OPTIMAL ORDERING RULE FOR A PERISHABLE PRODUCT WITH A DYNAMIC PRICING POLICY

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Abstract   This paper considers the problem of determining the optimal ordering quantity for a perishable product to be sold over a finite sales horizon with a dynamic pricing policy. To date, most models developed to study a dynamic pricing policy for perishable products assume that the salvage value is nonnegative. In this model we allow the salvage value to be either a nonnegative or negative value. We derive the conditions under which the optimal ordering quantity takes either a zero value or finite value greater than zero. Moreover, we demonstrate the existence of a shortest sales horizon under a condition for which ordering of the product is profitable if the seller's planned sales horizon is longer than the shortest sales horizon.

1. Introduction

In this paper we consider a product that must be sold within a specified time horizon. Typical examples include perishable fresh food products which physically decay as time elapses, fashion garments which become outdated due to shifts in consumer preferences, and personal computers which become obsolete due to rapid technological changes. With such products, sellers will normally lower their prices gradually as time passes rather than adopt a fixed price throughout the sales horizon. Since the actual demand for such products is unpredictable, the seller faces the risk of having leftover items at the end of the sales horizon, that is the deadline, if the quantity ordered exceeds the actual demand over the sales horizon. Items remaining unsold at the deadline may be salvaged at a giveaway price. Similarly, some of the products such as fresh food products which become unsafe for consumption by the deadline will have to be discarded as waste by paying a disposal cost. This situation gives rise to the problem of determining the optimal ordering quantity at the start of the sales horizon and the optimal prices to charge dynamically over the entire horizon so as to maximize the total expected net profit gained.

There is ample literature on the problem of determining the optimal price and ordering quantity for selling perishable products. Here we restrict our attention to the literature on the integration of dynamic pricing and ordering quantity in which replenishment is not permitted, and refer the interested reader to Chan et al. [4] and Elmaghraby and Keskinocak [8] for discussions of the research on the joint pricing and inventory problem with replenishment. Gupta et al. [10] divided the literature on the integration of dynamic pricing and ordering quantity into two groups based on the approach used to model the demand for the perishable product. In the first group demand functions such as additive and multiplicative demand functions which depend on the selling price are given [2, 13], and the optimal ordering quantity is derived using these demand functions. The literature in the second group assumes that each arriving buyer has his own reservation price and the demand for the

product materializes if the reservation price of the arriving buyer is higher than the price offered by the seller [1, 3, 5, 7, 9, 10, 12, 14, 15]. Research papers in this group focus mainly on finding the optimal pricing policy. Brief discussions on the determination of the optimal ordering quantity can be found in [1, 5, 9].

The pricing problem presented in this paper is in line with those in the second group. That is, we model the demand for a product using the concept of the buyer's reservation price.

Most of the literature [1, 3, 5, 9, 10, 12, 15] that studied multi-period dynamic pricing models for perishable products assumes that the salvage value of unsold items at the deadline is nonnegative. In other words, revenue may be earned by selling the unsold items at the end of the sales horizon at a discounted price and no penalty is imposed. For instance, Monahan et al. [13] demonstrated the existence of the optimal ordering quantity for the finite planning horizon models where the salvage price is zero. Although many end-of-life perishable products can be sold at a discounted price or donated to charity, some sellers have to pay a fee such as a waste collection fee or composting fee to dispose of the unsaleable perishable products. The payment of the disposal fee is a cost to the seller and hence can be expected to affect both the optimal ordering quantity and profit gained. Of the past research papers, Ee and Ikuta [7] and You [14] addressed the dynamic pricing problem for selling an asset or a perishable product without replenishment in which the salvage value can take either a nonnegative or negative value. These two papers, however, did not discuss the derivation of the optimal ordering quantity. The model developed in this paper is a modification of the ones proposed in [7] and [14]. By assuming a salvage price which is not always nonnegative, we derive the optimal ordering rule for a perishable product without replenishment through the examination of the relationship between the optimal ordering quantity and the model parameters, such as, the buyer arriving probability, discount factor, salvage price, holding cost and purchasing cost.

Our main contribution is in considering the problem of determining the optimal ordering quantity when a dynamic pricing policy is employed and the salvage value can be either nonnegative or negative. We derive the conditions under which the ordering quantity takes either a zero value or finite value greater than zero. The ordering rule can then be used to assist the seller in determining whether or not ordering a product is profitable if the model parameters are known to the seller. If ordering a product is known to be unprofitable according to the conditions and for some reasons the seller would like to sell the product, the seller could then take steps to change the parameters to make the selling of the product profitable. In this case the conditions serve as a guideline in helping the seller determine how much effort would be required to make the selling of the product profitable. For example, a seller could contemplate volume buying by cooperating with other sellers to take advantage of quantity discount, thereby reducing the ordering cost per unit and increasing any profit that may be obtained.

