

## THE DYNAMIC PRICING FOR CALLABLE SECURITIES WITH MARKOV-MODULATED PRICES\*

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*Abstract* In this paper, we consider a model of valuing callable financial securities when the underlying asset price dynamic depends on a finite-state Markov chain. The callable securities enable both an issuer and an investor to exercise their rights to call. We formulate this problem as a coupled stochastic game for the optimal stopping problem with two stopping boundaries. Then, we show that there exists a unique optimal price of the callable contingent claim which is a unique fixed point of a contraction mapping. We derive analytical properties of optimal stopping rules of the issuer and the investor under general payoff functions by applying a contraction mapping approach. In particular, we derive specific stopping boundaries for the both players by specifying for the callable securities to be the callable American call and put options.

**Keywords:** Finance, optimal stopping, game option, markov chain, random environment, callable securities, fixed point

### 1. Introduction

The purpose of this paper is to develop a dynamic valuation framework for callable financial securities with general payoff function. Such examples of the callable financial security may include game options (Kifer [1], Kyprianou [2]), convertible bond (Yagi and Sawaki [3]), callable put and call options (Black and Scholes [4], Brennan and Schwartz [5], Geske and Johnson [6], McKean [7]). Kifer [8] provides a review of a research on optimal stopping games and discusses the work on Israeli (game) options and related derivatives securities. Most studies on these securities have focused on the pricing of the derivatives when the underlying asset price processes follow a Brownian motion defined on a single probability space. In other words the realizations of the price process come from the same source of the uncertainty over the planning horizon. However, the valuation of financial securities is subject to randomly changing environmental conditions that affect the price as well as payoff of the securities. The structural changes of the underlying asset prices are based on the macro-economic environment, fundamentals of the real economy and financial policies including international monetary cooperation. The term Markov-modulated dynamics is used to describe such the environmental changes. More specifically, the coefficients of price process are modulated by a finite-state Markov chains.

In Sato and Sawaki [9], we formulate the valuation model as a Markov decision process equipped with a contraction mapping. They show that there exists a unique price which is a fixed point of the contraction mapping. In addition, it is shown that there exists a pair of optimal stopping rules for the issuer and for the investor, and derive the value of the coupled game. Should the payoff functions be specified like options, some analytical properties of the optimal stopping rules and their values can be explored under the several assumptions. They derive the optimal stopping boundaries of the both of the issuer and the investor in

the cases of callable American put and call options, respectively. In this paper, we provide a numerical method to illustrate the optimal boundaries of callable American put option by using binomial tree model. It enables us to verify the analytical properties of optimal boundaries obtained in Sato and Sawaki [9].

The organization of our paper is as follows: In section 2, we formulate a discrete time valuation model for a callable contingent claim, which can be presented as a Markov decision process with two state variables, the underlying asset price and the state of the economy. And then we derive optimal policies and investigate their analytical properties by using contraction mappings. Section 3 discusses two special cases of the payoff functions to derive the specific stop and continue regions for callable put and call, respectively. In Section 4 we present numerical results for the American callable put option using binomial model. Finally, last section concludes the paper with further comments. It summarizes results of this paper and raises further directions for future research.

## 2. A Genetic Model of Callable-Puttable Financial Commodities

In this section we formulate the valuation of callable securities as an optimal stopping problem in discrete time. Let  $\mathcal{T}$  be the time index set  $\{0, 1, \dots\}$ . We consider a complete probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ , where  $\mathcal{P}$  is a real-world probability. We suppose that the uncertainties of an asset price depend on its fluctuation and the economic states which are described by the probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ . Let  $\{1, 2, \dots, N\}$  be the set of states of the economy and  $i$  or  $j$  denote one of these states. We denote  $Z := \{Z_t\}_{t \in \mathcal{T}}$  be the finite Markov chain with transition probability  $P_{ij} = \Pr\{Z_{t+1} = j \mid Z_t = i\}$ . We assume that the decision maker can completely observe the environmental state<sup>†</sup>. Let  $r$  be the market interest rate of the bank account. We suppose that the price dynamics  $B := \{B_t\}_{t \in \mathcal{T}}$  of the bank account is given by

$$B_t = B_{t-1}e^r, \quad B_0 = 1.$$

Let  $S := \{S_t\}_{t \in \mathcal{T}}$  be the asset price at time  $t$ . We suppose that  $\{X_t^i\}$  be a sequence of i.i.d. random variable having mean  $E[X_t^i] = \mu_i < \infty$  for all  $i$  with the probability distribution  $F_i(\cdot)$  and its parameters depend on the state of the economy modeled by  $Z$ . Here, the sequence  $\{X_t^i\}$  and  $\{Z_t\}$  are assumed to be independent. Then, the asset price is defined as

$$S_{t+1} = S_t X_t^i. \quad (2.1)$$

The Esscher transform is well-known tool to determine an equivalent martingale measure for the valuation of options in an incomplete market (Elliott et al. [12] and Ching et al. [15]). Ching et al. [15] define the regime-switching Esscher transform in discrete time and apply it to determine an equivalent martingale measure when the price dynamics is modeled by high-order Markov chain.

We define  $Y_t^i = \log X_t^i$  and  $Y := \{Y_t\}_{t \in \mathcal{T}}$ . By Jensen's inequality, we have  $E[Y_t^i] = E[\log X_t^i] \leq \log E[X_t^i] < \infty$  for all  $i$ . Let  $\mathcal{F}_t^Z$  and  $\mathcal{F}_t^Y$  denote the  $\sigma$ -algebras generated by the values of  $Z$  and  $Y$ , respectively. We set  $\mathcal{G} = \mathcal{F}_t^Z \vee \mathcal{F}_t^Y$  for  $t \in \mathcal{T}$ . We assume that  $\theta_t$

<sup>†</sup>In recent years, a *regime switching model* have gotten attention to represent the state-dependent price process in which the state of the economy is unobservable. The regime switching model is driven by a hidden Markov chain process and applied to option pricing problem (e.g. Naik [10], Guo [11], Elliott et al. [12], Guo and Zhang [13], Le and Wang [14], Ching et al. [15]). Although the unobservability of the state seems reasonable in practice, but the assumption is useful for analytical purposes. We leave the relaxing assumption for future work.

be a  $\mathcal{F}_T^Z$ -measurable random variable for each  $t = 1, 2, \dots$ . It is interpreted as the regime-switching Esscher parameter at time  $t$  conditional on  $\mathcal{F}_T^Z$ . Let  $M_Y(t, \theta_t)$  denote the moment generating function of  $Y_t^i$  given  $\mathcal{F}_T^Z$  under  $\mathcal{P}$ , that is,  $M_Y(t, \theta_t) := E[e^{\theta_t Y_t^i} | \mathcal{F}_T^Z]$ . We define  $\mathcal{P}^\theta$  as a equivalent martingale measure for  $\mathcal{P}$  on  $\mathcal{G}_T$  associated with  $(\theta_1, \theta_2, \dots, \theta_T)$ .

The next proposition follows from Ching et al. [15].

**Proposition 2.1.** *The discounted price process  $\{S_t/B_t\}_{t \in \mathcal{T}}$  is a  $(\mathcal{G}, \mathcal{P}^\theta)$ -martingale if and only if  $\theta_t$  satisfies*

$$\frac{M_Y(t+1, \theta_{t+1} + 1)}{M_Y(t+1, \theta_{t+1})} = e^r. \tag{2.2}$$

If the dynamics  $Y$  is governed by the following Markov-modulated binomial model:

$$P(Y_t^i = y) = \begin{cases} p(i), & \text{if } y = b(i), \\ 1 - p(i), & \text{if } y = a(i), \end{cases} \tag{2.3}$$

then the following proposition provides the Esscher transform of this process. For simplicity of notation, we write  $p_t$ ,  $a_t$  and  $b_t$  instead of  $p(i)$ ,  $a(i)$  and  $b(i)$ , respectively.

