Journal of the Operations Research Society of Japan
Vol. 55, No. 4, December 2012, pp. 225–234

A NOTE ON RELAXED DIVISOR METHODS

Tetsuo Ichimori
Osaka Institute of Technology

(Received April 6, 2012; Revised August 3, 2012)

Abstract The purpose of this note is to add some important properties to the results obtained in [2]. Specifically, it is shown that (i) an apportionment for relaxed divisor methods remains unchanged over an interval and (ii) any relaxed divisor method approaches the Webster method as the house size increases.

Keywords: Discrete optimization, apportionment, divisor methods

1. Introduction

Balinski and Young [1] shows that an apportionment method is population monotone if and only if it is a divisor method. Because the population monotonicity is admitted to be most reflective of proportionality in apportionment, divisor methods are indispensable. However, it is true that some undesirable methods are divisor methods. For example, the Jefferson method favors large states and the Adams method favors small states, although they are divisor methods. On the other hand, there are some desirable divisor methods as Webster's and Hill's. Hence it might be useful to exclude such undesirable methods from the class of divisor methods. This is exactly the motivation for relaxed divisor methods. Relaxed divisor methods are derived from the concept of "relaxed proportionality", see [2] for details.

Because the rounding criterion of any relaxed divisor method is describable by way of a parameter, it might be expected that we could easily handle it mathematically and hence this might be most helpful in discovering the best method of apportionment.

In this note we add some important properties to the results obtained in [2]. Specifically, we will show that (i) an apportionment for relaxed divisor methods remains unchanged over an interval and (ii) any relaxed divisor method approaches the Webster method as the house size increases.

Although there are an infinite number of relaxed divisor methods based on parameters between any two different finite values, the first property (i) implies that there are only a finite number of apportionments between them. For example, for parameters between -10 and 10 we have only seven different apportionments, see Section 5. This will be helpful in the sense that we will find out the best one among a finite number of candidates rather than among an infinite number of candidates. In other words, the first property (i) could reduce the effort for finding the best one .

The second property (ii) theoretically shows that the rounding criterion of any method except for Webster's is not identical to that of Webster's for any finite house size. Moreover, computer simulations performed in [2] strongly suggest the Webster method is not biased for small or large states for any of the house size between 200 and 43,500. Therefore, the second property (ii) shows that any relaxed divisor method except for the Webster method is biased for small or large states. This means that the Webster method is the one and only unbiased relaxed divisor method, or it is the best.

In Section 2 we review divisor methods and relaxed divisor methods. In Section 3 we discuss the invariance of an apportionment over an interval. In Section 4 we show the limitation of any relaxed divisor method is the Webster method as the house size approaches infinity. Finally in Section 5 we give some apportionments for the 2010 U.S. Census.

2. Apportionment Methods

In this section, the definitions of divisor methods and relaxed divisor methods are given.

2.1. Divisor methods

Let $\mathbb{N}$ and $\mathbb{N}_+$ denote the sets of non-negative integers and positive integers, respectively. Define a real valued function $d(a)$ for $a \in \mathbb{N}$, known as a rounding criterion. The function $d(a)$ is to be defined as a strictly increasing function in a. It satisfies $a \le d(a) \le a+1$ for $a \in \mathbb{N}$ and moreover satisfies $d(b) = b$ and $d(c) = c+1$ for no pair of integers $b \in \mathbb{N}_+$ and $c \in \mathbb{N}$.

Let z be a positive real number and let $[z]$ denote an integer defined by the following rule:

1. If $z < d(0)$, then $[z] = 0$.
2. If $d(a) < z < d(a+1)$ for some $a \in \mathbb{N}$, then $[z] = a+1$.
3. If $z = d(a)$ for some $a \in \mathbb{N}$, then $[z] = a$ or $a+1$.

Let s denote the number of states and $h \ge s$ the house size. Let $p_i > 0$ denote the population of state i. If the equality $\sum_{i=1}^{s}[p_i/x] = h$ is achieved for some divisor $x > 0$, then the number of seats to which state i is entitled is $a_i = [p_i/x]$; s-vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_s)$ is referred to as an " apportionment of h." Such an apportionment of h is determined through $d(a), a \in \mathbb{N}$. If an apportionment method is defined with $d(a)$, then the method is called a divisor method. The following divisor methods are especially well known and defined with respective $d(a)$:

- the Adams method with $d(a) = a$,
- the Dean method with $d(a) = a(a+1)/(a+0.5)$,
- the Hill method with $d(a) = \sqrt{a(a+1)}$,
- the Webster method with $d(a) = a+0.5$,
- the Jefferson method with $d(a+1) = a+1$.

