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AN ALTERNATIVE PROOF OF LOVÁSZ'S CATHEDRAL THEOREM

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Abstract A graph G with a perfect matching is called saturated if $G + e$ has more perfect matchings than G for any edge e that is not in G. Lovász gave a characterization of the saturated graphs called the cathedral theorem, with some applications to the enumeration problem of perfect matchings, and later Szigeti gave another proof. In this paper, we give a new proof with our preceding works which revealed canonical structures of general graphs with perfect matchings. Here, the cathedral theorem is derived in quite a natural way, providing more refined or generalized properties. Moreover, the new proof shows that it can be proved without using the Gallai-Edmonds structure theorem.

Keywords: Graph theory, combinatorial optimization, matching theory, perfect matchings, the cathedral theorem

1. Introduction

A graph with a perfect matching is called *factorizable*. A factorizable graph G, with the edge set $E(G)$, is called *saturated* if $G + e$ has more perfect matchings than G for any edge $e \notin E(G)$. There is a constructive characterization of the saturated graphs known as the *cathedral theorem* [12, 13, 15, 16]. Counting the number of perfect matchings is one of the most fundamental enumeration problems, which has applications to physical science, and the cathedral theorem is known to be useful for such a counting problem. For a given factorizable graph, we can obtain a saturated graph which possesses the same family of perfect matchings by adding appropriate edges repeatedly. Many matching-theoretic structural properties are preserved by this procedure. Therefore, we can find several properties on perfect matchings of factorizable graphs using the cathedral theorem, such as relationships between the number of perfect matchings of a given factorizable graph and its structural properties such as its connectivity [13] or the numbers of vertices and edges [3].

The cathedral theorem was originally given by Lovász [12] (see also [13]), and later another proof was given by Szigeti [15, 16]. Lovász's proof is based on the Gallai-Edmonds structure theorem [13], which is one of the most powerful theorem in matching theory. Any graph G has a partition of its vertices into three parts, some of which might be empty, so-called $D(G), A(G)$, and $C(G)$ [13], which we call in this paper the *Gallai-Edmonds partition*. The property that $A(G)$ forms a barrier with certain special properties is called the *Gallai-Edmonds structure theorem* [13]. The Gallai-Edmonds structure theorem tells non-trivial structures only for non-factorizable graphs, because it treats factorizable graphs as irreducible. Thus, Lovász proved the cathedral theorem by applying the Gallai-Edmonds structure theorem to non-factorizable subgraphs of saturated graphs.

Szigeti's proof is based on some results on the *optimal ear-decompositions* by Frank [2], which is also based on the Gallai-Edmonds structure theorem and is not a "matching-theory-closed" notion, while the cathedral theorem itself is closed.

The cathedral theorem is outlined as follows:

- There is a constructive characterization of the saturated graphs with an operation called the *cathedral construction*.
- A set of edges of a saturated graph is a perfect matching if and only if it is a disjoint union of perfect matchings of each "component part" of the cathedral construction that creates the saturated graph.
- For each saturated graph, the way to construct it by the cathedral construction uniquely exists.
- There is a relationship between the cathedral construction and the Gallai-Edmonds partition.

In our preceding works [4–7], we introduced canonical structure theorems which tells non-trivial structures for general factorizable graphs. Based on these results, we provide yet another proof of the cathedral theorem in this paper. The features of the new proof are the following: First, it is quite natural and provides new facts as by-products. The notion of "saturated" is defined by edge-maximality. By considering this edge-maximality over the canonical structures of factorizable graphs, we obtain the new proof in quite a natural way. Therefore, our proof reveals the essential structure that underlies the cathedral theorem, and provides a bit more refined or generalized statements from the point of view of the canonical structure of general factorizable graphs.

Second, it shows that the cathedral theorem can be proved without the Gallai-Edmonds structure theorem nor the notion of barriers, since our previous works, as well as the proofs presented in this paper, are obtained without them. Even the portion of the statements of the cathedral theorem stating its relationship to the Gallai-Edmonds partition can be obtained without them.

In Section 2, we give notations, definitions, and some preliminary facts on matchings used in this paper. In Section 3 we present an outline of how we give the new proof of the cathedral theorem. Section 4 is to present our previous works [4, 5]: the canonical structure theorems for general factorizable graphs. In Section 5, we further consider the theorems in Section 4 and show one of the new theorems, which later turns out to provide a generalized version of the part of the cathedral theorem regarding the Gallai-Edmonds partition. In Section 6, we complete the new proof of the cathedral theorem. Finally, in Section 7, we conclude this paper. Some basic properties on matchings, i.e., Properties A1, A2, A3, A4, and A5 are presented in Appendix.

2. Preliminaries

2.1. Notations and definitions

Here, we list some standard notations and definitions, most of which are given by Schrijver [14]. For general accounts on matchings, see also Lovász and Plummer [13].

We define the *symmetric difference* of two sets A and B as $(A \setminus B) \cup (B \setminus A)$ and denote it by $A \triangle B$. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$ and write $G = (V(G), E(G))$. Hereafter for a while let G be a graph and let $X \subseteq V(G)$. The subgraph of G induced by X is denoted by $G[X]$, and $G - X$ means $G[V(G) \setminus X]$. We define the *contraction* of G by X as the graph arising from contracting each edge of $E(G[X])$ into one vertex, and denote it by G/X. We denote the subgraph of G determined by $F \subseteq E(G)$ by $G.F$.

Let G be a subgraph of a graph $\hat{G}$, and let $e = xy \in E(\hat{G})$. If G does not have an edge joining x and y, then we call xy a *complement edge* of G. The graph $G + e$

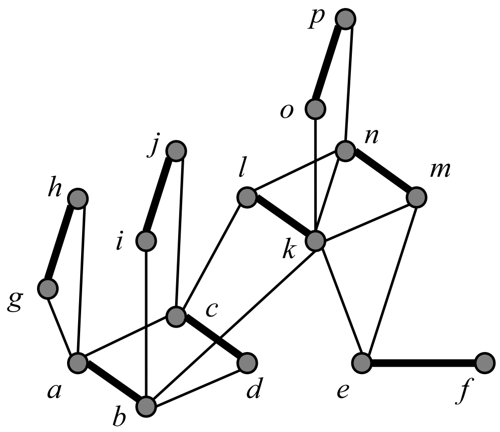

Figure 1: A graph with a (perfect) matching

denotes the graph $(V(G) \cup \{x, y\}, E(G) \cup \{e\})$, and $G - e$ the graph $(V(G), E(G) \setminus \{e\})$. For $F = \{e_1, \ldots, e_k\} \subseteq E(\hat{G})$, we define $G + F := G + e_1 + \cdots + e_k$ and $G - F := G - e_1 - \cdots - e_k$.

We define the set of *neighbors* of X as the vertices in $V(G) \setminus X$ that are joined to some vertex of X, and denote it by $N_G(X)$. Given $Y, Z \subseteq V(G)$, the set $E_G[Y, Z]$ denotes the edges joining Y and Z, and $\delta_G(Y)$ denotes $E_G[Y, V(G) \setminus Y]$.

A set of edges is called a *matching* if no two of them share end vertices. A matching of cardinality $|V(G)|/2$ (resp. $|V(G)|/2 - 1$) is called a *perfect matching* (resp. a *near-perfect matching*). Hereafter for a while let M be a matching of a graph G. We say M *exposes* a vertex $v \in V(G)$ if $\delta_G(v) \cap M = \emptyset$.

In this paper, we treat paths and circuits as graphs. For a path or circuit Q of G, Q is M-*alternating* if $E(Q) \setminus M$ is a matching of Q; in other words, if edges of M and $E(Q) \setminus M$ appear alternately in Q. Let P be an M-alternating path of G with end vertices u and v. If P has an even number of edges and $M \cap E(P)$ is a near-perfect matching of P exposing only v, we call it an M-*balanced path* from u to v. We regard a trivial path, that is, a path composed of one vertex and no edges as an M-balanced path. If P has an odd number of edges and $M \cap E(P)$ (resp. $E(P) \setminus M$) is a perfect matching of P, we call it M-*saturated* (resp. M-*exposed*).

