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A SMUGGLING GAME WITH THE SECRECY OF SMUGGLER'S INFORMATION

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Abstract This paper deals with a smuggling game with multiple stages, in which Customs and a smuggler participate. Customs and the smuggler are allowed to take an action of patrol and smuggling, respectively, within the limited number of chances. Customs obtains reward by the capture of the smuggler and the smuggler gets reward by the success of smuggling. The reward or the payoff of the game is brought at each stage and is assumed to be zero-sum. Almost all past researches modeled their games by the so-called complete information game and they assumed that each player knows the past strategies taken by his opponent or never knows them. Recently, we recognize that information is crucial to the results of the games. In this paper, we deal with a smuggling game with incomplete information, where information acquisition is asymmetric between players and is disadvantageous to Customs, and we evaluate the value of information by developing a computational methodology to derive Bayesian equilibrium.

Keywords: Game theory, linear programming, smuggling game, incomplete information

1. Introduction

The paper deals with an inspection game or a smuggling game with multiple stages, in which Customs and a smuggler participate. Dresher [6] formulated a compliance problem for the treaty of arms reduction and started studying the inspection game, where a violator wishes to violate the treaty in secret for his benefit and an inspector wants to prevent the illegal behavior of the violator. Maschler [15] generalized Dresher's problem.

Their research has two branches as its extension. One is the problem of the arms-reduction treaty and the international inspection of nuclear plants by the International Atomic Energy Agency (IAEA). Avenhaus et al. [2] surveyed past studies on the compliance with regulations and treaties. Canty et al. [5] modeled a sampling problem of inspecting nuclear materials as a sequential game model and proposed an efficient inspection strategy to compel an inspectee to the Treaty on the Non-Proliferation of Nuclear Weapons and related treaties. Avenhaus and Canty [1] analyzed two types of errors in the inspection by a sequential game model. Avenhaus and Kilgour [3] discussed a nonzero-sum one-shot game with an inspector and two inspectee countries as an optimization problem of inspection resource, where the inspector distributes his inspection resources to two countries and each inspectee decides to take a legal or an illegal action for his interest. Hohzaki [11] extended their model to the inspection game with many inspectees and derived an optimal dispatching plan of inspection staffs to facilities in the inspectee countries.

The other branch stemming from Dresher's research is the smuggling game with Customs and a smuggler. Thomas and Nisgav [17] would be the first research that dealt with the smuggling game with multiple stages. They proposed a numerical algorithm of repeatedly solving a matrix game stage by stage to derive the value of the game. Baston and Bostock [4] gave a closed form of solution for the game similar to Thomas-Nisgav's model. At the early stage of this research field, many researchers adopted the so-called perfect-capture assumption that Customs certainly captures the smuggler when both players meet. Baston and Bostock modeled the smuggling problem into the imperfect-capture model, where Customs captures the smuggler with some probability depending on the number of patrol boats. The smuggler was still assumed to have at most one opportunity to ship contraband, though. Garnaev [8] handled a model with three patrol boats. Sakaguchi [16] first assumed that the smuggler might take an action several times by the perfect-capture smuggling model. Sakaguchi model was extended by Ferguson and Melolidakis [7], who assumed that the smuggler can get rid of the capture by means of side payment but he must pay penalty on his capture. Hohzaki et al. [13] and Hohzaki [10] also extended Sakaguchi's model such that the capture of the smuggler terminates the game and the encounter of the smuggler and Customs stochastically results in one of capture, success of smuggling or no-event. Almost all researches so far treated a two-option strategy of smuggling or non-smuggling as a smuggler's strategy. Hohzaki [12] first analyzed the smuggling game with the Customs' decision on the amount of contraband.

Recently, we recognize that information is crucial to the results of the game. The concept of complete information and incomplete information was proposed by Harsanyi [9] and it has been applied to a variety of game models. In the smuggling game, players would acquisition information about their opponents in an asymmetric manner. Customs is generally thought to be a public organization and the smuggler would be a secret society. Therefore, the behavior of Customs is comparatively open to outside but the smuggler's information tends to be kept in secret. Considering the practical situation, the information acquisition must be asymmetrical between players. In the past models, they never thought of the asymmetrical information. They assumed that players know the behavior their opponents took in the past or that information about their opponents is perfectly hidden to competitors. Hohzaki and Maehara [14] considered the latter model by a one-shot game. In all models surveyed so far, the information acquisition is symmetric between players. In this paper, however, we deal with a smuggling game with incomplete information and asymmetric information, where Customs decides to patrol or not to patrol and the smuggler chooses one of smuggling or not smuggling. In terms of the assumption about information acquisition, this paper is located between two references [13] with disclosure of information and [14] with perfect secrecy of information. This paper carries the same assumptions as in [13] and [14] except for the system of information and adopts a multi-stage game model as in [13]. So we could evaluate the value of information by comparing our results with ones of [13]. The evaluation of the information value is a main purpose of this paper.

In the next section, we model our smuggling problem on a time horizon with multiple stages or multiple days, and formulate it by elucidating the difference between the states which two players can observe. In Section 3, we develop the system of equations to solve the optimization problem formulated in the previous section and propose a numerical algorithm to derive Bayesian equilibrium point in a general case. At the same time, we derive analytical forms of equilibrium points in some special cases in Section 4. In Section 5, we analyze optimal strategies of players by some numerical examples and evaluate the value of information by comparing this model with the previous one.

2. Modeling and Formulation

We consider the following smuggling model, which Customs and a smuggler play.

A1. There are N stages or N days on a time horizon. The number of stage is represented by the number of residual days so that time passes like $N,\ N-1,\ldots,1$.
A2. After the first stage $n = N$, Customs has K chances of patrol and the smuggler has L opportunities of smuggling.
A3. At each stage, Customs has a strategy with two options of patrol (P) or no-patrol (NP), and the smuggler also has a two-option strategy with smuggling (S) or no-smuggling (NS). They can take their strategies once per stage. That is the reason why excess chances of patrol and smuggling are discarded at any stage if the number of residual chances is larger than the number of stage.
A4. If Customs patrols and the smuggler smuggles, there can be two events of the capture of the smuggler by Customs or the success of smuggling, or no-event, exclusively. The first event happens with probability p_1 and the second one does with probability p_2 and the no-event has the residual probability $1-(p_1+p_2)$, where $p_1+p_2 \leq 1$. If the smuggler tries to smuggle when not patrolling, he certainly succeeds in smuggling.
A5. On capture of the smuggler, Customs earns reward $\alpha > 0$ and the game is terminated. On success of smuggling, the smuggler gets reward 1. The payoff of the game is assumed to be zero-sum, that is, a player loses the same amount of payoff as the other player's reward. We define the payoff of the game as the reward on the Customs' side. We suppose an inequality $\gamma \equiv \alpha p_1 - p_2 > 0$ such that the smuggler does not dare to smuggle on the patrol day.
A6. The game transfers to the next day if there occurs no capture of the smuggler. At the beginning of the next stage, the smuggler knows the strategy taken by Customs at the previous stage but the smuggler's past strategy is in secret to Customs. The initial state of the game, (N, K, L), is common information for both players.
A7. On capture of the smuggler or expiration of stages, the game ends.

By (n, k, l), we denote the state in which Customs keeps k chances of patrol and the smuggler has l chances of smuggling at Stage n. At the beginning of the stage n, the smuggler recognizes the state (n, k, l). However, Customs only knows (n, k) and anticipates the number of the residual chances of smuggling, l, by his belief on it. We denote his belief by $q_n = \{q_n(l), l = \underline{l}_n, \underline{l}_n+1, \cdots, \bar{l}_n\}$, which is a probability distribution with respect to l. $q_n(l)$ is the probability that the smuggler has l chances of smuggling left and has to satisfy the condition $\sum_{l=\underline{l}_n}^{\bar{l}_n} q_n(l) = 1,\ q_n(l) \geq 0$ for all l. We call the smuggler with l chances of smuggling as the l-type smuggler. The lower bound $\underline{l}_n$ of l is estimated by everyday smuggling and the upper bound $\bar{l}_n$ is calculated by no-smuggling, as follows:

$$\underline{l}_n = \max\{L - N + n, 0\}, \quad \bar{l}_n = \min\{L, n\}. \tag{1}$$

Because all players are supposed to know initial state (N, K, L) from Assumption (A6), Customs has a faultless belief of

$$q_N(L) = 1,\ q_N(l) = 0\ (l \neq L) \tag{2}$$

at the first stage N. We can also derive $\underline{l}_N = \bar{l}_N = L$ by applying $n = N$ to Equation (1).

Let us denote the strategies of players in State (n, k, l) by x_n and $y_n(l)$. Let x_n and $1-x_n$ be the respective probabilities of taking P and NP by Customs. Let $y_n(l)$ and $1 - y_n(l)$ be the probabilities that the l-type smuggler takes S and NS, respectively. We denote a whole set of strategies for all types of smuggler by $y_n \equiv \{y_n(l),\ l \in [\underline{l}_n, \bar{l}_n]\}$. Please note that x_n or $y_n(l)$ ought to be zero for $k = 0$ or $l = 0$, respectively, because Customs or the smuggler is forced to take Strategy NP or NS in such cases. We illustrate the stage game with the state (n, k, l) by Table 1.

Table 1: A table of payoffs in State (n,k,l)

Belief		$q_n(\underline{l}_n)$		$q_n(l)$		$q_n(\bar{l}_n)$
			$\cdots$	$y_n(l),\ 1-y_n(l)$	$\cdots$	
	$C\backslash S$	$S,\ NS$	$\cdots$	$S,\ NS$		$S,\ NS$
x_n	P		$\cdots$	(1), (2)	$\cdots$	
$1-x_n$	NP		$\cdots$	(3), (4)	$\cdots$	

For the stage game at Stage n, a combination of strategies of both players, (X,Y), $X\in\{P,NP\}$, $Y\in\{S,NS\}$, brings an expected payoff $R_l(X,Y)$, which is defined by the following equations and is to replace (1)–(4) in the table:

$$\text{(1)}\quad R_l(P,S)=\gamma+(1-p_1)v(n-1,k-1,l-1;\Gamma_P(q_n)), \tag{3}$$

$$\text{(2)}\quad R_l(P,NS)=v(n-1,k-1,l;\Gamma_P(q_n)), \tag{4}$$

$$\text{(3)}\quad R_l(NP,S)=-1+v(n-1,k,l-1;\Gamma_N(q_n)), \tag{5}$$

$$\text{(4)}\quad R_l(NP,NS)=v(n-1,k,l;\Gamma_N(q_n)), \tag{6}$$

where $\Gamma_P(q_n)$ or $\Gamma_N(q_n)$ are the revised beliefs of q_n for the next stage $n-1$, depending on the Customs' patrol strategy P or NP at the current stage n. By $v(n,k,l;q_n)$, we denote the expected payoff which the smuggler expects from State (n,k,l) to the end of the game. The payoff is realized by an equilibrium solution of both players. Anyway, please note that the players have some limits to their strategies in the special cases of $k=0$ and $l=0$, as mentioned above.

