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SOLUTION OF NONSMOOTH GENERALIZED COMPLEMENTARITY PROBLEMS

Mohamed Aly Tawhid
Thompson Rivers University

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Abstract We consider an unconstrained minimization reformulation of the generalized complementarity problem GCP(f, g) when the underlying functions f and g are H-differentiable. We describe H-differentials of some GCP functions based on the min function and the penalized Fischer-Burmeister function, and their merit functions. Under appropriate semimonotone ($\mathbf{E_0}$), strictly semimonotone ($\mathbf{E}$) regularity-conditions on the H-differentials of f and g, we show that a local/global minimum of a merit function (or a 'stationary point' of a merit function) is coincident with the solution of the given generalized complementarity problem. When specialized GCP(f, g) to the nonlinear complementarity problems, our results not only give new results but also extend/unify various similar results proved for C^1, semismooth, and locally Lipschitzian.

Keywords: Optimization, generalized complementarity problem, merit function, regularity conditions, locally Lipschitzian, semismooth-functions, H-differentiability

1. Introduction

In this article, we consider the nonsmooth generalized complementarity problem, denoted by the GCP(f, g), which is to find $\bar{x} \in R^n$ satisfying the conditions

$$g(\bar{x}) \geq 0\,, \quad f(\bar{x}) \geq 0 \quad \text{and } \langle f(\bar{x}), g(\bar{x}) \rangle = 0 \tag{1.1}$$

where $f : R^n \to R^n$ and $f : R^n \to R^n$ are given H-differentiable functions not necessarily locally Lipschitzian nor directionally differentiable. For the applications, numerical methods and formulation of GCP(f, g), see [16, 17], and the references cited therein. If $g(x) = x - T(x)$ with some $T : R^n \to R^n$, then GCP(f, g) is known as the quasi/implicit complementarity problem, see e.g., [17, 24, 25]. Also, if $g(x) = x$, then GCP(f, g) reduces to the nonlinear complementarity problem NCP(f). By taking in NCP(f) $f(x) = Mx + q$ with $M \in R^{n \times n}$ and a vector $q \in R^n$, then NCP(f) is called a linear complementarity problem LCP(M, q).

These problems have many interesting applications in optimization, engineering, economics and other areas has been well documented in the literature, see e.g., [4, 10, 15], and the references therein.

Andreani et al. [1] formulated the GCP(f, g, K) where K is a nonempty closed convex cone as an equivalent bound-constrained smooth optimization problem in the sense that a global minimizer with zero objective function value is a solution of the GCP. Also Andreani et al. [1] established conditions for proving that stationary points of the minimization problems are global minimizers and, consequently, solutions of the GCP. An unconstrained minimization reformulation of the GCP is considered such that the merit function is differentiable when $K = R^n_+$ in [18, 20] and K is a cone in [38].

In this paper, we study a nonsmooth generalized complementarity problem GCP(f, g) when the underlying functions f and g are H-differentiable (not necessarily locally Lipschitzian/ directionally differentiable). Our approach is to reformulate GCP(f, g) as an unconstrained optimization problem through some merit function. We construct a merit function via a GCP function $\phi : R^2 \to R$:

$$\phi(a, b) = 0 \Leftrightarrow ab = 0, a \geq 0, \, b \geq 0.$$

For the problem GCP(f, g), we define

$$\Phi(x) = \left[\; \phi(f_1(x), g_1(x)) \ldots \phi(f_i(x), g_i(x)) \ldots \phi(f_n(x), g_n(x)) \; \right]^T \tag{1.2}$$

and, we call $\Phi(x)$ a GCP function for GCP(f, g). A function $\Psi : R^n \to [0, \infty)$ is said to be a merit function for GCP(f, g) provided that the global minima of Ψ are coincident with the solutions of the original GCP(f, g). We consider a GCP function $\Phi : R^n \to R^n$ associated with GCP(f, g) and its merit function

$$\Psi(x) := \frac{1}{2} ||\Phi(x)||^2, \tag{1.3}$$

so that

$$\bar{x} \text{ solves GCP}(f, g) \; \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \Psi(\bar{x}) = 0.$$

The organization of the paper is as follows. We state some basic definitions and preliminary results. We describe H-differentials of some GCP functions based on the min-function and the penalized Fischer-Burmeister function, and their merit functions. We show that under appropriate semimonotone ($\mathbf{E_0}$), strictly semimonotone ($\mathbf{E}$) regularity-conditions on the H-differentials of f and g, local/global minimum of a merit function (or a 'stationary point' of a merit function) based on the min function and the penalized Fischer-Burmeister function coincides with the solution of the given generalized complementarity problem. Also, we consider GCP functions on the basis of the min-function and the penalized Fischer-Burmeister function which seem to be new.

