i, j) \in E$, $x^s(e)$ is interpreted as an offer of the number of commodities by seller i and $x^b(e)$ as a demand on the number of commodities by buyer j. That is, $x^b \leq x^s$ means that demands of buyers are less than or equal to offers of sellers. The algorithm updates the subvectors $x^s_{\delta^+(i)}$ and $x^b_{\delta^-(i)}$ for each $i = 1, 2, \ldots, n$ preserving $x^b \leq x^s$ until $x^b = x^s$. The algorithm is described as follows.

FIND_STABLEALLO
input: value functions f_v with SSS-CSC property on $(\delta^+(v), \delta^-(v))$ for $v \in V$;
output: a feasible allocation $x^* \in \mathbb{Z}^E$ and $\overline{z}, \underline{z} \in (\mathbb{Z} \cup \{+\infty\})^E$ satisfying (4.1) and (4.2);

$x^s := \overline{z} := (L, L, \ldots, L)$, $x^b := \underline{z} := (0, 0, \ldots, 0)$;
repeat {
  for $i := 1$ **to** n **do**
    let $(x^s_{\delta^+(i)}, x^b_{\delta^-(i)})$ be an element in $\mathrm{argmax}\left\{ f_i(y) \;\middle|\; \begin{array}{l} x^b_{\delta^+(i)} \leq y_{\delta^+(i)} \leq \overline{z}_{\delta^+(i)}, \\ y_{\delta^-(i)} \leq x^s_{\delta^-(i)} \end{array} \right\}$;
  for each $e \in E$ with $x^s(e) > x^b(e)$ **do**
    $\overline{z}(e) := x^b(e)$ and $\underline{z}(e) := +\infty$;
} **until** $x^s = x^b$;
replace all components of $\overline{z}$ with L by $+\infty$;
return $(x^s, \overline{z}, \underline{z} \vee x^s)$.

It should be noted here that FIND_STABLEALLO terminates after at most nL repeat-iterations, because $\sum_{e \in E} \overline{z}(e)$ strictly decreases during each repeat-iteration. Let $x^{s(k)}$, $x^{b(k)}$, $\overline{z}^{(k)}$ and $\underline{z}^{(k)}$ denote x^s, x^b, $\overline{z}$ and $\underline{z}$ obtained after the k-th repeat-iteration for $k = 1, 2, \ldots, t$, where t is the last to get the outputs. For convenience, let us assume that $x^{s(0)}$, $x^{b(0)}$, $\overline{z}^{(0)}$ and $\underline{z}^{(0)}$ are the vectors defined before the repeat-loop. Since the vertices are in topological order, at the k-th repeat-iteration, $x^{s(k)}$ and $x^{b(k)}$ are determined by

$$(x^{s(k)}_{\delta^+(i)}, x^{b(k)}_{\delta^-(i)}) \in \mathrm{argmax}\left\{ f_i(y) \mid x^{b(k-1)}_{\delta^+(i)} \leq y_{\delta^+(i)} \leq \overline{z}^{(k-1)}_{\delta^+(i)},\ y_{\delta^-(i)} \leq x^{s(k)}_{\delta^-(i)} \right\} \quad (i \in V). \quad (5.1)$$

Update (5.1) can be interpreted as follows: broker i determines offers $x^{s(k)}_{\delta^+(i)}$ on the numbers of commodities to sell, and demands $x^{b(k)}_{\delta^-(i)}$ on the commodities to buy, so that his/her utility is maximized under the conditions:

- offers on the number of commodities for sale are greater than or equal to the demands $x^{b(k-1)}_{\delta^+(i)}$ made by consumers in the previous iteration, and less than or equal to the upper bounds $\overline{z}^{(k-1)}_{\delta^+(i)}$ determined in the previous iteration,
- demands on buying commodities are less than or equal to the current offers $x^{s(k)}_{\delta^-(i)}$ made in the same iteration.