Here let us confirm initial values and boundary values of $v(n,k,l;q_n)$ in some special cases of (n,k,l), as follows.

(i) If the stage expires or the game ends, there is no payoff:

$$v(0,k,l;q_n)=0. \tag{7}$$

(ii) In the cases of $k>n$ or $l>n$, excess chances are discarded:

$$v(n,k,l;q_n)=v(n,n,l;q_n)\text{ if }k>n,\quad v(n,k,l;q_n)=v(n,k,n;q_n)\text{ if }l>n. \tag{8}$$

(iii) In the case of $k=0$, the smuggler certainly succeeds in smuggling at any stage because he knows the current situation:

$$v(n,0,l;q_n)=-l\text{ for }n>0\text{ and }l\le n. \tag{9}$$

(iv) If there is no chance of trying smuggling, no payoff occurs:

$$v(n,k,0;q_n)=0. \tag{10}$$

After State (n,k), Customs with belief q_n expects to get the payoff $P(x_n,y_n)$ given by the following expression, depending on the players' strategies x_n and y_n. Customs tries to maximize the payoff $P(x_n,y_n)$ by changing his strategy x_n:

$$\begin{aligned}P(x_n,y_n) &= \sum_{l=\underline{l}_n,l\neq 0}^{\bar{l}_n} q_n(l)\left[x_n\{y_n(l)R_l(P,S)+(1-y_n(l))R_l(P,NS)\}\right.\\ &\qquad\left.+(1-x_n)\{y_n(l)R_l(NP,S)+(1-y_n(l))R_l(NP,NS)\}\right].\end{aligned} \tag{11}$$

The smuggler of type l ($l \in [\underline{l}_n, \bar{l}_n],\ l \neq 0$) recognizes the state (n, k, l) and correctly anticipates the Customs' belief q_n. Without knowing the Customs' strategy x_n, the l-type smuggler wants to minimize the following expected payoff:

$$\begin{aligned} V(x_n, y_n(l)) \;=\; & x_n \{y_n(l)R_l(P,S) + (1-y_n(l))R_l(P,NS)\} \\ & +(1-x_n)\{y_n(l)R_l(NP,S) + (1-y_n(l))R_l(NP,NS)\}. \end{aligned} \tag{12}$$

We can exclude $l = 0$ from the enumeration of index l in Equation (11) because the smuggler of type $l = 0$ never tries smuggling and the no-smuggling strategy has no effect on the payoff afterwards. Therefore we can limit the region of the type of smuggler, l, to $\Lambda_n \equiv \{l \mid \underline{l}_n \leq l \leq \bar{l}_n,\ l \neq 0\}$ at Stage n.

Noting that the minimization of the function (12) with respect to $y_n(l)$ for all $l \in \Lambda_n$ is equivalent to the minimization of the function (11) with respect to y_n. Because of that, we can regard the smuggler as a minimizer for the expected payoff $P(x_n, y_n)$. Therefore, the stage game in State (n, k, l) is a two-person zero-sum game between Customs as a maximizer and the smuggler as a minimizer for the expected payoff $P(x_n, y_n)$. We can derive an equilibrium point of x_n^* and y_n^* by solving a minimax optimization problem of $P(x_n, y_n)$. Then we can calculate the value $v(\cdot)$ by using Equation (12), as follows:

$$\begin{aligned} & v(n,k,l;q_n) \\ & = \min\{x_n^* R_l(P,S) + (1-x_n^*)R_l(NP,S),\ x_n^* R_l(P,NS) + (1-x_n^*)R_l(NP,NS)\}. \end{aligned} \tag{13}$$

We aim to obtain value $v(N, K, L; q_N)$ in the initial state (N, K, L), which is just the value of the game, and optimal strategies x_n^* and y_n^* in any state $(n, k, l; q_n)$ on an equilibrium path by repeatedly solving a sequence of stage games from the last stage $n = 1$ to the initial one $n = N$.

3. A Computational Method for Bayesian Equilibrium Point

Because the expression (11) is bilinear for variables x_n and $y_n = \{y_n(l)\}$, we can obtain its minimax value and its maximin value comparatively easily. By the maximization of (11) with respect to x_n, we have

$$\begin{aligned} \max_{0 \leq x_n \leq 1} P(x_n, y_n) \;=\; \max \Bigg\{ & \sum_{l \in \Lambda_n} q_n(l) \{y_n(l)R_l(P,S) + (1-y_n(l))R_l(P,NS)\}, \\ & \sum_{l \in \Lambda_n} q_n(l) \{y_n(l)R_l(NP,S) + (1-y_n(l))R_l(NP,NS)\} \Bigg\} \end{aligned} \tag{14}$$

and, by the minimization of Equation (14) with respect to y_n, we have the following linear programming problem (P_S):

$$(P_S) \quad \min_{\eta, \{y_n(l), l \in \Lambda_n\}} \ \eta$$

$$\text{s.t.} \quad \sum_{l \in \Lambda_n} q_n(l) \{y_n(l)R_l(P,S) + (1-y_n(l))R_l(P,NS)\} \leq \eta, \tag{15}$$

$$\sum_{l \in \Lambda_n} q_n(l) \{y_n(l)R_l(NP,S) + (1-y_n(l))R_l(NP,NS)\} \leq \eta, \tag{16}$$

$$0 \leq y_n(l) \leq 1,\ l \in \Lambda_n.$$

The problem (P_S) gives us the value of the stage game in State $(n,k,l;q_n)$ and an optimal strategy of the smuggler y_n^*.

We also have the following transformation,

$$\min_{\{0\le y_n(l)\le 1, l\in\Lambda_n\}} P(x_n,y_n) = \sum_{l\in\Lambda_n} q_n(l)\min\{x_n R_l(P,S)+(1-x_n)R_l(NP,S),$$
$$x_n R_l(P,NS)+(1-x_n)R_l(NP,NS)\},$$

and the following linear programming problem through the maximization of the above expression:

$$(P_C)\quad \max_{x_n,\{\nu(l),l\in\Lambda_n\}} \sum_{l\in\Lambda_n} q_n(l)\nu(l)$$
$$\text{s.t.}\quad x_n R_l(P,S)+(1-x_n)R_l(NP,S)\ge \nu(l),\ l\in\Lambda_n, \qquad (17)$$
$$x_n R_l(P,NS)+(1-x_n)R_l(NP,NS)\ge \nu(l),\ l\in\Lambda_n, \qquad (18)$$
$$0\le x_n\le 1.$$

By solving the problem (P_C), we obtain the maximin value and an optimal patrol strategy x_n^*. Comparing Equations (17) and (18) with (13), we can see that an optimal value of $\nu(l)$ in Problem (P_C) is just the value $v(n,k,l;q_n)$, i.e.,

$$v(n,k,l;q_n)=\nu^*(l),\ l\in\Lambda_n. \qquad (19)$$

The two problems (P_S) and (P_C) are dual to each other as there is duality between the minimax optimization and the maximin optimization problems for an ordinary matrix game. If we set two kinds of dual variables $z_1(l)\ge 0$ and $z_2(l)\ge 0$ corresponding to the equations (17) and (18), respectively, we have $y_n^*(l)=z_1^*(l)/q_n(l)$ and $1-y_n^*(l)=z_2^*(l)/q_n(l)$. We also have the other relation that x_n^* and $1-x_n^*$ become optimal dual variables to two conditions (15) and (16). So we can obtain a set of optimal strategies $x_n^*,\ y_n^*$ and the value $v(n,k,l;q_n)$ by solving one of Problems (P_S) or (P_C).

We calculate $R_l(\cdot)$ in Problems (P_S) or (P_C) at Stage n by applying $v(n-1,\cdot)$ to the equations (3)–(6). Thus we can derive a full set of Bayesian equilibrium solution at all stages and the value of the game $v(N,K,L;q_N)$ in the following manner, theoretically. We start from the initial value (7) at $n=0$ and recursively solve Problems (P_S) or (P_C) in the order of $n=1,\cdots,N$, taking account of boundary conditions (8), (9) and (10). However, Customs has to revise his belief stage by stage. To embed the revision into the recursive calculation, we need some algorithmic idea to reach the value of the game. We mention the idea later, though.

Here we are going to discuss the operators of the revision Γ_P and Γ_N. Depending on the strategy P or NP taken at the present stage n, Customs revises his belief at the beginning of the next stage $n-1$. We can enumerate several cases that the smuggler would have l chances of smuggling left at hand at Stage $n-1$. The first case is the ordinary case that $l+1$ chances decrease to l by smuggling at Stage n or l chances remain unchanged by no-smuggling. We have other cases as follows:

(a) Case of $n-1\ge L$: l becomes $\bar{l}_{n-1}=\min\{L,n-1\}=L$ at Stage $n-1$ only if $\bar{l}_n=L$ residual chances is kept the same by no-smuggling at Stage n.

(b) Case of $L\ge n$: l becomes $\bar{l}_{n-1}=\min\{L,n-1\}=n-1$ at Stage $n-1$ if one chance of $\bar{l}_n=n$ is discarded by no-smuggling at Stage n besides the ordinary case mentioned above.

(c) Case of $L-N+n>0$: l becomes $\underline{l}_{n-1}=\max\{0,L-N+(n-1)\}=L-N+n-1$ at Stage $n-1$ only if $l=\underline{l}_n=L-N+n$ chances decrease by one by smuggling at Stage n.

Customs can carry out the revision of belief by taking account of the transition of l enumerated above and the optimal smuggler's strategy y_n^*. An exception is the case of $L-N+n\le 0$, where the number of smuggling chances $l=\underline{l}_n=\max\{0,L-N+n\}=0$ at Stage n remains $l=\underline{l}_{n-1}=\max\{0,L-N+(n-1)\}=0$ at Stage $n-1$ with certainty because the smuggler of type $l=0$ cannot smuggle at all. In addition to the cases above, Customs has to consider the condition of no capture of the smuggler at Stage n to revise his belief because the transition of the game to the next stage is conditioned by the no-capture. The capture could happen for the Customs' strategy P but never happens for NP. Considering all above, we have the following evaluation for the revision operators Γ_P and Γ_N.