In addition, we demonstrate that a shortest sales horizon exists under a condition for which the ordering of the product will result in a profitable return for the seller if its planned sales horizon is longer than the shortest sales horizon. The length of the seller's planned sales horizon of a product often depends on the product's market life or useful life. For instance, the planned sales horizon of ski apparel may cover the time horizon starting from the time when the ski apparel is on sale to the time when it has to exit the market at the end of the ski season. Therefore, if the planned sales horizon is shorter than the shortest sales horizon, the seller should not engage in selling the perishable product. Moreover, using a numerical example, we demonstrate that both optimal ordering quantity and maximum

total expected net profit gained reduce when the salvage value changes from a positive to negative value.

The rest of the paper is organized as follows. Section 2 presents the basic assumptions of the model. In Section 3 we derive the optimality equations for the model and in Section 4 we clarify the properties of the optimal ordering quantity. Finally, in Section 5 we present the overall conclusions of our research and suggest some further work.

2. Model Formulation

The model discussed in this paper is defined on the following assumptions:

(a) Consider the following discrete-time sequential stochastic decision problem of purchasing a certain quantity of items at a certain point in time and then selling them at a certain point in time that follows. The points in time are numbered backward from the final point in time of the planning horizon, time 0 (the deadline) as $0, 1, \cdots$ and so on. Accordingly, if time t is the present point in time, the two adjacent times $t+1$ and $t-1$ are the previous and next points in time, respectively. Let the time interval between times t and $t-1$ be called the *period t*, which is small enough so that only one buyer who requests one unit of the product may appear.

(b) A buyer who requests a unit of the product arrives with a probability λ $(0 < \lambda < 1)$ and the seller offers a selling price z. In this paper we assume that the appearing buyer does not possess prior information about the selling price, so his decision to visit a store is not influenced by the selling price. Accordingly, λ is independent of z.

(c) By w let us denote the reservation price of a buyer, implying that the buyer is willing to buy an item if and only if the selling price z offered for the item by the seller is lower than or equal to w, i.e., $z \leq w$. Here, assume that subsequent buyers' reservation prices w, $w', \cdots$ are independent, identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ. Also, let $f(w)$ denote its probability density function, which is truncated on both sides, and assume that

$$f(w) = 0, \ w < a, \qquad f(w) > 0, \ a \leq w \leq b, \qquad f(w) = 0, \ b < w,$$

for certain given numbers a and b such that $0 < a < b < \infty$; clearly $a < \mu < b$. Thus, the probability of an arriving buyer buying the item, provided that a price z is offered by the seller, is given by $p(z) = \Pr\{z \leq w\}$, where $0 \leq p(z) \leq 1$. Hence, $\lambda p(z)$ represents the probability of a buyer appearing and purchasing the item at the selling price z.

(d) Let $c > 0$ be the purchasing price per item and set $c < b$ as a natural assumption.

(e) Let $h \geq 0$ be the inventory holding cost per item remaining unsold for a period.

(f) An item remaining unsold at time 0, the deadline, can be sold at a salvage price $\rho \in (-\infty, c)$. Here, $\rho < 0$ implies the per-unit disposal cost to discard an unsold item.

(g) $\beta \in (0, 1)$ denotes the discount factor, where the monetary value of one unit a period hence is equivalent to β units at the present point in time.

The decision rules of the model consist of:

1. *Ordering rule* prescribing how many items to order at the time when the process starts.
2. *Pricing rule* prescribing what price to offer to an arriving buyer at each point in time.

The objective of the model is to find the optimal decision rule to maximize the total expected present discounted net profit over the planning horizon, i.e., the total expected

present discounted revenue *minus* the total expected present discounted purchasing costs *minus* the total expected present discounted holding cost *plus* the total expected present discounted salvage value.

3. Optimality Equations

Suppose that a certain quantity of a product had been purchased at a certain past point in time and that i items remain unsold at a time t after that. Let $u_t(i, 0)$ and $u_t(i, 1)$ be the maximum total expected present discounted profits, respectively, with no buyer and with a buyer. Then, clearly

$$u_0(i, 0) = \rho i, \qquad u_t(0, 0) = u_t(0, 1) = 0, \quad t \geq 0, \quad i \geq 0, \tag{3.1}$$

and

$$u_t(i, 0) = \beta\big(\lambda u_{t-1}(i, 1) + (1 - \lambda)u_{t-1}(i, 0)\big) - hi, \qquad t \geq 1, \quad i \geq 0. \tag{3.2}$$

$$u_t(i, 1) = \max_z \big\{p(z)\big(z + u_t(i - 1, 0)\big) + (1 - p(z))u_t(i, 0)\big\}, \qquad t \geq 0, \quad i \geq 1. \tag{3.3}$$

Optimal selling price   *The optimal selling price at time $t \geq 0$ with $i \geq 1$ items remaining unsold is given by the z attaining the maximum of the right hand side of Equation (3.3) if it exists, denoted by $z_t(i)$.*

By $v_t(i)$ let us define the maximum of the total expected present discounted net profit, provided that i items are ordered at time t with no buyer. Here, assume that there is no lead time between ordering and receiving a product, i.e., the delivery is instantaneous and that the items are sold at the optimal selling price at each point in time that follows up to time 0. Then we have

$$v_t(i) = u_t(i, 0) - ci, \quad \text{or equivalently,} \qquad u_t(i, 0) = v_t(i) + ci, \qquad t \geq 0, \quad i \geq 0, \tag{3.4}$$

where

$$v_t(0) = 0, \qquad v_0(i) = (\rho - c)i, \qquad t \geq 0, \quad i \geq 0. \tag{3.5}$$

Also, define $v(i) = \lim_{t \to \infty} v_t(i)$ for any given $i \geq 1$ if it exists.

