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MATROID INTERSECTION WITH PRIORITY CONSTRAINTS

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Abstract In this paper, we consider the following variant of the matroid intersection problem. We are given two matroids $\mathbf{M}_1, \mathbf{M}_2$ on the same ground set E and a subset A of E. Our goal is to find a common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I \cap A|$ is maximum among all common independent sets of $\mathbf{M}_1, \mathbf{M}_2$ and such that (secondly) $|I|$ is maximum among all common independent sets of $\mathbf{M}_1, \mathbf{M}_2$ satisfying the first condition. This problem is a matroid-generalization of the simplest case of the rank-maximal matching problem introduced by Irving, Kavitha, Mehlhorn, Michail and Paluch (2006). In this paper, we extend the "combinatorial" algorithm of Irving *et al.* for the rank-maximal matching problem to our problem by using a Dulmage-Mendelsohn type decomposition for the matroid intersection problem.

Keywords: Discrete optimization, matroid, matching, Dulmage-Mendelsohn decomposition

1. Introduction

Consider the following constrained bipartite matching problem. We are given a finite bipartite graph G and a subset A of the edge set of G. We want to find a matching M in G such that $|M \cap A|$ is maximum among all matchings in G and such that (secondly) $|M|$ is maximum among all matchings in G satisfying the first condition. This problem is the simplest case of the rank-maximal matching problem introduced by Irving, Kavitha, Mehlhorn, Michail and Paluch [7]. (For the formal definition of the rank-maximal matching problem, see Section 6). This problem models the situation in which we want to make a matching between two groups as large as possible and some possible pairs have a higher priority than other possible pairs. It is not difficult to see that this problem can be solved in polynomial time by reducing it to the maximum-weight matching problem. In [7], the authors proposed a "combinatorial" algorithm that can solve the rank-maximal matching problem without reduction to the maximum-weight matching problem. This algorithm is generally more efficient than a straightforward reduction to the maximum-weight matching problem. (For a more efficient reduction to the maximum-weight matching problem, see [10]).

In this paper, we consider the following matroid-generalization of the above constrained matching problem. We are given two matroids $\mathbf{M}_1, \mathbf{M}_2$ on the same ground set E and a subset A of E. (For the definition of matroids, see Section 2.) Our goal is to find a common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I \cap A|$ is maximum among all common independent sets of $\mathbf{M}_1, \mathbf{M}_2$ and such that (secondly) $|I|$ is maximum among all common independent sets of $\mathbf{M}_1, \mathbf{M}_2$ satisfying the first condition. We should remark that a common independent set of two matroids can represent not only a matching in a bipartite graph but also, e.g., a branching in a directed graph which is a directed analogue of a forest in an undirected

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graph. (For the definition of branchings, see [13].) This problem can be solved in polynomial time by reducing it to the maximum-weight matroid intersection problem.

The aim of this paper is to extend the "combinatorial" algorithm of Irving, Kavitha, Mehlhorn, Michail and Paluch [7] for the rank-maximal matching problem to our problem. The motivation for our study is the extendibility of algorithms for bipartite matching problems to those for matroid intersection problems. The augmenting-path algorithm for the maximum-size matching problem and the Hungarian method for the maximum-weight matching problem can be naturally extended to the matroid intersection setting. The above constrained matching problem can be regarded as an intermediate problem of the maximum-size matching problem and the maximum-weight matching problem. So, the following question naturally arises. Can we extend the "combinatorial" algorithm for the rank-maximal matching problem to our problem? If possible, how can we do that? In this paper, we prove that the algorithm in [7] for the rank-maximal matching problem can be extended to our problem by using the idea of a Dulmage-Mendelsohn decomposition [3].

The algorithm of [7] partitions the vertices of a given bipartite graph into three categories by using a maximum-size matching on A. However, in our matroid setting, there is no object corresponding to vertices. Elements of matroids correspond to edges in a bipartite graph. So, it seems that the algorithm of [7] cannot be straightforwardly extended to our problem. In this paper, by using a Dulmage-Mendelsohn type decomposition for the maximum-size matroid intersection problem, we prove an "augmentability" theorem for our problem (see Theorem 5.1) and we extend the algorithm of [7] to our problem. This theorem can be viewed as an intermediate theorem of corresponding theorems for the maximum-size matroid intersection problem and the maximum-weight matroid intersection problem (see Theorems 3.1 and 3.2).

The rest of this paper is organized as follows. In Section 2, we give necessary definitions and notation. In Section 3, we explain known results about the matroid intersection problem. In Section 4, we give definitions and some structural results about a Dulmage-Mendelsohn type decomposition for the maximum-size matroid intersection problem. In Section 5, we give our algorithm. Section 6 concludes this paper with some remarks.

2. Preliminaries

For each finite set X and each element x, we write $X + x$ and $X - x$ instead of $X \cup \{x\}$ and $X \setminus \{x\}$, respectively. For each finite sets X and Y, define

$$X \triangle Y := (X \setminus Y) \cup (Y \setminus X).$$

A pair $(E, \mathcal{I})$ is called a **matroid**, if E is a finite set and $\mathcal{I}$ is a nonempty family of subsets of E satisfying the following conditions.

(I1) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.

(I2) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I + x \in \mathcal{I}$ for some element x of $J \setminus I$.

Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid. A subset I of E is called an **independent set of M**, if $I \in \mathcal{I}$. A subset C of E is called a **circuit**, if $C \notin \mathcal{I}$, but any proper subset C' of C is an independent set of $\mathbf{M}$. It is known [12] that if I is an independent set of $\mathbf{M}$ and x is an element of $E \setminus I$ such that $I + x \notin \mathcal{I}$, then $I + x$ contains the unique circuit C and $x \in C$. The circuit C is called the **basic circuit of x with respect to I in M**. It is easy to see that the basic circuit C of x with respect to I in $\mathbf{M}$ is the set of all elements y of $I + x$ such that $I - y + x \in \mathcal{I}$. For each subset X of E, a subset B of X is called a **base of X**, if B is an inclusionwise maximal independent subset of X. By (I2), every two bases of a subset X

of E have the same size, which is called the **rank of** X and denoted by $\rho_{\mathbf{M}}(X)$. For each subset X of E, define

$$\mathcal{I}|X := \{I \in \mathcal{I} \mid I \subseteq X\}, \qquad \mathbf{M}|X := (X, \mathcal{I}|X).$$

It is not difficult to see that $\mathbf{M}|X$ is a matroid.

Now we formally define our problem, called the **matroid intersection problem with priority constraints** (the MIwPC problem for short). Throughout this paper, we assume that two matroids $\mathbf{M}_1 = (E, \mathcal{I}_1)$ and $\mathbf{M}_2 = (E, \mathcal{I}_2)$ are given. Furthermore, we are given a subset A of E. A common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ is said to be **feasible**, if $|I \cap A|$ is maximum among all common independent sets of $\mathbf{M}_1, \mathbf{M}_2$. Namely, I is said to be feasible, if $I \cap A$ is a maximum-size common independent set of $\mathbf{M}_1|A$ and $\mathbf{M}_2|A$. Then, the MIwPC problem asks for finding a feasible common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I|$ is maximum among all feasible common independent sets of $\mathbf{M}_1, \mathbf{M}_2$.

3. Matroid Intersection Problems

For each common independent set I of $\mathbf{M}_1, \mathbf{M}_2$, define a directed graph $D_{\mathbf{M}_1\mathbf{M}_2}(I)$, with a vertex set E, as follows. For each elements x of $E \setminus I$ and y of I,

$$\begin{aligned}&(y, x) \text{ is an arc of } D_{\mathbf{M}_1\mathbf{M}_2}(I) \text{ if and only if } I - y + x \in \mathcal{I}_1,\\ &(x, y) \text{ is an arc of } D_{\mathbf{M}_1\mathbf{M}_2}(I) \text{ if and only if } I - y + x \in \mathcal{I}_2.\end{aligned}$$

These are all arcs of $D_{\mathbf{M}_1\mathbf{M}_2}(I)$. Notice that $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ is a bipartite graph with vertex classes $I, E \setminus I$. We may not distinguish a simple directed path P (that is, no vertices in P are repeated) in $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ and its vertex set. The size of a path P of $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ is defined by the number of the vertices traversed by P. Define

$$\begin{aligned}S_{\mathbf{M}_1}(I) &:= \{x \in E \setminus I \mid I + x \in \mathcal{I}_1\},\\ S_{\mathbf{M}_2}(I) &:= \{x \in E \setminus I \mid I + x \in \mathcal{I}_2\}.\end{aligned}$$

Let $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ be the set of directed paths from $S_{\mathbf{M}_1}(I)$ to $S_{\mathbf{M}_2}(I)$ in $D_{\mathbf{M}_1\mathbf{M}_2}(I)$. The following theorem is well-known (also, see [13, Section 41.2]).

Theorem 3.1 ([1, 9]). *For each common independent set I of $\mathbf{M}_1, \mathbf{M}_2$, there exists a common independent set I' of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I'| = |I| + 1$ if and only if $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I) \neq \emptyset$. Moreover, if $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I) \neq \emptyset$, then for each minimum-size path P of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$, $I \triangle P$ is a common independent set of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I \triangle P| = |I| + 1$.*

Next we consider the weighted version of Theorem 3.1. Suppose that we are given a weight function $w\colon E \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The weight $w(X)$ of a non-empty subset X of S is defined by $w(X) := \sum_{x \in X} w(x)$, and define $w(\emptyset) := 0$. A common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ is said to be **extreme**, if $w(I) \geq w(J)$ for any common independent set J of $\mathbf{M}_1, \mathbf{M}_2$ such that $|J| = |I|$. For each element x of E, define the **length** $l(x)$ by

$$l(x) := \begin{cases} w(x) & \text{if } x \in I,\\ -w(x) & \text{if } x \notin I.\end{cases} \tag{3.1}$$

The length of a directed path P in $D_{\mathbf{M}_1\mathbf{M}_2}(I)$, denoted by $l(P)$, is equal to the sum of the lengths of the vertices traversed by P. Although the length of some element may be negative in $D_{\mathbf{M}_1\mathbf{M}_2}(I)$, it is known [5, 8] that I is extreme if and only if $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ has no directed cycle of negative length. So, we can find a minimum-length path among all paths of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ in polynomial time with, e.g., the Bellman-Ford method. The following weighted version of "augmentability" theorem is well-known (also, see [13, Section 41.3]).