Moreover, when specialized GCP(f, g) to the nonlinear complementarity problems, our results not only give new results but also extend/unify various similar results proved for C^1, semismooth, and locally Lipschitzian [2, 7, 19, 21].

2. Preliminaries

Throughout this paper, vector inequalities are interpreted componentwise. We regard vectors in R^n as column vectors. For a vector $x \in R^n$, x_i denotes the i-th component of x; x_+ denotes the vector with components $\max\{0, x_i\}, 1 \leq i \leq n$. Inequalities such as $x \geq 0$ and $x \geq 0$, $\min\{x, y\}$, and $\sqrt{x}$ are defined componentwise. For a matrix A, A_i denotes the i-th row of A. A diagonal matrix is a square matrix in which the entries outside the main diagonal are all zero and denoted $diag(a_1, \cdots, a_n) = diag(a_i) \; \forall i = 1, \cdots, n$, the diagonal entries themselves may or may not be zero. We denote the inner-product between two vectors x and y in R^n by either $x^T y$ or $\langle x, y \rangle$. For a differentiable function $f : R^n \to R^m$, $\nabla f(\bar{x})$ denotes the Jacobian matrix of f at $\bar{x}$. For a set $E \subseteq R^n$, $co\, E$ denotes the convex hull of E and $\overline{co}\, E$ denotes the closure of $co\, E$.

We need the following definition from [4].

Definition 2.1. *A matrix $A \in R^{n \times n}$ is called*

(a) $\mathbf{P_0}$ *(**P**) if $\forall x \in R^n, x \neq 0$, there exists i such that $x_i \neq 0$ and $x_i \, (Ax)_i \geq 0 \; (> 0)$. Equivalently, every principle minor of A is nonnegative (respectively, positive). (A typical* *principle minor of* A *is given by the determinant of the principle submatrix* $A_{\alpha\alpha}$ *where* $\alpha \subseteq \{1, 2, \cdots, n\}$*.)*

(b) *semimonotone* ($\mathbf{E_0}$) *(strictly semimonotone* ($\mathbf{E}$)*) if*

$$\forall x \in R^n_+,\ x \neq 0, \quad \text{there exists } i \text{ such that } \ x_i\,(Ax)_i \geq 0 \quad (> 0).$$

Definition 2.2. *For a function* $f : R^n \to R^n$*, we say that* f *is a*

(i) *monotone if*

$$\langle f(x) - f(y), x - y \rangle \geq 0 \quad \text{for all } x, y \in R^n.$$

(ii) $\mathbf{P_0}(\mathbf{P})$*-function if, for any* $x \neq y$ *in* R^n*,*

$$\max_{\{i : x_i \neq y_i\}} (x - y)_i [f(x) - f(y)]_i \geq 0 \ (> 0). \tag{2.1}$$

We note that every monotone (strictly monotone) function is a $\mathbf{P}_0(\mathbf{P})$-function.

H-differentiability and H-differentials

In [14], the authors introduced the concepts of the H-differentiability and H-differential for a function $f : R^n \to R^n$. They showed that the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [3], the Bouligand subdifferential of a semismooth function [22, 26, 28], and the C-differential of a C-differentiable function [27] are instances of H-differentials.

These concepts give useful and unified treatments for many problems when the underlying functions are not necessarily locally Lipschitzian or semismooth (see e.g., [12–14, 30, 32, 33, 36, 37]).

We first recall the following definition and examples from [14].

Definition 2.3. *Given a function* $f : \Omega \subseteq R^n \to R^m$ *where* Ω *is an open set in* R^n *and* $x^* \in \Omega$*, we say that a nonempty subset* $T_f(x^*)$ *of* $R^{m\times n}$ *is an* H*-differential of* f *at* x^* *if for every sequence* $\{x^k\} \subseteq \Omega$ *converging to* x^**, there exist a subsequence* $\{x^{k_j}\}$ *and a matrix* $A \in T_f(x^*)$ *such that*

$$f(x^{k_j}) - f(x^*) - A(x^{k_j} - x^*) = o(||x^{k_j} - x^*||). \tag{2.2}$$

We say that f *is* H*-differentiable at* x^* *if* f *has an* H*-differential at* x^**.*

A useful equivalent definition of an H-differential $T_f(x^*)$ is: For any sequence $x^k := x^* + t_k d^k$ with $t_k \downarrow 0$ and $||d^k|| = 1$ for all k, there exist convergent subsequences $t_{k_j} \downarrow 0$ and $d^{k_j} \to d$, and $A \in T_f(x^*)$ such that

$$\lim_{j\to\infty} \frac{f(x^* + t_{k_j} d^{k_j}) - f(x^*)}{t_{k_j}} = Ad.$$

Example 1 (Fréchet differentiability)

Let $F : R^n \to R^m$ be Fréchet differentiable at $x^* \in R^n$ with Fréchet derivative matrix (= Jacobian matrix derivative) $\{\nabla F(x^*)\}$ such that

$$F(x) - F(x^*) - \nabla F(x^*)(x - x^*) = o(||x - x^*||).$$

Then F is H-differentiable with $\{\nabla F(x^*)\}$ as an H-differential.