2.2. Relaxed divisor methods

Let $\mathbb{R}_+$ and $\mathbb{R}$ denote the sets of positive real numbers and real numbers, respectively, where it should be noted that infinities $\pm\infty$ are not included in $\mathbb{R}$ in a customary way.

For a real number $\theta \in \mathbb{R}$ and a non-negative integer $a \in \mathbb{N}_+$, define the rounding criterion $d_\theta(a)$ as follows: *

$$d_\theta(a) = \begin{cases} \dfrac{1}{e}\dfrac{(a+1)^{a+1}}{a^a} & \text{if } \theta = 1, \\ \dfrac{1}{\log\frac{a+1}{a}} & \text{if } \theta = 0, \\ \left(\dfrac{(a+1)^\theta - a^\theta}{\theta}\right)^{\frac{1}{\theta-1}} & \text{if } \theta \ne 1, 0, \end{cases} \tag{2.1}$$

*In [2] $n+1$ is used instead of θ.

and for real $\theta \in \mathbb{R}$, define $d_\theta(0)$ as follows:

$$d_\theta(0) = \begin{cases} 0 & \text{if } \theta \le 0, \\ \dfrac{1}{e} \approx 0.37 & \text{if } \theta = 1, \\ \left(\dfrac{1}{\theta}\right)^{\frac{1}{\theta-1}} & \text{if } \theta > 0,\ \theta \ne 1. \end{cases} \tag{2.2}$$

An apportionment method is a relaxed divisor method if it is the divisor method defined with $d_\theta(a), a \in \mathbb{N}$. It is referred to as the relaxed divisor method based on θ.† For example, the relaxed divisor method based on $\theta = -1$ is the Hill method. Table 1 contains five methods based on $\theta \in \{-1, 0, 1, 2, 3\}$, see [2] for details.

Table 1: Relaxed divisor methods based on $\theta \in \{-1, 0, 1, 2, 3\}$

Method	θ
Hill	-1
T&S	0
Theil	1
Webster	2
"1/3"	3

Theorem 2.1. Let $\boldsymbol{a}$ be an apportionment of h with $\sum_{i\in J} a_i = h$ for the relaxed divisor method based on θ. Then

$$\max_{i\in I} \frac{d_\theta(a_i - 1)}{p_i} \le \min_{j\in J} \frac{d_\theta(a_j)}{p_j}$$

where $I = \{i \mid a_i > 0, i = 1, 2, \ldots, s\}$ and $J = \{1, 2, \ldots, s\}$.

Proof. See [1] for the proof. □

3. Invariance of an Apportionment over an Interval

For real $\theta, \omega \in \mathbb{R}$ with $\theta \ne 0$ and $\theta \ne \omega$, and for positive $x, y \in \mathbb{R}_+$ with $x \ne y$, let

$$u(x, y; \theta, \omega) = \left(\frac{\omega(x^\theta - y^\theta)}{\theta(x^\omega - y^\omega)}\right)^{\frac{1}{\theta-\omega}}.$$

It is easily seen that $u(x, y; \theta, \omega)$ can be extended by continuity to a function defined for all real θ and ω and all positive x and y with $x \ne y$. Namely, we have

$$u(x, y; \theta, \omega) = \begin{cases} \left(\frac{\omega(x^\theta - y^\theta)}{\theta(x^\omega - y^\omega)}\right)^{\frac{1}{\theta-\omega}} & \text{if } \theta\omega(\theta-\omega) \ne 0, \\ \left(\frac{x^\theta - y^\theta}{\theta(\log x - \log y)}\right)^{\frac{1}{\theta}} & \text{if } \theta \ne 0,\ \omega = 0, \\ \left(\frac{x^\omega - y^\omega}{\omega(\log x - \log y)}\right)^{\frac{1}{\omega}} & \text{if } \theta = 0,\ \omega \ne 0, \\ \exp\left(\frac{x^\theta \log x - y^\theta \log y}{x^\theta - y^\theta} - \frac{1}{\theta}\right) & \text{if } \theta = \omega \ne 0, \\ \sqrt{xy} & \text{if } \theta = \omega = 0. \end{cases} \tag{3.1}$$

†When we refer to the relaxed divisor method based on θ, the value of θ is assumed to be finite as in [2].