Theorem 3.2 ([9]). *For each extreme common independent set I of $\mathbf{M}_1, \mathbf{M}_2$, there exists an extreme common independent set I' of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I'| = |I| + 1$ if and only if $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I) \neq \emptyset$. Moreover, if $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I) \neq \emptyset$, then for each path P of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ such that $l(P)$ is minimum among all paths of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ and such that (secondly) the size of P is minimum among all minimum-length paths of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$, $I \triangle P$ is an extreme common independent set of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I \triangle P| = |I| + 1$ and $w(I \triangle P) = w(I) - l(P)$.*

4. Dulmage-Mendelsohn Type Decomposition

In this section, we explain a Dulmage-Mendelsohn type decomposition (a DM type decomposition for short) for the maximum-size matroid intersection problem. In the sequel, define $\mathbf{M}'_1 := \mathbf{M}_1|A$, $\mathcal{I}'_1 := \mathcal{I}_1|A$, $\mathbf{M}'_2 := \mathbf{M}_2|A$ and $\mathcal{I}'_2 := \mathcal{I}_2|A$,

Let I^* be a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$. Notice that by Theorem 3.1, $\mathcal{P}_{\mathbf{M}'_1\mathbf{M}'_2}(I^*) = \emptyset$. Let A^+ be the set of elements of A that are reachable from $S_{\mathbf{M}'_1}(I^*)$ in $D_{\mathbf{M}'_1\mathbf{M}'_2}(I^*)$, and let A^- be the set of elements of A from which $S_{\mathbf{M}'_2}(I^*)$ is reachable in $D_{\mathbf{M}'_1\mathbf{M}'_2}(I^*)$. Let D' be the directed graph obtained from $D_{\mathbf{M}'_1\mathbf{M}'_2}(I^*)$ by deleting vertices of $A^+ \cup A^-$. Let $A_1, \ldots, A_k$ be the strongly connected components of D' such that $i \leq j$ if A_i is reachable from A_j in D'. Define $A_0 := A^+$ and $A_{k+1} := A^-$. We call $\mathcal{A} = (A_0; A_1, \ldots, A_k; A_{k+1})$ a **DM type decomposition of $\mathbf{M}'_1, \mathbf{M}'_2$ for** I^*. For each $i \in \{0, 1, \ldots, k+1\}$, define $B_i := A_0 \cup A_1 \cup \cdots \cup A_i$. Although Lemmas 4.1 and 4.2 are easily obtained from well-known results (for example, see [6, 11]), we give their proofs for completeness.

Lemma 4.1. *For every $i \in \{0, 1, \ldots, k\}$,*

$$|I^* \setminus B_i| = \rho_{\mathbf{M}'_1}(A \setminus B_i), \qquad |I^* \cap B_i| = \rho_{\mathbf{M}'_2}(B_i).$$

Proof. For every $i \in \{0, 1, \ldots, k\}$, since $I^* \setminus B_i$ is an independent set of $\mathbf{M}'_1$,

$$|I^* \setminus B_i| \leq \rho_{\mathbf{M}'_1}(A \setminus B_i). \tag{4.1}$$

If Equation (4.1) is not satisfied by equality for some $i \in \{0, 1, \ldots, k\}$, then there exists a subset J of $A \setminus B_i$ such that $J \in \mathcal{I}'_1$ and $|J| > |I^* \setminus B_i|$. So, by (I2), there exists an element x of $A \setminus (B_i \cup I^*)$ such that $(I^* \setminus B_i) + x \in \mathcal{I}'_1$. Recall that $I^* + x \notin \mathcal{I}'_1$ since $S_{\mathbf{M}'_1}(I^*)$ is a subset of A_0. Let C be the basic circuit of x with respect to I^* in $\mathbf{M}'_1$. Since $\{x\} \in \mathcal{I}'_1$ follows from $(I^* \setminus B_i) + x \in \mathcal{I}'_1$ and (I1), $C - x$ is not empty. Moreover, by $(I^* \setminus B_i) + x \in \mathcal{I}'_1$, $C - x$ is not a subset of $I^* \setminus B_i$. So, there exists an element y of $C - x$ contained in $I^* \cap B_i$, and thus $I^* - y + x \in \mathcal{I}'_1$ by the property of the basic circuit. This implies that there exists an arc from B_i to $A \setminus B_i$, which contradicts the fact that $i \leq j$ if A_i is reachable from A_j in D'.

For every $i \in \{0, 1, \ldots, k\}$, since $I^* \cap B_i$ is an independent set of $\mathbf{M}'_2$,

$$|I^* \cap B_i| \leq \rho_{\mathbf{M}'_2}(B_i). \tag{4.2}$$

If Equation (4.2) is not satisfied with equality for some $i \in \{0, 1, \ldots, k\}$, then there exists a subset J of B_i such that $J \in \mathcal{I}'_2$ and $|J| > |I^* \cap B_i|$. So, by (I2), there exists an element x of $B_i \setminus I^*$ such that $(I^* \cap B_i) + x \in \mathcal{I}'_2$. Recall that $I^* + x \notin \mathcal{I}'_2$ since $S_{\mathbf{M}'_2}(I^*)$ is a subset of A_{k+1}. Let C be the basic circuit of x with respect to I^* in $\mathbf{M}'_2$. Since $\{x\} \in \mathcal{I}'_2$ follows from $(I^* \cap B_i) + x \in \mathcal{I}'_2$ and (I1), $C - x$ is not empty. Moreover, by $(I^* \cap B_i) + x \in \mathcal{I}'_2$, $C - x$ is not a subset of $I^* \cap B_i$. So, there exists an element y of $C - x$ contained in $I^* \setminus B_i$, and thus $I^* - y + x \in \mathcal{I}'_2$ by the property of the basic circuit. This implies that there exists an arc from B_i to $A \setminus B_i$, which contradicts the fact that $i \leq j$ if A_i is reachable from A_j in D'. $\square$

Lemma 4.2. *If I is a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$, then*

$$|I \setminus B_i| = \rho_{\mathbf{M}'_1}(A \setminus B_i), \qquad |I \cap B_i| = \rho_{\mathbf{M}'_2}(B_i)$$

for every $i \in \{0, 1, \ldots, k\}$.