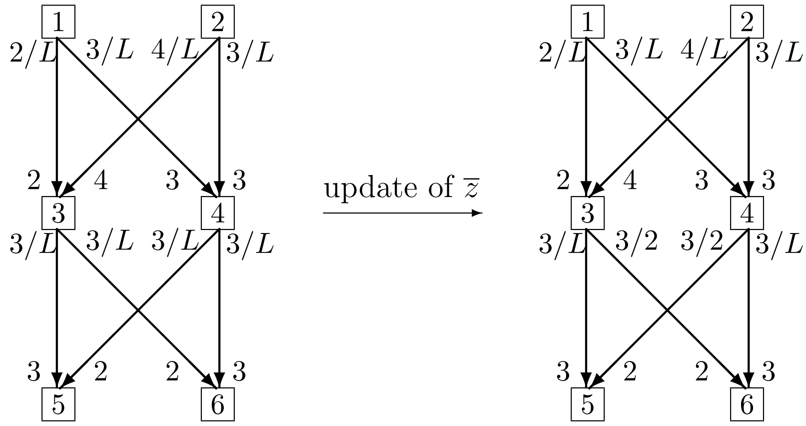

Figure 2: Movement of FIND_STABLEALLO at the first iteration.

The update of $\overline{z}$ can be interpreted as follows: for $e = (i, j) \in E$, if the demand $x^b(e)$ by buyer j is less than the offer $x^s(e)$ by seller i, then seller i resignedly accepts this, and updates $\overline{z}(e)$ by $x^b(e)$.

Example 5.1. *We apply* FIND_STABLEALLO *to the instance of Example 4.1. In the first iteration of* FIND_STABLEALLO, *we assume that each agent selects the allocation in the left-hand side figure in Figure 2 among the best allocations, where* $2/L$ *and* 2 *beside edge* $(1, 3)$ *mean that* $x^s_{13} = 2$, $\overline{z}_{13} = L$ *and* $x^b_{13} = 2$, *respectively before updates of* $\overline{z}$ *and* $\underline{z}$. *After updates of* $\overline{z}$ *and* $\underline{z}$ *in the first iteration, we obtain the allocation in the right-hand side of Figure 2. The vectors* x^s, $\overline{z}$ *and* x^b *given in Figure 3 show those obtained at the end of iterations 2 through 4. When we have flexibility in choice of* x^s *and* x^b, *we select those that are shown in the figures. For instance, in the third iteration, we choose* $(x^b_{14}, x^b_{24}) = (3, 2)$ *as the values for broker* 4, *although both* $(2, 3)$ *and* $(3, 2)$ *are candidates. We finally obtain the following outputs.*

	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,5)$	$(3,6)$	$(4,5)$	$(4,6)$
x^*	2	3	3	2	3	2	2	3
$\overline{z}$	$+\infty$	$+\infty$	3	2	3	2	2	3
$\underline{z}$	2	3	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

These satisfy (4.1) and (4.2), and hence, x^* *is a chain stable allocation. //*

We will show that the outputs $(x^*, \overline{z}, \underline{z})$ of FIND_STABLEALLO satisfy (4.1) and (4.2).

Lemma 5.1. *For each* $k = 1, 2, \ldots, t$, *we have*

$$(x^{s(k)}_{\delta^+(i)}, x^{b(k)}_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \mid y_{\delta^+(i)} \le \overline{z}^{(k-1)}_{\delta^+(i)},\ y_{\delta^-(i)} \le (\underline{z}^{(k)} \vee x^{s(k)})_{\delta^-(i)} \right\} \quad (i \in V). \tag{5.2}$$

Proof. We show (5.2) by induction on k. We first consider the case $k = 1$. By (5.1) and the definition of $x^{b(0)}$, $\overline{z}^{(0)}$ and $\underline{z}^{(0)}$, we have

$$(x^{s(1)}_{\delta^+(i)}, x^{b(1)}_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \mid y_{\delta^+(i)} \le \overline{z}^{(0)}_{\delta^+(i)},\ y_{\delta^-(i)} \le x^{s(1)}_{\delta^-(i)} \right\} \quad (i \in V). \tag{5.3}$$