(i) For l of $\underline{l}_{n-1}<l<\bar{l}_{n-1}$,

$$\begin{aligned}\Gamma_P(q_n)(l) &= \frac{q_n(l+1)y_n^*(l+1)(1-p_1)+q_n(l)(1-y_n^*(l))}{\sum_{s=\underline{l}_n}^{\bar{l}_n} q_n(s)(1-y_n^*(s)p_1)},\\ \Gamma_N(q_n)(l) &= q_n(l+1)y_n^*(l+1)+q_n(l)(1-y_n^*(l)).\end{aligned}$$

(ii) For $l=L$ in the case of $\bar{l}_{n-1}=L$,

$$\Gamma_P(q_n)(l)=\frac{q_n(l)(1-y_n^*(l))}{\sum_{s=\underline{l}_n}^{\bar{l}_n} q_n(s)(1-y_n^*(s)p_1)},\quad \Gamma_N(q_n)(l)=q_n(l)(1-y_n^*(l)).$$

(iii) For $l=\bar{l}_{n-1}=n-1$ in the case of $\bar{l}_n=n$,

$$\begin{aligned}\Gamma_P(q_n)(l) &= \frac{q_n(l+1)(1-y_n^*(l+1)p_1)+q_n(l)(1-y_n^*(l))}{\sum_{s=\underline{l}_n}^{\bar{l}_n} q_n(s)(1-y_n^*(s)p_1)},\\ \Gamma_N(q_n)(l) &= q_n(l+1)+q_n(l)(1-y_n^*(l)).\end{aligned}$$

(iv) For $l=\underline{l}_{n-1}=\underline{l}_n-1$ in the case of $\underline{l}_n>0$ and $\bar{l}_{n-1}>\underline{l}_{n-1}$,

$$\Gamma_P(q_n)(l)=\frac{q_n(l+1)y_n^*(l+1)(1-p_1)}{\sum_{s=\underline{l}_n}^{\bar{l}_n} q_n(s)(1-y_n^*(s)p_1)},\quad \Gamma_N(q_n)(l)=q_n(l+1)y_n^*(l+1).$$

(v) For $l=\underline{l}_{n-1}=0$ in the case of $\underline{l}_n=0$,

$$\Gamma_P(q_n)(l)=\frac{q_n(0)+q_n(1)y_n^*(1)(1-p_1)}{q_n(0)+\sum_{s=1}^{\bar{l}_n} q_n(s)(1-y_n^*(s)p_1)},\quad \Gamma_N(q_n)(l)=q_n(0)+q_n(1)y_n^*(1).$$

Let us finally recall the initial value of belief of Equation (2), i.e.,

$$q_N(L)=1,\ q_N(l)=0\ (l=0,1,\dots,L-1). \tag{20}$$

As seen above, the revision of belief starts from the initial value of Equation (20) at Stage N and the belief q_n at Stage n is revised to $\Gamma_P(q_n)$ or $\Gamma_N(q_n)$ at the next stage $n-1$ by taking account of the smuggler's optimal strategy y_n^*. So the revision proceeds in the order of $n=N,N-1,\dots,1$. On the other hand, the stage game at each stage n is solved by Problem (P_S) or (P_C) in the order of $n=1,2,\dots,N$ in contradiction to the order of the belief revision. Let us discretize the belief $\{q_n(l), l \in \Lambda_n\}$ in order to resolve the contradiction. We assign a continuous $q_n(l)$ a discrete number in a set $\Phi \equiv \left\{0, \frac{1}{m}, \ldots, \frac{k}{m}, \ldots, 1\right\}$, based on which interval of $\left[0, \frac{1}{2m}\right)$, $\left[\frac{1}{2m}, \frac{3}{2m}\right)$, $\ldots$, $\left[\frac{2k-1}{2m}, \frac{2k+1}{2m}\right)$, $\ldots$, $\left[\frac{2m-1}{2m}, 1\right]$ the value $q_n(l)$ belongs to. The belief q_n or the probability distribution is represented by a vector with the discretized numbers, as

$$q_n \in \Psi_n \equiv \left\{ q_n = (q_n(0), \ldots, q_n(L)) \mid q_n(l) \in \Phi (l = 0, \ldots, L), \right.$$
$$\left. q_n(l) = 0 \ (l \notin \Lambda_n), \sum_{l \in \Lambda_n} q_n(l) = 1 \right\}.$$

We can build a computational algorithm to derive a Bayesian equilibrium point, as follows. Using the discretized belief, we solve the stage games sequentially in the order of stages $n = 1, 2, \ldots, N$ and obtain Bayesian equilibrium. Then we check the optimality of the equilibrium while calculating continuous beliefs in an exact manner and sometimes modify the equilibrium to keep the consistency of optimal strategies with the beliefs in the order of $n = N, N-1, \ldots, 1$. In the algorithm, we use the newly defined notation: $\underline{k}_n \equiv \max\{K - N + n, 0\}$, $\bar{k}_n \equiv \min\{K, n\}$ and $\Delta_n \equiv [\underline{k}_n, \bar{k}_n]$.

(Disc_Bf) Derivation algorithm for Bayesian equilibrium under discretized belief

(S1) Initialize $v(n, k, l; q_n)$ to be zero for all k, l, q_n at Stage $n = 0$. Set $n = 1$.

(S2) If $n = N$, execute (S3) by applying $k = K$, $q_N(L) = 1$ and $q_N(l) = 0$ $(l \neq L-1)$, and stop the algorithm. The obtained $v(N, K, L; q_N)$ is the value of the game.
Else if $n < N$, execute (S3) for all $k \in \Delta_n$ and $q_n \in \Psi_n$.

(S3) Using $v(n-1, \cdot; q')$ and $v(n-1, \cdot; q'')$ by substituting randomly selected q' and $q'' \in \Psi_{n-1}$ for $\Gamma_P(q_n)$ and $\Gamma_N(q_n)$, calculate the expressions (3)–(6) and solve Problem (P_S) or (P_C) to obtain optimal strategies x_n^*, $y_n^*(l)$ and the value $v(n, k, l; q_n)$ $(l \in \Lambda_n)$ for the stage game at Stage n. Revise the belief q_n to $\Gamma_P(q_n)$ and $\Gamma_N(q_n)$ in accordance with Cases (i)–(v) above. Furthermore, discretize the two revised beliefs.
If $q' = \Gamma_P(q_n)$ and $q'' = \Gamma_N(q_n)$, we save x_n^* as an optimal Customs' strategy for the information set $(n, k; q_n)$. We also save $y_n^*(l)$ as an optimal smuggler's strategy and value $v(n, k, l; q_n)$ for the information set $(n, k, l; q_n)$. Otherwise, we select other discrete beliefs q' and q'' from Ψ_{n-1} and repeat the above process.
If we cannot find the above coincidence between (q', q'') and $(\Gamma_P(q_n), \Gamma_N(q_n))$ for all q', $q'' \in \Psi_{n-1}$, we notify that the information set $(n, k; q_n)$ is off path of equilibrium.

(S4) Increase n by one, $n = n + 1$, and go back to (S2).

In the algorithm below, we calculate exact beliefs using the value of the stage game $v(n, \cdot)$ obtained in the above algorithm, in the order of $n = N, N-1, \ldots, 1$. If the newly computed value $v(n, \cdot)$ is different from the original value at a stage n, we go back to the previous stage $n + 1$ and redo the calculation. This type of calculation with repetition possibly does not converge to a solution. The algorithm results in two states; success and failure of confirming the optimality of Bayesian equilibrium.

(Conf_Sol) Algorithm of calculating continuous belief and rebuilding Bayesian equilibrium

(B1) Call Subroutine $\Theta(n, k; q_n)$ by setting $(n, k, l) = (N, K, L)$ and initial belief $q_n = q_N$. If it returns with a normal state of flag, we have obtained an exact Bayesian equilibrium.

If with an abnormal flag, it means the incompletion of confirming the optimality of Bayesian equilibrium. If the subroutine comes back with a revision flag, we repeat calling the subroutine $\Theta(N, K; q_N)$ until the allowed number of times. The failure of all callings means the failure of confirmation of the optimality.

(B2) Subroutine $\Theta(n, k; q_n)$:

(1) In some special cases of $n = 0$, $k > n$ or $k = 0$, we return the values given by initial conditions or boundary conditions with a normal flag.
If $n = N$, we calculate revised beliefs $\Gamma_P(q_n)$ and $\Gamma_N(q_n)$ by saved optimal strategies x_n^*, $\{y_n^*(l), l \in \Lambda_n\}$ and a fixed belief q_N at the initial stage N. Go to (3).

(2) Solve the problems (P_S) or (P_C) using saved values $v(n-1, \cdot)$ to obtain optimal solutions x_n^*, $\{y_n^*(l), l \in \Lambda_n\}$ and $\{v(n, k, l; q_n), l \in \Lambda_n\}$.
  (i) If obtained $\{v(n, k, l; q_n), l \in \Lambda_n\}$ is different from saved value $v(n, \cdot)$, replace $v(n, \cdot)$ with new one and return with a revision flag.
  (ii) If both values of $v(n, k, l; q_n)$ coincide, revise the present belief q_n into $\Gamma_P(q_n)$ and $\Gamma_N(q_n)$ using optimal strategies.

(3) Recursively call subroutines $\Theta(n-1, k-1; \Gamma_P(q_n))$ and $\Theta(n-1, k; \Gamma_N(q_n))$.
  (i) If any of these subroutines returns with an abnormal flag, return with an abnormal flag from the current subroutine $\Theta(n, k; q_n)$.
  (ii) If any subroutine returns with a revision flag, go back to (2).
  (iii) If all subroutines have normal flags, return with a normal flag from the current subroutine.

4. Analytical Form of Bayesian Equilibrium Point in Some Special Cases

Here we are going to derive an equilibrium point in an analytic manner in some special cases.

4.1. Optimal strategies in some special cases

In this subsection, we find some analytical forms of solutions in special cases of $k = n$ and $L = 1$.

Lemma 1. *(i) The value $v(\cdot)$, including the value of the game $v(N, K, L; q_N)$, is nonnegative:*

$$v(n, k, l; q_n) \leq 0. \tag{21}$$

(ii) For State (n, n, l) with $k = n$, $v(n, n, l; q_n) = 0$ for all l and q_n. An optimal patrol strategy is an arbitrary x_n^* satisfying $x_n^* \geq 1/(\gamma+1)$ and an optimal smuggling strategy is no-smuggling, $y_n^* = 0$.

Proof. (i) It is self-evident because always-no-smuggling strategy brings any stage game zero payoff regardless of Customs' strategy.

(ii) Let us prove this by mathematical induction. This statement is valid for $n = k = 0$ because of $v(0, k, l; q) = 0$. By the assumption of $v(n-1, n-1, l; q_{n-1}) = 0$ for Stage $n-1$, we have $R_l(P, S) = \gamma$, $R_l(NP, S) = -1$ and $R_l(P, NS) = R_l(NP, NS) = 0$ by the equations (3)–(6). Therefore, we can see

$$\nu(l) = \min\{(\gamma+1)x_n - 1, 0\}$$

for any $l \in \Lambda_n$ in Problem (P_C), from the conditions (17) and (18). Noting that, in the brace $\{\ \}$ on the right-hand side, the first expression equals the second for $x_n = \widehat{x}^* \equiv 1/(\gamma+1)$, $\nu(l)$ is $(\gamma+1)x - 1$ for $0 \leq x \leq \widehat{x}^*$ and 0 for $\widehat{x}^* < x_n \leq 1$. Since this fact is valid for any $l \in \Lambda_n$, the objective function of (P_C) is $\left(\sum_{l \in \Lambda_n} q_n(l)\right)\{(\gamma+1)x_n - 1\}$ for $0 \leq x_n \leq \widehat{x}^*$

and 0 for $\widehat{x}^* < x_n \le 1$. Therefore, any x_n^* of $\widehat{x}^* \le x_n^*$ is optimal and then we have $v(n,n,l;q_n) = \nu^*(l) = 0$.