Example 2.1. In Figure 1 a graph is given with one of its matchings (actually, a perfect matching), say M, whose edges are indicated in bold lines. In this graph, for example, the subgraph determined by the edges $\{ba, ac, cd\}$ is an M-saturated path between b and d, the one determined by $\{ac, cd, db\}$ is an M-exposed path between a and b, and the one determined by $\{mn, nk, kl, lc\}$ is an M-balanced path from m to c. ■

A path P of G is an *ear relative to* X if both end vertices of P are in X while internal vertices are not. Also, a circuit C is an ear relative to X if exactly one vertex of C is in X. For simplicity, we call the vertices of $V(P) \cap X$ *end vertices* of P, even if P is a circuit. For an ear P of G relative to X, we call it an M-*ear* if $P - X$ is an M-saturated path.

A graph is called *factorizable* if it has a perfect matching. A graph is called *factor-critical* if a deletion of an arbitrary vertex results in a factorizable graph. For convenience, we regard a graph with only one vertex as factor-critical.

We sometimes regard a graph as the set of its vertices. For example, given a subgraph

H of G, we denote $N_G(V(H))$ by $N_G(H)$. For simplicity, regarding the operations of the contraction or taking the union of graphs, we identify vertices, edges, and subgraphs of the newly created graph with those of old graphs that naturally correspond to them.

Let G be a factorizable graph. An edge $e \in E(G)$ is called *allowed* if there is a perfect matching of G containing e, and each connected component of the subgraph of G determined by the set of all the allowed edges is called an *elementary component* of G. A factorizable graph which has exactly one elementary component is called *elementary.* For an elementary component H of G, we call $G[V(H)]$ a *factor-connected component* of G, and denote the set of all factor-connected components of G by $\mathcal{G}(G)$. Hence, any factor-connected component is elementary and a factorizable graph is composed of its factor-connected components and additional edges joining distinct factor-connected components.

For example, consider the graph G in Figure 2. The allowed edges of G are those indicated by bold lines in Figure 3. Hence, it has these six factor-connected components $G_1 = G[\{a, b, c, d\}]$, $G_2 = G[\{e, f\}]$, $G_3 = G[\{g, h\}]$, $G_4 = G[\{i, j\}]$, $G_5 = G[\{k, l, m, n\}]$, and $G_6 = G[\{o, p\}]$.

2.2. The Gallai-Edmonds partition

Given a graph G, we define $D(G)$ as

$$D(G) := \{v \in V(G) : \text{there is a maximum matching that exposes } v\}.$$

We also define $A(G)$ as $N_G(D(G))$ and $C(G)$ as $V(G) \setminus (D(G) \cup A(G))$. We call in this paper this partition of $V(G) = D(G) \dot\cup A(G) \dot\cup C(G)$ into three parts the *Gallai-Edmonds partition.* It is known as the *Gallai-Edmonds structure theorem* that $A(G)$ forms a barrier with special properties, which is one of the most powerful theorem in matching theory [13].

In this section, we present a proposition which shows another property of the Gallai-Edmonds partition that is different from the Gallai-Edmonds structure theorem. This proposition is a well-known fact that connects the Gallai-Edmonds structure theorem and Edmonds' maximum matching algorithm, and we can find it in [1,8]. However, this proposition can be proved in an elementary way without using them, nor the notion of barriers. In the following we present it with a proof to confirm it. Note that Proposition 2.2 itself is NOT the Gallai-Edmonds structure theorem.

Proposition 2.2. *Let G be a graph, M be a maximum matching of G, and S be the set of vertices that are exposed by M. Then, the following hold:*

(i) *A vertex u is in $D(G)$ if and only if there exists $v \in S$ such that there is an M-balanced path from u to v.*

(ii) *A vertex u is in $A(G)$ if and only if there is no M-balanced path from u to any vertex of S, while there exists $v \in S$ such that there is an M-exposed path between u and v.*

(iii) *A vertex u is in $C(G)$ if and only if for any $v \in S$ there is neither an M-balanced path from any u to v nor an M-exposed path between u and v.*

Proof. For the necessity part of (i), let P be the M-balanced path from u to v. Then, $M \triangle E(P)$ is a maximum matching of G that exposes u. Thus, $u \in D(G)$.

Now we move on to the sufficiency part of (i). If $u \in D(G) \cap S$, the trivial M-balanced path $(\{u\}, \emptyset)$ satisfies the property. Otherwise, that is, if $u \in D(G) \setminus S$, by the definition of $D(G)$ there is a maximum matching M' of G that exposes u. Then, $G.M \triangle M'$ has a connected component which is an M-balanced path from u to some vertex in S. Hence, we are done for (i).

For (ii), we first prove the necessity part. Let P be the M-exposed path between u and v, and $w \in V(P)$ be such that $uw \in E(P)$. Then, $P - u$ is an M-balanced path from w

to v, which means $w \in D(G)$ by (i). Then, we have $u \in A(G)$, since the first part of the condition on P yields $u \notin D(G)$ by (i).

Now we move on to the sufficiency part of (ii). Note that the first part of the conclusion follows by (i). By the definition of $A(G)$, there exists $w \in D(G)$ such that $wu \in E(G)$. By (i), there is an M-balanced path Q from w to a vertex $v \in S$. If $u \in V(Q)$, then since $u \notin D(G)$, the subpath of Q from v to u is an M-exposed path between v and u by (i). Thus, the claim follows. Otherwise, that is, if $u \notin V(Q)$, then $Q + wu$ forms an M-exposed path between v and u. Therefore, again the claim follows. Thus, we are done for (ii).

Since we obtain (i) and (ii), consequently (iii) follows. □

The next proposition is also known (see [1]) and is easily obtained from Proposition 2.2.

Proposition 2.3. *Let G be a factorizable graph and M be a perfect matching of G. Then, for any $x \in V(G)$, the following hold:*

(i) *A vertex u is in $D(G - x)$ if and only if there is an M-saturated path between x and u.*

(ii) *A vertex u is in $A(G - x) \cup \{x\}$ if and only if there is no M-saturated path between x and u, while there is an M-balanced path from x to u.*

(iii) *A vertex u is in $C(G - x)$ if and only if there is neither an M-saturated path between u and x nor an M-balanced path from x to u.*

Proof. Let $x' \in V(G)$ be such that $xx' \in M$. Let $G' := G - x$ and $M' := M \setminus \{xx'\}$. Note that apparently

M' is a maximum matching of G', exposing only x'.

By Propositions 2.2, $u \in D(G')$ if and only if there is an M'-balanced path from u to x'. Additionally, the following apparently holds: there is an M'-balanced path from u to x' in G' if and only if there is an M-saturated path between u and x in G. Thus, we obtain (i). The other claims, (ii) and (iii), also follow by similar arguments. □

Proposition 2.3 associates factorizable graphs with the Gallai-Edmonds partition, and it will be used later in the proof of Theorem 5.1. Hence it will contribute to the new proof of the cathedral theorem.

3. Outline of the New Proof

Here we give an outline of how we give a new proof of the cathedral theorem together with backgrounds of the theorem. In our previous work [4,5], we revealed canonical structures of factorizable graphs. The key points of them are as follows. (We shall explain them in detail in Section 4.)

(a) For a factorizable graph G, a partial order $\lhd$ can be defined on the factor-connected components $\mathcal{G}(G)$ (Theorem 4.3).

(b) An equivalence relation $\sim_G$ based on factor-connected components can be defined on $V(G)$ (Theorem 4.5). The equivalence classes by $\sim_G$ can be regarded as a generalization of Kotzig's canonical partition [9–11].

(c) These two notions $\lhd$ and $\sim_G$ are related each other in the sense that for $H \in \mathcal{G}(G)$ a relationship between H and its strict upper bounds in the poset $(\mathcal{G}(G), \lhd)$ can be described using $\sim_G$ (Theorem 4.9).