**Proposition 2.2.** *The Esscher transform of the Markov modulated binomial model with parameter  $p_t$  is again a binomial model with the parameter  $\frac{e^r - e^{a_t}}{e^{b_t} - e^{a_t}}$ .*

*Proof.* The moment generating function of  $Y_t^i$  defined by equations (2.3) is given by

$$M_Y(t+1, \theta_{t+1}) = (1 - p_{t+1})e^{a_{t+1}\theta_{t+1}} + p_{t+1}e^{b_{t+1}\theta_{t+1}}. \tag{2.4}$$

Thus, we have

$$\frac{M_Y(t+1, \theta_{t+1} + 1)}{M_Y(t+1, \theta_{t+1})} = \pi(\theta_{t+1})e^{b_{t+1}} + (1 - \pi(\theta_{t+1}))e^{a_{t+1}}, \tag{2.5}$$

where

$$\pi(\theta_{t+1}) = \frac{p_{t+1}e^{b_{t+1}\theta_{t+1}}}{(1 - p_{t+1})e^{a_{t+1}\theta_{t+1}} + p_{t+1}e^{b_{t+1}\theta_{t+1}}}. \tag{2.6}$$

It follows from equation (2.2) that  $\pi(\theta_t) = \frac{e^r - e^{a_t}}{e^{b_t} - e^{a_t}}$ . By Lemma 3.2 of Ching et al. [15], we have

$$M_Y(t, \rho | \theta) = \frac{M_Y(t, \theta_t + \rho)}{M_Y(t, \theta_t)}, \tag{2.7}$$

where  $M_Y(t, \rho | \theta)$  is the moment generating function of  $Y_t$  given  $\mathcal{F}_T^Z$  under  $\mathcal{P}^\theta$  evaluated at  $\rho$ . Substituting  $\pi(\theta_t) = \frac{e^r - e^{a_t}}{e^{b_t} - e^{a_t}}$  into equation (2.7), we obtain

$$M_Y(t, \rho | \theta) = \pi(\theta_t)e^{b_t \rho} + (1 - \pi(\theta_t))e^{a_t \rho}. \tag{2.8}$$

This proves the proposition. □

A callable contingent claim is a contract between an issuer I and an investor II addressing the asset with a maturity  $T$ . The issuer can choose a stopping time  $\sigma$  to call back the claim with the payoff function  $f_\sigma$  and the investor can also choose a stopping time  $\tau$  to exercise his/her right with the payoff function  $g_\tau$  at any time before the maturity. Should neither of

them stop before the maturity, the payoff is  $h_T$ . The payoff always goes from the issuer to the investor. Here, we assume

$$0 \leq g_t \leq h_t \leq f_t, \quad 0 \leq t < T$$

and

$$g_T = h_T. \quad (2.9)$$

The investor wishes to exercise the right to maximize the expected payoff. On the other hand, the issuer wants to call the contract to minimize the payment to the investor. Then, for any pair of the stopping times  $(\sigma, \tau)$ , define the payoff function by

$$R(\sigma, \tau) = f_\sigma 1_{\{\sigma < \tau \leq T\}} + g_\tau 1_{\{\tau < \sigma \leq T\}} + h_T 1_{\{\sigma \wedge \tau = T\}}. \quad (2.10)$$

When the initial asset price  $S_0 = s$ , our stopping problem becomes the valuation of

$$v_0(s, i) = \min_{\sigma \in \mathcal{J}_{0,T}} \max_{\tau \in \mathcal{J}_{0,T}} E_{s,i}^\theta [\beta^{\sigma \wedge \tau} R(\sigma, \tau)], \quad (2.11)$$

where  $\beta \equiv e^{-r}$ ,  $0 < \beta < 1$  is the discount factor,  $\mathcal{J}$  is the finite set of stopping times taking values in  $\{0, 1, \dots, T\}$ , and  $E^\theta[\cdot]$  is an expectation under  $\mathcal{P}^\theta$ . Since the asset price process follows a random walk, the payoff processes of  $g_t$  and  $f_t$  are both Markov types. We consider this optimal stopping problem as a Markov decision process. Let  $v_n(s, i)$  be the price of the callable contingent claim when the asset price is  $s$ , the state is  $i$  and  $n$  denote the number of periods remaining. Here, the trading period moves backward in time indexed by  $n = 0, 1, 2, \dots, T$ . It is easy to see that  $v_n(s, i)$  satisfies

$$\begin{aligned} v_{n+1}(s, i) &\equiv (\mathcal{U}v_n)(s, i) \\ &\equiv \min \left\{ f_{n+1}(s, i), \max \left( g_{n+1}(s, i), \beta \sum_{j=1}^N P_{ij} \int_0^\infty v_n(sx, j) dF_i(x) \right) \right\} \end{aligned} \quad (2.12)$$

with the boundary conditions are  $v_0(s, i) = h_0(s, i)$  for any  $s, i$  and  $v_n(s, 0) \equiv 0$  for any  $n$  and  $s$ . Define the operator  $\mathcal{A}$  as follows:

$$(\mathcal{A}v_n)(s, i) \equiv \beta \sum_{j=1}^N P_{ij} \int_0^\infty v_n(sx, j) dF_i(x). \quad (2.13)$$

**Remark 2.1.** *The equation (2.12) can be reduced to the non-switching model when we set  $P_{ii} = 1$  for all  $i$ , or  $f_n(s, i) = f_n(s)$ ,  $g_n(s, i) = g_n(s)$ ,  $h_0(s, i) = h_0(s)$  and  $\mu_i = \mu$  for all  $i, n$  and  $s$ .*

Let  $V$  be the set of all bounded measurable functions with the norm  $\|v\| = \sup_{s \in (0, \infty)} |v(s, i)|$  for any  $i$ . For  $u, v \in V$ , we write  $u \leq v$  if  $u(s, i) \leq v(s, i)$  for all  $s \in (0, \infty)$ . A mapping  $\mathcal{U}$  is called a contraction mapping if

$$\|\mathcal{U}u - \mathcal{U}v\| \leq \beta \|u - v\|$$

for some  $\beta < 1$  and for all  $u, v \in V$ .