It is clear that the function $u(x,y;\theta,\omega)$ is symmetric with respect to x, y and θ, ω. Since $u(x,y;\theta,\omega)$ is a mean [‡] of positive x and y with $x \neq y$, we have

$$\min\{x,y\} < u(x,y;\theta,\omega) < \max\{x,y\}$$

for any $\theta, \omega \in \mathbb{R}$. For this function $u(x,y;\theta,\omega)$, Stolarsky [4] gives the following theorem:
Theorem 3.1. For positive x and y with $x \neq y$, the function $u(x,y;\theta,\omega)$ is strictly increasing in both θ and ω.

For positive $x \in \mathbb{R}_+$ and real $\theta \in \mathbb{R}$ define

$$u(x,\theta) = u(x+1, x; \theta, 1). \tag{3.2}$$

Lemma 3.1. For positive $x \in \mathbb{R}_+$

$$\lim_{\theta \to +\infty} u(x,\theta) = x+1, \quad \lim_{\theta \to -\infty} u(x,\theta) = x.$$

Proof. It is obvious from the definition of the Stolarsky mean, see [4]. □

Lemma 3.2. Let $u_x(x,\theta)$ be the derivative of $u(x,\theta)$ with respect to x. Then

$$\frac{u_x(x,\theta)}{u(x,\theta)} = \frac{1}{u(x+1,x;\theta,\theta-1)}.$$

Proof. Suppose $\theta \neq 1, 0$. Then we can get from (3.1) and (3.2)

$$u(x,\theta) = \left(\frac{(x+1)^\theta - x^\theta}{\theta}\right)^{\frac{1}{\theta-1}}. \tag{3.3}$$

Take the logarithm of each side of (3.3), then

$$\log u(x,\theta) = \frac{1}{\theta-1}\log((x+1)^\theta - x^\theta) - \frac{1}{\theta-1}\log\theta.$$

Differentiate with respect to x, then we get

$$\begin{aligned}\frac{u_x(x,\theta)}{u(x,\theta)} &= \frac{\theta}{\theta-1} \times \frac{(x+1)^{\theta-1} - x^{\theta-1}}{(x+1)^\theta - x^\theta} \\ &= \frac{1}{\frac{\theta-1}{\theta}\frac{(x+1)^\theta - x^\theta}{(x+1)^{\theta-1} - x^{\theta-1}}}.\end{aligned}$$

Note that

$$u(x+1,x;\theta,\theta-1) = \left(\frac{(\theta-1)\big((x+1)^\theta - x^\theta\big)}{\theta\big((x+1)^{\theta-1} - x^{\theta-1}\big)}\right)^{\frac{1}{\theta-(\theta-1)}},$$

then the equality can be obtained:

$$\frac{u_x(x,\theta)}{u(x,\theta)} = \frac{1}{u(x+1,x;\theta,\theta-1)}.$$

[‡] $u(x,y;\theta,\omega)$ is the so-called Stolarsky mean. An apportionment method based on $u(x,y;\theta,\omega)$ is proposed in [3]. Note that it is a divisor method but not a relaxed divisor method.

Next suppose $\theta = 1$. Then we have

$$u(x,1) = \exp\left(\frac{(x+1)\log(x+1) - x\log x}{(x+1)-x} - 1\right)$$
$$= \exp\left((x+1)\log(x+1) - x\log x - 1\right),$$

or

$$\log u(x,1) = (x+1)\log(x+1) - x\log x - 1.$$

Again differentiate with respect to x, then

$$\frac{u_x(x,1)}{u(x,1)} = \log(x+1) + 1 - \log x - 1 = \log\frac{x+1}{x}.$$

Note that

$$u(x+1,x;1,0) = \frac{(x+1)-x}{\log(x+1) - \log x} = \frac{1}{\log\frac{x+1}{x}}.$$

Then we obtain

$$\frac{u_x(x,1)}{u(x,1)} = \frac{1}{u(x+1,x;1,0)}.$$

Finally consider the case for $\theta = 0$. But the proof is parallel to that for $\theta = 1$, hence omitted.