We have $\eta = \sum_l q_n(l)\gamma y_n(l)$ in Problem (P_S) from the conditions (15) and (16), and an optimal smuggler's strategy $y_n^*(l) = 0$ for any $l \in \Lambda_n$. □

Because, in the case of $n = k$, Customs can patrol every day, the smuggler inevitably takes no-smuggling strategy and then the value $v(n,n,l;q_n)$ should be 0. Lemma 1 tells us that the probability of patrolling more than $\widehat{x}^*$ deters any smuggling.

Lemma 2. *In the case of $L = 1$, we have*

$$v(n,k,1;q_n) = -\binom{n-1}{k} \Big/ \sum_{s=0}^{k} \gamma^{k-s} \binom{n}{s} \tag{22}$$

for any state $(n,k,1)$ of $n > k$.

Proof. In this case, Problem (P_C) becomes the maximization problem of $q_n(1)\nu(1)$ under conditions $0 \le x_n \le 1$ and

$$(\gamma+1)x_n - 1 \ge \nu(1), \tag{23}$$

$$-\{v(n-1,k,1;\Gamma_N(q_n)) - v(n-1,k-1,1;\Gamma_P(q_n))\}x_n + v(n-1,k,1;\Gamma_N(q_n)) \ge \nu(1) \tag{24}$$

by applying $\Lambda_n = \{1\}$ and Equation (10). The problem has only two variables $\nu(1)$ and x_n, and it is easy to solve. First, we can see $v(n-1,k-1,1;q) < \gamma$ and $-1 \le v(n-1,k,1;q)$ for any belief q, and $v(n-1,k,1;\Gamma_N(q)) \ge v(n-1,k-1,1;\Gamma_P(q))$ since Customs had better not go patrolling if he never meets the smuggling.

Considering the inequalities above, an optimal strategy x_n^* of (P_C) is given by equalizing the left-hand sides of the conditions (23) and (24) to be

$$x_n^* = \frac{1 + v(n-1,k,1;\Gamma_N(q))}{\gamma + 1 + v(n-1,k,1;\Gamma_N(q)) - v(n-1,k-1,1;\Gamma_P(q))}.$$

Then we have

$$v(n,k,1;q) = \nu^*(1) = \frac{\gamma v(n-1,k,1;\Gamma_N(q)) + v(n-1,k-1,1;\Gamma_P(q))}{\gamma + 1 + v(n-1,k,1;\Gamma_N(q)) - v(n-1,k-1,1;\Gamma_P(q))}. \tag{25}$$

Because the values $v(1,1,1;q) = 0$ and $v(1,0,1;q) = -1$ at Stage 1 are determined regardless of q, $v(n,k,1;q)$ does not rely on $q_n(1)$ at Stage $n = 2$. Thus the value $v(n,k,1;q)$ calculated by the recursive equation (25) does not depend on q, in general. We delete q and argument 1 from $v(n,k,1;q)$ to create a new symbol $u(n,k)$ as a substitute for $v(n,k,1;q)$. In the result, we have a recursive formula to derive the value $u(n,k)$, as follows:

$$u(n,k) = \frac{\gamma u(n-1,k) + u(n-1,k-1)}{\gamma + 1 + u(n-1,k) - u(n-1,k-1)}, \tag{26}$$

$$\text{Initial value}: \ u(1,1) = 0, \ u(1,0) = -1. \tag{27}$$

We could solve the difference equation (26) to derive the analytic form (22) for $u(n,k)$ or $v(n,k,1;q_n)$. But readers can refer to Theorem 1 in Reference [13], where the same difference equation is discussed and the formula (22) is derived.

In a similar manner, we can derive an optimal smuggler's strategy

$$y_n^* = \frac{v(n-1,k,1;\Gamma_N(q)) - v(n-1,k-1,1;\Gamma_P(q))}{\gamma + 1 + v(n-1,k,1;\Gamma_N(q)) - v(n-1,k-1,1;\Gamma_P(q))}$$

from Problem (P_S), which is solved by connecting the left-hand sides of Equations (15) and (16) with an equality. □

In the situation of Lemma 2, the type of smuggler is $l = 0$ or 1 at any stage. Because the smuggler of type $l = 0$ cannot take an action of smuggling, the payoff should be zero for any Customs' strategy of P and NP. Therefore, Customs might presume only the smuggler of type $l = 1$. We see that the objective function of (P_C) is essentially $\nu(1)$, as shown in the proof above, so that Customs does not need any information about the type of smuggler in the case of $L = 1$.

4.2. A basic procedure for the analytical form of equilibrium point and the special case of $N = 3$

The problems (P_S) and (P_C) look comparatively simple in terms of the number of variables and conditions. It encourages us to try to derive Bayesian equilibrium point in an analytical manner. Here we discuss a basic procedure for the solution and try to find all equilibrium points for all K and L $(0 \le K,\ L \le 3)$ in the case of $N = 3$. Before discussing the procedure, let us estimate some qualitative relation among four values of $R_l(\cdot)$ in Problem (P_C).

If Customs patrols, the smuggler would prefer no-smuggling. Getting reward 1 by the smuggling on the no-patrol day is the best for the smuggler. If the patrol never meets the smuggling, Customs had better keep a chance of patrol for later use. When the smuggler tries to smuggle, the best for the smuggler is to get reward 1 on the no-patrol day and the worst is to meet the patrol. The following inequalities represent the properties above:

$$\begin{aligned}
\gamma + (1-p_1)v(n-1,k-1,l-1;\Gamma_P(q_n)) &\ge v(n-1,k-1,l;\Gamma_P(q_n)), && (28)\\
-1 + v(n-1,k,l-1;\Gamma_N(q_n)) &\le v(n-1,k,l;\Gamma_N(q_n)), && (29)\\
v(n-1,k-1,l;\Gamma_P(q_n)) &\le v(n-1,k,l;\Gamma_N(q_n)), && (30)\\
\gamma + (1-p_1)v(n-1,k-1,l-1;\Gamma_P(q_n)) &\ge -1 + v(n-1,k,l-1;\Gamma_N(q_n)). && (31)
\end{aligned}$$

These inequalities can be replaced with the values of $R_l(\cdot)$ from the definitions (3)–(6), as follows:

$$R_l(P,S) \ge R_l(P,NS),\ R_l(NP,S) \le R_l(NP,NS),$$
$$R_l(P,NS) \le R_l(NP,NS),\ R_l(P,S) \ge R_l(NP,S).$$

We have not proved yet that these inequalities are always valid. In practice, we have to make sure of the validity in the process of calculating solutions at each stage.

Recalling that the conditions (17) and (18),

$$(R_l(P,S) - R_l(NP,S))x_n + R_l(NP,S) \ge \nu(l),$$
$$-(R_l(NP,NS) - R_l(P,NS))x_n + R_l(NP,NS) \ge \nu(l),$$

and denoting the left-hand side expression of the first inequality and the second one by $h_1^l(x_n)$ and $h_2^l(x_n)$, respectively, $\nu(l)$ is given by $\nu(l) = h_1^l(x_n)$ for $0 \le x_n \le \widehat{x}_l^*$ and $\nu(l) = h_2^l(x_n)$

for $\widehat{x}_l^* \le x_n \le 1$ with a boundary point $\widehat{x}_l^*$ because we have $h_1^l(0) \le h_2^l(0)$ for $x_n = 0$ and $h_1^l(1) \ge h_2^l(1)$ for $x_n = 1$. The boundary $\widehat{x}_l^*$ is

$$\begin{aligned}\widehat{x}_l^* &= \frac{R_l(NP,NS) - R_l(NP,S)}{R_l(P,S) - R_l(NP,S) + R_l(NP,NS) - R_l(P,NS)} \\ &= \{v(n-1,k,l;\Gamma_N(q_n)) + 1 - v(n-1,k,l-1;\Gamma_N(q_n))\}/ \\ &\quad \{\gamma + 1 + (1-p_1)v(n-1,k-1,l-1;\Gamma_P(q_n)) - v(n-1,k,l-1;\Gamma_N(q_n)) \\ &\quad + v(n-1,k,l;\Gamma_N(q_n)) - v(n-1,k-1,l;\Gamma_P(q_n))\}\,. \end{aligned} \tag{32}$$

Since the function $h_1^l(x_n)$ is linearly non-decreasing for x_n and $h_2^l(x_n)$ is linearly non-increasing, $\nu(l)$ is maximized by $x_n = \widehat{x}_l^*$. Therefore, the objective function of (P_C), $\sum_l q_n(l)\nu(l)$, is piecewise linear for x_n and then an optimal x_n^* coincides with $\widehat{x}_l^*$ for some $l \in \Lambda_n$. An optimal value of $\nu^*(l)$ or $v(n,k,l;q_n)$ belongs to the following interval:

$$\max\{R_l(P,NS),\ R_l(NP,S)\} \le \nu^*(l) = v(n,k,l;q_n) \le \min\{R_l(P,S),\ R_l(NP,NS)\}.$$

Using the basic procedure to derive x_n^* and $v(n,k,l;q_n)$ explained above, we could solve the game with small number of stages in an analytical manner. As an example, we find the values of the games for all parameters $(N,K,L) = (3,K,L),\ K,L = 1,2,3$. In Appendix A, we enumerate the results of them but omit the process of the calculation and self-evident equilibrium points for the sake of simplicity. We can also check that the derived solutions coincide with the solutions given by Lemma 2 in the case of $L = 1$ and additionally, we can say that the inequalities (28)–(31) are always valid in this case.

5. Numerical Examples

In this section, we first analyze the game with parameters $(N,K,L) = (3,2,2)$, using the solutions we have already obtained in Appendix A.

Figure 1 shows the tree of the game branching from a root node or the first information set $(N,K,L) = (3,2,2)$. At each stage, there are a pair of two branches with Customs' strategies $\{P, NP\}$ and the smuggler's strategies $\{S, NS\}$. A stage game consists of a pair of these two sets of branches. Customs decides his strategy based on the history of his past strategies without any information about his opponent's strategy. On the other hand, the smuggler takes his strategy without the current strategy of Customs, knowing past Customs' strategies. The asymmetric acquisition of information generates complicated information sets, which are depicted by ovals at Stages 3 and 2, and lines connecting some nodes or moves at Stage 1 in Figure 1. The player cannot discriminate the difference among the nodes contained in the information set. In the ovals or besides the lines, we write a pair (n,k) or a triplet (n,k,l) indicating the Customs' or the smuggler's information set, respectively, where n, k and l are the numbers of stage, the residual chances of patrol and the residual chances of smuggling. For the combination of strategies P and S, there could be the capture of the smuggler by Customs, which is illustrated by a square on an arc. From the assumption, the capture occurs and the game ends with probability p_1 but the game proceeds to the next stage with probability $1-p_1$.