In Section 5, we begin to present new results in this paper. We further consider the structures given by (a) (b) (c) and show a relationship between the structures and the Gallai-Edmonds partition:

If the poset $(\mathcal{G}(G), \lhd)$ of a factorizable graph G has the minimum element G_0, then $V(G_0) = V(G) \setminus \bigcup_{x \in V(G)} C(G - x)$ (Theorem 5.1).

This theorem later plays a crucial role in the new proof of the cathedral theorem.

In Section 6, we consider saturated graphs and present a new proof of the cathedral theorem. Given a saturated elementary graph and a family of saturated graphs satisfying a certain condition, we can define an operation, the *cathedral construction*, that creates a new graph obtained from the given graphs by adding new edges. Here the given graphs are called the *foundation* and the family of *towers*, respectively. We consider the structures by (a) (b) (c) for saturated graphs and obtain the following:

If G is a saturated graph, then the poset $(\mathcal{G}(G), \lhd)$ has the minimum element G_0 (Lemma 6.5).

Moreover, G_0 and all connected components of $G - V(G_0)$ are saturated and they are well-defined as a foundation and towers (Lemmas 6.7 and 6.9). We show that G is the graph obtained from them by the cathedral construction (Theorem 6.3).

Conversely, if a graph G obtained by the cathedral construction from a foundation G_0 and some towers is saturated, and G_0 is the minimum element of the poset $(\mathcal{G}(G), \lhd)$ (Theorem 6.4).

By Theorems 6.3 and 6.4, the constructive characterization of the saturated graphs—the most important part of the cathedral theorem—is obtained. Additionally, the other parts of the cathedral theorem follow quite smoothly by Theorem 5.1 and the natures of the structures given by (a) (b) (c).

4. Canonical Structures of Factorizable Graphs

In this section we introduce canonical structures of factorizable graphs, which will later turn out to be the underlying structure of the cathedral theorem. They are composed mainly of three parts: a partial order on factor-connected components (Theorem 4.3), a generalization of the canonical partition (Theorem 4.5), and a relationship between them (Theorem 4.9).

Definition 4.1. Let G be a factorizable graph. A set $X \subseteq V(G)$ is *separating* if any $H \in \mathcal{G}(G)$ satisfies $V(H) \subseteq X$ or $V(H) \cap X = \emptyset$.

It is easy to see that the following four statements are equivalent for a factorizable graph G and $X \subseteq V(G)$:

(i) The set X is separating.
(ii) Either $X = \emptyset$ or there exist $H_1, \ldots, H_k \in \mathcal{G}(G)$ with $X = V(H_1)\dot\cup \cdots \dot\cup V(H_k)$.
(iii) For any perfect matching M of G, M contains a perfect matching of $G[X]$.
(iv) For any perfect matching M of G, $\delta_G(X) \cap M = \emptyset$.

Definition 4.2. Let G be a factorizable graph. We define a binary relation $\lhd$ on $\mathcal{G}(G)$ as follows: For $G_1, G_2 \in \mathcal{G}(G)$, $G_1 \lhd G_2$ if there exists $X \subseteq V(G)$ such that

1. X is separating,
2. $V(G_1) \cup V(G_2) \subseteq X$, and
3. $G[X]/V(G_1)$ is factor-critical.

For the above relation, the following theorem is known:

Theorem 4.3 (Kita [4,5])**.** *For any factorizable graph G, the binary relation $\lhd$ is a partial order on $\mathcal{G}(G)$.*

For example, consider the factorizable graph G in Figure 2. The factor-connected components form the poset $(\mathcal{G}(G) = \{G_1, \ldots, G_6\}, \lhd)$, given by the Hasse diagram in Figure 4.

Definition 4.4. Let G be a factorizable graph. We define a binary relation $\sim_G$ on $V(G)$ as follows: For $u, v \in V(G)$, $u \sim_G v$ if

1. u and v are contained in the same factor-connected component and
2. either u and v are identical, or $G - u - v$ is not factorizable.

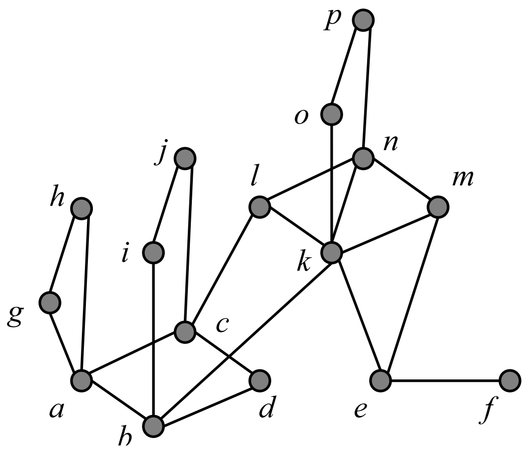

Figure 2: A factorizable graph G

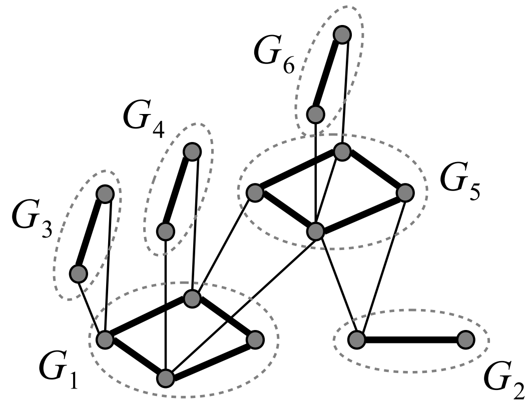

Figure 3: The factor-connected components of G

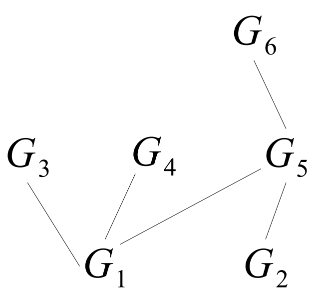

Figure 4: The Hasse diagram of $(\mathcal{G}(G), \triangleleft)$

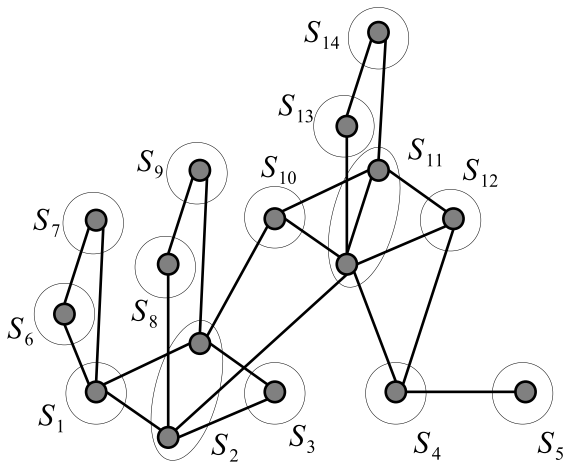

Figure 5: The generalized canonical partition of G

We also have a theorem for the relation $\sim_G$:

Theorem 4.5 (Kita [4, 5])**.** *For any factorizable graph G, the binary relation $\sim_G$ is an equivalence relation on $V(G)$.*

If a graph G is elementary, then the family of equivalence classes by $\sim_G$, i.e., $V(G)/\sim_G$ coincides with Kotzig's *canonical partition* [9–11, 13] (see [4, 5]). Therefore, given a factorizable graph G, we call $V(G)/\sim_G$ the *generalized canonical partition*, and denote it by $\mathcal{P}(G)$. By the definition of $\sim_G$, each member of $\mathcal{P}(G)$ is contained in some factor-connected component. Therefore, $\mathcal{P}_G(H) := \{S \in \mathcal{P}(G) : S \subseteq V(H)\}$ forms a partition of $V(H)$ for each $H \in \mathcal{G}(G)$.