**Lemma 2.1.** *For any  $0 < c_2 < c_1$  and  $0 < b \leq a$ , we have*

$$\min\{a, \max(b, c_1)\} - \min\{a, \max(b, c_2)\} \leq \min(a, c_1) - \max(b, c_2). \quad (2.14)$$

*Proof.* Put  $\psi = \min\{a, \max(b, c_1)\} - \min\{a, \max(b, c_2)\}$ . If  $c_2 < c_1 < b$ , then  $\psi = 0$ . If  $c_2 < b < c_1$ , then  $\psi = \min(a, c_1) - b$ . If  $b \leq a < c_2 < c_1$ , then  $\psi = 0$ . If  $b < c_2 < \min(a, c_1)$ , then  $\psi = \min(a, c_1) - c_2$ . Thus, we have  $\psi \leq \min(a, c_1) - \max(b, c_2)$ .  $\square$

**Lemma 2.2.** *The mapping  $\mathcal{U}$  as defined by equation (2.12) is a contraction mapping.*

*Proof.* For any  $u_n, v_n \in V$ , we have

$$\begin{aligned} (\mathcal{U}u_n)(s, i) - (\mathcal{U}v_n)(s, i) &= \min\{f_{n+1}(s, i), \max(g_{n+1}(s, i), \mathcal{A}u_n)\} \\ &\quad - \min\{f_{n+1}(s, i), \max(g_{n+1}(s, i), \mathcal{A}v_n)\} \\ &\leq \min\{f_{n+1}(s, i), \mathcal{A}u_n\} - \max\{g_{n+1}(s, i), \mathcal{A}v_n\} \\ &\leq \mathcal{A}u_n - \mathcal{A}v_n \\ &\leq \beta \sum_{j=1}^N P_{ij} \int_0^\infty \sup(u_n(sx, j) - v_n(sx, j)) dF_i(x) \\ &\leq \beta \|u_n - v_n\| \end{aligned}$$

where the first inequality follows from Lemma 2.1. Hence, we obtain

$$\sup_{s \in (0, \infty)} \{(\mathcal{U}u)(s, i) - (\mathcal{U}v)(s, i)\} \leq \beta \|u - v\|. \quad (2.15)$$

By taking the roles of  $u$  and  $v$  reversely, we have

$$\sup_{s \in (0, \infty)} \{(\mathcal{U}v)(s, i) - (\mathcal{U}u)(s, i)\} \leq \beta \|v - u\|. \quad (2.16)$$

Putting equations (2.15) and (2.16) together, we obtain

$$\|\mathcal{U}u - \mathcal{U}v\| \leq \beta \|u - v\|.$$

$\square$

**Corollary 2.1.** *There exists a unique function  $v \in V$  such that*

$$(\mathcal{U}v)(s, i) = v(s, i) \quad \text{for all } s, i. \quad (2.17)$$

Furthermore, for all  $u \in V$ ,

$$(\mathcal{U}^T u)(s, i) \rightarrow v(s, i) \quad \text{as } T \rightarrow \infty,$$

where  $v(s, i)$  is equal to the fixed point defined by equation (2.17), that is,  $v(s, i)$  is a unique solution to

$$v(s, i) = \min\{f(s, i), \max(g(s, i), \mathcal{A}v)\}.$$

Since  $\mathcal{U}$  is a contraction mapping from Corollary 2.1, the optimal value function  $v$  for the perpetual contingent claim can be obtained as the limit by successively applying an operator  $\mathcal{U}$  to any initial value function  $v$  for a finite lived contingent claim.

To establish an optimal policy, we make some assumptions;

**Assumption 2.1.**

(i)  $F_1(x) \geq F_2(x) \geq \dots \geq F_N(x)$  for all  $x$ .

(ii)  $f_n(s, i)$ ,  $g_n(s, i)$  and  $h_n(s, i)$  are monotone in  $s$  for each  $i$  and  $n$ , and are non-decreasing in  $n$  for each  $s$  and  $i$ .

(iii) For each  $k \leq N$ ,  $\sum_{j=k}^N P_{ij}$  is non-decreasing in  $i$ .

Assumption (i) means  $X_n^{i+1}$  first order stochastically dominates  $X_n^i$  for any  $i$  and  $n$ . That is, as the state  $i$  increases, the economy is going well. Thus, the state  $N$  represents that the most "Good" economy. Assumption (ii) implies that the payoff values decreases as the maturity approaches. Assumption (iii) asserts that the probability of a transition into any block of states  $\{k, k+1, \dots\}$  is an increasing function of the present state.

**Lemma 2.3.** *Suppose Assumption 2.1 holds.*

(i) For each  $i$ ,  $(\mathcal{U}^n v)(s, i)$  is monotone in  $s$  for  $v \in V$ .

(ii)  $v$  satisfying  $v = \mathcal{U}v$  is monotone in  $s$ .

(iii) If  $f_n(s, i)$ ,  $g_n(s, i)$  and  $h_n(s, i)$  are non-decreasing in  $i$  for each  $s$  and  $n$ , and  $v_n(s, i)$  is non-decreasing in  $s$  for each  $i$  and  $n$ , then  $v_n(s, i)$  is non-decreasing in  $i$ .

(iv) If  $f_n(s, i)$ ,  $g_n(s, i)$  and  $h_n(s, i)$  are non-increasing in  $i$  for each  $s$  and  $n$ , and  $v_n(s, i)$  is non-increasing in  $s$  for each  $i$  and  $n$ , then  $v_n(s, i)$  is non-increasing in  $i$ .

(v)  $v_n(s, i)$  is non-decreasing in  $n$  for each  $s$  and  $i$ .

(vi) For each  $i$ , there exists a pair  $(s_n^*(i), s_n^{**}(i))$ ,  $s_n^{**}(i) < s_n^*(i)$ , of the optimal boundaries such that

$$v_n(s, i) \equiv (\mathcal{U}v_{n-1})(s) = \begin{cases} f_n(s, i), & \text{if } s_n^*(i) \leq s, \\ \mathcal{A}v_{n-1}, & \text{if } s_n^{**}(i) < s < s_n^*(i), n = 1, 2, \dots, T, \\ g_n(s, i), & \text{if } s \leq s_n^{**}(i), \end{cases}$$

with  $v_0(s, i) = h_0(s, i)$ .

*Proof.*

(i) The proof follows by induction on  $n$ . For  $n = 1$ , we have

$$(\mathcal{U}^1 v)(s, i) = \min \left\{ f_1(s, i), \max \left( g_1(s, i), \beta \sum_{j=1}^N P_{ij} \int_0^\infty h_0(sx, j) dF_i(x) \right) \right\} \quad (2.18)$$

which, since Assumption 2.1 (ii), implies that  $(\mathcal{U}^1 v)(s, i)$  is monotone in  $s$ . Suppose that  $(\mathcal{U}^n v)(s, i)$  is monotone for  $n > 1$ . Then, we have

$$(\mathcal{U}^{n+1} v)(s, i) = \min \left\{ f_{n+1}(s, i), \max \left( g_{n+1}(s, i), \beta \sum_{i=1}^n P_{ij} \int_0^\infty (\mathcal{U}^n v)(sx, j) dF_i(x) \right) \right\} \quad (2.19)$$

which is again monotone in  $s$ .

(ii) Since  $\lim_{n \rightarrow \infty} (\mathcal{U}^n v)(s, i)$  point-wisely converges to the limit  $v(s, i)$  from Corollary 2.1, the limit function  $v(s, i)$  is also monotone in  $s$ .

(iii) For  $n = 0$ , it follows from Assumption 2.1 (ii) that  $v_0(s, i) = h_0(s, i)$  is non-decreasing in  $i$ . Suppose (iii) holds for  $n$ . If  $v_n(s, i)$  is non-decreasing in  $s$ , then  $v_n(sx, i)$  is also

non-decreasing in  $x$  for each  $s$ . Then, from Assumption 2.1 (i), we obtain

$$\begin{aligned}
 \beta \sum_{j=1}^N P_{ij} \int_0^\infty v_n(sx, j) dF_i(x) &\leq \beta \sum_{j=1}^N P_{ij} \int_0^\infty v_n(sx, j) dF_{i+1}(x) \\
 &= \beta \int_0^\infty \sum_{k=1}^N (v_n(sx, k) - v_n(sx, k-1)) \sum_{j=k}^N P_{ij} dF_{i+1}(x) \\
 &\leq \beta \int_0^\infty \sum_{k=1}^N (v_n(sx, k) - v_n(sx, k-1)) \sum_{j=k}^N P_{i+1j} dF_{i+1}(x) \\
 &= \beta \sum_{j=1}^N P_{i+1j} \int_0^\infty v_n(sx, j) dF_{i+1}(x),
 \end{aligned}$$

where the second inequality follows from Assumption 2.1 (iii). Hence, we obtain

$$\begin{aligned}
 v_{n+1}(s, i) &= \min\{f_{n+1}(s, i), \max(g_{n+1}(s, i), \mathcal{A}v_n(s, i))\} \\
 &\leq \min\{f_{n+1}(s, i+1), \max(g_{n+1}(s, i+1), \mathcal{A}v_n(s, i+1))\} \\
 &= v_{n+1}(s, i+1).
 \end{aligned} \tag{2.20}$$

(iv) The proof is similar to the proof of (iii).