Example 2 (Locally Lipschitzian function)

Let $F : \Omega \subseteq R^n \to R^m$ be locally Lipschitzian at each point of an open set Ω. For $x^* \in \Omega$, define the Bouligand subdifferential of F at x^* by

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \to x^*, x^k \in \Omega_F\}$$

where Ω_F is the set of all points in Ω where F is Fréchet differentiable. Then, the (Clarke) generalized Jacobian [3]

$$\partial F(x^*) = co\partial_B F(x^*)$$

is an H-differential of F at x^*.

Example 3 (Semismooth function)

Consider a locally Lipschitzian function $F : \Omega \subseteq R^n \to R^m$ that is semismooth at $x^* \in \Omega$ [22, 26, 28]. This means for any sequence $x^k \to x^*$, and for $V_k \in \partial F(x^k)$,

$$F(x^k) - F(x^*) - V_k(x^k - x^*) = o(||x^k - x^*||).$$

Then the Bouligand subdifferential

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \to x^*, x^k \in \Omega_F\}$$

is an H-differential of F at x^*. In particular, this holds if F is piecewise smooth, i.e., there exist continuously differentiable functions $F_j : R^n \to R^m$ such that

$$F(x) \in \{F_1(x), F_2(x), \dots, F_J(x)\} \quad \forall\, x \in R^n.$$

Example 4 (C-differentiability)

Let $F : R^n \to R^n$ be C-differentiable [27] in a neighborhood D of x^*. This means that there is a compact upper semicontinuous multivalued mapping $x \mapsto T(x)$ with $x \in D$ and $T(x) \subset R^{n\times n}$ satisfying the following condition at any $a \in D$: For $V \in T(x)$,

$$F(x) - F(a) - V(x - a) = o(||x - a||).$$

Then, F is H-differentiable at x^* with $T(x^*)$ as an H-differential.

Remarks.

• Any superset of an H-differential is an H-differential, H-differentiability implies continuity, and H-differentials enjoy simple sum, product and chain rules, see [30].

• The authors in [37] noted that if a function $f : \Omega \subseteq R^n \to R^m$ is H-differentiable at a point $\bar{x}$, then there exist a constant $L > 0$ and a neighbourhood $B(\bar{x}, \delta)$ of $\bar{x}$ with

$$||f(x) - f(\bar{x})|| \leq L||x - \bar{x}||, \quad \forall x \in B(\bar{x}, \delta). \tag{2.3}$$

Conversely, if condition (2.3) holds, then $T_f(\bar{x}) := R^{m\times n}$ can be taken as an H-differential of f at $\bar{x}$. We thus have, in (2.3), an alternate description of H-differentiability.

Clearly any function locally Lipschitzian at $\bar{x}$ will satisfy (2.3). For real valued functions, condition (2.3) is known as the 'calmness' of f at $\bar{x}$ (see [29], Chapter 8).

• While the Fréchet derivative of a differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [3], the Bouligand differential of a semismooth function [26], and the C-differential of a C-differentiable function [27] are particular instances of H-differential, the following simple example, is taken from [12], shows that an H-differentiable function need not be locally Lipschitzian nor directionally differentiable. Consider on R,

$$f(x) = x \sin\left(\frac{1}{x}\right) \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

Then f is H-differentiable on R with

$$T_f(0) = [-1, 1] \text{ and } T_f(c) = \left\{\sin\left(\frac{1}{c}\right) - \frac{1}{c}\cos\left(\frac{1}{c}\right)\right\} \text{ for } c \neq 0.$$

We note that f is not locally Lipschitzian around zero. We also see that f is neither Fréchet differentiable nor directionally differentiable.

3. Main Results

3.1. H-differentials of some GCP functions associated with H-differentiable functions

In this section, we compute the H-differentials of some GCP functions based on the min function and the penalized Fischer-Burmeister function.