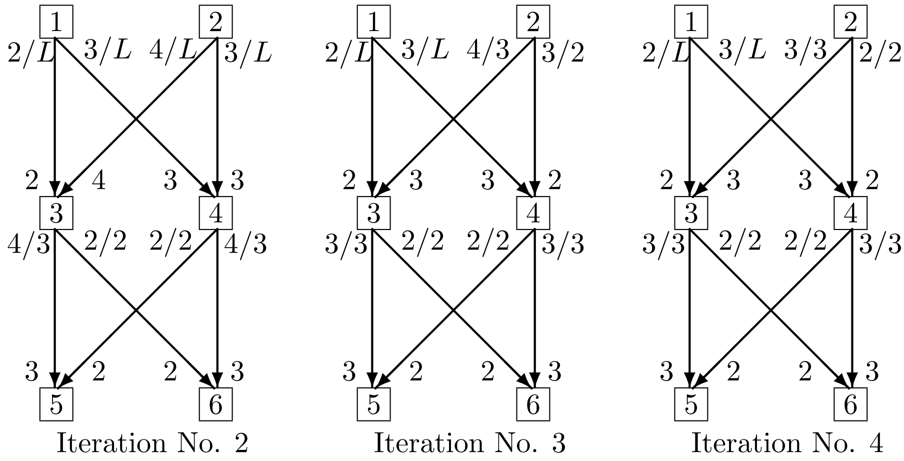

Figure 3: Movement of FIND_STABLEALLO after the first iteration.

For each $e \in \delta^-(i)$, if $x^{s(1)}(e) > x^{b(1)}(e)$ then $\underline{z}^{(1)}(e) \vee x^{s(1)}(e) = +\infty$; otherwise $\underline{z}^{(1)}(e) \vee x^{s(1)}(e) = x^{s(1)}(e)$. By Lemma 2.2 and (5.3), we have

$$(x^{s(1)}_{\delta^+(i)}, x^{b(1)}_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \mid y_{\delta^+(i)} \le \overline{z}^{(0)}_{\delta^+(i)},\ y_{\delta^-(i)} \le (\underline{z}^{(1)} \vee x^{s(1)})_{\delta^-(i)} \right\} \quad (i \in V),$$

which is (5.2) for $k = 1$.

We next assume that for some l with $1 \le l < t$, (5.2) holds, and show (5.2) for $l+1$. That is, $x^{s(l)}$ and $x^{b(l)}$ satisfy

$$(x^{s(l)}_{\delta^+(i)}, x^{b(l)}_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \mid y_{\delta^+(i)} \le \overline{z}^{(l-1)}_{\delta^+(i)},\ y_{\delta^-(i)} \le (\underline{z}^{(l)} \vee x^{s(l)})_{\delta^-(i)} \right\} \quad (i \in V).$$

Since $\overline{z}^{(l-1)} \ge \overline{z}^{(l)}$, by (a) in SSS-CSC property, there exist $\widehat{x}^s, \widehat{x}^b \in \mathbb{Z}^E$ such that

$$\begin{array}{l} (\widehat{x}^s_{\delta^+(i)}, \widehat{x}^b_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \mid y_{\delta^+(i)} \le \overline{z}^{(l)}_{\delta^+(i)},\ y_{\delta^-(i)} \le (\underline{z}^{(l)} \vee x^{s(l)})_{\delta^-(i)} \right\}, \\ (\overline{z}^{(l)} \wedge x^{s(l)})_{\delta^+(i)} \le \widehat{x}^s_{\delta^+(i)} \quad \text{and} \quad \widehat{x}^b_{\delta^-(i)} \le x^{b(l)}_{\delta^-(i)} \end{array} \quad (i \in V).$$