For example, consider the factorizable graph G in Figure 2. Its generalized canonical partition is shown in Figure 5. Here the set of all the vertices is partitioned into fourteen parts $\{S_1, \ldots, S_{14}\} = \mathcal{P}(G)$, and $\{S_1, S_2, S_3\} = \mathcal{P}_G(G_1)$ forms a partition of $V(G_1)$, $\{S_4, S_5\} = \mathcal{P}_G(G_2)$ of $V(G_2)$, $\{S_6, S_7\} = \mathcal{P}_G(G_3)$ of $V(G_3)$, $\{S_8, S_9\} = \mathcal{P}_G(G_4)$ of $V(G_4)$, $\{S_{10}, S_{11}, S_{12}\} = \mathcal{P}_G(G_5)$ of $V(G_5)$, and $\{S_{13}, S_{14}\} = \mathcal{P}_G(G_6)$ of $V(G_6)$.

Note also the following, which are also stated in [4, 5]:

Fact 4.6. *Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$. Then, $\mathcal{P}_G(H)$ is a refinement of $\mathcal{P}(H) = \mathcal{P}_H(H)$; that is, if $u, v \in V(H)$ satisfies $u \sim_G v$, then $u \sim_H v$ holds.*

Proof. We prove the contrapositive. Let $u, v \in V(H)$ be such that $u \not\sim_H v$, which is equivalent to u and v satisfying $u \neq v$ and $H - u - v$ is factorizable. Let M be a perfect matching of $H - u - v$. Since $G - V(H)$ is also factorizable, by letting M' be a perfect matching of it, we can construct a perfect matching of $G - u - v$, namely, $M \cup M'$. Therefore, $u \not\sim_G v$. □

For example, consider the elementary subgraph G_1, which is given as one of the factor-connected components in Figure 3. It is a circuit with four vertices $\{a, b, c, d\}$, and it is easy to see that its canonical partition, $\mathcal{P}(G_1)$, is composed of two sets: $\mathcal{P}(G_1) = \{\{a, d\}, \{b, c\}\}$. Therefore, compared with Figure 5, the partition $\mathcal{P}_G(G_1)$, which equals $\{\{a\}, \{b, c\}, \{d\}\}$, indeed gives a refinement (actually a proper refinement in this case) of $\mathcal{P}(G_1)$.

The following fact can be immediately obtained by Property A3.

Fact 4.7. *Let G be a factorizable graph, and M be a perfect matching of G. Let $u, v \in V(G)$ be vertices contained in the same factor-connected component of G. Then, $u \sim_G v$ if and only if there is no M-saturated path between u and v.*

Definition 4.8. Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$. We denote the upper bounds of H in the poset $(\mathcal{G}(G), \triangleleft)$ by $\mathcal{U}^*_G(H)$; that is, $\mathcal{U}^*_G(H) := \{H' \in \mathcal{G}(G) : H \triangleleft H'\}$. We define $\mathcal{U}_G(H) := \mathcal{U}^*_G(H) \setminus \{H\}$, and the vertices contained in $\mathcal{U}^*_G(H)$ (resp. $\mathcal{U}_G(H)$) as $U^*_G(H)$ (resp. $U_G(H)$); i.e., $U^*_G(H) := \bigcup_{H' \in \mathcal{U}^*_G(H)} V(H')$ and $U_G(H) := \bigcup_{H' \in \mathcal{U}_G(H)} V(H')$. We often omit the subscripts "G" if they are apparent from contexts.

There is a relationship between the partial order and the generalized canonical partition:

Theorem 4.9 (Kita [4, 5])**.** *Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$. Let K be one of the connected components of $G[U(H)]$. Then, there exists $S_K \in \mathcal{P}_G(H)$ such that $N_G(K) \cap V(H) \subseteq S_K$.*

In the above theorem, $U(H)$ and $\mathcal{P}_G(H)$ are notions determined by $\triangleleft$ and $\sim_G$, respectively. Therefore, Theorem 4.9 describes a relationship between $\triangleleft$ and $\sim_G$.

Example 4.10. Consider the factor-connected component G_1 in Figure 3. We have $\mathcal{U}^*(G_1) = \{G_1, G_3, G_4, G_5, G_6\}$ and $\mathcal{U}(G_1) = \{G_3, G_4, G_5, G_6\}$, while $U^*(G_1) = \{a, \ldots, d, g, \ldots p\}$ and $U(G_1) = \{g, \ldots, p\}$. Therefore, as indicated in Figure 6, $G[U(G_1)]$ has three connected

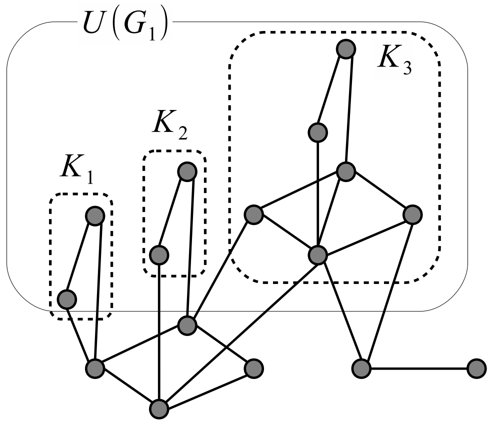

Figure 6: The strict upper bounds of G_1

components K_1, K_2, and K_3, and they satisfy $N(K_1) \cap V(G_1) \subseteq S_1$, $N(K_2) \cap V(G_1) \subseteq S_2$, and $N(K_3) \cap V(G_1) \subseteq S_2$. ■

Let us add some propositions used later in this paper.

Proposition 4.11 (Kita [4,5]). *Let G be a factorizable graph and M be a perfect matching of G, and let $H \in \mathcal{G}(G)$. Then, for any M-ear P relative to H, its end vertices $u, v \in V(H)$ satisfy $u \sim_G v$.*

Theorem 4.12 (Kita [4,5]). *Let G be a factorizable graph, $G_1 \in \mathcal{G}(G)$ be a minimal element of the poset $(\mathcal{G}(G), \lhd)$, and $G_2 \in \mathcal{G}(G)$ be such that $G_1 \lhd G_2$ does not hold. Then, G has (possibly identical) complement edges e, f joining $V(G_1)$ and $V(G_2)$ such that $\mathcal{G}(G+e+f) = \mathcal{G}(G)$ and $G_1 \lhd G_2$ in $(\mathcal{G}(G+e+f), \lhd)$.*

Theorem 4.12 will play a crucial role in Section 6 when we show that the poset by $\lhd$ has the minimum element if a given graph is saturated (Lemma 6.5).

5. Factorizable Graphs through the Gallai-Edmonds Partition

In this section, we present a new result on a relationship between the Gallai-Edmonds partition and the canonical structures of factorizable graphs in Section 4. As we later see in Section 6, Theorem 5.1 can be regarded as a generalization of a part of the statements of the cathedral theorem.

Theorem 5.1. *Let G be a factorizable graph such that the poset $(\mathcal{G}(G), \lhd)$ has the minimum element G_0. Then, $V(G_0)$ is exactly the set of vertices that is disjoint from $C(G-x)$ for any $x \in V(G)$; that is, $V(G_0) = V(G) \setminus \bigcup_{x \in V(G)} C(G-x)$.*

To show Theorem 5.1, we give some definitions and lemmas. Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Based on Theorem 4.9, we denote the set of all the strict upper bounds of H "assigned" to S by $\mathcal{U}_G(S)$; that is to say, $H' \in \mathcal{U}_G(S)$ if and only if $H' \in \mathcal{U}(H)$ and there is a connected component K of $G[U(H)]$ such that $V(H') \subseteq V(K)$ and $N_G(K) \cap V(H) \subseteq S$. We define $U_G(S) := \bigcup_{H' \in \mathcal{U}_G(S)} V(H')$ and $U^*_G(S) := U_G(S) \cup S$. We often omit the subscripts "G" if they are apparent from contexts. Note that $\mathcal{U}(H) = \bigcup_{S \in \mathcal{P}_G(H)} \mathcal{U}(S)$. For example, consider the graph given in Figure 2; we have $\mathcal{U}(S_2) = \{G_4, G_5, G_6\}$ and $U(S_2) = \{i, \ldots, p\}$.

Next, we present fundamental results on factorizable graphs.