If the pure strategy is optimal, we draw the arc of the optimal strategy with a bold red line. At Stage 1, all states have optimal pure strategies and then all other arcs except the optimal ones are deleted for the sake of simplicity. After information set $(n,k) = (2,2)$, for example, the optimal Customs' strategy is always-patrol P and the optimal smuggler's strategy is always-no-smuggling NS at every stage. So Customs does not need any information about the smuggler's strategy to make his strategy. However, for information set

$(n, k) = (2, 1)$, Customs' decision making depends on his anticipation or belief on how many chances of $l = 1$ or $l = 2$ the smuggler keeps for smuggling.

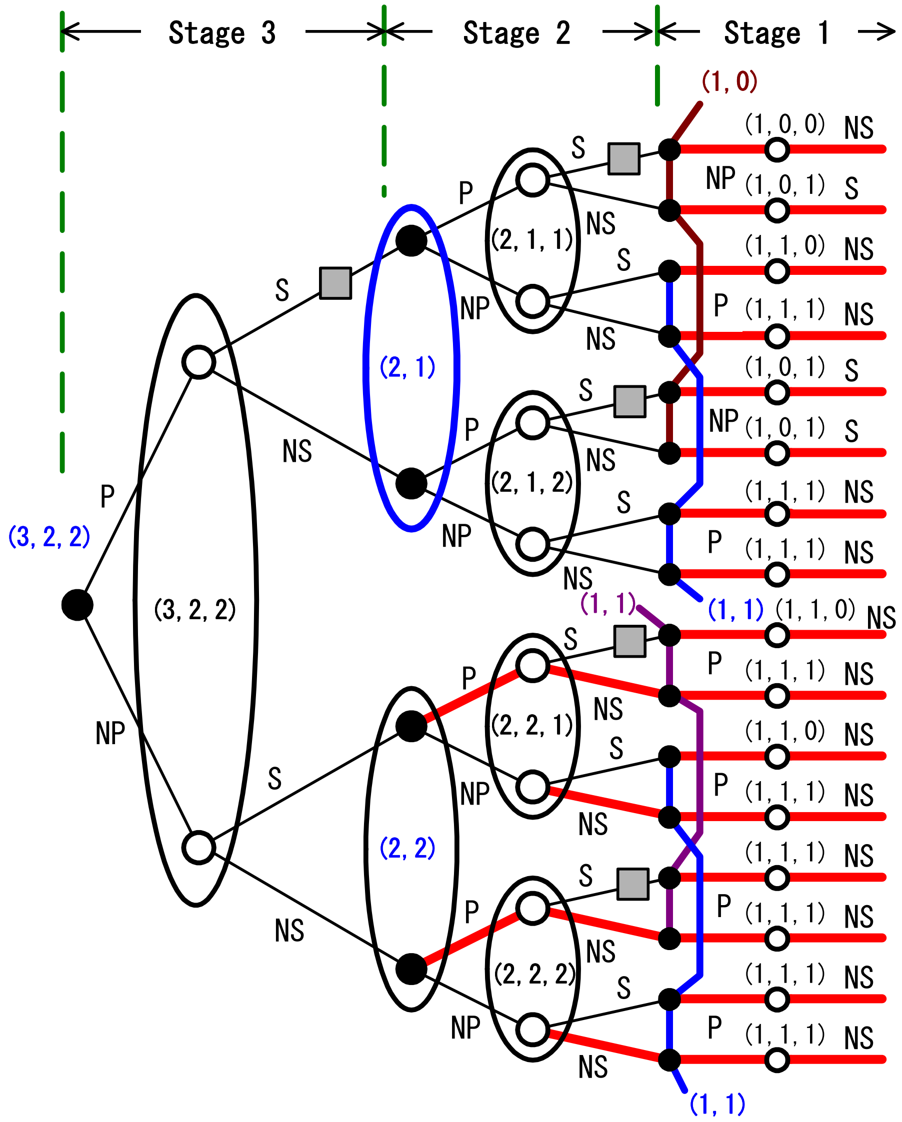

Figure 1: Game tree in the case of $(N, K, L) = (3, 2, 2)$

(1) Dependency of equilibrium point on parameter setting and Customs' belief

Here we set $\alpha = 3$ for all examples taken here. The game has some natural properties that the value of the game, $v(3, 2, 2; q_3)$, monotonically increases and the optimal probability of smuggling at Stage 3, y_3^*, decreases as the capture probability p_1 becomes larger. We can also easily understand that the value of game decrease and y_3^* increases as the success probability of smuggling, p_2, becomes larger. The value of the game and y_3^* are illustrated on a $p_1 - p_2$ plane in Figures 2 and 3, respectively. Please note that p_1 and p_2 have their feasible regions of $p_1 + p_2 \leq 1$, $\gamma = \alpha p_1 - p_2 > 0$ and $p_1, \ p_2 \geq 0$ on the plane.

Next we are going to analyze optimal patrol strategy x_3^* at Stage 3. Figure 4 illustrates the change of x_3^* for p_1 and p_2. Figure 5 shows its profile for p_1 while fixing p_2 to 0.2. The reward of Customs is directly born by α on capture of the smuggler and indirectly born by deterring the smuggler from earning reward 1 on success of the smuggling. In both cases, it

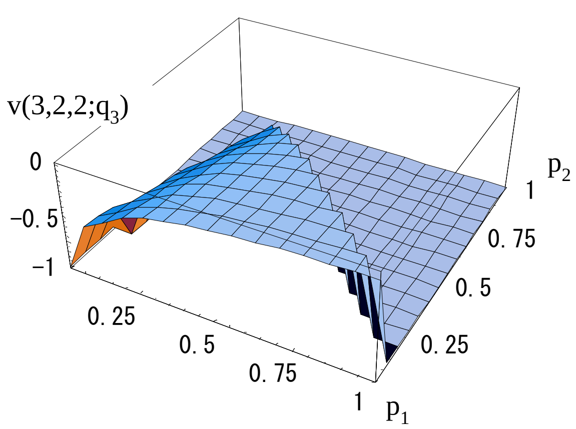

Figure 2: Value of the game

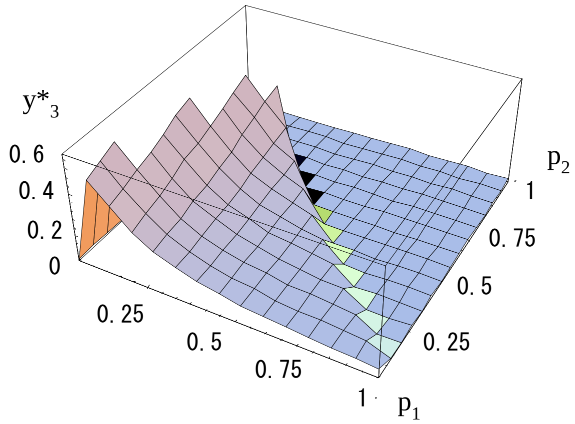

Figure 3: Optimal smuggling probability

is crucial that the patrol meets the smuggling. Therefore, it is reasonable that x_3^* decreases corresponding to smaller y_3^* as p_1 becomes larger. However Figures 4 and 5 show us the discontinuity of x_3^* along a curve in Figure 4 and at a point in Figure 5. Let us analyze the discontinuity by Customs' strategy x_2^* in State $(n, k) = (2, 1)$.

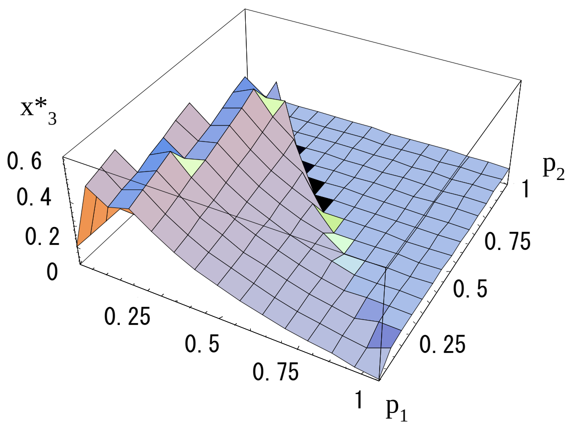

Figure 4: Optimal patrol probability

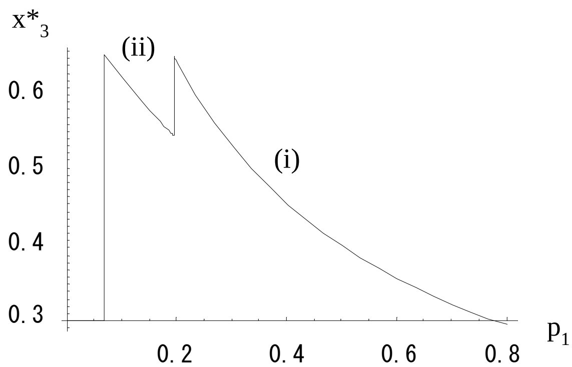

Figure 5: Optimal patrol probability

At Stage $n = 2$, we have $\Lambda_2 = \{1, 2\}$ and the objective function $\sum_{l \in \Lambda_2} q_2(l)\nu(l)$ of Problem (P_C) becomes neutral on the curve of

$$-\Gamma_P(q_3)(1) + (\gamma + p_1)\Gamma_P(q_3)(2) = 0 \tag{33}$$

from Equation (A4) in Appendix A, that is, the objective function does not change even though x_2 varies on the curve. The curve partitions a whole space into the region of $-\Gamma_P(q_3)(1) + (\gamma + p_1)\Gamma_P(q_3)(2) > 0$, say Region (i), and the other region of $-\Gamma_P(q_3)(1) + (\gamma + p_1)\Gamma_P(q_3)(2) < 0$, say Region (ii). In Region (i), Customs estimates that the smuggler has $l = 2$ chances of smuggling with more probabilities and then the smuggler is more likely to take an action of smuggling. The estimation makes Customs expect larger payoff by his patrol. That is why Customs is more likely to patrol in Region (i) than (ii). From the equations (A1)–(A3), we have the following solution:

$$x_2^* = \frac{1}{\gamma + 1 + p_1}, \tag{34}$$

$$v(2,1,1;q) = -\frac{1}{\gamma + 1 + p_1},\ v(2,1,2;q) = -\frac{1}{\gamma + 1 + p_1} \tag{35}$$

in Region (i) and

$$x_2^* = \frac{1}{\gamma + 2}, \tag{36}$$

$$v(2,1,1;q) = -\frac{1}{\gamma + 2},\ v(2,1,2;q) = -\frac{2 - p_1}{\gamma + 2} \tag{37}$$

in Region (ii). On the neutral curve, x_2^* might take an arbitrary value of $1/(\gamma + 1 + p_1) \geq x_2^* \geq 1/(\gamma + 2)$ between two values given by Equations (34) and (36). From Equations (35) and (37), we can prove that value $v(2,1,2;q)$ is larger in Region (i) than (ii) and $v(2,1,1;q)$ is smaller in (i) than (ii). We can qualitatively explain the fact, as follows. In Region (i), Customs has the belief that the smuggler more likely has $l = 2$ than $l = 1$. The value $v(2,1,2;q)$ with $l = 2$ is congruent with the belief and it becomes larger in Region (i) than (ii), but $v(2,1,1;q)$ with $l = 1$, which is not congruent with the belief, is smaller.