Proof. For every $i \in \{0, 1, \ldots, k\}$, $|I \setminus B_i| \leq \rho_{\mathbf{M}'_1}(A \setminus B_i)$ by $I \setminus B_i \in \mathcal{I}'_1$. If

$$|I \setminus B_j| < \rho_{\mathbf{M}'_1}(A \setminus B_j) \ \ (= |I^* \setminus B_j| \text{ by Lemma 4.1})$$

for some $j \in \{0, 1, \ldots, k\}$, then

$$|I \cap B_j| > |I^* \cap B_j| = \rho_{\mathbf{M}'_2}(B_j)$$

by $|I| = |I^*|$ and Lemma 4.1. This contradicts the fact that $I \cap B_j \in \mathcal{I}'_2$.

For every $i \in \{0, 1, \ldots, k\}$, $|I \cap B_i| \leq \rho_{\mathbf{M}'_2}(B_i)$ by $I \cap B_i \in \mathcal{I}'_2$. If

$$|I \cap B_j| < \rho_{\mathbf{M}'_2}(B_j) \ \ (= |I^* \cap B_j| \text{ by Lemma 4.1})$$

for some $j \in \{0, 1, \ldots, k\}$, then

$$|I \setminus B_j| > |I^* \setminus B_j| = \rho_{\mathbf{M}'_1}(A \setminus B_j)$$

by $|I| = |I^*|$ and Lemma 4.1. This contradicts the fact that $I \setminus B_j \in \mathcal{I}'_1$. □

5. Main Results

Throughout this section, let I^* be a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$. Let $\mathcal{A} = (A_0; A_1, \ldots, A_k; A_{k+1})$ be a DM type decomposition of $\mathbf{M}'_1, \mathbf{M}'_2$ for I^*. Recall that a common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ is said to be feasible, if $|I \cap A| = |I^*|$. For each feasible common independent set I of $\mathbf{M}'_1, \mathbf{M}'_2$, an arc (x, y) of $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ is said to be **bad**, if it satisfies one of the following three conditions (see Figure 1).

$$x \in A_i \text{ and } y \in A_j \text{ for some } i, j \in \{0, 1, \ldots, k+1\} \text{ such that } i \neq j, \text{ or} \tag{5.1}$$
$$x \in E \setminus (I \cup A) \text{ and } y \in (I \cap A) \setminus A_{k+1}, \text{ or} \tag{5.2}$$
$$x \in (I \cap A) \setminus A_0 \text{ and } y \in E \setminus (I \cup A). \tag{5.3}$$

For each feasible common independent set I of $\mathbf{M}_1, \mathbf{M}_2$, we denote by $H_{\mathbf{M}_1\mathbf{M}_2}(I)$ the directed graph obtained by removing all bad arcs from $D_{\mathbf{M}_1\mathbf{M}_2}(I)$. Let $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I)$ be the set of directed paths from from $S_{\mathbf{M}_1}(I)$ to $S_{\mathbf{M}_2}(I)$ in $H_{\mathbf{M}_1\mathbf{M}_2}(I)$. We are now ready to give our main theorem. We leave its proof to the next subsection.

Theorem 5.1. *For each feasible common independent set I of $\mathbf{M}_1, \mathbf{M}_2$, there exists a feasible common independent set I' of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I'| = |I| + 1$ if and only if $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I) \neq \emptyset$. Moreover, if $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I) \neq \emptyset$, then for each minimum-size path P of $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I)$, $I \triangle P$ is a feasible common independent set of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I \triangle P| = |I| + 1$.*

5.1. Proof of Theorem 5.1

Before proving Theorem 5.1, we give necessary lemmas.

Lemma 5.1. *For each feasible common independent set I of $\mathbf{M}_1, \mathbf{M}_2$,*

$$S_{\mathbf{M}_1}(I) \cap A \subseteq A_0, \qquad S_{\mathbf{M}_2}(I) \cap A \subseteq A_{k+1}.$$

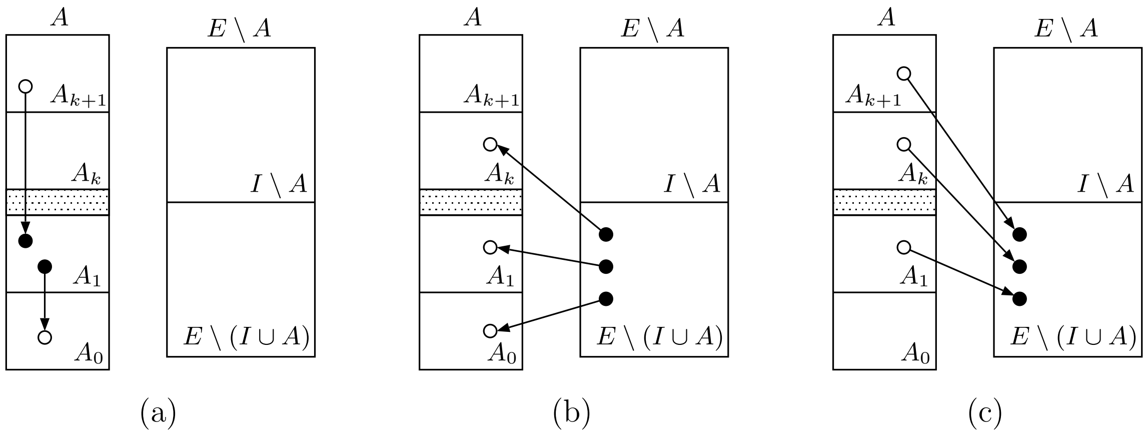

Figure 1: White and black points represent elements of I and $E \setminus I$, respectively (a) Arcs satisfy Equation (5.1) (b) Arcs satisfy Equation (5.2) (c) Arcs satisfy Equation (5.3)