Theorem 3.1. *For H-differentiable functions $f : R^n \to R^n$ and $g : R^n \to R^n$, consider the GCP function*

$$\Phi(x) = \min\{f(x), g(x)\}. \tag{3.1}$$

Then Φ has an H-differential at $\bar{x}$ given by

$$\begin{aligned} T_\Phi(\bar{x}) = \ & \{VA + WB : A \in T_f(\bar{x}), B \in T_g(\bar{x}), V = diag(v_i), W = diag(w_i), \text{ with } v_i, w_i \in \{0,1\}, \\ & V + W = I\}. \end{aligned} \tag{3.2}$$

Proof. To see this claim, let $x^k \to \bar{x}$. By the H-differentiability of f and g, there exist a subsequence of $\{x^k\}$, which we continue to write as $\{x^k\}$ for simplicity, a matrix $A \in T_f(\bar{x})$ and $B \in T_g(\bar{x})$ such that $f(x^k) - f(\bar{x}) - A\ (x^k - \bar{x}) = o(||x^k - \bar{x}||)$ and $g(x^k) - g(\bar{x}) - B\ (x^k - \bar{x}) = o(||x^k - \bar{x}||)$, respectively. By considering a suitable subsequence, if necessary, we may write $\{1, \cdots, n\}$ as a disjoint union of sets α and β where

$$\alpha = \{i : \Phi_i(x^k) = f_i(x^k)\ \forall\, k\} \ \text{ and } \ \beta = \{i : \Phi_i(x^k) = g_i(x^k)\ \forall\, k\}.$$

Put

$$v_i = \begin{cases} 1 & \text{if } i \in \alpha \\ 0 & \text{if } i \in \beta \end{cases}, \quad w_i = \begin{cases} 0 & \text{if } i \in \alpha \\ 1 & \text{if } i \in \beta \end{cases},$$

$$V = diag(v_i),\ W = diag(w_i), \text{ and } \ C := VA + WB.$$

We show that $\Phi(x^k) - \Phi(\bar{x}) - C(x^k - \bar{x}) = o(||x^k - \bar{x}||)$. To see this, we fix an index j and show that $\Phi_j(x^k) - \Phi_j(\bar{x}) - [C(x^k - \bar{x})]_j = o(||x^k - \bar{x}||)$. Let $j = 1$ (for simplicity). We have two cases:

Case (1): $1 \in \alpha$.

$$\begin{aligned} \left[\Phi(x^k) - \Phi(\bar{x}) - (VA + WB)(x^k - \bar{x})\right]_1 &= f_1(x^k) - f_1(\bar{x}) - [VA(x^k - \bar{x})]_1 \\ -[WB(x^k - \bar{x})]_1 &= \left[f(x^k) - f(\bar{x}) - A\ (x^k - \bar{x})\right]_1 = o(||x^k - \bar{x}||). \end{aligned}$$

Case (2): $1 \in \beta$. It is easy to verify that $\Phi_1(x^k) - \Phi_1(\bar{x}) - [C(x^k - \bar{x})]_1 = o(||x^k - \bar{x}||)$. This proves the above claim. $\square$

Theorem 3.2. *The following GCP function is based on the so-called the penalized Fischer-Burmeister function [2]*

$$\Phi_\lambda(x) := \lambda[f(x) + g(x) - \sqrt{f(x)^2 + g(x)^2}] + (1 - \lambda) f(x)_+\, g(x)_+ \tag{3.3}$$

where $x_+ = \max\{0, x\}$ and $\lambda \in (0,1)$ is a fixed parameter. Let

$$J(\bar{x}) = \{i : f_i(\bar{x}) = 0 = g_i(\bar{x})\} \ \text{ and } \ K(\bar{x}) = \{i : f_i(\bar{x}) > 0,\ g_i(\bar{x}) > 0\}.$$

Then Φ_λ in (3.3) has an H-differential at $\bar{x}$ given by is given by

$$T_\Phi(\bar{x}) = \{VA + WB :\ (A, B, V, W, d) \in \Gamma\},$$

where Γ *is the set of all quintuples* (A, B, V, W, d) *with* $A \in T_f(\bar{x})$, $B \in T_g(\bar{x})$, $||d|| = 1$, $V = diag(v_i)$ *and* $W = diag(w_i)$ *are diagonal matrices with*

$$v_i = \begin{cases} \lambda\left(1 - \frac{f_i(\bar{x})}{\sqrt{f_i(\bar{x})^2 + g_i(\bar{x})^2}}\right) + (1-\lambda) g_i(\bar{x}) & \text{when } i \in K(\bar{x}) \\ \lambda\left(1 - \frac{A_i d}{\sqrt{(A_i d)^2 + (B_i d)^2}}\right) & \text{when } i \in J(\bar{x}) \text{ and } (A_i d)^2 + (B_i d)^2 > 0 \\ \lambda\left(1 - \frac{f_i(\bar{x})}{\sqrt{f_i(\bar{x})^2 + g_i(\bar{x})^2}}\right) & \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \textit{arbitrary} & \text{when } i \in J(\bar{x}) \textit{ and } (A_i d)^2 + (B_i d)^2 = 0, \end{cases} \tag{3.4}$$