We make use of the analysis above for State $(n,k) = (2,1)$ to inspect the change of x_3^* at Stage 3. The initial state $(N,K,L) = (3,2,2)$ transfers to State $(n,k) = (2,1)$ by patrol strategy P and to $(n,k) = (2,2)$ by no-patrol strategy NP. From Lemma 1(ii), we have $v(2,2,l;q) = 0$ for all l in State $(2,2)$. Now we solve Problem (P_S) using the value of the game $v(2,\cdot;q)$ at Stage 2 and obtain an optimal smuggling strategy y_3^* at Stage 3. Using the revised belief Γ_P calculated by y_3^*, the neutral curve of Equation (33) becomes $\gamma = \widehat{\gamma}$, where $\widehat{\gamma}$ is defined by Equation (A6) in Appendix A. The curve can be written down to

$$p_2^2 - \{(2\alpha + 1)p_1 + 1\}\, p_2 + \alpha(\alpha + 1)p_1^2 + (\alpha + 2)p_1 - 1 = 0 \tag{38}$$

by variables p_1 and p_2. With feasibility conditions $p_1 + p_2 \leq 1$ and $\gamma = \alpha p_1 - p_2 > 0$, the curve separates a whole space into two regions. Region (i) is placed in the area with comparatively larger p_1 because larger p_1 makes the smuggler negative for smuggling and the smuggling probability y_3^* smaller at Stage 3, and the tendency causes larger $\Gamma_P(q_3)(2)$ with $l = 2$ left chances of smuggling at Stage 2. These coincide with the properties analyzed above for State $(n,k) = (2,1)$. Similarly, Region (ii) is assigned to the area with smaller p_1. In Figure 6, the neutral curve $\gamma = \widehat{\gamma}$ and two additional curves of $p_1 + p_2 = 1$ and $\gamma = 0$ are drawn on the $p_1 - p_2$ space.

The value $v(2,\cdot)$ discontinuously changes between Regions (i) and (ii), as seen by Equations (35) and (37), for both states $(2,1,1)$ and $(2,1,2)$ at Stage 2. Let us figure out how the discontinuity at Stage 2 is inherited to Stage 3 in Regions (i) and (ii). Both players know the initial state $(N,K,L) = (3,2,2)$ and the stage game has the following payoff matrix at the first stage $n = 3$:

$$\begin{array}{c} \\ P \\ NP \end{array} \begin{array}{c} \begin{array}{cc} \qquad S \qquad\qquad\qquad & NS \end{array} \\ \begin{pmatrix} \gamma + (1 - p_1)v(2,1,1;\Gamma_P(q_3)) & v(2,1,2;\Gamma_P(q_3)) \\ -1 + v(2,2,1;\Gamma_N(q_3)) & v(2,2,2;\Gamma_N(q_3)) \end{pmatrix} \end{array}.$$

Two elements in the second row are -1 and 0 but two elements in the first row change depending on which region of (i) and (ii) a point (p_1,p_2) belongs to. Denoting the two

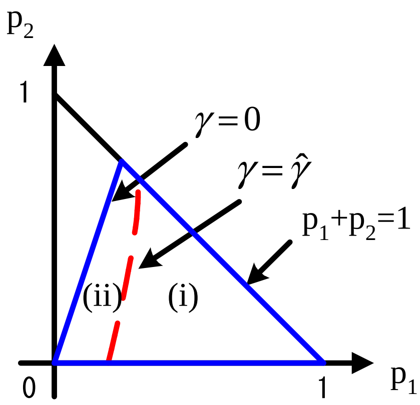

Figure 6: Regions (i) and (ii)

elements by a and b from left, the following relation holds in the payoff matrix:

$$\begin{pmatrix} a & > & b \\ \vee & & \wedge \\ -1 & < & 0 \end{pmatrix}.$$

Therefore, mixed strategies are optimal for both players and the optimal probability of patrolling is given by

$$x_3^* = \frac{1}{a-b+1} \tag{39}$$

at Stage 3 after solving the equation of $ax-(1-x)=bx$. From the previous analysis on the largeness of $v(2,1,1;\Gamma_P(q_3))$ and $v(2,1,2;\Gamma_P(q_3))$ in Regions (i) and (ii), the value a is smaller in Region (i) than (ii) and b is larger in (i) than (ii). That is why x_3^* becomes larger in Region (i) than (ii) from Equation (39) and the discontinuity appears on the neutral curve separating Regions (i) and (ii) in Figures 4 and 5.

In Figure 5 with the fixed parameter $p_2=0.2$, the discontinuity appears at $p_1 \approx 0.1953$, which is derived as a solution of the equation (38). If we substitute Equations (35) and (37) into (39) and calculate an optimal patrol probability x_3^*, the probability changes from $(\gamma+1+p_1)/((\gamma+1)^2+p_1(\gamma+1)+p_1)$ in Region (i) to $(\gamma+2)/((\gamma+1)(\gamma+2)+1)$ in Region (ii). On the neutral curve of the boundary between Regions (i) and (ii), Customs might patrol with probability $x_3^*=1/(\gamma+2-(\gamma+1)z)$ at Stage 3 under the condition that he would take a probability z of $1/(\gamma+2) \le z \le 1/(\gamma+1+p_1)$ as the patrol probability in the coming state $(n,k)=(2,1)$. We can see only the results of the discussion above by the equations (A7)–(A9) in Appendix A.

(2) Value of information

In this paper, we deal with the asymmetric system of information that the smuggler knows all past Customs' strategies but Customs does not know the type of the smuggler. We already have the published analysis on the symmetric model [13], where both players get information about opponents' past strategies. We could estimate the value of the information about the smuggler's type by comparing this model with the symmetric one. We name the asymmetric model of this paper M2 and the symmetric one M1.

Setting parameter $\alpha=2$, $p_1=0.5$, $p_2=0.3$ and $m=5$, we compute the values of the games for all combinations (N,K,L) of the number of stages $N=3,4,5$, the number of permitted patrols K $(\leq N)$ and the number of feasible smuggling L $(\leq N)$ for Models M1 and M2, and show them in Table 2. Upper figures are for M1 and lower ones for M2. We omit them in such self-evident cases as $v(N, N, L; q_N) = 0$, $v(N, 0, L; q_N) = -L$ and $v(N, K, 0; q_N) = 0$. Table 3 shows the values of the games with the same parameters as Table 2 except for $p_1 = 0.2$.

Algorithm (Disc_Bf) proposed in Section 3 always provides us a unique Bayesian equilibrium, that is, a unique value of the game but Algorithm (Conf_Sol) fails to confirm the optimality of the equilibrium in three cases for $p_1 = 0.5$ and in one case for $p_1 = 0.2$. In such cases with no convergence, Algorithm (Conf_Sol) vibrates between two values of $v(N, K, L; q_N)$. Two values are close to each other and one of them is what (Disc_Bf) gives. We show the figures that (Disc_Bf) gives us in Tables 2 and 3, but write the other figure below the tables with symbol "*". The two tables show us some properties of the value of the game.

(1) In the case of $L = 1$ or $L = N$, we have $\Lambda_n \subseteq \{0, 1\}$ or $\Lambda_n = \{n\}$ at any stage n, respectively, and Customs does not need any information about l for his decision making. Therefore, the values of the games of Models M1 and M2 coincides.
(2) It is natural that the value of the game for M2 is equal to or smaller than for M1. Its monotonically non-decreasingness or its monotonically non-increasingness is seen with respect to K or L, respectively, for both models. As N becomes larger while fixing K and L, the estimation on the type of smuggler l is getting more difficult for Customs and it puts down the value of the game for both models.

Table 2: Value of the game ($p_1 = 0.5$, Upper: M1, Lower: M2)

N	K	L=1	2	3	4	5
3	1	−0.54	−0.91	−1.03		
		−0.54	−0.94	−1.03		
	2	−0.18	−0.23	−0.24		
		−0.18	−0.24	−0.24		
4	1	−0.64	−1.15	−1.51	−1.64	
		−0.64	−1.19*1	−1.55	−1.64	
	2	−0.32	−0.51	−0.58	−0.59	
		−0.32	−0.54	−0.59	−0.59	
	3	−0.10	−0.13	−0.13	−0.13	
		−0.10	−0.13	−0.13	−0.13	
5	1	−0.70	−1.31	−1.80	−2.15	−2.28
		−0.70	−1.35*2	−1.83	−2.19	−2.28
	2	−0.43	−0.73	−0.92	−1.00	−1.01
		−0.43	−0.77*3	−0.95	−1.01	−1.01
	3	−0.20	−0.31	−0.35	−0.36	−0.36
		−0.20	−0.33	−0.36	−0.36	−0.36
	4	−0.05	−0.07	−0.07	−0.07	−0.07
		−0.05	−0.07	−0.07	−0.07	−0.07

*1: −1.17, *2: −1.32, *3: −0.76

To check the change of the value of the game between Models M1 and M2, we compare the increasing ratios of the value of the game in two models. The increasing ratio is defined by

Table 3: Value of the game ($p_1 = 0.2$, Upper: M1, Lower: M2)

N	K	L = 1	2	3	4	5
3	1	−0.65	−1.20	−1.59		
		−0.65	−1.23	−1.59		
	2	−0.30	−0.52	−0.61		
		−0.30	−0.54	−0.61		
4	1	−0.73	−1.41	−2.0	−2.42	
		−0.73	−1.42	−2.06	−2.42	
	2	−0.47	−0.86	−1.15	−1.30	
		−0.47	−0.87	−1.22*1	−1.30	
	3	−0.22	−0.38	−0.47	−0.50	
		−0.22	−0.39	−0.50	−0.50	
5	1	−0.78	−1.53	−2.21	−2.82	−3.25
		−0.79	−1.53	−2.24	−2.88	−3.25
	2	−0.57	−1.08	−1.51	−1.84	−2.01
		−0.57	−1.09	−1.55	−1.91	−2.01
	3	−0.36	−0.66	−0.89	−1.04	−1.09
		−0.36	−0.67	−0.93	−1.09	−1.09
	4	−0.16	−0.29	−0.37	−0.41	−0.42
		−0.16	−0.30	−0.40	−0.42	−0.42

*1: −1.23

$|w - v|/|v|$, where w and v are the values of the game of Models M1 and M2, respectively. Furthermore, in Table 4, we compare two cases of capture probability, $p_1 = 0.5$ and $p_1 = 0.2$, in terms of the ratio. Upper figures are for $p_1 = 0.5$ and lower ones for $p_1 = 0.2$. The ratio is a kind of indicator for the value of the information about the smuggler's type. We attach an asterisk "*" to the larger of two ratios. Let us use an adjective "strong" for Customs with larger p_1 and "weak" for Customs with smaller p_1. Thinking that the number K would rely on the budget of Customs for patrol as the political matter, we use words "rich" or "poor" for the large number or the small number of K. Similarly, we call larger L "in funds" or smaller L "out of funds" for the smuggler. From Table 4, we can see a common tendency that the information value of the smuggler's type is higher for strong Customs than weak Customs in the area of small K and L in Table 4. In the area of large K and L, the value is higher for weak Customs than strong Customs. From the tendency, we itemize some characteristics of the value of information, as follows.