By the modification of $\overline{z}$ in FIND_STABLEALLO, $\overline{z}^{(l)} \wedge x^{s(l)} = x^{b(l)}$ holds, and furthermore, by the above conditions, $x^{b(l)} \le \widehat{x}^s$ holds. It follows from the constraint $x^{b(k-1)}_{\delta^+(i)} \le y_{\delta^+(i)}$ in (5.1) that $x^{b(l)} \le x^{s(l+1)}$ holds. This together with $\widehat{x}^b \le x^{b(l)}$ implies $\widehat{x}^b \le x^{s(l+1)}$. By (b) in SSS-CSC property with respect to $(\overline{z}^{(l)}_{\delta^+(i)}, (\underline{z}^{(l)} \vee x^{s(l)})_{\delta^-(i)})$ and $(\overline{z}^{(l)}_{\delta^+(i)}, (\underline{z}^{(l)} \vee x^{s(l)} \vee x^{s(l+1)})_{\delta^-(i)})$ for each $i \in V$, there exist $\widetilde{x}^s, \widetilde{x}^b \in \mathbb{Z}^E$ such that

$$\begin{array}{l} (\widetilde{x}^s_{\delta^+(i)}, \widetilde{x}^b_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \;\middle|\; \begin{array}{l} y_{\delta^+(i)} \le \overline{z}^{(l)}_{\delta^+(i)}, \\ y_{\delta^-(i)} \le (\underline{z}^{(l)} \vee x^{s(l)} \vee x^{s(l+1)})_{\delta^-(i)} \end{array} \right\}, \\ ((\underline{z}^{(l)} \vee x^{s(l)}) \wedge \widetilde{x}^b)_{\delta^-(i)} \le \widehat{x}^b_{\delta^-(i)} \quad \text{and} \quad \widehat{x}^s_{\delta^+(i)} \le \widetilde{x}^s_{\delta^+(i)} \end{array} \quad (i \in V).$$

We have $x^{b(l)} \le \widetilde{x}^s$ because of $x^{b(l)} \le \widehat{x}^s$ and $\widehat{x}^s \le \widetilde{x}^s$. We also have $\widetilde{x}^b \le x^{s(l+1)}$ from $\widehat{x}^b \le x^{s(l+1)}$, $(\underline{z}^{(l)} \vee x^{s(l)}) \wedge \widetilde{x}^b \le \widehat{x}^b$ and $\widetilde{x}^b \le (\underline{z}^{(l)} \vee x^{s(l)}) \vee x^{s(l+1)}$ because, for each $e \in E$, if $\widetilde{x}^b(e) \le \widehat{x}^b(e)$ then the first inequality implies $\widetilde{x}^b(e) \le x^{s(l+1)}(e)$; otherwise, the second inequality guarantees $(\underline{z}^{(l)} \vee x^{s(l)})(e) \leq \widehat{x}^b(e) < \widetilde{x}^b(e)$ which together with the third inequality implies $\widetilde{x}^b(e) \leq x^{s(l+1)}(e)$. Thus, $\widetilde{x}^s$ and $\widetilde{x}^b$ satisfy

$$(\widetilde{x}^s_{\delta^+(i)}, \widetilde{x}^b_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \mid x^{b(l)}_{\delta^+(i)} \leq y_{\delta^+(i)} \leq \overline{z}^{(l)}_{\delta^+(i)},\ y_{\delta^-(i)} \leq x^{s(l+1)}_{\delta^-(i)} \right\} \quad (i \in V). \quad (5.4)$$

By (5.1), $(x^{s(l+1)}_{\delta^+(i)}, x^{b(l+1)}_{\delta^-(i)})$ is also a maximizer of (5.4), and hence, $x^{s(l+1)}$ and $x^{b(l+1)}$ satisfy

$$(x^{s(l+1)}_{\delta^+(i)}, x^{b(l+1)}_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \mid \begin{array}{l} y_{\delta^+(i)} \leq \overline{z}^{(l)}_{\delta^+(i)}, \\ y_{\delta^-(i)} \leq (\underline{z}^{(l)} \vee x^{s(l)} \vee x^{s(l+1)})_{\delta^-(i)} \end{array} \right\} \quad (i \in V).$$