Proposition 5.2 (Kita [4,5])**.** *If H is an elementary graph, then for any $u, v \in V(H)$ there is an M-saturated path between u and v, or there is an M-balanced path from u to v, where M is an arbitrary perfect matching of H.*

Lemma 5.3 (Kita [6,7])**.** *Let G be a factorizable graph, and M be a perfect matching of G. Let $H \in \mathcal{G}(G)$, $S \in \mathcal{P}_G(H)$, and $T \in \mathcal{P}_G(H) \setminus \{S\}$.*

(i) *For any $u \in U^*(S)$, there is an M-balanced path from u to some vertex $v \in S$ whose vertices except v are in $U(S)$.*
(ii) *For any $u \in S$ and $v \in U^*(T)$, there is an M-saturated path between u and v whose vertices are all contained in $U^*(H) \setminus U(S)$.*
(iii) *For any $u \in S$ and $v \in U(S)$, there are neither M-saturated paths between u and v nor M-balanced paths from u to v.*
(iv) *For any $u, v \in S$, there is no M-saturated path between u and v, while there is an M-balanced path from u to v.*

Proof. The statements (i), (ii), and (iii) are stated in [6,7]. The statement (iv) is immediately obtained by combining Fact 4.7 and Proposition 5.2. □

By Proposition 5.2 and Lemma 5.3, the next lemma follows.

Lemma 5.4. *Let G be a factorizable graph, and M be a perfect matching of G. Let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Then, the following hold:*

(i) *For any $u \in U(S)$ and $v \in U^*(H) \setminus U^*(S)$, there is an M-saturated path between u and v.*
(ii) *For any $u \in U(S)$ and $v \in S$, there is no M-saturated path between u and v; however, there is an M-balanced path from u to v.*
(iii) *For any $w \in S$ and $v \in U^*(H) \setminus U^*(S)$, there is an M-saturated path between w and v.*
(iv) *For any $w, v \in S$, there is no M-saturated path between w and v; however, there is an M-balanced path from w and v.*
(v) *For any $w \in S$ and $v \in U(S)$, there is neither an M-saturated path between w and v nor an M-balanced path from w to v.*

Proof. The statements (iii), (iv), and (v) are immediate from (ii), (iv), and (iii) of Lemma 5.3, respectively.

For (i), let P_1 be an M-balanced path from u to some vertex $x \in S$ such that $V(P_1) \setminus \{x\} \subseteq U(S)$, given by (i) of Lemma 5.3. By (ii) of Lemma 5.3, there is an M-saturated path P_2 between x and v such that $V(P_2) \subseteq U^*(H) \setminus U(S)$. Hence, the path obtained by adding P_1 and P_2 forms an M-saturated path between u and v, and (i) follows.

The first and the latter halves of (ii) are restatements of (iii) and (i) of Lemma 5.3, respectively. □

By comparing Proposition 2.3 and Lemma 5.4, the next lemma follows.

Lemma 5.5. *Let G be a factorizable graph such that the poset $(\mathcal{G}(G), \triangleleft)$ has the minimum element G_0. Let $S \in \mathcal{P}_G(G_0)$.*

(i) *If $x \in U(S)$, then $D(G-x) \supseteq U^*(G_0) \setminus U^*(S)$, $A(G-x) \cup \{x\} \supseteq S$, and $C(G-x) \subseteq U(S)$.*
(ii) *If $x \in S$, then $D(G-x) = U^*(G_0) \setminus U^*(S)$, $A(G-x) \cup \{x\} = S$, and $C(G-x) = U(S)$.*

Proof. The claims are all obtained by comparing the reachabilities of alternating paths regarding Proposition 2.3 and Lemma 5.4. Let $x \in U(S)$. By Proposition 2.3 (i) and Lemma 5.4 (i), we have $D(G-x) \supseteq U^*(G_0) \setminus U^*(S)$. It also follows that $A(G-x) \cup \{x\} \supseteq S$ by a similar argument, comparing Proposition 2.3 (ii) and Lemma 5.4 (ii). Therefore, since

$V(G) = D(G)\dot{\cup}A(G)\dot{\cup}C(G) = (U^*(G_0) \setminus U^*(S))\dot{\cup}S\dot{\cup}U(S)$, we have $C(G-x) \subseteq U(S)$, and we are done for (i). The statement (ii) also follows by similar arguments with Proposition 2.3 and Lemma 5.4 (iii) (iv) (v). □

Now we can prove Theorem 5.1 using Lemma 5.5.

Proof of Theorem 5.1.

Claim 5.6. *For any $x \in V(G)$, $V(G_0) \cap C(G-x) = \emptyset$.*

Proof. Let $u \in V(G_0)$ and let $S \in \mathcal{P}_G(G_0)$ be such that $u \in S$. By Lemma 5.5, if $x \in U^*(S)$ then $u \in A(G-x)$, and if $x \in U^*(G_0) \setminus U^*(S)$ then $u \in D(G-x)$. Thus, anyway we have $u \notin C(G-x)$, and the claim follows. □

Claim 5.7. *For any $u \in V(G) \setminus V(G_0)$, there exists $x \in V(G)$ such that $u \in C(G-x)$.*

Proof. Let $u \in V(G) \setminus V(G_0)$ and let $S \in \mathcal{P}_G(G_0)$ be such that $u \in U(S)$. Then, for any $x \in S$, we have $u \in C(G-x)$ by Lemma 5.5. Thus, we have the claim. □

By Claims 5.6 and 5.7, we obtain the theorem. □

As we mentioned in the outline given in Section 3, we will obtain in Section 6 that if a graph is saturated then the poset by $\triangleleft$ has the minimum element. Thus, the above theorem, Theorem 5.1, will turn out to be regarded as a generalized version of the part of the cathedral theorem related to the Gallai-Edmonds partition.

6. Another Proof of the Cathedral Theorem

6.1. The cathedral theorem

The *cathedral theorem* is a structure theorem of saturated graphs, originally given by Lovász [12, 13], and later Szigeti gave another proof [15, 16]. In this section, we give yet another proof as a consequence of the structures given in Section 4. For convenience, we treat empty graphs as factorizable and saturated.

Definition 6.1 (The Cathedral Construction)**.** Let G_0 be a saturated elementary graph and let $\{G_S\}_{S\in\mathcal{P}(G_0)}$ be a family of saturated graphs, some of which might be empty. For each $S \in \mathcal{P}(G_0)$, join every vertex in S and every vertex of G_S. We call this operation the *cathedral construction.* Here G_0 and $\{G_S\}_{S\in\mathcal{P}(G_0)}$ are respectively called the *foundation* and the family of *towers*.

Figures 7, 8, 9 show examples of the cathedral construction. In Figure 8, the graph G_0 is an elementary saturated graph with the canonical partition $\mathcal{P}(G_0) = \{S, T, R\}$, and the graphs G_S, G_T, G_R are saturated graphs such that G_S and G_R are respectively elementary and non-elementary while G_T is an empty graph. If we conduct the cathedral construction with the foundation G_0 and the family of towers $\mathcal{T} = \{G_S, G_T, G_R\}$, we obtain the saturated graph $\tilde{G}$ in Figure 7. Moreover, Figure 9 shows that if we conduct the cathedral construction with the foundation H_0 with $\mathcal{P}(H_0) = \{P, Q\}$ and the family of towers $\{H_P, H_Q\}$, where H_P is an elementary saturated graph and H_Q is an empty graph, then we obtain the saturated graph G_R. (Therefore, in other words, the graph $\tilde{G}$ is constructed by a repetition of the cathedral construction using the elementary saturated graphs H_0, H_P, G_0, and G_S as fundamental building blocks.)