(3) Strong Customs expects higher information value under the circumstance of smaller K or less budget than enough budget, and weak Customs gets higher information value using a lot of budget than a shortage of budget.

(4) Under the circumstance of the same budget, the information of the smuggler out of funds is more critical not for weak Customs but for strong Customs, and the type information of the smuggler in funds is more valuable for weak Customs than strong Customs.

Our common sense would support the characteristics above. If strong Customs has an abundance of budget, he is too strong for the smuggler to dare to take any action of smuggling. In this case, Customs could perfectly deter the smuggling and he does not have to depend on the information about the smuggler at all. Only if strong Customs does not have a lot

Table 4: Comparison of the increasing ratio (Upper: $p_1 = 0.5$, Lower: $p_1 = 0.2$)

N	K	L				
		1	2	3	4	5
3	1	0	0.031*	0		
		0	0.019	0		
	2	0	0.021	0		
		0	0.050*	0		
4	1	0	0.032*	0.023	0	
		0	0.007	0.030*	0	
	2	0	0.048*	0.022	0	
		0	0.016	0.056*	0	
	3	0	0.038*	0	0	
		0	0.031	0.060*	0	
5	1	0	0.030*	0.016*	0.017	0
		0	0.003	0.011	0.022*	0
	2	0	0.050*	0.026*	0.013	0
		0	0.007	0.023	0.039*	0
	3	0	0.058*	0.025	0.003	0
		0	0.013	0.042*	0.045*	0
	4	0	0.041*	0.014	0	0
		0	0.024	0.073*	0.019*	0

of budget, however, the smuggler sometimes tries to smuggle and the information about the smuggler brings strong Customs higher increase of his reward by capturing the smuggler more likely than weak Customs. If weak Customs does not have a lot of budget, he cannot be active enough to patrol and capture the smuggler because of a shortage of budget. With enough budget, weak Customs could make an effective patrol plan and expect the increase of his payoff by making use of the smuggler's information.

Before closing this section, we have to mention the accuracy of the computational algorithm proposed in Section 3. The algorithm shows us exact Bayesian equilibrium points in many cases but sometimes fails to confirm their optimality. Even in such fail cases, the proposed algorithm brings alternative values of game that are close to each other. We also derive the analytical forms of equilibrium points in some special cases of $N = 3$ or $L = 1$. In such cases, our numerical algorithm provides the same solutions as the analytical form of solutions.

6. Conclusion

In this paper, we analyze the smuggling game with asymmetric information about players, where the system of information acquisition is disadvantageous to Customs. We adopt the so-called Bayesian equilibrium point as the concept of solution. We derive analytical forms of solutions in some special cases and propose a computational algorithm to derive a solution in a general case. By some examples, we quantitatively clarify the characteristics of optimal strategies and the value of the information about the smuggler's type, which could be supported by our common sense. We also numerically analyze problems with the number of stages $N = 5$ at most. If the smuggler makes a smuggling plan every month, these examples correspond to the analyses on the time span of half a year. As a form of the game, we handle the extensive form or the game tree as Figure 1 and develop a computational methodology to derive Bayesian equilibrium on the tree. However, the form is inadequate for computation because the tree with the whole options or the whole braches of players' behaviors expands on an exponential scale. Therefore, we have to devise more efficient forms of representing the game as our future work. We also have the future topics of extending our model to the nonzero-sum smuggling game with incomplete information, where players have different criteria of payoff.

References

[1] R. Avenhaus and M.J. Canty: Playing for time: A sequential inspection game. *European Journal of Operational Research*, **167** (2005), 475–492.

[2] R. Avenhaus, M.J. Canty, D.M. Kilgour and B. von Stengel: Inspection games in arms control. *European Journal of Operational Research*, **90** (1996), 383–394.

[3] R. Avenhaus and D. Kilgour: Efficient distributions of arm-control inspection effort. *Naval Research Logistics*, **51** (2004), 1–27.

[4] V. Baston and F. Bostock: A generalized inspection game. *Naval Research Logistics*, **38** (1991), 171–182.

[5] M. Canty, D. Rothenstein, and R. Avenhaus: A sequential attribute sampling inspection game for item facilities. *Naval Research Logistics*, **48** (2001), 476–505.

[6] M. Dresher: A sampling inspection problem in arms control agreements: A game-theoretic analysis. Memorandum RM-2972-ARPA (The RAND Corporation, Santa Monica, 1962).

[7] T. Ferguson and C. Melolidakis: On the inspection game. *Naval Research Logistics*, **45** (1998), 327–334.

[8] A. Garnaev: A remark on the customs and smuggler game. *Naval Research Logistics*, **41** (1994), 287–293.

[9] J.C. Harsanyi: Games with incomplete information played by Bayesan players: Part I. *Management Science*, **8** (1967), 159–182.

[10] R. Hohzaki: A compulsory smuggling model of inspection game taking account of fulfillment probability of players' aims. *Journal of the Operations Research Society of Japan*, **49** (2006), 306–318.

[11] R. Hohzaki: An inspection game with multiple inspectees. *European Journal of Operational Research*, **178** (2007), 894–906.

[12] R. Hohzaki: An inspection game with smuggler's decision on the amount of contraband. *Journal of the Operations Research Society of Japan*, **54** (2011), 25–45.

[13] R. Hohzaki, D. Kudoh, and T. Komiya: An inspection game: Taking account of fulfillment probabilities of players' aims, *Naval Research Logistics*, **53** (2006), 761–771.

[14] R. Hohzaki and H. Maehara: A single-shot game of multi-period inspection. *European Journal of Operational Research*, **207** (2010), 1410–1418.

[15] M. Maschler: A price leadership method for solving the inspection's non-constant-sum game. *Naval Research Logistics Quarterly*, **13** (1966), 11–33.

[16] M. Sakaguchi: A sequential game of multi-opportunity infiltration. *Mathematica Janonica*, **39** (1994), 157–166.

[17] M. Thomas and Y. Nisgav:, An infiltration game with time dependent payoff. *Naval Research Logistics Quarterly*, **23** (1976), 297–302.

A. Bayesial Equilibrium Points in the Case of $N = 3$

Case of $(N, L) = (3, 3)$

(1) At Stage $n = 1$, $v(1, 1, l; q) = 0$, $v(1, 0, l; q) = -l$ $(l \leq 1)$.

(2) At Stage $n = 2$, $\Lambda_2 = \{2\}$.

(i) For $k = 2$, $v(2, 2, 2; q) = 0$.

(ii) For $k = 1$, $h_1^2(x) = (\gamma + p_1)x - 1$, $h_2^2(x) = -x$,

$$x_2^* = \widehat{x}_2^* = \frac{1}{\gamma + 1 + p_1},\ y_2^* = \frac{1}{\gamma + 1 + p_1},\ v(2, 1, 2; q) = -\frac{1}{\gamma + 1 + p_1}.$$

(3) At Stage $n = N = 3$, $\Lambda_3 = \{3\}$.

(i) For $K = 3$, $v(3, 3, 3; q) = 0$.

(ii) For $K = 2$, $h_1^3(x) = \left(\gamma + 1 - \dfrac{1 - p_1}{\gamma + 1 + p_1}\right)x - 1$, $h_2^3(x) = -\dfrac{x}{\gamma + 1 + p_1}$,

$$x_3^* = \widehat{x}_3^* = \frac{\gamma + 1 + p_1}{(\gamma + 1)^2 + p_1(\gamma + 1) + p_1},\ y_3^* = \frac{1}{(\gamma + 1)^2 + p_1(\gamma + 1) + p_1},$$

$$v(3, 2, 3; q) = -\frac{1}{(\gamma + 1)^2 + p_1(\gamma + 1) + p_1}.$$

(iii) For $K = 1$, $h_1^3(x) = \left(\gamma - 1 + 2p_1 + \dfrac{1}{\gamma + 1 + p_1}\right)x - \dfrac{\gamma + 2 + p_1}{\gamma + 1 + p_1}$,

$$h_2^3(x) = -\frac{2\gamma + 1 + 2p_1}{\gamma + 1 + p_1}x - \frac{1}{\gamma + 1 + p_1},$$

$$x_3^* = \widehat{x}_3^* = \frac{1}{\gamma + 1 + 2p_1},\ y_3^* = \frac{2\gamma + 1 + 2p_1}{(\gamma + 1 + p_1)(\gamma + 1 + 2p_1)},$$

$$v(3, 1, 3; q) = -\frac{3\gamma + 2 + 4p_1}{(\gamma + 1 + p_1)(\gamma + 1 + 2p_1)}.$$

Case of $(N, L) = (3, 2)$

(1) At Stage $n = 1$, $v(1, 1, l; q) = 0$, $v(1, 0, l; q) = -l$ $(l \leq 1)$.

(2) At Stage $n = 2$, $\Lambda_2 = \{1, 2\}$.

(i) For $k = 2$, $v(2, 2, l; q) = 0$ $(l = 1, 2)$.

(ii) For $k = 1$, $h_1^1(x) = (\gamma + 1)x - 1$, $h_2^1(x) = -x$, $\widehat{x}_1^* = \dfrac{1}{\gamma + 2}$,

$$h_1^2(x) = (\gamma + p_1)x - 1,\ h_2^2(x) = -x,\ \widehat{x}_2^* = \frac{1}{\gamma + 1 + p_1}.$$

The objective function of (P_C), $q(1)\min\{h_1^1(x), h_2^1(x)\} + q(2)\min\{h_1^2(x), h_2^2(x)\}$, is maximized by the following solutions.