(v) For  $n = 1$  in equation (2.12), it follows from Assumption 2.1 (ii) that

$$\begin{aligned}
 v_1(s, i) &= \min\{f_1(s, i), \max(g_1(s, i), \mathcal{A}v_0)\} \\
 &\geq \min\{f_1(s, i), g_1(s, i)\} = g_1(s, i) \geq g_0(s, i) = v_0(s, i).
 \end{aligned}$$

Suppose (v) holds for  $n$ . We obtain

$$\begin{aligned}
 v_{n+1}(s, i) &= \min\{f_{n+1}(s, i), \max(g_{n+1}(s, i), \mathcal{A}v_n)\} \\
 &\geq \min\{f_n(s, i), \max(g_n(s, i), \mathcal{A}v_{n-1})\} \\
 &= v_n(s, i).
 \end{aligned}$$

(vi) Should  $v_n(s, i) = (\mathcal{U}^{n-1}v)(s, i)$  be monotone in  $s$ , then there exists at least one pair of boundary values  $s_n^*(i)$  and  $s_n^{**}(i)$  such that

$$v_n(s, i) = \begin{cases} f_n(s, i), & \text{if } s \geq s_n^*(i), \\ \max(g_n(s, i), \mathcal{A}v_{n-1}), & \text{otherwise,} \end{cases}$$

and

$$\max(g_n(s, i), \mathcal{A}v_{n-1}) = \begin{cases} g_n(s, i), & \text{for } s \leq s_n^{**}(i), \\ \mathcal{A}v_{n-1}, & \text{otherwise.} \end{cases}$$

□

**Corollary 2.2.** *The relationship between  $g_n$ ,  $f_n$  and  $v_n(s, i)$  is given by*

$$g_n(s, i) \leq v_n(s, i) \leq f_n(s, i).$$

*Proof.* The proof directly follows from equation (2.12). □

We define the stopping regions  $S^I$  for the issuer and  $S^{II}$  for the investor as

$$S_n^I(i) = \{(s, n, i) \mid v_n(s, i) \geq f_n(s, i)\}, \quad (2.21)$$

$$S_n^{II}(i) = \{(s, n, i) \mid v_n(s, i) \leq g_n(s, i)\}. \quad (2.22)$$

Moreover, the optimal exercise boundaries for the issuer and the investor are defined as

$$s_n^*(i) = \inf\{s \in S_n^I(i)\}, \quad (2.23)$$

$$s_n^{**}(i) = \inf\{s \in S_n^{II}(i)\}. \quad (2.24)$$

### 3. A Simple Callable American Option with Markov-modulated Prices

Interesting results can be obtained for the special cases when the payoff functions are specified. In this section we consider callable American options whose payoff functions are specified as a special case of callable contingent claim. If the issuer call back the claim in period  $n$ , the issuer must pay to the investor  $g_n(s, i) + \delta_n^i$ . Note that  $\delta_n^i$  is the compensate for the contract cancellation, and varies depending on the state and the time period. If the investor exercises his/her right at any time before the maturity, the investor receives the amount  $g_n(s, i)$ . In the following subsections, we discuss the optimal cancel and exercise policies both for the issuer and investor and show the analytical properties under some conditions.

#### 3.1. Callable Call Option

We consider the case of a callable call option where  $g_n(s, i) = (s - K^i)^+$  and  $f_n(s, i) = g_n(s, i) + \delta_n^i$ ,  $0 < \delta_n^i < K^i$ . Here,  $K^i$  is the strike price on the state  $i$ . We set out the assumptions to show the analytical properties of the optimal exercise policies.

##### Assumption 3.1.

- (i)  $\beta\mu_N \leq 1$
- (ii)  $K^1 \geq K^2 \geq \dots \geq K^N \geq 0$ .
- (iii)  $0 \leq \delta_n^1 \leq \delta_n^2 \leq \dots \leq \delta_n^N$  for each  $n$ .
- (iv)  $\delta_0^i = 0$  and  $\delta_n^i$  is non-decreasing and concave in  $n > 0$  for each  $i$ .
- (v)  $\beta \sum_{j=1}^N P_{ij} \delta_n^j - \delta_n^i$  is non-increasing in  $i$  for each  $n$ .

Assumption (i) means the expected rate of variability for the asset price is less than or equal to  $\frac{1}{\beta} = e^r$ . Assumption (ii) and (iii) imply that the strike price decreases and the compensate increases as the economy is getting better. Assumption (iv) shows that the compensate becomes smaller and smaller as the maturity approaches. Assumption (v) implies that the value of the postponement for cancellation decreases as the economy is getting better.

**Remark 3.1.** For example,  $\delta_n^i = \delta^i e^{-r(T-n)} = \frac{\delta^i}{(1+r)^{T-n}}$  satisfies Assumption 3.1(iv).

By the form of payoff function, the value function  $v_n$  is not bounded. To apply the result of Corollary 2.1, we assume that the issuer has to call back the claim when the payoff value exceeds a value  $M > K^1$ . Define  $\tilde{s}_n^i \equiv \inf\{s \mid f_n(s, i) \geq M\}$ . Since  $f_n(s, i)$  is increasing in  $s$  and  $i$  for any  $n$ , we have  $\tilde{s}_n^1 > \tilde{s}_n^i$  for any  $i, n$  and  $\tilde{s}_n^1 = M + K^1 - \delta_n^1$  for any  $n$ .

The stopping regions for the issuer  $S_n^I(i)$  and investor  $S_n^{II}(i)$  with respect to the callable call option are given by

$$\begin{cases} S_n^I(i) = \{s \mid v_n(s, i) \geq (s - K^i)^+ + \delta_n^i\} \cup \{\tilde{s}_n^1\}, & \text{for } n = 1, \dots, T, \\ S_n^I(i) = \phi, & \text{for } n = 0, \\ S_n^{II}(i) = \{s \mid v_n(s, i) \leq (s - K^i)^+\}, & \text{for } n = 0, 1, \dots, T. \end{cases}$$



For each  $i$  and  $n$ , we define the thresholds for the callable call option as

$$\begin{aligned} s_n^*(i) &= \inf\{s \mid v_n(s, i) = (s - K^i)^+ + \delta_n^i\} \wedge \tilde{s}_n^1, \\ s_n^{**}(i) &= \inf\{s \mid v_n(s, i) = (s - K^i)^+\}. \end{aligned}$$

The following lemma represents the well known result that American call options are identical to the corresponding European call options.