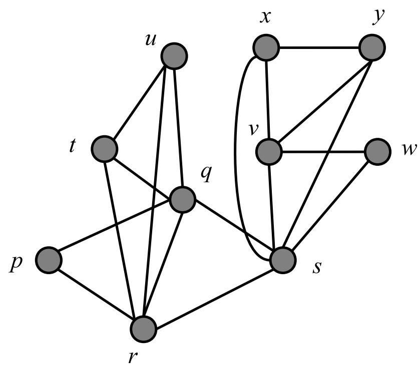

Figure 7: A saturated graph $\tilde{G}$

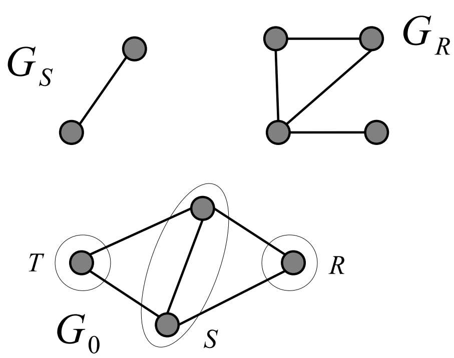

Figure 8: The foundation and the towers that create $\tilde{G}$

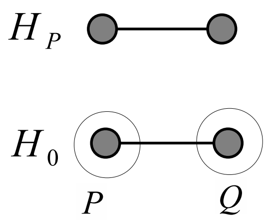

Figure 9: The foundation and the towers that create G_R

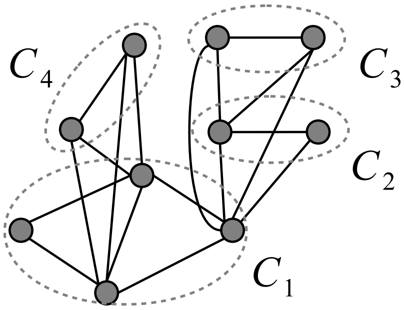

Figure 10: The factor-connected components of $\tilde{G}$

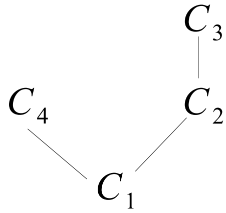

Figure 11: The Hasse diagram of $(\mathcal{G}(\tilde{G}), \triangleleft)$

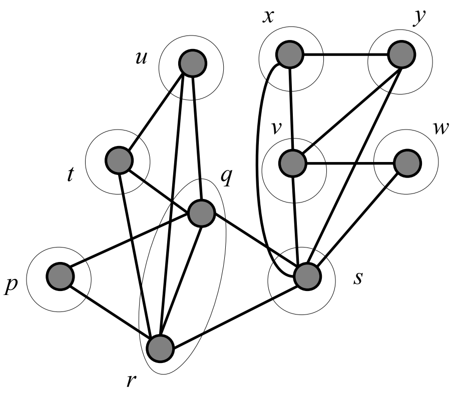

Figure 12: The generalized canonical partition of $\tilde{G}$

Theorem 6.2 (The Cathedral Theorem [12, 13])**.** *A factorizable graph G is saturated if and only if it is constructed from smaller saturated graphs by the cathedral construction. In other words, if a factorizable graph G is saturated, then there is a subgraph G_0 and a family of subgraphs $\mathcal{T}$ of G which are well-defined as a foundation and a family of towers, and G is the graph constructed from G_0 and $\mathcal{T}$ by the cathedral construction; conversely, if G is a graph obtained from a foundation and towers by the cathedral construction, then G is saturated.*

Additionally, if G is a saturated graph obtained from a foundation G_0 and a family of towers $\mathcal{T} = \{G_S\}_{S\in\mathcal{P}(G_0)}$ by the cathedral construction, then,

(i) *$e \in E(G)$ is allowed if and only if it is an allowed edge of G_0 or G_S for some $S \in \mathcal{P}(G_0)$,*

(ii) *such G_0 uniquely exists; that is, if G can be obtained from a foundation G_0' and a family of towers $\mathcal{T}'$ by the cathedral construction, then $V(G_0) = V(G_0')$ holds, and*

(iii) *$V(G_0)$ is exactly the set of vertices that is disjoint from $C(G-x)$ for any $x \in V(G)$.*

In the cathedral construction, each tower is saturated. Therefore, the first sentence of Theorem 6.2 reveals a nested or inductive structure and gives a constructive characterization of the saturated graphs by the cathedral construction. In this characterization, the elementary saturated graphs are the fundamental building blocks. Theorem 6.2 (i) tells that a set of edges in a saturated graph G is a perfect matching if and only if it is a disjoint union of perfect matchings of the foundation and the towers that create G. Theorem 6.2 (ii) tells that for each saturated graph, the way to construct it uniquely exists, and (iii) shows a relationship between the cathedral construction and the Gallai-Edmonds partition.

In the new proof, the following two theorems, Theorems 6.3 and 6.4, together with Theorem 5.1, will serve as nuclei, referring to the special features of the poset and the canonical partition for saturated graphs.

Theorem 6.3. *If a factorizable graph G is saturated, then the poset $(\mathcal{G}(G), \lhd)$ has the minimum element, say G_0, and it satisfies $\mathcal{P}_G(G_0) = \mathcal{P}(G_0) =: \mathcal{P}_0$. Additionally, for each $S \in \mathcal{P}_0$, the connected component G_S of $G - V(G_0)$ such that $N_G(G_S) \subseteq S$ exists uniquely or is an empty graph, and G is the graph obtained from the foundation G_0 and the family of towers $\mathcal{T} := \{G_S\}_{S\in\mathcal{P}_0}$ by the cathedral construction.*

Theorem 6.4. *Let G_0 be a saturated elementary graph, and $\mathcal{T} := \{G_S\}_{S\in\mathcal{P}(G_0)}$ be a family* *of saturated graphs. Let G be the graph obtained from the foundation G_0 and the family of towers $\mathcal{T}$ by the cathedral construction. Then, G is saturated, G_0 forms a factor-connected component of G, that is, $G[V(G_0)] \in \mathcal{G}(G)$, and it is the minimum element of the poset $(\mathcal{G}(G), \lhd)$.*

In the remaining part of this paper, we are going to prove Theorem 6.3 and Theorem 6.4 and then obtain Theorem 6.2. With Theorem 6.3 and Theorem 6.4, we obtain the constructive characterization of the saturated graphs. We also obtain a new characterization of foundations and families of towers, which gives a clear comprehension of saturated graphs by the canonical structures of factorizable graphs in Section 4. Thanks to this new characterization, the remaining statements of the cathedral theorem will be obtained quite smoothly.

6.2. Proof of Theorem 6.3

Here we show some lemmas etc. to show that any saturated graph is constructed by the cathedral construction and prove Theorem 6.3.

Lemma 6.5. *If a factorizable graph G is saturated, then the poset $(\mathcal{G}(G), \lhd)$ has the minimum element.*

Proof. Suppose the claim fails, that is, the poset has distinct minimal elements $G_1, G_2 \in \mathcal{G}(G)$. Then, by Theorem 4.12, there exist possibly identical complement edges e, f joining $V(G_1)$ and $V(G_2)$ such that $\mathcal{G}(G+e+f) = \mathcal{G}(G)$. This means that adding e or f to G does not create any new perfect matchings, which contradicts G being saturated. □

In order to obtain Theorem 6.3, by letting G be a saturated graph, we show in the following that the minimum element G_0 of the poset by $\lhd$ and the connected components of $G - V(G_0)$ are well-defined as a foundation and towers of the cathedral construction and G is the graph obtained by the cathedral construction with them.

The next fact is easy to see from Fact 4.7 and Property A5. We will use this fact in the proofs of Lemma 6.7 and Lemma 6.9 later.

Fact 6.6. *Let G be a saturated graph, and let $H \in \mathcal{G}(G)$. Then, for any $u, v \in V(H)$ with $u \sim_G v$, $uv \in E(G)$.*

Next, we give the following lemma, which will contribute to the proofs of both of Theorems 6.3 and 6.4, actually.

Lemma 6.7. *Let G be a saturated graph, and let $G_0 \in \mathcal{G}(G)$. Then, $\mathcal{P}_G(G_0) = \mathcal{P}(G_0)$.*

Proof. Since we know by Fact 4.6 that $\mathcal{P}_G(G_0)$ is a refinement of $\mathcal{P}(G_0)$, it suffices to prove that $\mathcal{P}(G_0)$ is a refinement of $\mathcal{P}_G(G_0)$, that is, if $u \sim_{G_0} v$, then $u \sim_G v$. We prove the contrapositive of this.