Optimal ordering quantity   *The optimal ordering quantity when the process starts from time t is given by the smallest i maximizing $v_t(i)$ on $i \geq 0$ if it exists, denoted by i_t^*, that is, $v_t(i_t^*) = \max_{i \geq 0} v_t(i)$. If i_t^* does not exist, then let $i_t^* = \infty$ for explanatory convenience.*

4. Analysis

The properties of the optimal pricing rule are very similar to those in [14], thus they are omitted for brevity. Since our focus in this paper is the optimal ordering rule, we will examine only its properties in this section. We refer the interested reader to You [14] for the discussion on the properties of the optimal pricing rule. Before proceeding to the analysis of the optimal ordering rule, we introduce some functions used in the subsequent analysis and present their properties.

4.1. Preliminaries

This section introduces the functions that will be used to describe the optimality equations of the model in Section 3. The properties of the functions given in this subsection will be applied to the analysis of the model in the subsections that follow. For any x define

$$T(x) = \max_z p(z)(z - x), \tag{4.1}$$

and by $z(x)$ let us designate the z attaining the maximum of the right hand side of Equation (4.1) if it exists, i.e., $T(x) = p(z(x))(z(x) - x)$. This T-function is the same as the $T(\nu)$ function defined in [14]. Using the function $T(x)$, we shall define the following two functions:

$$K(x) = \lambda\beta T(x) - (1 - \beta)x, \tag{4.2}$$

$$N(x) = \lambda\beta T(x) + \beta x - c - h. \tag{4.3}$$

Here, by x^*, x_h and x_N let us denote the solutions of, respectively, $K(x) = 0$, $K(x) = h$, and $N(x) = 0$ if they exist, i.e.,

$$K(x^*) = 0, \qquad K(x_h) = h, \qquad N(x_N) = 0, \tag{4.4}$$

and if $K(x) = 0$, $K(x) = h$ and $N(x) = 0$ have multiple solutions, then let us define the *smallest* solution of $K(x) = 0$ and $K(x) = h$, respectively, by x^* and x_h, and the *largest* solution of $N(x) = 0$ by x_N.

The following are the properties of $T(x)$, $K(x)$, and $N(x)$.

Lemma 4.1.
(a) $T(x) > 0$ on $(-\infty, b)$ and $T(x) = 0$ on $[b, \infty)$.
(b) $T(x)$ is continuous and nonincreasing on $(-\infty, \infty)$.
(c) If $x \le (\ge) y$, then $T(x) - T(y) \le (\ge) (y - x)$.
(d) $\lambda\beta T(x) + \beta x$ is strictly increasing on $(-\infty, \infty)$.
(e) $\lim_{x \to -\infty} T(x) = \infty$ and $\lim_{x \to -\infty} \lambda\beta T(x) + \beta x = -\infty$.

Proof. See Appendix A. □

Lemma 4.2.
(a) $K(x)$ is continuous and strictly decreasing on $(-\infty, \infty)$.
(b) $K(x) + x$ is strictly increasing in $x \in (-\infty, \infty)$.
(c) For any x and y we have $|K(x) + x - K(y) - y| \le \beta|x - y|$.
(d) x^* uniquely exists with $0 < x^* < b$ where $x < (= (>)) x^* \Leftrightarrow K(x) > (= (<)) 0$.
(e) x_h uniquely exists with $x < (= (>)) x_h \Leftrightarrow K(x) > (= (<)) h$.

Proof. See Appendix B. □

Lemma 4.3.
(a) $x < (= (>)) x_N \Leftrightarrow N(x) < (= (>)) 0$.
(b) $x_h > (= (<)) c \Leftrightarrow x_N < (= (>)) x_h$.

Proof. See Appendix C. □

4.2. Properties of $U_t(i)$

For convenience of analysis in the following subsections, let us transform the optimality equations defined in Section 3. First, let us define, for $i \ge 1$ and $t \ge 0$

$$U_t(i) = u_t(i, 0) - u_t(i - 1, 0) \tag{4.5}$$

where $U_t(i)$ is the marginal total expected present discounted profit by increasing the quantity i for one additional unit. In addition, let $U(i) = \lim_{t \to \infty} U_t(i)$ if it exists. Noting Equation (3.1), we have

$$U_0(i) = \rho, \qquad i \ge 1. \tag{4.6}$$

Using the T-function, we can rewrite Equation (3.3) multiplied by $\lambda\beta$ as follows.

$$\lambda\beta u_t(i,1) = \lambda\beta T(U_t(i)) + \lambda\beta u_t(i,0), \qquad t \geq 0, \quad i \geq 1. \tag{4.7}$$