**Lemma 3.1.** *Callable call option with the maturity  $T < \infty$  can be degenerated into callable European, that is  $S_n^{II}(i) = \phi$  for  $n > 0$  and  $S_0^{II}(i) = \{K^i\}$  for each  $i$ .*

*Proof.* Since the discounted price process  $\{S_t/B_t\}_{t \in \mathcal{T}}$  is  $(\mathcal{G}, \mathcal{P}^\theta)$ -martingale,  $\beta^{\sigma \wedge \tau} g_t(S_{\sigma \wedge \tau}, i) = \beta^{\sigma \wedge \tau} \max(S_{\sigma \wedge \tau} - K^i, 0)$  is a  $(\mathcal{G}, \mathcal{P}^\theta)$ -submartingale. Applying the Optional Sampling Theorem, we obtain that

$$\begin{aligned} v_t(s, i) &= \min_{\sigma \in \mathcal{J}_{t, T}} \max_{\tau \in \mathcal{J}_{t, T}} E_s^\theta[\beta^{\sigma \wedge \tau} R(\sigma, \tau)] \\ &= \min_{\sigma \in \mathcal{J}_{t, T}} \max_{\tau \in \mathcal{J}_{t, T}} E_s^\theta[\beta^{\sigma \wedge \tau} (f_t(S_{\sigma \wedge \tau}, i) 1_{\{\sigma < \tau\}} + g_t(S_{\sigma \wedge \tau}, i) 1_{\{\tau < \sigma\}} + h_T 1_{\{\sigma \wedge \tau = T\}})] \\ &= \min_{\sigma \in \mathcal{J}_{t, T}} E_s^\theta[\beta^\sigma f_t(S_\sigma, i) 1_{\{\sigma < T\}} + \beta^T h_T 1_{\{\sigma = T\}}]. \end{aligned} \quad (3.1)$$

This completes the proof.  $\square$

It implies that it is optimal for the investor not to exercise his/her puttable right before the maturity. However, the issuer should choose an optimal call stopping time so as to minimize the expected payoff function.

**Lemma 3.2.** *If Assumption 3.1 (i) holds, then  $v_n(s, i) - s$  is decreasing in  $s$  for  $s > K^i$ , and  $v_n(s, i)$  is non-decreasing in  $s$  for  $s \leq K^i$  for each  $n, i$ .*

*Proof.* We prove it by induction. For  $n = 0$ , the claim certainly holds. It is sufficient to prove for the case of  $s > K^i$ . Suppose the claim holds for  $n$ , then we have

$$\begin{aligned} v_{n+1}(s, i) - s &= \min\{s - K^i + \delta_{n+1}^i, \max(s - K^i, \mathcal{A}v_n)\} - s \\ &= \min\left\{-K^i + \delta_{n+1}^i, \max\left(-K^i, \beta \sum_{j=1}^N P_{ij} \int_0^\infty (v_n(sx, j) - sx) dF_i(x) + (\beta\mu_i - 1)s\right)\right\}. \end{aligned}$$

Since the statement is true for  $n$ ,  $v_n(sx, j) - sx$  is decreasing in  $s$  for  $x > K^i$ . Assumption 2.1 (i) implies that  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_N$ . If  $\mu_N \leq \frac{1}{\beta}$ , then  $(\beta\mu_i - 1)s$  is non-increasing in  $s$ . Hence,  $v_{n+1}(s, i) - s$  is decreasing in  $s$  for  $s > K^i$ .  $\square$

**Lemma 3.3.**

(i) *Suppose that  $n_i^* = \inf\{n \mid \delta_n^i < v_n^a(K^i, i)\}$ , where  $v_n^a(s, i) = \max\{(s - K^i)^+, \mathcal{A}v_{n-1}(s, i)\}$  and  $v_0^a(s, i) = (s - K^i)^+$ . If  $n_i^* \leq n \leq T$ , we have  $S_n^I(i) = \{K^i\}$ . If  $0 \leq n < n_i^*$ , we have  $S_n^I(i) = \{\tilde{s}_n^1\}$ .*

(ii)  *$n_i^*$  is non-decreasing in  $i$ .*

*Proof.*

(i) Let  $\Psi_n^I(s, i) = v_n(s, i) - (s - K^i)^+ - \delta_n^i$ . For  $s = K^i$ , we have

$$\begin{aligned}\Psi_n^I(K^i, i) &= v_n(K^i, i) - \delta_n^i \\ &= \min\{0, \max\{0, \mathcal{A}v_{n-1}(K^i, i)\} - \delta_n^i\} \\ &= \min\{0, v_n^a(K^i, i) - \delta_n^i\}.\end{aligned}$$

If  $v_n^a(K^i, i) > \delta_n^i$ , then  $\Psi_n^I(K^i, i) = 0$  for any  $i$  and  $n$ . Since  $\delta_n^i$  is non-decreasing and concave in  $n$  by Assumption 2.1 (iii) and  $v_n(s, i)$  is non-decreasing in  $n$  by Assumption 3.1 (iv), there exists at least one value  $n_i^*$  such that  $n_i^* = \inf\{n \mid \delta_n^i < v_n^a(K^i, i)\}$ .

By Lemma 3.2, the function  $\Psi_n^I(s, i)$  is non-decreasing for  $s \leq K^i$  and is decreasing for  $K^i < s$ . It implies that it is unimodal function in  $s$ , and  $K^i$  is a maximizer of  $\Psi_n^I(s, i)$ . Thus,  $v_n(s, i) < (s - K^i)^+ + \delta_n^i$  if  $s \neq K^i$ . Moreover,  $\tilde{s}_n^1 = M + K^1 - \delta_n^1 > K^i$  for any  $i$ . Therefore,  $S_n^I(i) = \{K^i\}$  for  $n_i^* \leq n \leq T$ . For  $0 \leq n < n_i^*$ , since  $\delta_n^i > v_n^a(K^i, i)$  for each  $i$  and  $n$ , we have

$$v_n(K^i, i) = \min\{0, v_n^a(K^i, i) - \delta_n^i\} + \delta_n^i = v_n^a(K^i, i) < \delta_n^i \leq (s - K^i)^+ + \delta_n^i.$$

Hence, we have  $\Psi_n^I(K^i, i) < 0$ , so  $S_n^I(i) = \{\tilde{s}_n^1\}$ .

(ii) For  $n = 0$ , by Assumption 3.1 (iii),  $v_0^a(K^i, i) - \delta_0^i$  is non-increasing in  $i$ . We suppose that  $v_{n-1}^a(K^i, i) - \delta_{n-1}^i$  is non-increasing in  $i$ . Since  $v_n^a(K^i, i) - \delta_n^i = \max\{-\delta_n^i, \mathcal{A}v_{n-1}(K^i, i) - \delta_n^i\}$ , it is enough to show that  $\mathcal{A}v_{n-1}(K^i, i) - \delta_n^i$  is non-increasing in  $i$ . From Assumption 3.1 (v), we have

$$\begin{aligned}\mathcal{A}v_{n-1}(K^i, i) - \delta_n^i &= \beta \sum_{j=1}^N P_{ij} \int_0^\infty v_{n-1}(sK^i, j) dF_i(x) - \delta_n^i \\ &= \beta \sum_{j=1}^N P_{ij} v_{n-1}(sK^i, j) - \delta_n^i \\ &\geq \beta \sum_{j=1}^N P_{ij} (v_{n-1}(sK^i, j) - \delta_n^j) + \beta \sum_{j=1}^N P_{ij} \delta_n^j - \delta_n^i \\ &\geq \beta \sum_{j=1}^N P_{i+1j} v_{n-1}(sK^i, j) - \delta_n^{i+1} \\ &= \mathcal{A}v_{n-1}(K^i, i+1) - \delta_n^{i+1}\end{aligned}$$

So,  $v_n^a(K^i, i) - \delta_n^i$  is non-increasing in  $i$ . Since  $v_n^a(K^i, i) - \delta_n^i$  is non-decreasing in  $n$ , the value  $n_i^*$  is non-decreasing in  $i$ .  $\square$

**Theorem 3.1.** *Suppose that Assumption 3.1 (i)-(v) holds. The stopping regions for the issuer and investor can be obtained as follows;*

(i) *The optimal stopping region for the issuer:*

$$\begin{cases} S_n^I(i) = \{K^i\}, & \text{if } n_i^* \leq n \leq T, \\ S_n^I(i) = \{\tilde{s}_n^1\}, & \text{if } 0 \leq n < n_i^*, \end{cases} \quad (3.2)$$

where  $K^1 \geq K^2 \geq \dots \geq K^N \geq 0$ , and  $n_i^* \equiv \inf\{n \mid \delta_n^i \leq v_n^a(K^i, i)\}$  which is non-decreasing in  $i$ . Here,  $v_n^a(s, i) = \max\{(s - K^i)^+, \mathcal{A}v_{n-1}(s, i)\}$ .