Proof. Suppose that there exists an element x of $E \setminus I$ such that $x \in S_{\mathbf{M}_1}(I) \cap A$ and $x \notin A_0$. Obviously,

$$(I + x) \cap (A \setminus A_0) = ((I \cap A) \setminus A_0) + x \in \mathcal{I}'_1. \tag{5.4}$$

However, since $I \cap A$ is a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$,

$$\rho_{\mathbf{M}'_1}(A \setminus A_0) = |(I \cap A) \setminus A_0| < |((I \cap A) \setminus A_0) + x|$$

by Lemma 4.2. This contradicts Equation (5.4) and the definition of $\rho_{\mathbf{M}'_1}$.

Suppose that there exists an element x of $E \setminus I$ such that $x \in S_{\mathbf{M}_2}(I) \cap A$ and $x \notin A_{k+1}$. Obviously,

$$(I + x) \cap B_k = (I \cap B_k) + x \in \mathcal{I}'_2. \tag{5.5}$$

However, since $I \cap A$ is a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$,

$$\rho_{\mathbf{M}'_2}(B_k) = |I \cap B_k| < |(I \cap B_k) + x|$$

by Lemma 4.2. This contradicts Equation (5.5) and the definition of $\rho_{\mathbf{M}'_2}$. □

Lemma 5.2. *For each feasible common independent set I of $\mathbf{M}_1, \mathbf{M}_2$, there exists no arc from A_i to A_j in $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ for any $i, j \in \{0, 1, \ldots, k+1\}$ such that $i < j$.*

Proof. Suppose that there exists an arc (x, y) from A_i to A_j in $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ for some $i, j \in \{0, 1, \ldots, k+1\}$ such that $i < j$.

If $x \in I$ and $y \notin I$, then $I - x + y \in \mathcal{I}_1$. So,

$$(I - x + y) \cap (A \setminus B_i) = ((I \cap A) \setminus B_i) + y \in \mathcal{I}'_1. \tag{5.6}$$

However, since $I \cap A$ is a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$,

$$\rho_{\mathbf{M}'_1}(A \setminus B_i) = |(I \cap A) \setminus B_i| < |((I \cap A) \setminus B_i) + y|$$

by Lemma 4.2. This contradicts Equation (5.6) and the definition of $\rho_{\mathbf{M}'_1}$.

If $x \notin I$ and $y \in I$, then $I - y + x \in \mathcal{I}_2$. So,

$$(I - y + x) \cap B_i = (I \cap B_i) + x \in \mathcal{I}'_2. \tag{5.7}$$

However, since $I \cap A$ is a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$,

$$\rho_{\mathbf{M}'_2}(B_i) = |I \cap B_i| < |(I \cap B_i) + x|$$

by Lemma 4.2. This contradicts Equation (5.7) and the definition of $\rho_{\mathbf{M}'_2}$. □

Lemma 5.3. *Assume that I is a feasible common independent set of $\mathbf{M}_1, \mathbf{M}_2$, x is an element of $A \setminus I$ and y is an element of $I \setminus A$.*

$$\text{If } (y, x) \text{ is an arc of } D_{\mathbf{M}_1\mathbf{M}_2}(I), \text{ then } x \in A_0.$$
$$\text{If } (x, y) \text{ is an arc of } D_{\mathbf{M}_1\mathbf{M}_2}(I), \text{ then } x \in A_{k+1}.$$

Proof. Assume that there exists an arc (y, x) of $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ such that $x \in A_i$ for some $i \neq 0$. Since $I - y + x \in \mathcal{I}_1$,

$$(I - y + x) \cap (A \setminus B_{i-1}) = ((I \cap A) \setminus B_{i-1}) + x \in \mathcal{I}'_1. \tag{5.8}$$

However, since $I \cap A$ is a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$,

$$\rho_{\mathbf{M}'_1}(A \setminus B_{i-1}) = |(I \cap A) \setminus B_{i-1}| < |(I \cap A) \setminus B_{i-1}) + x|$$

by Lemma 4.2 and $i \neq 0$. This contradicts Equation (5.8) and the definition of $\rho_{\mathbf{M}'_1}$.