In addition, let

$$U_t(0) = M, \qquad t \geq 0 \tag{4.8}$$

for a sufficiently large $M > b > \rho$. Noting $\lambda\beta T(U_t(0)) + \lambda\beta u_t(0,0) = \lambda\beta T(M) = 0 = \lambda\beta u_t(0,1)$ due to Equation (3.1) and Lemma 4.1(a), we see that Equation (4.7) also holds for $i \geq 0$ instead of $i \geq 1$. Thus, from Equation (4.7) we can rewrite Equation (3.2) as follows.

$$u_t(i,0) = \lambda\beta T(U_{t-1}(i)) + \beta u_{t-1}(i,0) - hi, \qquad t \geq 1, \quad i \geq 0. \tag{4.9}$$

Accordingly, we can express $U_t(i)$ for $t \geq 1$ and $i \geq 1$ as follows.

$$U_t(i) = \lambda\beta T(U_{t-1}(i)) - \lambda\beta T(U_{t-1}(i-1)) + \beta U_{t-1}(i) - h. \tag{4.10}$$

Now let us examine the properties of $U_t(i)$, which will be used to derive the optimal ordering rule.

Lemma 4.4.
(a) $U_t(i) \leq M$ for $t \geq 0$ and $i \geq 0$.
(b) $U_t(i)$ is nonincreasing in $i \geq 0$ for $t \geq 0$.
(c) $U_t(i)$ is bounded in t for $i \geq 0$.
(d) If $\rho \leq (>) \, x_h$, then $U_t(1)$ is nondecreasing (strictly decreasing) in $t \geq 0$.

Proof. The properties of $U_t(i)$ stated in (a,b) are proven using the approach similar to the one employed in [7, 14]. However, due to the differences that exist between the optimality equations in this paper and those in the aforementioned past research, it becomes necessary to provide the proof of these properties below.

(a) The assertion holds for $i = 0$ since $U_t(0) = M$ for $t \geq 0$ from Equation (4.8). Then from this result, Equation (4.6) and the assumption of $\rho < M$ we have $U_0(i) \leq M$ for $i \geq 0$. Suppose $U_{t-1}(i) \leq M \cdots (1^*)$ for $i \geq 0$, so $U_{t-1}(i-1) \leq M \cdots (2^*)$ for $i \geq 1$. Note that $\lambda\beta T(M) = 0 \cdots (3^*)$ due to Lemma 4.1(a) and the assumption of $M > b$. Then from (1^*) and Lemma 4.1(d), we have $\lambda\beta T(U_{t-1}(i)) + \beta U_{t-1}(i) \leq \lambda\beta T(M) + \beta M = \beta M \leq M$ for $i \geq 0$. In addition, $\lambda\beta T(U_{t-1}(i-1)) \geq \lambda\beta T(M) = 0$ for $i \geq 1$ from (2^*) and Lemma 4.1(b). Hence from Equation (4.10) and the above results it follows that $U_t(i) \leq M - h \leq M$ for $i \geq 1$, thus for $i \geq 0$.

(b) Since $U_0(0) = M > \rho = U_0(i)$ for $i \geq 1$ by assumption, Equations (4.8) and (4.6), the assertion is clearly true for $t = 0$. Let $t \geq 1$. Suppose $U_{t-1}(i)$ is nonincreasing in $i \geq 0$ as the induction hypothesis. Then $U_{t-1}(i) \leq U_{t-1}(i-1) \leq U_{t-1}(i-2) \cdots (4^*)$ for $i \geq 2$. From Equation (4.10) for $i \geq 2$ we get

$$U_t(i) - U_t(i-1) = \lambda\beta(T(U_{t-1}(i)) - T(U_{t-1}(i-1))) + \beta(U_{t-1}(i) - U_{t-1}(i-1))$$
$$- \lambda\beta(T(U_{t-1}(i-1)) - T(U_{t-1}(i-2))). \tag{4.11}$$

Moreover, from (4^*) and Lemma 4.1(c), we get $T(U_{t-1}(i)) - T(U_{t-1}(i-1)) \leq U_{t-1}(i-1) - U_{t-1}(i) \cdots (5^*)$. Also, $T(U_{t-1}(i-1)) - T(U_{t-1}(i-2)) \geq 0 \cdots (6^*)$ due to Lemma 4.1(b). Noting (6^*) and (4^*), and substituting (5^*) into Equation (4.11), we obtain

$$U_t(i) - U_t(i-1) \leq \lambda\beta(U_{t-1}(i-1) - U_{t-1}(i)) + \beta(U_{t-1}(i) - U_{t-1}(i-1))$$
$$= \beta(1-\lambda)(U_t(i) - U_t(i-1)) \leq 0,$$

i.e., $U_t(i) \leq U_t(i-1)$ for $i \geq 2$. Since $U_t(1) \leq M = U_t(0)$ from (a), it follows that $U_t(i) \leq U_t(i-1)$ for $i \geq 1$, which completes the proof.