(ii) The optimal stopping region for the investor:

$$\begin{cases} S_n^{II}(i) = \phi, & \text{if } n > 0, \\ S_0^{II}(i) = \{K^i\}, & \text{if } n = 0. \end{cases} \quad (3.3)$$

Moreover, the thresholds for the issuer and investor are  $s_n^*(i) = K^i$  for  $n_i^* \leq n \leq T$  and  $s_0^{**}(i) = K^i$ , respectively.

*Proof.* Part (i) follows from Lemma 3.3. Part (ii) is obtained from Lemma 3.1. In addition, since  $s_n^{**}(i) = \inf\{s \mid (s - K^i)^+ \leq s - K^i\} = K^i$  for  $n = 0$ , we obtain  $S_0^{II}(i) = \{K^i\}$ .  $\square$

### 3.2. Callable Put Option

We consider the case of a callable put option where  $g_n(s, i) = \max\{K^i - s, 0\}$  and  $f_n(s, i) = g_n(s, i) + \delta_n^i$ . The stopping regions for the issuer  $S_n^I(i)$  and the investor  $S_n^{II}(i)$  with respect to the callable put option are given by

$$\begin{cases} S_n^I(i) = \{s \mid v_n(s, i) \geq (K^i - s)^+ + \delta_n^i\}, & \text{for } n = 1, \dots, T, \\ S_n^I(i) = \phi, & \text{for } n = 0, \\ S_n^{II}(i) = \{s \mid v_n(s, i) \leq (K^i - s)^+\}, & \text{for } n = 0, 1, \dots, T. \end{cases}$$

For each  $i$  and  $n$ , we define the optimal exercise boundaries for the issuer  $\tilde{s}_n^*(i)$  and the investor  $\tilde{s}_n^{**}(i)$  as

$$\tilde{s}_n^*(i) = \inf\{s \mid v_n(s, i) = (K^i - s)^+ + \delta_n^i\}, \quad (3.4)$$

$$\tilde{s}_n^{**}(i) = \inf\{s \mid v_n(s, i) = (K^i - s)^+\}. \quad (3.5)$$

#### Assumption 3.2.

- (i)  $\beta\mu_N \leq 1$
- (ii)  $K^1 \geq K^2 \geq \dots \geq K^N \geq 0$ .
- (iii)  $\delta_n^1 \geq \delta_n^2 \geq \dots \geq \delta_n^N \geq 0$  for each  $n$ .
- (iv)  $\delta_0^i = 0$  and  $\delta_n^i$  is non-decreasing and concave in  $n > 0$  for each  $i$ .
- (v)  $\beta \sum_{j=1}^N P_{ij}K^j - K^i$  is non-increasing in  $i$ .
- (vi)  $\beta \sum_{j=1}^N P_{ij}\delta_n^j - \delta_n^i$  is non-decreasing in  $i$  for each  $n$ .

Assumptions (i), (ii) and (iv) is the same as that of call option in Section 3.1. Assumption (iii) implies that the compensate decreases as the economy is getting better, because good economy decreases the likelihood of exercising the put option by investor. Assumption (v) asserts that the difference between the discounted expected value of a strike price when the state transits to any state and the strike price of present state is a non-increasing function of the present state. In other words, the value of the postponement for exercise becomes smaller as the economy is getting better. Assumption (vi) means the value of the postponement for cancellation increases as the economy is getting better.

**Lemma 3.4.** *If Assumption 3.2 (i) holds, then  $v_n(s, i) + s$  is increasing in  $s$  for  $s < K^i$ , and  $v_n(s, i)$  is non-increasing in  $s$  for  $K^i \leq s$ .*

*Proof.* It is sufficient to prove for the case of  $s < K^i$ . The claim holds for  $n = 0$ . Suppose the assertion holds for  $n$ . Then, we have

$$\begin{aligned} v_{n+1}(s, i) + s &= \min\{K^i - s + \delta_{n+1}^i, \max(K^i - s, \mathcal{A}v_n)\} + s \\ &= \min\left\{K^i + \delta_{n+1}^i, \max\left(K^i, \beta \sum_{j=1}^N P_{ij} \int_0^\infty (v_n(sx, j) + sx)dF_i(x) + (1 - \beta\mu_i)s\right)\right\}. \end{aligned}$$

Hence, from Assumption 3.2 (i),  $v_{n+1}(s, i) + s$  is increasing in  $s$  for  $s < K^i$ .  $\square$

**Lemma 3.5.**  $v_n(s, i) - K^i$  is non-increasing in  $i$  for each  $s < K^i$  and  $n$ .

*Proof.* When  $n = 0$ , the claim holds. Suppose the claim holds for  $n$ . For  $s < K^i$ , we have

$$\begin{aligned} v_{n+1}(s, i) - K^i &= \min\{K^i - s + \delta_{n+1}^i, \max(K^i - s, \mathcal{A}v_n(s, i))\} - K^i \\ &= \min\{-s + \delta_{n+1}^i, \max(-s, \mathcal{A}v_n(s, i) - K^i)\} \end{aligned}$$

By Lemma 2.3 (iv),  $v_n(x, i)$  is non-increasing in  $i$ . Thus, we have

$$\begin{aligned} \mathcal{A}v_n(s, i) - K^i &= \beta \sum_{j=1}^N P_{ij} \int_0^\infty v_n(sx, j) dF_i(x) - K^i \\ &\geq \beta \sum_{j=1}^N P_{ij} \int_0^\infty v_n(sx, j) dF_{i+1}(x) - K^i \\ &= \beta \sum_{j=1}^N P_{ij} \int_0^\infty (v_n(sx, j) - K^j) dF_{i+1}(x) + \beta \sum_{j=1}^N P_{ij} K^j - K^i \\ &\geq \mathcal{A}v_n(s, i+1) - \beta \sum_{j=1}^N (P_{i+1j} - P_{ij}) K^j - K^i \\ &\geq \mathcal{A}v_n(s, i+1) - K^{i+1}. \end{aligned}$$

The last inequality comes from Assumption 3.2 (v). Since  $\delta_{n+1}^i \geq \delta_{n+1}^{i+1}$ , this leads to  $v_{n+1}(s, i) - K^i \leq v_{n+1}(s, i+1) - K^{i+1}$ .  $\square$

**Lemma 3.6.**

(i) There exists a time  $n_i^*$  for each  $i$  such that  $n_i^* \equiv \inf\{n \mid \delta_n^i \leq v_n^a(K^i, i)\}$ , where  $v_n^a(s, i) = \max\{(K^i - s)^+, \mathcal{A}v_{n-1}(s, i)\}$ . Moreover, if  $n_i^* \leq n \leq T$ , we have  $S_n^I(i) = \{K^i\}$ . If  $0 \leq n < n_i^*$ , we have  $S_n^I(i) = \phi$ .