Assume that there exists an arc (x, y) of $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ such that $x \in A_i$ for some $i \neq k+1$. Since $I - y + x \in \mathcal{I}_2$,

$$(I - y + x) \cap B_i = (I \cap B_i) + x \in \mathcal{I}'_2. \tag{5.9}$$

However, since $I \cap A$ is a maximum-size common independent set of $\mathbf{M}'_1, \mathbf{M}'_2$,

$$\rho_{\mathbf{M}'_2}(B_i) = |I \cap B_i| < |(I \cap B_i) + x|$$

by Lemma 4.2 and $i \neq k+1$. This contradicts Equation (5.9) and the definition of $\rho_{\mathbf{M}'_2}$. □

By Lemmas 5.2, 5.3 and the fact that there exists no bad arc in $H_{\mathbf{M}_1\mathbf{M}_2}(I)$, every arc (x, y) of $H_{\mathbf{M}_1\mathbf{M}_2}(I)$ satisfies one of the following conditions for each feasible common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ (see Figure 2).

$$x, y \in A_i \text{ for some } i \in \{0, 1, \ldots, k+1\}, \text{ or } x, y \in E \setminus A. \tag{5.10}$$
$$x \in A_{k+1} \setminus I \text{ and } y \in I \setminus A, \text{ or } x \in I \setminus A \text{ and } y \in A_0 \setminus I. \tag{5.11}$$
$$x \in A_0 \cap I \text{ and } y \in E \setminus (I \cup A), \text{ or } x \in E \setminus (I \cup A) \text{ and } y \in A_{k+1} \cap I. \tag{5.12}$$

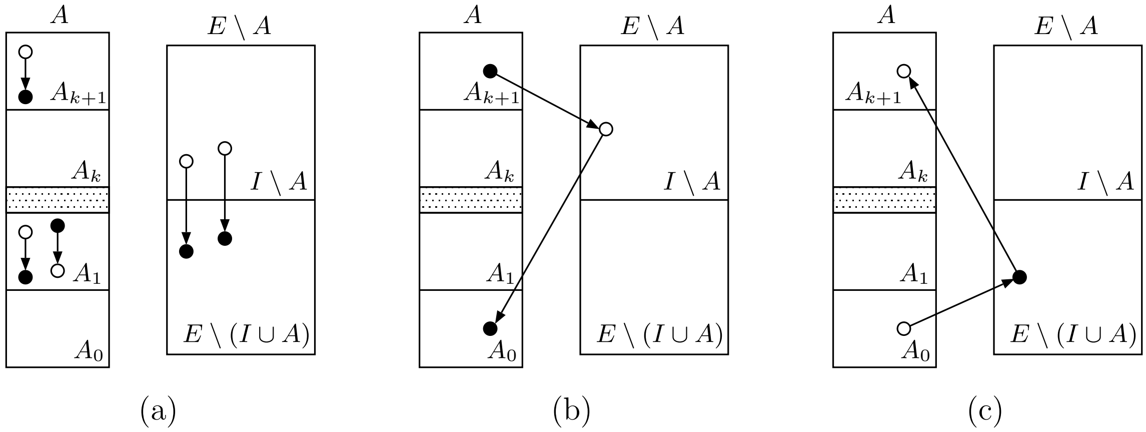

Figure 2: White and black points represent elements of I and $E \setminus I$, respectively (a) Arcs satisfy Equation (5.10) (b) Arcs satisfy Equation (5.11) (c) Arcs satisfy Equation (5.12)

In the sequel, define the weight of each element x of E by

$$w(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{5.13}$$

Recall that the length of each element of E is defined by Equation (3.1).

Lemma 5.4. *For each feasible common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ and each path P of $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I)$, we have $l(P) = 0$ and $I \triangle P$ is feasible.*

Proof. We first consider the case where P traverses no vertex of A. In this case, $I \triangle P$ is feasible since $|(I \triangle P) \cap A| = |I \cap A|$. Furthermore, by Equation (5.13), we have $l(P) = 0$.

Next we consider the case where P traverses at least one vertex of A. Assume that P traverses vertices $x_1, x_2, \ldots, x_m$ in this order. Since P traverses at least one vertex of A, there exist $p, q \in \{1, 2, \ldots, m\}$ such that

$$p \leq q, \qquad x_p, x_{p+1}, \ldots, x_q \in A, \qquad p = 1 \text{ or } x_{p-1} \notin A, \qquad q = m \text{ or } x_{q+1} \notin A. \tag{5.14}$$

For proving the lemma, we need the following claim.

Claim 5.1. *Exactly one of x_p, x_q is contained in I.*

Proof. We first consider the case of $p = 1$ or $x_{p-1} \in I$. If $p = 1$, then $x_p \in S_{\mathbf{M}_1}(I)$, i.e., $x_p \notin I$. Moreover, $x_p \in A_0$ follows from Lemma 5.1. If $p \neq 1$ and $x_{p-1} \in I$, then $x_p \notin I$. Moreover, since all arcs from $E \setminus A$ to x_p in $H_{\mathbf{M}_1\mathbf{M}_2}(I)$ satisfy Equation (5.11), we have $x_p \in A_0$. In both cases, $x_q \in A_0$ follows from $x_p \in A_0$ and Lemma 5.2. This implies that $x_q \notin S_{\mathbf{M}_2}(I)$ by Lemma 5.1. So, since all arcs from x_q to $E \setminus A$ in $H_{\mathbf{M}_1\mathbf{M}_2}(I)$ satisfy Equation (5.12), we have $x_q \in I$.