(c) Note that $U_t(i)$ is upper bounded in t for any given $i \geq 0$ from (a). First, it is clear from Equation (4.8) that the assertion is true for $i = 0$. Since $U_{t-1}(i) \leq U_{t-1}(i-1)$ due to (b), we have $T(U_{t-1}(i)) - T(U_{t-1}(i-1)) \geq 0$ from Lemma 4.1(b). Then from Equation (4.10) we get $U_t(i) \geq \beta U_{t-1}(i) - h$ for $t \geq 1$ and $i \geq 1$. Noting that $U_1(i) \geq \beta U_0(i) - h = \beta\rho - h$ for $i \geq 1$, we immediately obtain $U_t(i) \geq \beta^t\rho - (1 + \beta + \cdots + \beta^{t-1})h$. Moreover, since $\beta < 1$, we get $U_t(i) \geq \beta^t\rho - h/(1-\beta)$ for $t \geq 1$ and $i \geq 1$. If $\rho \geq 0$, then $U_t(i) \geq -h/(1-\beta)$, and if $\rho < 0$, then $U_t(i) \geq \rho - h/(1-\beta)$. Hence, from these results and Equation (4.6) $U_t(i)$ is lower bounded in t for any $i \geq 1$.

(d) Since $\lambda\beta T(U_{t-1}(0)) = \lambda\beta T(M) = 0$ for $t \geq 1$ due to Equation (4.8) and (3^*) in the proof of (a), from Equation (4.10) and the fact that $K(x) + x = \lambda\beta T(x) + \beta x$ (see Equation (4.2)) we immediately obtain

$$U_t(1) = K(U_{t-1}(1)) + U_{t-1}(1) - h, \qquad t \geq 1. \tag{4.12}$$

Let $t = 1$. Then $U_1(1) - U_0(1) = K(\rho) - h$ from Equations (4.12) and (4.6). If $\rho \leq (>) x_h$, we get $U_1(1) \geq (<) U_0(1)$ due to Lemma 4.2(e). Assume $U_{t-1}(1) \geq (<) U_{t-2}(1)$. Then from Equation (4.12) and Lemma 4.2(b) we have $U_t(1) \geq (<) K(U_{t-2}(1)) + U_{t-2}(1) - h = U_{t-1}(1)$ for $t \geq 2$. In other words, $U_t(1)$ is nondecreasing in $t \geq 0$ if $\rho \leq x_h$, and strict decreasing in $t \geq 0$ if $\rho > x_h$. $\qquad\square$

4.3. Optimal ordering rule

In this section we first examine the properties of $v_t(i)$ which then leads us to the derivation of the optimal ordering rule. For notation simplicity, let us define $\nabla v_t(i) = v_t(i) - v_t(i-1)$. Then from Equations (4.5) and (3.4) we get

$$U_t(i) = \nabla v_t(i) + c, \qquad t \geq 0, \quad i \geq 1. \tag{4.13}$$

Note that $\nabla v_t(1) = v_t(1) - v_t(0) = v_t(1)$ from Equation (3.5). Then Equation (4.13) can be rewritten as

$$U_t(1) = v_t(1) + c, \quad \text{or equivalently}, \quad v_t(1) = U_t(1) - c, \qquad t \geq 0. \tag{4.14}$$

Lemma 4.5.

(a) *If $v_t(1) \leq 0$ for a certain $t \geq 0$, then $v_t(i)$ is nonincreasing in $i \geq 0$ with $v_t(i) \leq 0$ for $i \geq 0$.*

(b) *$v_t(i)$ is strictly decreasing in $i \geq t \geq 0$.*

(c) *If $\rho \leq (>) x_h$, then $v_t(1)$ is nondecreasing (strictly decreasing) in $t \geq 0$.*

(d) *$v_t(1)$ converges to a finite $v(1)$ with $v(1) = x_h - c$.*

Proof. (a) The proof of this assertion consists of two steps. First, we show that $v_t(i)$ is concave in $i \geq 0$ for all $t \geq 0$. Then, we prove the monotonicity property of $v_t(i)$ in $i \geq 0$.

In general, let a series a_x, $x = 0, 1, \cdots$, be said to be concave in x if the difference $a_x - a_{x-1}$ is nonincreasing in x. $v_0(i)$ is concave in $i \geq 0$ due to Equation (3.5). Moreover, $U_{t-1}(i) \leq U_{t-1}(i-1) \leq U_{t-1}(i-2) \cdots (1^*)$ for any given $t \geq 1$ and $i \geq 2$ from Lemma 4.4(b). Then from Equations (4.13) and (4.10) for $i \geq 1$ we get:

$$\nabla v_t(i) = \lambda\beta T(U_{t-1}(i)) - \lambda\beta T(U_{t-1}(i-1)) + \beta\nabla v_{t-1}(i) - (1-\beta)c - h. \tag{4.15}$$

Note that $T(U_{t-1}(i)) - T(U_{t-1}(i-1)) \leq U_{t-1}(i-1) - U_{t-1}(i) \cdots (2^*)$ due to (1^*) and Lemma 4.1(c) and that $T(U_{t-1}(i-1)) - T(U_{t-1}(i-2)) \geq 0$ due to Lemma 4.1(b). Then from Equation (4.15) and the above results for $i \geq 2$ we get

$$
\begin{aligned}
\nabla v_t(i) &- \nabla v_t(i-1) \\
&= \lambda\beta(T(U_{t-1}(i)) - T(U_{t-1}(i-1))) - \lambda\beta(T(U_{t-1}(i-1)) - T(U_{t-1}(i-2))) \\
&\quad + \beta(\nabla v_{t-1}(i) - \nabla v_{t-1}(i-1)) \\
&\leq \lambda\beta(U_{t-1}(i-1) - U_{t-1}(i)) + \beta(\nabla v_{t-1}(i) - \nabla v_{t-1}(i-1)) \cdots (3^*)
\end{aligned}
$$