Therefore, the theorem is proved. $\square$

For positive $x \in \mathbb{R}_+$ and real $\theta, \omega \in \mathbb{R}$ with $\theta < \omega$, define

$$g(x) = \frac{u(x,\theta)}{u(x,\omega)}. \tag{3.4}$$

Lemma 3.3. For $x > 0$, $g(x)$ is strictly increasing in x.[§]

Proof. Take the logarithm of each side of (3.4) and differentiate with respect to x, then it follows from Lemma 3.2 that

$$\frac{g'(x)}{g(x)} = \frac{1}{u(x+1,x;\theta,\theta-1)} - \frac{1}{u(x+1,x;\omega,\omega-1)}.$$

Notice $g(x) > 0$, then we have $g'(x) > 0$ because Theorem 3.1 implies $u(x+1,x;\theta,\theta-1) < u(x+1,x;\omega,\omega-1)$ for $\theta < \omega$. Hence the function $g(x)$ is strictly increasing in x. $\square$

Lemma 3.4. For positive $x, y \in \mathbb{R}_+$ and real $\theta \in \mathbb{R}$, let

$$r(\theta) = \frac{u(x,\theta)}{u(y,\theta)}.$$

If $x = y$, then $r(\theta) \equiv 1$; if $x < y$, then $r(\theta)$ is strictly increasing and $x/y < r(\theta) < (x+1)/(y+1)$; if $x > y$, then it is strictly decreasing and $x/y > r(\theta) > (x+1)/(y+1)$.

[§]This lemma appears in [3].

Proof. It is obvious that $r(\theta) \equiv 1$ for all θ if $x = y$. Assume $x < y$. Then, it follows from Lemma 3.3 that

$$\frac{u(x,\theta)}{u(x,\omega)} < \frac{u(y,\theta)}{u(y,\omega)}$$

for any x, y, θ, ω with $0 < x < y$ and $\theta < \omega$. Notice all $u(\ ,\ ) > 0$. Then we can cross-multiply to get

$$\frac{u(x,\theta)}{u(y,\theta)} < \frac{u(x,\omega)}{u(y,\omega)},$$

namely, $r(\theta) < r(\omega)$ for $\theta < \omega$. In other words, $r(\theta)$ is strictly increasing in θ. From Lemma 3.1 we can get the relations $\lim_{\theta\to+\infty} r(\theta) = (x+1)/(y+1)$ and $\lim_{\theta\to-\infty} r(\theta) = x/y$. Hence we obtain $x/y < r(\theta) < (x+1)/(y+1)$.

The proof for the case where $x > y$ is analogous to the proof of the case where $x < y$ and hence omitted. $\square$

Lemma 3.5. Let $x, y > 0$. If $u(x,\theta)/u(y,\theta)$ is strictly increasing in θ, then $x < y$.

Proof. We will prove by contrapositive. Let $r(\theta) = u(x,\theta)/u(y,\theta)$ and assume $x \geq y > 0$. Then Lemma 3.4 claims that $r(\theta) \equiv 1$ or $r(\theta)$ is strictly decreasing, namely, it is not 'strictly increasing'. Hence the lemma. $\square$

Extend $u(x,\theta)$ defined for positive $x > 0$ to $u(x,\theta)$ for non-negative $x \geq 0$ by defining $u(0,\theta) = d_\theta(0)$. Then we can obtain the following lemma similar to Lemma 3.5:

Lemma 3.6. Let $x, y \geq 0$ and $\theta > 0$. If $u(x,\theta)/u(y,\theta)$ is strictly increasing in θ, then $x < y$.

Proof. Omitted because it is similar to that of Lemma 3.5. $\square$

Lemma 3.7. For $a \in \mathbb{N}$,

$$d_\theta(a) = u(a,\theta).$$

Proof. We can easily obtain the lemma by comparing (2.1) with (3.1). $\square$

Theorem 3.2. If $\boldsymbol{a}$ is an apportionment of h for two relaxed divisor methods, one based on θ_1 and the other on θ_2 with $\theta_1 < \theta_2$, then it is also an apportionment of h for all relaxed divisor methods based on θ, where $\theta_1 \leq \theta \leq \theta_2$.

Proof. Assume $\boldsymbol{a}$ is not an apportionment of h for the relaxed divisor method based on some $\bar\theta$ where $\theta_1 < \bar\theta < \theta_2$. Then Theorem 2.1 means that there must exist two different states i and j satisfying

$$\frac{d_{\bar\theta}(a_i - 1)}{p_i} > \frac{d_{\bar\theta}(a_j)}{p_j} \tag{3.5}$$

where $a_i > 0$ and $a_j \geq 0$.