By the modification of $\underline{z}$, we have $x^{s(l+1)}(e) > x^{b(l+1)}(e)$ if $\underline{z}^{(l)}(e) < \underline{z}^{(l+1)}(e)$. Hence, Lemma 2.2 guarantees that $x^{s(l+1)}$ and $x^{b(l+1)}$ satisfy

$$(x^{s(l+1)}_{\delta^+(i)}, x^{b(l+1)}_{\delta^-(i)}) \in \operatorname{argmax}\left\{ f_i(y) \mid \begin{array}{l} y_{\delta^+(i)} \leq \overline{z}^{(l)}_{\delta^+(i)}, \\ y_{\delta^-(i)} \leq (\underline{z}^{(l+1)} \vee x^{s(l)} \vee x^{s(l+1)})_{\delta^-(i)} \end{array} \right\} \quad (i \in V).$$

Since $x^{b(l+1)} \leq x^{s(l+1)}$, we can replace the upper bound $(\underline{z}^{(l+1)} \vee x^{s(l)} \vee x^{s(l+1)})$ by $(\underline{z}^{(l+1)} \vee x^{s(l+1)})$ in the above formulation preserving the optimality of $x^{s(l+1)}$ and $x^{b(l+1)}$, and hence, (5.2) for $l+1$ holds. □

Theorem 5.1. *The outputs* $(x^*, \overline{z}, \underline{z})$ *of* FIND_STABLEALLO *satisfy (4.1) and (4.2), that is,* x^* *is a chain stable allocation.*

Proof. Since $x^* = x^{s(t)} = x^{b(t)}$ and $\overline{z}^{(t)} = \overline{z}^{(t-1)}$ hold at the end of the t-th repeat-iteration, from Lemma 5.1, we have

$$x^*_{\delta(i)} \in \operatorname{argmax}\left\{ f_i(y) \mid y_{\delta^+(i)} \leq \overline{z}^{(t)}_{\delta^+(i)},\ y_{\delta^-(i)} \leq (\underline{z}^{(t)} \vee x^{s(t)})_{\delta^-(i)} \right\} \quad (i \in V),$$

which is equivalent to (4.1) because the replacement of elements of $\overline{z}^{(t)}$ has no effect on the optimality of x^* and $\underline{z} = \underline{z}^{(t)} \vee x^{s(t)}$. The way of modifying $\overline{z}^{(k)}$ and $\underline{z}^{(k)}$ obviously guarantees (4.2). Hence, Lemma 4.1 says that x^* is chain stable. □

6. Time Complexities of Algorithms

In this section, we discuss the time complexities of STABILITY_CHECK and FIND_STABLEALLO when each value function f_v is a twisted M$^\natural$-concave functions on $(\delta^+(v), \delta^-(v))$.

We assume that an acyclic directed graph $G = (V, E)$ with $n = |V|$ and $m = |E|$, and twisted M$^\natural$-concave functions f_v on $(\delta^+(v), \delta^-(v))$ with $\operatorname{dom} f_v \subseteq [\mathbf{0}, (L-1)\mathbf{1}]$ for $v \in V$ are given, and that the function value $f_v(y)$ for each $v \in V$ can be calculated in constant time for each point y.

The maximization problem of a twisted M$^\natural$-concave function in two algorithms can be executed by solving the M-convex function minimization problem. For a given M-convex function $f : \mathbb{Z}^d \to \mathbb{R} \cup \{+\infty\}$ with $D = ||\operatorname{dom} f||_\infty$, the minimization problem of f can be solved in $\mathrm{O}(d^3 \log(D/d))$-time by algorithms of Shioura [25] and Tamura [27], and in $\mathrm{O}((d^3 + d^2 \log(D/d))(\log(D/d)/\log d))$-time by an algorithm of Shioura [25]. In our context, for each twisted M$^\natural$-concave function f_v, a corresponding M-convex function has $|\delta(v)| + 1$ variables and at most $L \times |\delta(v)|$ diameter. Thus, the maximization problem

$$\max\left\{ f_i(y) \mid x^b_{\delta^+(i)} \leq y_{\delta^+(i)} \leq \overline{z}_{\delta^+(i)},\ y_{\delta^-(i)} \leq x^s_{\delta^-(i)} \right\}$$