Using Equation (4.13) and (1^*), we can rewrite (3^*) as $\nabla v_t(i) - \nabla v_t(i-1) \leq \beta(1-\lambda)(U_{t-1}(i) - U_{t-1}(i-1)) \leq 0$, i.e., $\nabla v_t(i) \leq \nabla v_t(i-1) \cdots (4^*)$ for all $t \geq 0$ and $i \geq 2$. So, $v_t(i)$ is concave in $i \geq 0$ for all $t \geq 0$.

Now, let $v_t(1) \leq 0$ for a certain $t \geq 0$. Then $\nabla v_t(1) = v_t(1) - v_t(0) = v_t(1) \leq 0 = v_t(0)$ due to Equation (3.5). Suppose $v_t(i-1) \leq 0$ and $\nabla v_t(i-1) \leq 0$. Then from (4^*) we have $\nabla v_t(i) \leq \nabla v_t(i-1) \leq 0$, so $v_t(i) \leq v_t(i-1) \leq 0$. Accordingly, by induction we have $v_t(i) \leq v_t(i-1) \leq 0$ for $t \geq 0$, hence the assertion holds.

(b) Clearly, $v_0(i) - v_0(i-1) = \rho - c < 0$ for $i \geq 1$ from Equation (3.5) and the assumption of $\rho < c$, i.e., $v_0(i) < v_0(i-1)$ for $i \geq 1$, hence the assertion holds for $t = 0$. Let $t \geq 1$. Suppose $v_{t-1}(i) < v_{t-1}(i-1) \cdots (5^*)$ for $i \geq t$, or equivalently, $v_{t-1}(i-1) < v_{t-1}(i-2) \cdots (6^*)$ for $i \geq t+1$. Then from Equations (4.15), (2^*) and (4.13) we get

$$
\begin{aligned}
\nabla v_t(i) &\leq \lambda\beta(U_{t-1}(i-1) - U_{t-1}(i)) + \beta\nabla v_{t-1}(i) - (1-\beta)c - h \\
&= \lambda\beta(\nabla v_{t-1}(i-1) - \nabla v_{t-1}(i)) + \beta\nabla v_{t-1}(i) - (1-\beta)c - h \\
&= \beta(1-\lambda)\nabla v_{t-1}(i) + \lambda\beta\nabla v_{t-1}(i-1) - (1-\beta)c - h \\
&= \beta(1-\lambda)(v_{t-1}(i) - v_{t-1}(i-1)) + \lambda\beta(v_{t-1}(i-1) - v_{t-1}(i-2)) - (1-\beta)c - h.
\end{aligned}
$$

From (5^*), (6^*) and the fact that $(1-\beta)c + h > 0$ it follows that $\nabla v_t(i) < 0$, i.e., $v_t(i) < v_t(i-1)$ for $i \geq t+1$. Thus, by induction it follows that $v_t(i)$ is strictly decreasing in $i \geq t$ for $t \geq 0$.

(c) Immediate from Equation (4.14) and Lemma 4.4(d).

(d) Since $U_t(1)$ is bounded in t for $i \geq 0$ due to Lemma 4.4(c) and monotone in t due to Lemma 4.4(d), it follows that $U_t(1)$ converges to a finite $U(1)$ as $t \to \infty$. Hence $v(1) = U(1) - c \cdots (7^*)$ from Equation (4.14). Furthermore, we can easily show that $U_t(1)$ converges to $K(U(1)) + U(1) - h$ using the same approach as the one in Lemma 12.2(c) in [11]. From Equation (4.12) and Lemma 4.2 (c) we have $|U_t(1) - K(U(1)) - U(1) + h| \leq \beta|U_{t-1}(1)) - U(1)|$, which converges to 0 as $t \to \infty$. Thus, $U(1) = K(U(1)) + U(1) - h$, so $K(U(1)) = h$. Since $U(1) = x_h$ from Lemma 4.2(e), $v(1) = x_h - c$ due to (7^*). $\qquad\square$

Lemma 4.5 summarizes the properties of $v_t(i)$. Using this lemma and the properties of $N(x)$, we derive the ordering rule in the following theorem.