(ii)  $n_i^*$  is non-increasing in  $i$ .

*Proof.* The proof can be done similarly as in the case of the call option in Lemma 3.3.  $\square$

**Lemma 3.7.** Suppose Assumption 3.2 (i) holds. Then, there exists an optimal exercise policy for the both players, and  $\tilde{s}_n^{**}(i) < \tilde{s}_n^*(i)$  such that the investor exercises the option if  $s \leq \tilde{s}_n^{**}(i)$  and the issuer exercises the option if  $s_n^*(i) \leq s$ .

*Proof.* We first consider the optimal exercise policy for the investor. Let  $\Psi_n^{II}(s, i) \equiv v_n(s, i) - (K^i - s)^+$ . The investor does not exercise the option when  $s > K^i$  because he/she wishes to exercise the right so as to maximize the expected payoff. For  $s \leq K^i$ ,  $\Psi_n^{II}(s, i)$  is increasing in  $s$  by Lemma 3.4. Since  $v_n(K^i, i) \geq 0$ , there exists a value  $\tilde{s}_n^{**}(i)$  satisfying (3.5). For  $s \leq \tilde{s}_n^{**}(i)$ ,  $v_n(s, i) \leq (s - K^i)^+$ . Hence, it is optimal for the investor to exercise the option when  $s \leq \tilde{s}_n^{**}(i)$ .

It follows from Lemma 3.6 (i) that the optimal exercise policy for the issuer is  $\tilde{s}_n^*(i) = K^i$  for  $n_i^* \leq n \leq T$  and  $\tilde{s}_n^*(i) = \infty$  for  $0 \leq n < n_i^*$ . Since  $\Psi_n^{II}(s, i)$  is increasing in  $s$  for  $s \leq K^i$ , we have  $\tilde{s}_n^{**}(i) < \tilde{s}_n^*(i)$  for each  $i$  and  $n \in [n_i^*, T]$ .  $\square$

**Lemma 3.8.**

- (i)  $\tilde{s}_n^{**}(i)$  is non-decreasing in  $i$  for each  $n$ .  
 (ii)  $\tilde{s}_n^{**}(i)$  is non-increasing in  $n$  for each  $i$ .

*Proof.* We only consider the case of  $K^i > s$ .

- (i) By Lemma 3.4 and Lemma 3.5,  $v_n(s, i) + s$  is increasing in  $s$  for  $K^i > s$ , and  $v_n(s, i) - K^i$  is non-increasing in  $i$ . Hence, we have

$$\begin{aligned}\tilde{s}_n^{**}(i) &= \inf\{s \mid v_n(s, i) + s = K^i\} \\ &\leq \inf\{s \mid v_n(s, i+1) + s = K^{i+1}\} \\ &= \tilde{s}_n^{**}(i+1).\end{aligned}$$

- (ii) By Lemma 2.3 (v),  $v_n(s, i)$  is non-decreasing in  $n$ , so we have

$$\begin{aligned}\tilde{s}_n^{**}(i) &= \inf\{s \mid v_n(s, i) + s = K^i\} \\ &\geq \inf\{s \mid v_{n+1}(s, i) + s = K^i\} \\ &= \tilde{s}_{n+1}^{**}(i).\end{aligned}$$

□

**Theorem 3.2.** *Suppose that Assumption 3.2 (i)-(vi) holds. The stopping regions for the issuer and investor can be obtained as follows;*

- (i) *The optimal stopping region for the issuer:*

$$\begin{cases} S_n^I(i) = \{K^i\}, & \text{if } n_i^* \leq n \leq T, \\ S_n^I(i) = \phi, & \text{if } 0 \leq n < n_i^*, \end{cases} \quad (3.6)$$

where  $K^1 \geq K^2 \geq \dots \geq K^N \geq 0$ , and  $n_i^* \equiv \inf\{n \mid \delta_n^i \leq v_n^a(K^i, i)\}$  which is non-increasing in  $i$ . Here,  $v_n^a(s, i) = \max\{(K^i - s)^+, \mathcal{A}v_{n-1}(s, i)\}$ .

- (ii) *The optimal stopping region for the investor:*

$$\begin{cases} S_n^{II}(i) = [0, \tilde{s}_n^{**}(i)], & \text{if } n > 0, \\ S_0^{II}(i) = \{K^i\}, & \text{if } n = 0, \end{cases} \quad (3.7)$$

where  $\tilde{s}_n^{**}(i)$  is non-decreasing in  $i$  and non-increasing in  $n$ . Moreover,  $\tilde{s}_n^{**}(i) \leq K^i$  for each  $i$  and  $n$ .

*Proof.* Part (i) follows from Lemma 3.6. Part (ii) can be obtained by Lemma 3.7 and 3.8. For  $n = 0$ , since  $\tilde{s}_n^{**}(i) = \inf\{s \mid (s - K^i)^+ \leq s - K^i\} = K^i$ , we have  $S_0^{II}(i) = \{K^i\}$ . □

#### 4. Numerical Examples

In this section we provide a numerical example for a callable American option by using the binomial tree model. We assume that the transition probability matrix is given by

$$\mathbf{P} = \begin{pmatrix} p_1 & 1 - p_1 \\ 1 - p_2 & p_2 \end{pmatrix}. \quad (4.1)$$

For a fixed  $T$ , let us divide the interval  $[0, T]$  into  $M$  subintervals such that  $T = hM$ . Suppose that  $b(Z_t) = b_i$  and  $a(Z_t) = -b_i$ ,  $i = 1, 2$ , in equation (2.3). By Proposition 2.2, the probability of upward in the state  $i$  is given by

$$q_i = \frac{e^{rh} - d_i}{u_i - d_i}, \quad i = 1, 2, \quad (4.2)$$

where  $u_i = e^{b_i}$  and  $d_i = e^{-b_i}$ . Let  $u_{i,j}$  and  $d_{i,j}$  be the upward and downward rate when the state changes from  $i$  to  $j$ , respectively. The probability distribution function of  $X_t^i$  is described by

$$P(X_t^i = x) = \begin{cases} q_i p_i, & \text{if } x = u_{i,i}, \\ q_i(1 - p_i), & \text{if } x = u_{i,j}, \\ (1 - q_i)p_i, & \text{if } x = d_{i,i}, \\ (1 - q_i)(1 - p_i), & \text{if } x = d_{i,j}, \end{cases} \quad (4.3)$$

where  $i = 1, 2$ ,  $i \neq j$ . It is easy to show that the process is a martingale. The asset price after  $n$  periods on tree can be obtained by

$$S_n = S_0 u_1^{n_1} u_2^{n_2} d_1^{n_3} d_2^{n_4} \quad (4.4)$$

where  $n_1 + n_2 + n_3 + n_4 = n$ .

**Remark 4.1.** Aingworth *et al.* [16] show that the number of distinct underlying prices at period  $n$  is  ${}_{n+2N-1}C_{2N-1}$ . Here,  $N$  is the number of the state of the economy.