in FIND_STABLEALLO, is solved in $\mathrm{O}(m^3 \log L)$-time or in $\mathrm{O}((m^3+m^2 \log L)(\log L/\log m))$-time. FIND_STABLEALLO terminates after at most nL repeat-iterations and twisted M$^\natural$-concave function maximization problems with $|\delta(v)|$ variables for $v \in V$ are solved in each repeat-iteration. Because time for solving n small problems is less than that for solving the problem with m variables, each iteration of FIND_STABLEALLO requires $\mathrm{O}(m^3 \log L)$-time or $\mathrm{O}((m^3+m^2 \log L)(\log L/\log m))$-time. Hence, the time complexity of FIND_STABLEALLO is $\mathrm{O}(nL(m^3 \log L))$ or $\mathrm{O}(nL(m^3+m^2 \log L)(\log L/\log m))$.

Lemma 6.1. *Given an instance $(G=(V,E),\{f_v \mid v \in V\})$ of our model with twisted M$^\natural$-concave value functions, one can find a chain stable allocation in $\mathrm{O}(nL(m^3 \log L))$-time or in $\mathrm{O}(nL(m^3+m^2 \log L)(\log L/\log m))$-time.*

We now analyze the time complexity of STABILITY_CHECK. The crucial parts of STABILITY_CHECK are the problems of deciding for $e=(v,w)$, whether the following inequalities hold or not:

$$f_v(x^*_{\delta(v)}) < \max\{f_v(y) \mid y \le (x^*+\chi_e)_{\delta(v)}\},$$
$$f_w(x^*_{\delta(w)}) < \max\{f_w(y) \mid y \le (x^*+\chi_e)_{\delta(w)}\},$$
$$f_v(x^*_{\delta^+(v)}) < \max\{f_v(y) \mid y \le (x^*+\chi_{I^{k-1}\cap\delta^-(v)}+\chi_e)_{\delta(v)}\}.$$

By Example 3.2, these can be decided by checking whether or not $x^*_{\delta(v)}$ is a maximizer of f for a (modified) twisted M$^\natural$concave function $f:\mathbb{Z}^{\delta(v)} \to \mathbb{R}\cup\{+\infty\}$. This can be accomplished in $\mathrm{O}(|\delta(v)|^2)$-time, because we have

$$x^*_{\delta(v)} \in \mathrm{argmax}\, f \iff f(x^*_{\delta(v)}) \ge f(\mathrm{tw}(\mathrm{tw}(x^*_{\delta(v)})+\chi_e-\chi_{e'})) \quad (e,e' \in \delta(v)\cup\{0\}),$$

from the optimality criterion for M$^\natural$-convex functions [20]. By solving these decision problems at most m times, either I_k strictly increases or STABILITY_CHECK terminates. Thus, time complexity of the crucial parts is $\mathrm{O}(m^3)$. The other parts do not effect the time complexity as below. In STABILITY_CHECK, we must also check whether or not

$$f_v(x^*_{\delta(v)}) < \max\{f_v(y) \mid y \le (x^*+\chi_{e'}+\chi_e)_{\delta(v)},\ y(e')=x^*(e')+1,\ y(e)=x^*(e)+1\}$$

holds. It is not difficult to show that this can be done by checking whether or not $f_v(x^*_{\delta(v)}) < f_v((x^*+\chi_{e'}+\chi_e)_{\delta(v)})$, by applying (M$^\natural$-EXC). Moreover, the feasibility of a vector in $\mathbb{Z}^E$ can be checked in $\mathrm{O}(n)$-time, and the individual rationality of a feasible allocation can be checked in $\mathrm{O}(nm^2)$ time by the above optimality criterion. Summing up the above arguments, we have the following lemma.

Lemma 6.2. *Given an instance $(G=(V,E),\{f_v \mid v \in V\})$ of our model with twisted M$^\natural$-concave value functions and $x^* \in \mathbb{Z}^E$, the chain stability of x^* can be checked in $\mathrm{O}(m^3)$-time.*

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