(a) If $-q(1) + (\gamma + p_1)q(2) > 0$,

$$x_2^* = \widehat{x}_2^* = \frac{1}{\gamma + 1 + p_1}, \tag{A1}$$

$$v(2, 1, 1; q) = -\frac{1}{\gamma + 1 + p_1},\ v(2, 1, 2; q) = -\frac{1}{\gamma + 1 + p_1}. \tag{A2}$$

(b) If $-q(1) + (\gamma + p_1)q(2) < 0$,

$$x_2^* = \widehat{x}_1^* = \frac{1}{\gamma + 2},\ v(2, 1, 1; q) = -\frac{1}{\gamma + 2},\ v(2, 1, 2; q) = -\frac{2 - p_1}{\gamma + 2}. \tag{A3}$$

(c) If $-q(1)+(\gamma+p_1)q(2)=0$,

$$x_2^* = {}^\forall x \in \left[\frac{1}{\gamma+2}, \frac{1}{\gamma+1+p_1}\right], \tag{A4}$$
$$v(2,1,1;q) = -x_2^*,\ v(2,1,2;q) = (\gamma+p_1)x_2^* - 1.$$

Noting that $q(1)+q(2)=1$, Case (c) occurs only for the belief

$$q(1) = \frac{\gamma+p_1}{\gamma+1+p_1},\ q(2) = \frac{1}{\gamma+1+p_1}. \tag{A5}$$

(3) At Stage $n=N=3$, $\Lambda_3=\{2\}$.
 (i) For $K=3$, $v(3,3,2;q)=0$.
 (ii) For $K=2$, we sort out solutions in accordance with the classifications (a)–(c) for $(n,k)=(2,1)$, as follows:
 (a) If $-\Gamma_P(q)(1)+(\gamma+p_1)\Gamma_P(q)(2)>0$,

$$h_1^2(x) = \left(\gamma+1-\frac{1-p_1}{\gamma+1+p_1}\right)x-1,\ h_2^2(x) = -\frac{1}{\gamma+1+p_1}x,$$
$$x_3^* = \widehat{x}_2^* = \frac{\gamma+1+p_1}{(\gamma+1)^2+p_1(\gamma+1)+p_1},\ y_3^* = \frac{1}{(\gamma+1)^2+p_1(\gamma+1)+p_1},$$
$$v(3,2,2;q) = -\frac{1}{(\gamma+1)^2+p_1(\gamma+1)+p_1}.$$

 (b) If $-\Gamma_P(q)(1)+(\gamma+p_1)\Gamma_P(q)(2)<0$,

$$h_1^2(x) = \left(\gamma+1-\frac{1-p_1}{\gamma+2}\right)x-1,\ h_2^2(x) = -\frac{2-p_1}{\gamma+2}x,$$
$$x_3^* = \widehat{x}_2^* = \frac{\gamma+2}{(\gamma+1)(\gamma+2)+1},\ y_3^* = \frac{2-p_1}{(\gamma+1)(\gamma+2)+1},$$
$$v(3,2,2;q) = -\frac{2-p_1}{(\gamma+1)(\gamma+2)+1}.$$

 (c) If $-\Gamma_P(q)(1)+(\gamma+p_1)\Gamma_P(q)(2)=0$, Customs might decide to take a number z in $[1/(\gamma+2), 1/(\gamma+1+p_1)]$ as his patrol probability when he is in State $(n,k)=(2,1)$ at the next stage 2 and we have

$$h_1^2(x) = (\gamma-(1-p_1)z+1)\,x-1,\ h_2^2(x) = -\left(1-(\gamma+p_1)z\right)x,$$
$$x_3^* = \widehat{x}_2^* = \frac{1}{\gamma+2-(\gamma+1)z},\ y_3^* = \frac{1-(\gamma+p_1)z}{(\gamma+1)(1-z)+1},$$
$$v(3,2,2;q) = -\frac{1-(\gamma+p_1)z}{\gamma+2-(\gamma+1)z},$$

 depending on z.
 Furthermore, Customs would select z to maximize the value of the game $v(3,2,2;q)$. An optimal value of z could be given by $\max_z v(3,2,2;q)$.

Applying y_3^* to Customs' belief q at Stage $N=3$, we generate the revised belief $\Gamma_P(q)$ and $\Gamma_N(q)$, and check the conditions of the classifications (a)–(c) above. The check gives us the relation between parameters γ and p_1 which validates the conditions of (a)–(c). Before we itemize the relation, we have to say that y_3^* is uniquely determined in the result although, in the classification (c), the smuggler's strategy y_3^* seemingly depends on the strategy z in Customs' hands. We use notation $\widehat{\gamma}$ as a solution of the equation $\gamma^2 + (1+p_1)\gamma + 2p_1 - 1 = 0$, which is given by

$$\widehat{\gamma} \equiv \frac{1}{2}\left\{-(1+p_1) + \sqrt{(5-p_1)(1-p_1)}\right\}. \tag{A6}$$

In the case of $(N, K, L) = (3, 2, 2)$

(a) If $\gamma > \widehat{\gamma}$,

$$x_3^* = \frac{\gamma+1+p_1}{(\gamma+1)^2 + p_1(\gamma+1) + p_1},\ y_3^* = \frac{1}{(\gamma+1)^2 + p_1(\gamma+1) + p_1}, \tag{A7}$$
$$v(3,2,2;q) = -\frac{1}{(\gamma+1)^2 + p_1(\gamma+1) + p_1}.$$

(b) If $\gamma < \widehat{\gamma}$,

$$x_3^* = \frac{\gamma+2}{(\gamma+1)(\gamma+2)+1},\ y_3^* = \frac{2-p_1}{(\gamma+1)(\gamma+2)+1}, \tag{A8}$$
$$v(3,2,2;q) = -\frac{2-p_1}{(\gamma+1)(\gamma+2)+1}.$$

(c) If $\gamma = \widehat{\gamma}$,

$$x_3^* = \frac{1}{\gamma+2-(\gamma+1)z},\ y_3^* = \frac{1}{\gamma+2},\ v(3,2,2;q) = -\frac{1}{\gamma+2}, \tag{A9}$$

where z is an arbitrary number in interval $1/(\gamma+2) \le z \le 1/(\gamma+1+p_1)$ and Customs is supposed to patrol with the probability z in State $(n,k) = (2,1)$ at Stage $n = 2$.

(iii) For $K = 1$, we also sort out solutions in accordance with the classifications (a)–(c) for $(n,k) = (2,1)$, as follows:

(a) If $-\Gamma_N(q)(1) + (\gamma+p_1)\Gamma_N(q)(2) > 0$,

$$h_1^2(x) = \left(\gamma + p_1 + \frac{1}{\gamma+1+p_1}\right)x - \frac{\gamma+2+p_1}{\gamma+1+p_1},$$
$$h_2^2(x) = -\frac{2\gamma+1+2p_1}{\gamma+1+p_1}x - \frac{1}{\gamma+1+p_1},$$
$$x_3^* = \widehat{x}_2^* = \frac{1}{\gamma+2+p_1},\ y_3^* = \frac{2\gamma+1+2p_1}{(\gamma+1+p_1)(\gamma+2+p_1)},$$
$$v(3,1,2;q) = -\frac{3}{\gamma+2+p_1}.$$

(b) If $-\Gamma_N(q)(1) + (\gamma+p_1)\Gamma_N(q)(2) < 0$,

$$h_1^2(x) = \left(\gamma+p_1+\frac{1}{\gamma+2}\right)x - \frac{\gamma+3}{\gamma+2},\ h_2^2(x) = -\frac{2\gamma+2+p_1}{\gamma+2}x - \frac{2-p_1}{\gamma+2},$$
$$x_3^* = \widehat{x}_2^* = \frac{1}{\gamma+3},\ y_3^* = \frac{2\gamma+2+p_1}{(\gamma+3)(\gamma+1+p_1)},\ v(3,1,2;q) = -\frac{4-p_1}{\gamma+3}.$$

(c) If $-\Gamma_N(q)(1) + (\gamma + p_1)\Gamma_N(q)(2) = 0$, Customs decides to take a number z in $[1/(\gamma+2), 1/(\gamma+1+p_1)]$ as his patrol probability when he is in State $(n, k) = (2, 1)$ at the next stage and we have

$$h_1^2(x) = (\gamma + p_1 + z)\,x - (1+z),$$
$$h_2^2(x) = -\left(1 + (\gamma+p_1)z\right)x - \left(1 - (\gamma+p_1)z\right),$$
$$x_3^* = \widehat{x}_2^* = \frac{z}{z+1},\, y_3^* = \frac{1+(\gamma+p_1)z}{(\gamma+1+p_1)(z+1)},\, v(3,1,2;q) = -\frac{1+(2-\gamma-p_1)z}{z+1}.$$

Checking the conditions of the classifications (a), (b) and (c) above by y_3^*, we obtain the following results. y_3^* is also determined uniquely in the classification (c).

In the case of $(N, K, L) = (3, 1, 2)$

(a) If $\gamma + p_1 > 1$, $x_3^* = \dfrac{1}{\gamma+2+p_1}$, $y_3^* = \dfrac{2\gamma+1+2p_1}{(\gamma+1+p_1)(\gamma+2+p_1)}$,

$$v(3,1,2;q) = -\frac{3}{\gamma+2+p_1}.$$

(b) If $\gamma + p_1 < 1$,

$$x_3^* = \frac{1}{\gamma+3},\ y_3^* = \frac{2\gamma+2+p_1}{(\gamma+3)(\gamma+1+p_1)},\ v(3,1,2;q) = -\frac{4-p_1}{\gamma+3}.$$

(c) If $\gamma + p_1 = 1$, $x_3^* = \dfrac{z}{z+1}$, $y_3^* = \dfrac{1}{2}$, $v(3,1,2;q) = -1$,
where z is an arbitrary number in interval $1/(\gamma+2) \le z \le 1/2$ and Customs is supposed to patrol with the probability z in State $(n,k) = (2,1)$ at Stage $n = 2$.

Case of $(N, L) = (3, 1)$

(1) At Stage $n = 1$, $v(1,1,l;q) = 0$, $v(1,0,l;q) = -l\ (l \le 1)$.

(2) At Stage $n = 2$, $\Lambda_2 = \{1\}$.

(i) For $k = 2$, $v(2,2,1;q) = 0$.

(ii) For $k = 1$, $h_1^1(x) = (\gamma+1)x - 1$, $h_2^1(x) = -x$, $x_2^* = \widehat{x}_1^* = \dfrac{1}{\gamma+2}$, $y_2^* = \dfrac{1}{\gamma+2}$,

$$v(2,1,1;q) = -\frac{1}{\gamma+2}.$$

(3) At Stage $n = N = 3$, $\Lambda_3 = \{1\}$.

(i) For $K = 3$, $v(3,3,1;q) = 0$.

(ii) For $K = 2$, $h_1^1(x) = (\gamma+1)x - 1$, $h_2^1(x) = -\dfrac{x}{\gamma+2}$,

$$x_3^* = \widehat{x}_1^* = \frac{\gamma+2}{\gamma^2+3\gamma+3},\ y_3^* = \frac{1}{\gamma^2+3\gamma+3},\ v(3,2,1;q) = -\frac{1}{\gamma^2+3\gamma+3}.$$

(iii) For $K = 1$, $h_1^1(x) = (\gamma+1)x - 1$, $h_2^1(x) = -\dfrac{\gamma+1}{\gamma+2}x - \dfrac{1}{\gamma+2}$,

$$x_3^* = \widehat{x}_1^* = \frac{1}{\gamma+3},\ y_3^* = \frac{1}{\gamma+3},\ v(3,1,1;q) = -\frac{2}{\gamma+3}.$$

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