Because $\boldsymbol{a}$ is an apportionment of h for two relaxed divisor methods based on θ_1 and θ_2, these two states i and j have the following relations:

$$\frac{d_{\theta_1}(a_i - 1)}{p_i} \leq \frac{d_{\theta_1}(a_j)}{p_j} \tag{3.6}$$

and

$$\frac{d_{\theta_2}(a_i-1)}{p_i} \le \frac{d_{\theta_2}(a_j)}{p_j}. \tag{3.7}$$

We have two cases to be considered: the case where $\theta_1 \le 0$ and the case where $\theta_1 > 0$. First consider the case where $\theta_1 \le 0$. Then the first line in (2.2) gives us $d_{\theta_1}(0) = 0$. Since we assume $h \ge s$, the equality $d_{\theta_1}(0) = 0$ demands that every state should receive at least one seat. Hence the state j naturally receives $a_j \ge 1$ seats, or $d_{\theta_1}(a_j)$ and $d_{\bar{\theta}}(a_j)$ are positive. In addition, (3.5) demands $a_i \ge 2$ for the state i, or $d_{\bar{\theta}}(a_i-1)$ and $d_{\theta_2}(a_i-1)$ are positive.

It follows from (3.5) and (3.6) that

$$\frac{d_{\theta_1}(a_i-1)}{d_{\theta_1}(a_j)} < \frac{d_{\bar{\theta}}(a_i-1)}{d_{\bar{\theta}}(a_j)}.$$

Lemmas 3.5 and 3.7 give the relation: $a_i - 1 < a_j$. On the other hand, it follows from (3.5) and (3.7) that

$$\frac{d_{\bar{\theta}}(a_j)}{d_{\bar{\theta}}(a_i-1)} < \frac{d_{\theta_2}(a_j)}{d_{\theta_2}(a_i-1)}.$$

Again, Lemmas 3.5 and 3.7 give the relation: $a_j < a_i - 1$ and hence we have a contradiction.

In the latter case, where $0 < \theta_1 < \bar{\theta} < \theta_2$, note that all $d_{\theta_1}(a_j)$, $d_{\bar{\theta}}(a_j)$, $d_{\bar{\theta}}(a_i-1)$ and $d_{\theta_2}(a_i-1)$ are positive. Use Lemmas 3.6 and 3.7. Then the remaining proof is parallel to the proof for the first case. Hence the theorem. $\square$

Combine Theorem 3.2 with Theorem 4.1 in Section 4, then we have the following:

Corollary 3.1. Let $\boldsymbol{a}$ be an apportionment of h for two divisor methods, one is the Jefferson method and the other is the relaxed divisor method based on θ_J, then it is also an apportionment of h for all relaxed divisor methods based on $\theta \ge \theta_J$

Corollary 3.2. Let $\boldsymbol{a}$ be an apportionment of h for two divisor methods, one is the Adams method and the other is the relaxed divisor method based on θ_A, then it is also an apportionment of h for all relaxed divisor methods based on $\theta \le \theta_A$

4. Limit of the Relaxed Divisor Method Based on θ

In this section, we first investigate the limits of the relaxed divisor method based on θ as θ approaches $\pm\infty$.

Theorem 4.1. The limit of the relaxed divisor method based on θ is the Jefferson method as θ approaches $+\infty$ and the limit of it is the Adams method as θ approaches $-\infty$.

Proof. Note that Lemma 3.7 shows $d_\theta(a) = u(a, \theta)$ for non-negative $a \in \mathbb{N}$. Since Lemma 3.1 shows $\lim_{\theta\to+\infty} u(a, \theta) = a + 1$ for positive a, we get $d_\theta(a) = a + 1$ for positive $a \in \mathbb{N}_+$. In addition, from the definition (2.2) of $d_\theta(0)$, we can get $\lim_{\theta\to+\infty} d_\theta(0) = 1$. Hence we have the relation: $\lim_{\theta\to+\infty} d(a) = a+1$ for each non-negative $a \in \mathbb{N}$ which is Jefferson's rounding criterion. The proof for the Adams method can be carried out in a similar manner. $\square$

Comment: This proof shows that the methods of Jefferson and Adams are not relaxed divisor methods. In addition, the Dean method is not a relaxed divisor method because it is obviously impossible to find a value of θ satisfying $u(a, \theta) = a(a + 1)/(a + 0.5)$ for all non-negative integers $a \in \mathbb{N}$.