Theorem 4.1. *Let $t \geq 1$.*

(a) *Let $x_h \leq c$. Then $i_t^* = 0$.*

(b) *Let $x_h > c$.*

    1. *If $\rho \leq x_N$, then there exists a finite $t^* \geq 1$ such that if $1 \leq t \leq t^*$, then $i_t^* = 0$, or else $1 \leq i_t^* \leq t$.*

    2. *If $\rho > x_N$, then $1 \leq i_t^* \leq t$.*

Proof. Noting that $0 = v_t(0) = v(0) \cdots (1^*)$ for all $t \geq 0$ due to Equation (3.5), from Lemma 4.5(d) we get

$$v(1) - v(0) = x_h - c. \qquad (4.16)$$

(a) Let $x_h \leq c$. Then $v(1) \leq 0$ from Equation (4.16). In addition, $v_0(1) = \rho - c < 0$ $\cdots (2^*)$ from Equation (3.5) and the assumption of $\rho < c$. Noting Lemma 4.5(c), if $\rho \leq x_h$, we have $v_t(1) \leq 0$ for $t \geq 0$ due to $v(1) \leq 0$, and if $\rho > x_h$, then $v_t(1) \leq 0$ for $t \geq 0$ due to (2^*). Accordingly, $v_t(1) \leq 0 = v_t(0)$ for $t \geq 0$ whether $\rho \leq x_h$ or $\rho > x_h$. Thus, it follows from Lemma 4.5(a) that $v_t(i) \leq 0$ for $i \geq 0$, implying that $i_t^* = 0$.

(b) Let $x_h > c$. Then $x_N < x_h$ due to Lemma 4.3(b), and $v(1) > v(0) = 0 \cdots (3^*)$ from Equation (4.16). Also, we have $\rho < x_h$ due to the assumption of $\rho < c$, so $v_t(1)$ is nondecreasing in $t \geq 0$ due to Lemma 4.5(c). Moreover, from Lemma 4.5(b) and the definition of i_t^* it follows that $i_t^* \leq t \cdots (4^*)$ for $t \geq 1$.

Noting that $U_0(1) = u_0(1, 0)$ due to Equations (4.5) and (3.1), from Equations (3.4), (4.9), (4.3) and (4.6) we obtain $v_1(1) = \lambda \beta T(U_0(1)) + \beta U_0(1) - h - c = N(U_0(1)) = N(\rho) \cdots (5^*)$.

(b1) Let $\rho \leq x_N$. Then $N(\rho) \leq 0$ due to Lemma 4.3(a), so $v_1(1) \leq 0 = v_1(0)$ from (5^*) and (1^*). In addition, since $v(1) > 0$ due to (3^*), from the monotonicity of $v_t(1)$ in $t \geq 0$ it follows that there exists a $t^* \geq 1$ such that $v_t(1) > 0 = v_t(0)$ for $t > t^*$ and $v_t(1) \leq 0 = v_t(0)$ for $1 \leq t \leq t^*$. This implies that $1 \leq i_t^* \leq t$ if $t > t^*$ due to (4^*), or else $i_t^* = 0$ due to Lemma 4.5(a).

(b2) Let $\rho > x_N$. Since $N(\rho) > 0$ due to Lemma 4.3(a), we have $v_1(1) > 0$ from (5^*) and (1^*). Hence $v_t(1) > 0 = v_t(0)$ for $t \geq 1$ due to the monotonicity of $v_t(1)$ in $t \geq 0$, implying that $1 \leq i_t^* \leq t$ for $t \geq 1$. $\qquad \square$

With reference to Theorem 4.1, we can summarize the conditions for the optimal ordering quantity as follows:

1. Suppose $x_h \leq c$. Then the optimal ordering quantity is zero, i.e., do not order any quantity for all $t \geq 1$.

2. Suppose $x_h > c$. If $\rho > x_N$, then the optimal ordering quantity is a finite value greater than zero. On the other hand, if $\rho \leq x_N$, then if $t \leq t^*$, the optimal ordering quantity is zero, or else the optimal ordering quantity is a finite value greater than zero. Hence in this case a shortest sales horizon indicated by t^* exists. If the shelf life of the product or the seller's planned sales horizon is shorter than or equal to t^*, the product should not be ordered.

Numerical examples. A clothing retailer is considering the ordering of a certain quantity of a fashion blouse for sale in its store. Suppose the blouse can be purchased at a cost of \$20 per unit (i.e., $c = 20$) at the start of the planned sales horizon. Let the buyer arriving probability be $\lambda = 0.6$, discount factor $\beta = 0.999$, and holding cost $h = 0.15$. The distribution of the buyer's reservation price $F(w)$ is assumed to be a uniform distribution on $[15, 45]$, i.e., $a = 15$ and $b = 45$. Using Equations (4.2) to (4.4), we obtain the solution of $K(x) = h$ and $N(x) = 0$, $x_h \approx 38.8512$ and $x_N \approx 15.9509$, respectively. Hence, $c < x_h$ and this meets the condition in Theorem 4.1(b). Next, using two different values of the salvage price ρ, we illustrate the following two scenarios:

1. Assume that the clothing retailer can sell the remaining unsold blouses to a discount store for \$17.4 per unit (i.e., the salvage price $\rho = 17.4$) at the end of the planned sales horizon. Thus $x_N < \rho$. Then we have $1 \leq i_t^* \leq t$ for all $t \geq 1$, implying that

ordering the finite quantity i_t^* is optimal. From the numerical calculation we get the finite ordering quantity which is greater than zero, for example, $i_{50}^* = 10$ and $i_{80}^* = 14$, and the corresponding maximum total expected present discounted net profits if these finite quantities are ordered, that is, $v_{50}(10) = 89.0682$ and $v_{80}(14) = 114.5967$, respectively.