Next we consider the case of $p \neq 1$ and $x_{p-1} \notin I$, i.e., $x_p \in I$. In this case, since all arcs from $E \setminus A$ to x_p in $H_{\mathbf{M}_1\mathbf{M}_2}(I)$ satisfy Equation (5.12), we have $x_p \in A_{k+1}$. If $q = m$, then $x_q \in S_{\mathbf{M}_2}(I)$, i.e., $x_q \notin I$ and the proof is done. If $q \neq m$, then $x_q \in A_{k+1}$ follows from $x_p \in A_{k+1}$ and the fact that we remove all arcs satisfying Equation (5.1). So, since all arcs from x_q to $E \setminus A$ in $H_{\mathbf{M}_1\mathbf{M}_2}(I)$ satisfy Equation (5.11), we have $x_q \notin I$. □

Let Q be the path $(x_p, x_{p+1}, \ldots, x_q)$. Since $H_{\mathbf{M}_1\mathbf{M}_2}(I)$ is a bipartite graph with vertex classes $I, E \setminus I$, it follows from Claim 5.1 that $l(Q) = 0$ and $|(I \triangle Q) \cap A| = |I \cap A|$. There may exist several pairs $p, q \in \{1, 2, \ldots, m\}$ satisfying Equation (5.14). Let p_1, q_1 and p_2, q_2 be such pairs. Since we have $q_1 < p_2 + 1$ or $q_2 < p_1 + 1$, we can treat p_1, q_1 and p_2, q_2 independently. So, $l(P) = 0$ follows from Equation (5.13), and $|(I \triangle P) \cap A| = |I \cap A|$, i.e., $I \triangle P$ is feasible. □

Lemma 5.5. *For each feasible common independent set I of $\mathbf{M}_1, \mathbf{M}_2$ and each path P of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ using a bad arc, we have $l(P) \geq 1$.*

Proof. Assume that P traverses vertices $x_1, x_2, \ldots, x_m$ in this order. Since P uses a bad arc, P traverses at least one vertex of A. So, there exist $p, q \in \{1, 2, \ldots, m\}$

$$p \leq q, \qquad x_p, x_{p+1}, \ldots, x_q \in A, \qquad p = 1 \text{ or } x_{p-1} \notin A, \qquad q = m \text{ or } x_{q+1} \notin A. \tag{5.15}$$

If $p = 1$, then $x_p \in S_{\mathbf{M}_1}(I)$, i.e., $x_p \in A_0$ follows from Lemma 5.1. So, by Lemma 5.2, we have $x_q \in A_0$ and $q \neq m$. This implies that (x_i, x_{i+1}) is not a bad arc for any $i \in \{p, p+1, \ldots, q\}$. If $q = m$, then $x_q \in S_{\mathbf{M}_2}(I)$, i.e., $x_q \in A_{k+1}$ follows from Lemma 5.1. So, by Lemma 5.2, we have $x_p \in A_{k+1}$ and $p \neq 1$. This implies that (x_{i-1}, x_i) is not a bad arc for any $i \in \{p, p+1, \ldots, q\}$. By this observation and the fact that P use a bad arc, there exists $s, t \in \{1, 2, \ldots, m\}$ such that

$$1 < s \leq t < m, \qquad x_s, x_{s+1}, \ldots, x_t \in A, \qquad x_{s-1} \notin A, \qquad x_{t+1} \notin A,$$

and (x_i, x_{i+1}) is a bad arc for some $i \in \{s-1, s, \ldots, t\}$. For proving the lemma, we need the following claim.

Claim 5.2. *Both of x_s, x_t are contained in I.*

Proof. We first consider the case where (x_{s-1}, x_s) is a bad arc, i.e., $x_s \in I$ and $x_s \notin A_{k+1}$. By Lemma 5.2, we have $x_t \notin A_{k+1}$. If $x_t \notin I$, then $x_t \in A_{k+1}$ by Lemma 5.3. So, $x_t \in I$.

Next we consider the case where (x_{s-1}, x_s) is not a bad arc, but (x_t, x_{t+1}) is a bad arc. In this case, $x_t \in I$ and $x_t \notin A_0$. By Lemma 5.2, we have $x_s \notin A_0$. Since (x_{s-1}, x_s) is not a bad arc, if $x_s \notin I$, then $x_s \in A_0$ by Lemma 5.3. So, $x_s \in I$.

Finally, we consider the case where (x_{s-1}, x_s) and (x_t, x_{t+1}) are not bad arcs, but (x_i, x_{i+1}) is a bad arc for some $i \in \{s, s+1, \ldots, t-1\}$. By Lemma 5.2, $x_s \notin A_0$ and $x_t \notin A_{k+1}$. Since (x_{s-1}, x_s) and (x_t, x_{t+1}) are not bad arcs, then $x_s, x_t \in I$ follows from $x_s \notin A_0$, $x_t \notin A_{k+1}$ and Lemma 5.3. □

For $p, q \in \{1, 2, \ldots, m\}$ satisfying Equation (5.15), let Q be the path $(x_p, x_{p+1}, \ldots, x_q)$. In the same way as the proof of Lemma 5.4, we can prove that if (a) $p = 1$, or (b) $q = m$, or (c) $p \neq 1$ and $q \neq m$ and (x_i, x_{i+1}) is not a bad arc for any $i \in \{p-1, p, \ldots, q\}$, then $l(Q) = 0$. If $p \neq 1$ and $q \neq m$ and (x_i, x_{i+1}) is a bad arc for some $i \in \{p-1, p, \ldots, q\}$, then we have $l(Q) \geq 1$ by Claim 5.2. So, $l(P) \geq 1$ follows from Equation (5.13). □

We are now ready to prove Theorem 5.1.

Theorem 5.1. Suppose that $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I) \neq \emptyset$. By Lemmas 5.4 and 5.5, a path P of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ is a minimum-length path of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ if and only if $P \in \mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I)$. Let P be a minimum-size path of $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I)$. By Equation (5.13) and the feasibility of I, I is extreme. So, by Theorem 3.2, $I \triangle P$ is a common independent set such that $|I \triangle P| = |I| + 1$. Moreover, by Lemma 5.4, $I \triangle P$ is feasible.