$$w_i = \begin{cases} \lambda\left(1 - \frac{g_i(\bar{x})}{\sqrt{f_i(\bar{x})^2 + g_i(\bar{x})^2}}\right) + (1-\lambda) f_i(\bar{x}) & \text{when } i \in K(\bar{x}) \\ \lambda\left(1 - \frac{B_i d}{\sqrt{(A_i d)^2 + (B_i d)^2}}\right) & \text{when } i \in J(\bar{x}) \text{ and } (A_i d)^2 + (B_i d)^2 > 0 \\ \lambda\left(1 - \frac{g_i(\bar{x})}{\sqrt{f_i(\bar{x})^2 + g_i(\bar{x})^2}}\right) & \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \textit{arbitrary} & \text{when } i \in J(\bar{x}) \textit{ and } (A_i d)^2 + (B_i d)^2 = 0. \end{cases}$$

Proof. The proof is a straightforward calculation. □

Remark. The calculation in Theorem 3.2 relies on the observation that the following is an H-differential of the one variable function $s \mapsto s_+$ at any $\bar{s}$:

$$\Delta(\bar{s}) = \begin{cases} \{1\} & \text{if } \bar{s} > 0 \\ \{0, 1\} & \text{if } \bar{s} = 0 \\ \{0\} & \text{if } \bar{s} < 0. \end{cases}$$

The following theorem from [37] describes the H-differential of $\Psi := \frac{1}{2}||\Phi||^2$ where Φ is H-differentiable.

Theorem 3.3. *Suppose* $\Phi : R^n \to R^n$ *is* H*-differentiable at* $\bar{x}$ *with* $T_\Phi(\bar{x})$ *as an* H*-differential. Then* $\Psi : R^n \to R$, $\Psi := \frac{1}{2}||\Phi||^2$ *is* H*-differentiable at* $\bar{x}$ *with an* H*-differential given by*

$$T_\Psi(\bar{x}) = \{\Phi(\bar{x})^T C : C \in T_\Phi(\bar{x})\}.$$

3.2. Minimizing the merit function

In this section, we consider GCP function Φ and the corresponding merit function $\Psi = \frac{1}{2}||\Phi||^2$ when the underlying functions f and g are H-differentiable. It should be recalled that

$$\Psi(\bar{x}) = 0 \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \bar{x} \text{ solves GCP}(f, g).$$

We show that under appropriate conditions on the functions f and g, and their H-differentials, a vector $\bar{x}$ is a solution of the GCP(f, g) if and only if zero belongs to the set $T_\Psi(\bar{x})$.

3.2.1. Minimizing the merit function under regularity (strict regularity) conditions

We will minimize the merit function under regularity (strict regularity) conditions. But first we need to generalize the concept of a regular (strictly regular) point [5, 9, 21, 23]. For more details about the concept of a regular (strictly regular) vector, see the recent excellent monographs [8].

For given H-differentiable functions f and g, and $\bar{x} \in R^n$, we define the following subsets of $I = \{1, 2, \ldots, n\}$.

$$\mathcal{C}(\bar{x}) := \{i \in I : f_i(\bar{x}) \geq 0,\ g_i(\bar{x}) \geq 0,\ f_i(\bar{x})g_i(\bar{x}) = 0\}, \qquad \mathcal{R}(\bar{x}) := I \backslash \mathcal{C}(\bar{x}),$$
$$\mathcal{P}(\bar{x}) := \{i \in \mathcal{R}(\bar{x}) : f_i(\bar{x}) > 0,\ g_i(\bar{x}) > 0\}, \qquad \mathcal{N}(\bar{x}) := \mathcal{R}(\bar{x}) \backslash \mathcal{P}(\bar{x}).$$

Definition 3.1. *Consider f, g, $\bar{x}$, and the index sets as above. Let $T_f(\bar{x})$ and $T_g(\bar{x})$ be H-differentials of f and g at $\bar{x}$, respectively. Further, suppose that $T_g(\bar{x})$ consists of nonsingular matrices. Then the vector $\bar{x} \in R^n$ is called a relatively regular (strictly regular) point of f and g with respect to $T_f(\bar{x})$ and $T_g(\bar{x})$ if for every nonzero vector $z \in R^n$ such that*

$$z_{\mathcal{C}} = 0, \quad z_{\mathcal{P}} > 0, \quad z_{\mathcal{N}} < 0, \tag{3.5}$$

there exists a vector $s \in R^n$ such that

$$s_{\mathcal{P}} \geq 0, \quad s_{\mathcal{N}} \leq 0, \quad s_{\mathcal{R}} \neq 0, \ \text{and} \tag{3.6}$$

$$s^T(AB^{-1})^T z \geq 0 \ \ (> 0) \quad \text{for all } A \in T_f(\bar{x}), B \in T_g(\bar{x}). \tag{3.7}$$