Let $u, v \in V(G_0)$ with $u \not\sim_G v$. Let M be a perfect matching of G. By Fact 4.7, there are M-saturated paths between u and v; let P be a shortest one. Suppose $E(P) \setminus E(G_0) \neq \emptyset$, and let Q be one of the connected components of $P - E(G_0)$, with end vertices x and y. Since Q is an M-ear relative to G_0 by Property A4, $x \sim_G y$ follows by Proposition 4.11. Therefore, $xy \in E(G)$ by Fact 6.6, which means we can get a shorter M-saturated path between u and v by replacing Q by xy on P, a contradiction. Thus, we have $E(P) \setminus E(G_0) = \emptyset$; that is, P is a path of G_0. Accordingly, $u \not\sim_{G_0} v$ by Fact 4.7. □

As we mention in Fact 4.6, for a factorizable graph G and $H \in \mathcal{G}(G)$, $\mathcal{P}_G(H)$ is generally a refinement of $\mathcal{P}(H)$. However, the above lemma states that if G is a saturated graph then they coincide. Therefore this lemma associates the generalized canonical partition with the cathedral theorem.

Next, note the following fact, which we present to prove Lemma 6.9:

Fact 6.8. *If a factorizable graph G is saturated, G is connected.*

Proof. Suppose the claim fails, that is, G has two distinct connected components, K and L. Let $u \in V(K)$ and $v \in V(L)$, and let M be a perfect matching of G. By Property A5, there is an M-saturated path between u and v, contradicting the hypothesis that K and L are distinct. □

Before reading Lemma 6.9, note that if a factorizable graph G has the minimum element G_0 for the poset $(\mathcal{G}(G), \triangleleft)$, then for each connected component K of $G - V(G_0)$, $N_G(K) \subseteq V(G_0)$ holds.

Lemma 6.9. *Let G be a saturated graph, and G_0 be the minimum element of the poset $(\mathcal{G}(G), \triangleleft)$. Then, G_0 and the connected components of $G - V(G_0)$ are each saturated. Additionally, for each $S \in \mathcal{P}_G(G_0)$, a connected component K of $G - V(G_0)$ such that $N_G(K) \subseteq S$ exists uniquely or does not exist.*

Proof. We first prove that G_0 is saturated. Let $e = xy$ be a complement edge of G_0. By the contrapositive of Fact 6.6, $x \not\sim_G y$, which means $x \not\sim_{G_0} y$ by Lemma 6.7. Therefore, by Fact 4.7 and Property A5, the complement edge e creates a new perfect matching if it is added to G_0. Hence, G_0 is saturated.

Now we move on to the remaining claims. Take $S \in \mathcal{P}_G(G_0)$ arbitrarily, and let $K_1, \ldots, K_l$ be the connected components of $G - V(G_0)$ which satisfy $N_G(K_i) \subseteq S$ for each $i = 1, \ldots, l$. Let $\hat{K} := G[V(K_1)\dot{\cup} \cdots \dot{\cup} V(K_l)]$.

We are going to obtain the remaining claims by showing that $\hat{K}$ is saturated. Now let $e = xy$ be a complement edge of $\hat{K}$, i.e., $x, y \in V(\hat{K})$ and $xy \notin E(\hat{K})$. Let M be a perfect matching of G. With Property A5, in order to show that $\hat{K}$ is saturated it suffices to prove that there is an M-saturated path between x and y in $\hat{K}$. Since G is saturated, there is an M-saturated path P between x and y in G by Property A5.

Obviously by the definition, $N_G(\hat{K}) \subseteq S$; on the other hand, $V(G) \setminus V(\hat{K})$ is of course a separating set. Therefore, if $E(P) \setminus E(\hat{K}) \neq \emptyset$, each connected component of $P - V(\hat{K})$ is an M-saturated path, both of whose end vertices are contained in S, by Property A4. This contradicts Fact 4.7. Hence, $E(P) \subseteq E(\hat{K})$, which means $\hat{K}$ is itself saturated. Thus, by Fact 6.8, it follows $\hat{K}$ is connected, which is equivalent to $l = 1$. This completes the proof. □

By Lemma 6.7 and Lemma 6.9, it follows that G_0 is well-defined as a foundation and the connected components of $G - V(G_0)$ are well-defined as towers (of course if indices out of $\mathcal{P}(G_0)$ are assigned to them appropriately).

Lemma 6.10. *Let G be a saturated graph, and G_0 be the minimum element of the poset $(\mathcal{G}(G), \triangleleft)$, and let K be a connected component of $G - V(G_0)$, whose neighbors are in $S \in \mathcal{P}_G(G_0)$. Then, for any $u \in V(K)$ and for any $v \in S$, $uv \in E(G)$.*

Proof. Suppose the claim fails, that is, there are $u \in V(K)$ and $v \in S$ such that $uv \notin E(G)$. Then, by Property A5, there is an M-saturated path between u and v, where M is an arbitrary perfect matching of G. By the definitions, $V(K) \subseteq U(S)$; therefore, $u \in U(S)$. Hence, this contradicts (iii) of Lemma 5.3, and we have the claim. □

Now we are ready to prove Theorem 6.3:

Proof of Theorem 6.3. The first sentence of Theorem 6.3 is immediate from Lemma 6.5 and Lemma 6.7. The former of the second sentence is also immediate by Lemma 6.9.

For the remaining claim, first note that by Lemma 6.9, G_0 and any G_S are saturated. Therefore, G_0 and $\mathcal{T} = \{G_S\}_{S\in\mathcal{P}_0}$ are well-defined as a foundation and a family of towers of the cathedral construction.

By the definition, for each $S \in \mathcal{P}_0$, it follows that $N_G(G_S) \subseteq S$. Additionally by Lemma 6.10 every vertex of $V(G_S)$ and every vertex of S are joined. Therefore, it follows that G has a saturated subgraph G' obtained from G_0 and $\mathcal{T}$ by the cathedral construction. Moreover, by Theorem 4.9, for each connected component K of $G - V(G_0)$ there exists $S \in \mathcal{P}_0$ such that $N(K) \subseteq S$; in other words, K denotes the same subgraph of G as G_S. Hence, $V(G) = V(G_0) \cup \bigcup_{S\in\mathcal{P}_0} V(G_S)$ holds and actually G' is G. Thus, G is the graph obtained from G_0 and $\mathcal{T}$ by the cathedral construction. □

6.3. Proof of Theorem 6.4

Next we consider the graphs obtained by the cathedral construction and show Theorem 6.4, which states that the foundations of them are the minimum elements of the posets by $\lhd$.

Since the necessity of the first claim of Theorem 6.2, the next proposition, is not so hard (see [13]), we here present it without a proof.