Conversely, suppose that $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I) = \emptyset$. If $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2} = \emptyset$, then there exists no common independent set I' of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I'| = |I| + 1$ by Theorem 3.1, and thus the proof is done. So, assume that $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2} \neq \emptyset$. Let P be a path of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ such that $l(P)$ is minimum among all paths of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$ and such that (secondly) the size of P is minimum among all minimum-length paths of $\mathcal{P}_{\mathbf{M}_1\mathbf{M}_2}(I)$. By $\mathcal{Q}_{\mathbf{M}_1\mathbf{M}_2}(I) = \emptyset$, P uses a bad arc. So, by Lemma 5.5, $l(P) \geq 1$. By Equation (5.13) and the feasibility of I, I is extreme. So, by Theorem 3.2, $I \triangle P$ is an extreme common independent set of $\mathbf{M}_1, \mathbf{M}_2$ such that $|I \triangle P| = |I| + 1$ and $w(I \triangle P) = w(I) - l(P) < w(I)$. If there exists a feasible common independent set J of $\mathbf{M}_1, \mathbf{M}_2$ such that $|J| = |I| + 1$, then this contradicts the fact that $I \triangle P$ is extreme since $w(J) = w(I) > w(I \triangle P)$ by Equation (5.13). □

5.2. Algorithm and its time complexity

From Theorem 5.1, we can obtain the following Algorithm 1 for the MIwPC problem.

Algorithm 1

Input: matroids $\mathbf{M}_1 = (E, \mathcal{I}_1), \mathbf{M}_2 = (E, \mathcal{I}_2)$, and a subset A of E.
Output: an optimal solution for the MIwPC problem.

```
Find a maximum-size common independent set I* of M'_1, M'_2.
Find the DM type decomposition 𝒜 of M'_1, M'_2 for I*.
while Q_{M1M2}(I) ≠ ∅ do
   Find a minimum-size path P of Q_{M1M2}(I) and update I := I△P.
end while
return I
```

For each common independent set I of $\mathbf{M}_1, \mathbf{M}_2$, the time required to construct $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ heavily depends on matroids $\mathbf{M}_1, \mathbf{M}_2$. So, we denote by T the time required to construct $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ for any common independent set I of $\mathbf{M}_1, \mathbf{M}_2$.

Theorem 5.2. *Algorithm 1 solves the* MIwPC *problem in $O(\gamma T)$ time, where γ is the size of an optimal solution for the* MIwPC *problem.*

Proof. The correctness of Algorithm 1 immediately follows from Theorem 5.1. (Notice that if there exists no feasible common independent set of $\mathbf{M}_1, \mathbf{M}_2$ whose size is equal to N, then it follows from (I1) that there exists no feasible common independent set of $\mathbf{M}_1, \mathbf{M}_2$ whose size is more than N.) So, we consider its time complexity. Step 1 can be done in $O(\gamma T)$ time with Theorem 3.1 and breadth-first search. (Notice that breadth-first search can be done in $O(T)$ time since the size of $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ is $O(T)$.) Step 2 is essentially equivalent to finding the strongly connected components of $D_{\mathbf{M}_1\mathbf{M}_2}(I)$ for some common independent set I of $\mathbf{M}_1, \mathbf{M}_2$. So, this step can be done in $O(T)$ time. Since $H_{\mathbf{M}_1\mathbf{M}_2}(I)$ can be constructed in $O(T)$ time for any common independent set I of $\mathbf{M}_1, \mathbf{M}_2$, Steps 3 to 5 can be done in $O(\gamma T)$ time with Theorem 5.1 and breadth-first search. From these observations, the time complexity of Algorithm 1 is $O(\gamma T)$. □

As stated in Section 1, it is not difficult to see that the MIwPC problem can be reduced to the maximum-weight matroid intersection problem with the weight function defined in Equation (5.13). If we use the maximum-weight matroid intersection algorithm of [2, 4] with naive analysis, then the time complexity is $O(\gamma(T+n \log n))$, where define $n := |E|$ (also, see [13, Section 41.3]). However, it should be noted that the weight defined in Equation (5.13) is very restricted, and thus there is a possibility of achieving $O(\gamma T)$ time by using the algorithms of [2, 4] with more careful analysis.

6. Concluding Remarks

In this paper, we introduced the matroid intersection problem with priority constraints that is a matroid-generalization of the simplest case of the rank-maximal matching problem. Then, we proposed an algorithm for the matroid intersection problem with priority constraints by extending the "combinatorial" algorithm for the rank-maximal matching problem presented by Irving, Kavitha, Mehlhorn, Michail and Paluch [7]. We conclude this paper with a remark about the extendibility of the results of this paper to the "general" rank-maximal matching problem. The **rank-maximal matching problem** is formally defined as follows. We are given a finite bipartite graph G and a partition $E_1, E_2, \ldots, E_k$ of the edge set of G. Then, our goal is to find a matching M of G such that $|M \cap E_1|$ is maximized and given that $|M \cap E_1|$ is maximized, $|M \cap E_2|$ is also maximized, and so on. We can naturally generalize this problem to the matroid intersection setting and it is not difficult to see that this problem can be solved in polynomial time by reducing it to the maximum-weight matroid intersection problem. The problem studied in this paper is the case of $k = 2$. So, the following question naturally arises. Can we extend the "combinatorial" algorithm for the rank-maximal matching problem to its matroid-generalization? However, it does not seem that we can straightforwardly do this extension.

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