Remark. When f is C^1 and $g(x) = x$ (in which case we can let $T_f(\bar{x}) = \{\nabla f(\bar{x})\}$), if $\nabla f(\bar{x})$ is a positive semidefinite matrix, we can choose $s = z$ and directly obtained from Definition 3.1 that $\bar{x}$ is a regular vector.

In the following theorem, the proof under a relatively strictly regular point is similar to a relatively regular point, we omit the proof under a relatively strictly regular point.

Theorem 3.4. *Suppose $f : R^n \to R^n$ and $g : R^n \to R^n$ are H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Let Φ be a GCP function satisfying the following conditions:*

$$\begin{array}{l} i \in \mathcal{P}(\bar{x}) \Rightarrow \Phi_i(\bar{x}) > 0, \\ i \in \mathcal{N}(\bar{x}) \Rightarrow \Phi_i(\bar{x}) < 0, \\ i \in \mathcal{C}(\bar{x}) \Rightarrow \Phi_i(\bar{x}) = 0. \end{array} \tag{3.8}$$

Suppose Ψ is H-differentiable with an H-differential given by

$$\begin{array}{rl} T_\Psi(\bar{x}) = & \{\Phi(\bar{x})^T\,[VA + WB] : \ A \in T_f(\bar{x}), B \in T_g(\bar{x}), V = diag(v_i), \\ & W = diag(w_i), \ \text{with } v_i > 0,\ w_i > 0 (\geq 0) \ \text{whenever } \Phi(\bar{x})_i \neq 0\}. \end{array} \tag{3.9}$$

Further suppose that $T_g(\bar{x})$ consists of nonsingular matrices. Then $0 \in T_\Psi(\bar{x})$ and $\bar{x}$ is a relatively regular point (respectively, a relatively strictly regular point) if and only if $\bar{x}$ solves GCP(f, g).

Proof. The 'if' part of the theorem follows easily from the definitions. Now let us prove the 'only if' part of the theorem. Suppose that $0 \in T_\Psi(\bar{x})$ and $\bar{x}$ is a relatively regular point. Then for some

$\Phi(\bar{x})^T\ [VA + WB] \in T_\Psi(\bar{x})$,

$$0 = \Phi(\bar{x})^T VA + \Phi(\bar{x})^T WB. \tag{3.10}$$

Take the transpose of (3.10), we get

$$A^T V^T \Phi(\bar{x}) + B^T W^T \Phi(\bar{x}) = A^T z + B^T y = 0 \tag{3.11}$$

where $z = V^T \Phi(\bar{x})$ and $y = W^T \Phi(\bar{x})$.

Since $T_g(\bar{x})$ consists of nonsingular matrices, simple calculations (3.11) becomes

$$C^T z + y = 0 \text{ where } C := AB^{-1}. \tag{3.12}$$

Now, for any $s \in R^n$, (3.12) yields

$$s^T C^T z + s^T y = 0. \tag{3.13}$$

We claim that $\Phi(\bar{x}) = 0$. Assume the contrary that $\bar{x}$ is not a solution of GCP(f, g). Then $\mathcal{R} \neq \emptyset$ and $z_{\mathcal{C}} = 0, \ \ z_{\mathcal{P}} > 0, \ \ z_{\mathcal{N}} < 0$. Since $\bar{x}$ is a relatively regular point, and y and z have the same sign, by taking a vector $s \in R^n$ satisfying (3.6) and (3.7), we have

$$s^T C^T z \geq 0 \tag{3.14}$$

and

$$s^T y = s_{\mathcal{C}}^T y_{\mathcal{C}} + s_{\mathcal{P}}^T y_{\mathcal{P}} + s_{\mathcal{N}}^T y_{\mathcal{N}} > 0. \tag{3.15}$$

Clearly (3.14) and (3.15) contradict (3.13). Hence $\bar{x}$ is a solution to GCP(f, g). □

Remark. The GCP functions in theorems 3.1-3.2 satisfy the conditions in Theorem 3.4 so we can state Theorem 3.4 for GCP functions in theorems 3.1-3.2.