2. Suppose the clothing retailer has no other options to get rid of the unsaleable blouses, but to hire a recycling company to dispose of the blouses by paying a disposal fee of $1 per unit (i.e., the salvage price $\rho = -1$), hence $\rho < x_N$. Figure 1 gives the i_t^* for $0 \leq t \leq 83$. From this figure, we see that i_t^* is a step function with $i_t^* = 0$ for $t \leq 3$ and $1 \leq i_t^* \leq t$ for $t > 3$. Hence, by definition the shortest sales horizon is $t^* = 3$. In other words, ordering the fashion blouse will result in a profit for the retailer if the planned sales horizon for this blouse is greater than 3. In addition, we observe that the optimal ordering quantity may change as ρ changes. For example, in this scenario we get $i_{50}^* = 9$ and $i_{80}^* = 13$, which are smaller than $i_{50}^* = 10$ and $i_{80}^* = 14$ for the case of $\rho = 17.4$. The corresponding maximum total expected present discounted net profits for $i_{50}^* = 9$ and $i_{80}^* = 13$ in this scenario are $v_{50}(9) = 84.627$ and $v_{80}(13) = 112.7616$, respectively.
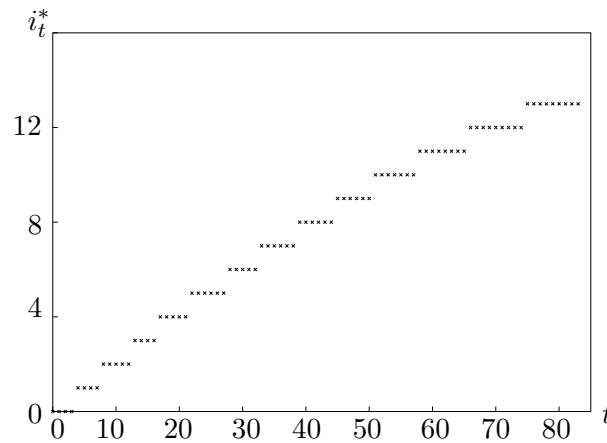

Figure 1: Optimal ordering quantity i_t^*

This result demonstrates that all the model parameters interrelate in determining the practicality of ordering a product for resale. By comparing the purchasing cost and salvage value against the two threshold values x_h and x_N, the seller can easily determine whether or not the ordering of a product is profitable. Furthermore, the result shows that both the optimal ordering quantity and maximum total expected net profit gained are smaller if the seller incurs a cost to dispose of the unsold items at the end of the sales horizon.

5. Conclusions

In this paper we examine the optimal ordering quantity for a perishable product with a dynamic pricing policy. Most research papers on the integration of dynamic pricing policy and ordering quantity assume that salvage value is nonnegative. By allowing the salvage value to be nonnegative or negative, we modified the models developed in [7, 14]. We derive the conditions that inform the sellers whether or not the product should be ordered. In addition, we show that under a certain condition a shortest sales horizon exists where the seller should engage in selling the product only if the seller's expected sales horizon or the shelf-life of the product is longer than the shortest sales horizon. Future research may consider the existence of a fixed advertising budget to search for new buyers and a case

where the buyer arriving probability is dependent on the price offered by the seller and the advertising budget.

Acknowledgements

Appendices

A. Proof of Lemma 4.1

(a,b) See Lemma 3.1(b,c1) in [14]. (c–e) see Lemmas 6.17, 6.14(e2) and 6.19(c,d) in [11].

B. Proof of Lemma 4.2

(a) Noting the assumption of $\beta < 1$, the assertion can be proven in the same manner as in Lemma 3.2(a) in [6].
(b) Note that $K(x)$ defined in this paper is similar to $K(x)$ of types S and P in [11]. Hence the assertion can be proven in the same manner as in Lemma 6.32(f) in [11].
(c) See Equation (6.62) in [11].
(d) The assertion can be proven in the same manner as in Lemma 3.2(f2) in [6].
(e) Note that $K(x) - h = 0$ and x_h defined in this paper are similar to $K_1(x) = 0$ and x_{K_1} in [6], respectively. Thus, the assertion can be proven in the same manner as in Lemma 3.2(e) in [6].

C. Proof of Lemma 4.3

(a) Note that $N(x)$ is strictly increasing on $(-\infty, \infty)$ from Equation (4.3) and Lemma 4.1(d). Since $\lambda \beta T(x) = 0$ for $x \geq b$ from Lemma 4.1(a), clearly $\lim_{x \to \infty} N(x) = \infty$. Further, from Lemma 4.1(e) we have $\lim_{x \to -\infty} N(x) = -\infty$. Accordingly, it must be that x_N uniquely exists. Hence the assertion holds due to the definition of x_N.
(b) Since $N(x) = K(x) + x - c - h$ due to Equation (4.2), from Equation (4.4) we have $N(x_h) = K(x_h) + x_h - c - h = x_h - c > (= (<)) \, 0$ if $x_h > (= (<)) \, c$. Hence the assertion holds due to (a).

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