Proposition 6.11 (Lovász [12, 13])**.** *Let G_0 be a saturated elementary graph, and $\mathcal{T} = \{G_S\}_{S\in\mathcal{P}(G_0)}$ be a family of saturated graphs. Then, the graph G obtained from the foundation G_0 and the family of towers $\mathcal{T}$ by the cathedral construction is saturated.*

We give one more lemma:

Lemma 6.12. *Let G be a saturated graph, obtained from the foundation G_0 and the family of towers $\{G_S\}_{S\in\mathcal{P}(G_0)}$ by the cathedral construction. Then, $G' := G/V(G_0)$ is factor-critical.*

Proof. Let M^S be a perfect matching of G_S for each $S \in \mathcal{P}(G_0)$, and let $M := \bigcup_{S\in\mathcal{P}(G_0)} M^S$. Then, M forms a near-perfect matching of G', exposing only the contracted vertex g_0 corresponding to $V(G_0)$. Take $u \in V(G') \setminus \{g_0\}$ arbitrarily and let u' be the vertex such that $uu' \in M$. Since $uu' \in M \cap E(G')$ and $u'g_0 \in E(G') \setminus M$, there is an M-balanced path from u to g_0 in G', namely, the one with edges $\{uu', u'g_0\}$. Thus, by Property A1, G' is factor-critical. □

Now we shall prove Theorem 6.4:

Proof of Theorem 6.4. By Proposition 6.11, G is saturated. Since we have Lemma 6.12, in order to complete the proof, it suffices to prove $G_0 \in \mathcal{G}(G)$. Let p be the number of non-empty graphs in $\mathcal{T}$. We proceed by induction on p. If $p = 0$, the claim obviously follows. Let $p > 0$ and suppose the claim is true for $p - 1$. Take a non-empty graph G_S from $\mathcal{T}$, and let $G' := G - V(G_S)$. Then, G' is the graph obtained by the cathedral construction with G_0 and $\mathcal{T} \setminus \{G_S\} \cup \{H_S\}$, where H_S is an empty graph. Therefore, Proposition 6.11 yields that G' is saturated, and the induction hypothesis yields that $G_0 \in \mathcal{G}(G')$ and G_0 is the minimum element of the poset $(\mathcal{G}(G'), \lhd)$. Thus, by Lemma 6.7,

Claim 6.13. $\mathcal{P}_{G'}(G_0) = \mathcal{P}(G_0)$.

Let M' be a perfect matching of G' and M^S be a perfect matching of G_S, and construct a perfect matching $M := M' \cup M^S$ of G.

Claim 6.14. *No edge of $E_G[S, V(G_S)]$ is allowed in G.*

Proof. Suppose the claim fails, that is, an edge $xy \in E_G[S, V(G_S)]$ is allowed in G. Then, there is an M-saturated path Q between x and y by Property A2, and $Q[V(G')]$ is an M-saturated path by Property A4. Moreover, since $N_G(G_S) \cap V(G') \subseteq S$, it follows that $Q[V(G')]$ is an M-saturated path of G' between two vertices in S. With Fact 4.7 this is a contradiction, because $S \in \mathcal{P}_{G'}(G_0)$ by Claim 6.13. Hence, we have the claim. □

By Claim 6.14, it follows that a set of edges is a perfect matching of G if and only if it is a disjoint union of a perfect matching of G' and G_S. Thus, G_0 forms a factor-connected component of G, and we are done. □

6.4. Proof of Theorem 6.2 and an example

Now we can prove the cathedral theorem, combining Theorems 6.3, 6.4, and 5.1:

Proof of Theorem 6.2. By Proposition 6.11 and Theorem 6.3, the first claim of Theorem 6.2 is proved. The statement (i) is by Theorem 6.4, since it states that $G_0 \in \mathcal{G}(G)$. The statement (ii) is also by Theorem 6.4, since the poset $(\mathcal{G}(G), \triangleleft)$ is a canonical notion. The statement (iii) is by combining Theorem 6.4 and Theorem 5.1. □

Example 6.15. The graph $\tilde{G}$ in Figure 7 consists of four factor-connected components, say $C_1, \ldots, C_4$ in Figure 10, and Figure 11 shows the Hasse diagram of $(\mathcal{G}(\tilde{G}), \triangleleft)$, which has the minimum element C_1, as stated in Lemma 6.5. Figure 12 indicates the generalized canonical partition of $\tilde{G}$:

$\mathcal{P}(\tilde{G}) = \{\{p\}, \{q, r\}, \{s\}, \{t\}, \{u\}, \{v\}, \{w\}, \{x\}, \{y\}\}$.

Here we have $\mathcal{P}_{\tilde{G}}(C_i) = \mathcal{P}(C_i)$ for each $i = 1, \ldots, 4$, as stated in Lemma 6.7. From these two figures we see examples for other statements on the saturated graphs in this section. ■

7. Concluding Remarks

Finally, we give some remarks.

Remark 7.1. Theorems 6.3 and 6.4 can be regarded as a refinement, and Theorem 5.1 as a generalization of Theorem 6.2, from the point of view of the canonical structures of Section 4. ■

Remark 7.2. The poset $(\mathcal{G}(G), \triangleleft)$ and $\mathcal{P}(G)$ can be computed in $O(|V(G)| \cdot |E(G)|)$ time [4, 5], where G is any factorizable graph. Therefore, given a saturated graph, we can also find how it is constructed by iterating the cathedral construction in the above time by computing the associated poset and the generalized canonical partition. ■

Remark 7.3. The canonical structures of general factorizable graphs in Section 4 can be obtained without the Gallai-Edmonds structure theorem nor the notion of barriers. The other properties we cite in this paper to prove the cathedral theorem are also obtained without them. Therefore, our proof shows that the cathedral theorem holds without assuming either of them. ■

With the whole proof, we can conclude that the structures in Section 4 is what essentially underlie the cathedral theorem. We see how a factorizable graph leads to a saturated graph having the same family of perfect matchings by sequentially adding complement edges. Our proof is quite a natural one because the cathedral theorem—a characterization of a class of graphs defined by a kind of edge-maximality "saturated"—is derived as a consequence of considering edge-maximality over the underlying general structure. We hope yet more would be found on the field of counting the number of prefect matchings with the results in this paper and [4–7].

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Note. The statements in [5] or [7] can be also found in [4] or [6], respectively.

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Appendix: Basic Properties on Matchings

Here we present some basic properties about matchings. These are easy to observe and some of them might be regarded as folklores.

Property A1. *Let M be a near-perfect matching of a graph G that exposes $v \in V(G)$. Then, G is factor-critical if and only if for any $u \in V(G)$ there exists an M-balanced path from u to v.*

Proof. Take $u \in V(G)$ arbitrarily. Since G is factor-critical, there is a near-perfect matching M' of G exposing only u. Then, $G.M\triangle M'$ is an M-balanced path from u to v, and the sufficiency part follows.

Now suppose there is an M-balanced path P from u to v. Then, $M\triangle E(P)$ is a near-perfect matching of G exposing u. Hence, the necessity part follows. □

Property A2. *Let G be a factorizable graph, M be a perfect matching of G, and $e = xy \in E(G)$ be such that $e \notin M$. The following three properties are equivalent:*

(i) *The edge e is allowed in G.*
(ii) *There is an M-alternating circuit C such that $e \in E(C)$.*
(iii) *There is an M-saturated path between x and y.*

Proof. We first show that (i) and (ii) are equivalent. Let M' be a perfect matching of G such that $e \in M'$. Then, $G.M\triangle M'$ has a connected component which is an M-alternating circuit containing e. Hence, (i) yields (ii).

Now let $L := M\triangle E(C)$. Then, L is a perfect matching of G such that $e \in L$. Hence, (ii) yields (i); consequently, they are equivalent.

Since (ii) and (iii) are obviously equivalent, now we are done. □

Property A3. *Let G be a factorizable graph and M be a perfect matching of G, and let $u, v \in V(G)$. Then, $G - u - v$ is factorizable if and only if there is an M-saturated path of G between u and v.*

Proof. For the sufficiency part, let M' be a perfect matching of $G - u - v$. Then, $G.M\triangle M'$ has a connected component which is an M-saturated path between u and v. For the necessity part, let P be an M-saturated path between u and v. Then, $M\triangle E(P)$ is a perfect matching of $G - u - v$, and we are done. □

The next one follows easily from the definition of the separating sets (Definition 4.1).

Property A4. *Let G be a factorizable graph and M be a perfect matching of G. Let $X \subseteq V(G)$ be a separating set and P be an M-saturated path. Then,*

(i) *each connected component of $P[X]$ is an M-saturated path, and*
(ii) *any connected component of $P - E(G[X])$ that does not contain any end vertices of P is an M-ear relative to X.*

The next property is immediate by Property A2 and is used frequently in Section 6.

Property A5. *Let G be a factorizable graph, M be a perfect matching, and $x, y \in V(G)$ be such that $xy \notin E(G)$. Then, the following properties are equivalent:*

(i) *The complement edge xy creates a new perfect matching in $G + xy$.*
(ii) *The edge xy is allowed in $G + xy$.*
(iii) *There is an M-saturated path between x and y in G.*

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