Now we will minimize the merit function under semimonotone($\mathbf{E_0}$) and strictly semimonotone ($\mathbf{E}$)-conditions.

3.2.2. Minimizing the merit function under semimonotone($\mathbf{E_0}$)-conditions

Before stating the results of this subsection, we call a vector $\bar{x}$ is said to be feasible (strictly feasible) for GCP(f, g) if $f(\bar{x}) \geq 0$ (> 0), and $g(\bar{x}) \geq 0$ (> 0). In the following theorem we will minimize the merit function under $\mathbf{E_0}$($\mathbf{E}$)-conditions. Since the proof of the following theorem under $\mathbf{E_0}$-conditions will be similar to the proof under $\mathbf{E}$-conditions, we will give only the proof under $\mathbf{E_0}$-conditions .

Theorem 3.5. *Suppose $f, g : R^n \to R^n$ are H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Suppose Φ is a GCP function of f and g. Assume that $\Psi := \frac{1}{2}||\Phi||^2$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$\begin{aligned} T_\Psi(\bar{x}) = \ & \{\Phi(\bar{x})^T\ [VA + WB] : A \in T_f(\bar{x}), B \in T_g(\bar{x}), V = diag(v_i),\ and \\ & W = diag(w_i),\ with\ v_i > 0,\ w_i > 0 (\geq 0)\ \ whenever\ \Phi_i(\bar{x}) \neq 0\}. \end{aligned}$$

Further suppose that $\bar{x}$ is a strictly feasible point(respectively, feasible point) of GCP(f, g) and $\Phi_i(\bar{x}) > 0$, $T_g(\bar{x})$ consists of nonsingular matrices, and $S(\bar{x})$ consists of $\mathbf{E_0}$($\mathbf{E}$)*-matrices where $S(\bar{x}) := \{AB^{-1} : A \in T_f(\bar{x}), B \in T_g(\bar{x})\}$. Then $\ 0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0$.*

Proof. Suppose $0 \in T_\Psi(\bar{x})$. Then , so that for some $\Phi(\bar{x})^T\ [VA + WB] \in T_\Psi(\bar{x})$,

$$0 = \Phi(\bar{x})^T VA + \Phi(\bar{x})^T WB$$

yielding

$$A^T y + B^T z = 0 \tag{3.16}$$

where $y = V^T \Phi(\bar{x})$ and $z = W^T \Phi(\bar{x})$. Since $T_g(\bar{x})$ consists of nonsingular matrices, (3.16) becomes

$$C^T y + z = 0 \text{ where } C := AB^{-1}.$$

Since $\bar{x}$ is a strictly feasible point to GCP(f, g), $\Phi_i(\bar{x}) > 0$, then for any index i, $\Phi_i(\bar{x}) \neq 0 \Leftrightarrow\ \ 0 < y_i \neq 0$ (because $y = V\Phi(\bar{x})$ and $v_i w_i > 0$ when $\Phi_i(\bar{x}) \neq 0$) in which case

$y_i(C^Ty)_i = -v_iw_i[\Phi_i(\bar{x})]^2 < 0$ contradicting the $\mathbf{E}_0$-property of C. We conclude that $\Phi(\bar{x}) = 0$. Conversely, if $\Phi(\bar{x}) = 0$, then $T_\Psi(\bar{x}) = \{0\}$ by the description of $T_\Psi(\bar{x})$. $\square$

Remark. We note that the GCP functions of Theorems 3.1-3.2 satisfy the conditions of Theorem 3.5, thus we can state Theorem 3.5 for GCP functions in Theorems 3.1-3.2.

Before we state the next results, we recall a definition from [31].

Definition 3.2. *Consider a nonempty set $\mathcal{C}$ in $R^{n\times n}$. We say that a matrix A is a row representative of $\mathcal{C}$ if for each index $i = 1, 2, \ldots, n$, the ith row of A is the ith row of some matrix $C \in \mathcal{C}$. We say that $\mathcal{C}$ has the row-$\mathbf{P}_0$-property (row-$\mathbf{P}$-property) if every row representative of $\mathcal{C}$ is a $\mathbf{P}_0$-matrix ($\mathbf{P}$-matrix). We say that $\mathcal{C}$ has the column-$\mathbf{P}_0$-property (column-$\mathbf{P}$-property) if $\mathcal{C}^T = \{A^T : A \in \mathcal{C}\}$ has the row-$\mathbf{P}_0$-property (row-$\mathbf{P}$-property).*

When specialized GCP(f, g) to the nonlinear complementarity problems, we state the next result for the penalized Fischer-Burmeister function Φ. It is possible to state a very general result for any NCP function Φ. For simplicity, we avoid dealing in such a generality.