Let  $\bar{v}_n^i(n_1, n_2, n_3)$  be the value of the callable American put at time period  $n$  when the number of the up moves in the state 1 is  $n_1$ , the number of the up moves in the state 2 is  $n_2$  and the number of the down moves in the state 1 is  $n_3$ . Then our optimal stopping problems can be rewritten as follows;

$$\bar{v}_0^i(0, 0, 0) = (K^i - s)^+, \quad i = 1, 2, \quad (4.5)$$

$$\begin{aligned} \bar{v}_n^1(n_1, n_2, n_3) = & \min\{(K^1 - S_n)^+ + \delta_n^1, \max\{(K^1 - S_n)^+, \beta\{p_1 q_1 \bar{v}_{n+1}^1(n_1 + 1, n_2, n_3) \\ & + p_1(1 - q_1)\bar{v}_{n+1}^1(n_1, n_2, n_3 + 1) + (1 - p_1)q_1 \bar{v}_{n+1}^2(n_1, n_2 + 1, n_3) \\ & + (1 - p_1)(1 - q_1)\bar{v}_{n+1}^2(n_1, n_2, n_3)\}\}\}, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \bar{v}_n^2(n_1, n_2, n_3) = & \min\{(K^2 - S_n)^+ + \delta_n^2, \max\{(K^2 - S_n)^+, \beta\{p_2 q_2 \bar{v}_{n+1}^2(n_1, n_2 + 1, n_3) \\ & + p_2(1 - q_2)\bar{v}_{n+1}^2(n_1, n_2, n_3) + (1 - p_2)q_2 \bar{v}_{n+1}^1(n_1 + 1, n_2, n_3) \\ & + (1 - p_2)(1 - q_2)\bar{v}_{n+1}^1(n_1, n_2, n_3 + 1)\}\}\}. \end{aligned} \quad (4.7)$$

We set the parameters as  $T = 1$ ,  $M = 300$ ,  $r = 0.1$ ,  $b_1 = 0.03$ ,  $b_2 = 0.01$ ,  $p_1 = 0.7$ ,  $p_2 = 0.8$ ,  $K^1 = K^2 = 100$ ,  $\delta_n^1 = 6$  and  $\delta_n^2 = 5$  for all  $n$ . These parameters satisfy Assumption 2.1 (i) and Assumption 3.2. The optimal exercise regions for the issuer and the investor is represented in Figure 1. We see that the properties of regions are consistent with Theorem 3.2. Next, we vary a parameter  $S_0$ ,  $T$ ,  $b_1$ ,  $p_2$  or  $\delta_1$  and keep all other parameters fixed. The resulting value for callable American put option are listed in Table 1. The option values are decreasing in  $S_0$ ,  $T$  and  $p_2$ , and non-decreasing in  $b_1$  and  $\delta_1$ .

## 5. Concluding Remarks

In this paper we consider the discrete time valuation model for callable contingent claims in which the asset price depends on a Markov environment process. It is shown that such valuation model with the Markov-modulated price dynamics can be formulated as a coupled optimal stopping problem of a two person game between the issuer and the investor. In particular, we show under some assumptions that there exists a simple optimal call policy for the issuer and optimal exercise policy for the investor which can be described by the control limit values. If the distributions of the state of the economy are stochastically ordered, then we investigate analytical properties of such optimal stopping rules for the issuer and the investor, respectively, possessing a monotone property.

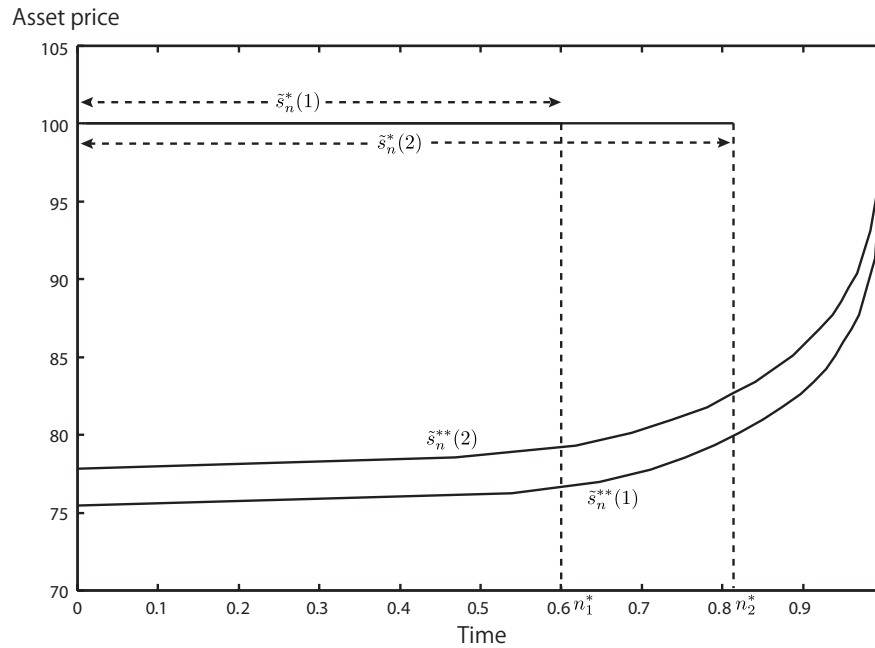


Figure 1: Optimal exercise boundaries for the callable American put.

We assume that the asset price follows a random walk with first order stochastic dominance constraint. We wish to extend it to the one with second order stochastic dominance. Moreover, it is of interest to extend it to the three person games among the issuer, investor and the third party like stake holders. If we can directly observe the state of the economy but be able to partially observable, a regime switching model can be formulated as a partially observable Markov one. We shall leave it for future research.

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Table 1: Sensitive analysis for callable American put option.

$S_0$	$T$	$b_1$	$b_2$	$p_1$	$p_2$	$\delta^1$	$\delta^1$	$v_T(s, 1)$	$v_T(s, 2)$
90	1	0.03	0.01	0.7	0.8	6	5	12.2286	12.1363
95	1	0.03	0.01	0.7	0.8	6	5	8.9324	8.8256
100	1	0.03	0.01	0.7	0.8	6	5	6.0000	5.0000
105	1	0.03	0.01	0.7	0.8	6	5	5.0045	4.9154
110	1	0.03	0.01	0.7	0.8	6	5	4.2822	4.2553
95	0.25	0.03	0.01	0.7	0.8	6	5	9.6775	9.5953
95	0.5	0.03	0.01	0.7	0.8	6	5	9.3985	9.3088
95	0.75	0.03	0.01	0.7	0.8	6	5	9.1472	9.0496
95	1	0.03	0.01	0.7	0.8	6	5	8.9324	8.8256
95	1.25	0.03	0.01	0.7	0.8	6	5	8.7497	8.6338
95	1	0.01	0.01	0.7	0.8	6	5	6.3252	6.3252
95	1	0.02	0.01	0.7	0.8	6	5	8.0394	7.9909
95	1	0.03	0.01	0.7	0.8	6	5	8.9324	8.8256
95	1	0.04	0.01	0.7	0.8	6	5	9.4728	9.2839
95	1	0.05	0.01	0.7	0.8	6	5	9.5570	9.4655
95	1	0.03	0.01	0.7	0.2	6	5	9.2762	9.2337
95	1	0.03	0.01	0.7	0.4	6	5	9.2308	9.1781
95	1	0.03	0.01	0.7	0.6	6	5	9.1523	9.0842
95	1	0.03	0.01	0.7	0.8	6	5	8.9324	8.8256
95	1	0.03	0.01	0.7	1	6	5	6.7308	6.3252
95	1	0.03	0.01	0.7	0.8	6	5	8.9324	8.8256
95	1	0.03	0.01	0.7	0.8	8	5	8.9649	8.8654
95	1	0.03	0.01	0.7	0.8	10	5	8.9649	8.8654
95	1	0.03	0.01	0.7	0.8	12	5	8.9649	8.8654

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