Next we study the limit of the rounding criterion $d_\theta(a)$ of a relaxed divisor method as the house size h approaches $+\infty$. Then we have the following:

Theorem 4.2.

$$\lim_{a \to \infty} \left| d_\theta(a) - \left(a + \frac{1}{2}\right) \right| = 0.$$

Proof. Let $v(x) = u(x, \theta) - x$. Then from Lemma 3.2 we get $v'(x) = u(x, \theta)/u(x+1, x; \theta, \theta - 1) - 1 = u(x+1, x; \theta, 1)/u(x+1, x; \theta, \theta-1) - 1$. It follows from Theorem 3.1 that if $\theta < 2$, then $v'(x) > 0$, if $\theta > 2$, then $v'(x) < 0$ and if $\theta = 2$, then $v'(x) = 0$.

Assume $\theta < 2$. Then the function $v(x)$ is strictly increasing in x and bounded from above since we have $v(x) = u(x, \theta) - x < u(x, 2) - x = 0.5$, where the last equality follows from $u(x, 2) = u(x+1, x; 2, 1) = ((x+1)^2 - x^2)/2 = x + 0.5$. Hence the function $v(x)$ must have a limit at positive infinity. Define the limit as $e = \lim_{x \to +\infty} v(x)$ where $e \leq 0.5$.

Moreover, assume $e < 0.5$. Then there must exist $\theta' < 2$ such that $u(x, \theta') = x + e$ since $u(x, \theta)$ is continuous and strictly increasing in θ, $\lim_{\theta \to -\infty} u(x, \theta) = x$ and $u(x, 2) = x + 0.5$. However, $u(x, \theta') - x$ is also strictly increasing in x, which means $\lim_{x \to +\infty} \big(u(x, \theta') - x\big) > e$. Here we have a contradiction. Therefore we obtain $\lim_{x \to +\infty} \big(u(x, \theta) - x\big) = 0.5$ for $\theta < 2$.

Next assume $\theta > 2$. Then $v(x) = u(x, \theta) - x$ is strictly decreasing and we have $u(x, \theta) - x > u(x, 2) - x = 0.5$. By doing in a similar way as before we can obtain $\lim_{x \to +\infty} \big(u(x, \theta) - x\big) = 0.5$ for $\theta > 2$.

For $\theta = 2$ there is nothing to do. Finally, choose $x = a \in \mathbb{N}$, then $d_\theta(a) = u(a, \theta)$. Hence we have the theorem. $\square$

Note that $d_2(a) = a + 0.5$ is Webster's rounding criterion. Then we have the following:

Corollary 4.1. Any relaxed divisor method approaches the Webster method as the house size increases.

This explains the results presented in Figure 1 of [2]. This author believes Corollary 4.1 is the most important in the theory of apportionment. In other words, he believes the Webster method to be the best among all relaxed divisor methods.

5. Some Apportionments for the 2010 U.S. Census

We consider all relaxed divisor methods based on θ where $-10 \leq \theta \leq 10$ for the 2010 U.S. Census. Then seven different apportionments $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_7$ are obtained for seven intervals $I_1, \ldots, I_7$, respectively. See Table 2 for the 7 intervals and the 4th to 10th columns of Table 3 for the 7 apportionments. In Table 3, the house size $h = 435$ is used and the 3rd column "Quota" represents the quota q_i (fair share) of state i where $q_i = h \times p_i / \sum_{j=1}^{s} p_j$.

Interval I_4 contains $\theta = -1, 0, 1$ which correspond to the methods of Hill, T&S and Theil, respectively. Interval I_5 contains $\theta = 2, 3$ which correspond to the methods of Webster and "1/3", respectively. Similar results are obtained for the 1950–2000 Censuses.

Table 2: Seven intervals for all relaxed divisor methods based on θ where $-10 \leq \theta \leq 10$

$I_1 = [-10.0000, -7.6931]$,	$I_2 = [-7.6931, -3.0885]$,	$I_3 = [-3.0885, -1.5265]$,
$I_4 = [-1.5265, 1.6147]$,	$I_5 = [1.6147, 7.0545]$,	$I_6 = [7.0545, 8.2831]$,
$I_7 = [8.2831, 10.0000]$		

Acknowledgements

The author would like to thank anonymous referees for their valuable comments.

Table 3: Seven different apportionments for intervals $I_1, \ldots, I_7$

State	Population	Quota	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{a}_7$
CA	37,341,989	52.538	52	52	53	53	53	53	53
TX	25,268,418	35.551	35	36	36	36	36	36	36
NY	19,421,055	27.324	27	27	27	27	27	28	28
FL	18,900,773	26.592	27	27	27	27	27	27	27
IL	12,864,380	18.099	18	18	18	18	18	18	18
PA	12,734,905	17.917	18	18	18	18	18	18	18
OH	11,568,495	16.276	16	16	16	16	16	16	16
MI	9,911,626	13.945	14	14	14	14	14	14	14
GA	9,727,566	13.686	14	14	14	14	14	14	14
NC	9,565,781	13.458	13	13	13	13	14	14	14
NJ	8,807,501	12.392	12	12	12	12	12	12	12
VA	8,037,736	11.309	11	11	11	11	11	11	11
WA	6,753,369	9.502	10	10	10	10	10	10	10
MA	6,559,644	9.229	9	9	9	9	9	9	9
IN	6,501,582	9.147	9	9	9	9	9	9	9
AZ	6,412,700	9.022	9	9	9	9	9	9	9
TN	6,375,431	8.970	9	9	9	9	9	9	9
MO	6,011,478	8.458	8	8	8	8	8	8	8
MD	5,789,929	8.146	8	8	8	8	8	8	8
WI	5,698,230	8.017	8	8	8	8	8	8	8
MN	5,314,879	7.478	8	8	7	8	8	7	8
CO	5,044,930	7.098	7	7	7	7	7	7	7
AL	4,802,982	6.757	7	7	7	7	7	7	7
SC	4,645,975	6.537	7	7	7	7	7	7	7
LA	4,553,962	6.407	6	6	6	6	6	6	6
KY	4,350,606	6.121	6	6	6	6	6	6	6
OR	3,848,606	5.415	5	5	5	5	5	5	5
OK	3,764,882	5.297	5	5	5	5	5	5	5
CT	3,581,628	5.039	5	5	5	5	5	5	5
IA	3,053,787	4.296	4	4	4	4	4	4	4
MS	2,978,240	4.190	4	4	4	4	4	4	4
AR	2,926,229	4.117	4	4	4	4	4	4	4
KS	2,863,813	4.029	4	4	4	4	4	4	4
UT	2,770,765	3.898	4	4	4	4	4	4	4
NV	2,709,432	3.812	4	4	4	4	4	4	4
NM	2,067,273	2.909	3	3	3	3	3	3	3
WV	1,859,815	2.617	3	3	3	3	3	3	3
NE	1,831,825	2.577	3	3	3	3	3	3	2
ID	1,573,499	2.214	2	2	2	2	2	2	2
HI	1,366,862	1.923	2	2	2	2	2	2	2
ME	1,333,074	1.876	2	2	2	2	2	2	2
NH	1,321,445	1.859	2	2	2	2	2	2	2
RI	1,055,247	1.485	2	2	2	2	1	1	1
MT	994,416	1.399	2	2	2	1	1	1	1
DE	900,877	1.267	2	1	1	1	1	1	1
SD	819,761	1.153	1	1	1	1	1	1	1
AK	721,523	1.015	1	1	1	1	1	1	1
ND	675,905	0.951	1	1	1	1	1	1	1
VT	630,337	0.887	1	1	1	1	1	1	1
WY	568,300	0.800	1	1	1	1	1	1	1
Totals	309,183,463	435.000	435	435	435	435	435	435	435

References

[1] M.L. Balinski and H.P. Young: *Fair Representation* 2nd edition (Brookings Institution Press, Washington DC, 2001).

[2] T. Ichimori: Relaxed divisor methods and their seat biases. *Journal of the Operations Research Society of Japan*, **55** (2012), 63–72.

[3] L. Lauwers and T. Van Puyenbroeck: Minimally disproportional representation: Generalized entropy and Stolarsky mean-divisor methods of apportionment. *Unpublished Discussion Paper* (2008).

[4] K.B. Stolarsky: Generalizations of the logarithmic mean. *Mathematics Magazine*, **48** (1975), 87–92.

Tetsuo Ichimori
Osaka Institute of Technology
1-79-1 Kitayama, Hirakata City
Osaka 573-0196, Japan
E-mail: ichimori@is.oit.ac.jp