Theorem 3.6. *Suppose $f : R^n \to R^n$ is H-differentiable at $\bar{x}$ with an H-differential $T(\bar{x})$ which is compact and having the row-$\mathbf{P_0}$-property. Let Φ be the penalized Fischer-Burmeister function as in Theorem 3.2 and $\Psi := \frac{1}{2}||\Phi||^2$. Let $T_\Phi(\bar{x})$ and $T_\Psi(\bar{x})$ be as in Theorem 3.2 and Theorem 3.3. Then the following are equivalent:*

(a) $\bar{x}$ is a local minimizer of Ψ.

(b) $0 \in \overline{co}\ T_\Psi(\bar{x})$.

(c) $\Phi(\bar{x}) = 0$, i.e., $\bar{x}$ solves NCP(f).

Proof. The proof is similar to that of Theorem 8 in [37]. $\square$

We now state consequences of the above theorems for the penalized Fischer-Burmeister function (for the sake of simplicity).

Corollary 3.1. *Let $f : R^n \to R^n$ be differentiable and $\Phi(x)$ be the penalized Fischer-Burmeister function and $\Psi(x) = \frac{1}{2}||\Phi||^2$. Then the equivalence*

$\bar{x}$ is a local minimizer to Ψ if and only if $\bar{x}$ solves NCP(f)

holds under each of the following conditions.

(a) f is monotone function.

(b) f is $\mathbf{P_0}$-function.

This corollary is seen from the above theorem by taking $T(\bar{x}) = \{\nabla f(\bar{x})\}$. If we assume the continuous differentiability of f in the above corollary, we get the following result: For a continuously differentiable $\mathbf{P}_0$-function f, every stationary point of Ψ solves NCP(f). (This is because, when f is C^1, Ψ becomes continuously differentiable, the proof of this statement is similar to that of Prop. 3.4 in [6].)

Corollary 3.2. *Let $f : R^n \to R^n$ be locally Lipschitzian. Let Φ be the penalized Fischer-Burmeister function and $\Psi(\bar{x}) = \frac{1}{2}||\Phi||^2$. If $\partial_B f(\bar{x})$ has the row-$\mathbf{P_0}$-property, then*

$$0 \in \partial\Psi(\bar{x}) \Leftrightarrow \Psi(\bar{x}) = 0.$$

Proof. By Corollary 1 in [37], every matrix in $\partial f(\bar{x}) = co\,\partial_B f(\bar{x})$ is a $\mathbf{P}_0$-matrix and noting $\partial\Psi(x) \subseteq T_\Psi(x)$ for all x. Now we have the stated equivalence. $\square$

Remark The above corollary might be especially useful when the function f is piecewise smooth in which case $\partial_B f(\bar{x})$ consists of a finite number of matrices.

Theorem 3.7. *Suppose $f : R^n \to R^n$ and $g : R^n \to R^n$ are H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Let Φ be a GCP function satisfying the conditions in (3.8). Suppose Ψ is H-differentiable with an H-differential given by*

$$\begin{array}{rl} T_\Psi(\bar{x}) = & \{\Phi(\bar{x})^T [VA + WB] : A \in T_f(\bar{x}), B \in T_g(\bar{x}), V = diag(v_i), \\ & W = diag(w_i), \text{ with } v_i > 0, w_i \geq 0 \text{ whenever } \Phi(\bar{x})_i \neq 0\}. \end{array} \quad (3.17)$$

Further suppose that $T_g(\bar{x})$ consists of nonsingular matrices and $S(\bar{x})$ as described in Theorem 3.4 has the column-$\mathbf{P}$-property. Then

$$0 \in T_\Psi(\bar{x}) \text{ if and only if } \bar{x} \text{ solves } GCP(f, g).$$

Proof. In view of Theorem 3.4, it is enough to show $\bar{x}$ is a relatively strictly regular point. To see this, let v be a nonzero vector satisfying (3.5). Since $S(\bar{x})$ has the column-$\mathbf{P}$-property, by Theorem 2 in [31], there exists an index j such that $v_j \left[C^T v\right]_j > 0 \ \ \forall C \in S(\bar{x})$. Choose $s \in R^n$ so that $s_j = v_j$ and $s_i = 0$ for all $i \neq j$. Then

$$s^T C^T v = v_j \left[C^T v\right]_j > 0 \ \forall C \in S(\bar{x}).$$

Hence $\bar{x}$ is a relatively strictly regular point. □

For penalized Fischer-Burmeister function, when f and g are C^1 (in which case we can let $T_f(\bar{x}) = \{\nabla f(\bar{x})\})$ and $T_g(\bar{x}) = \{\nabla g(\bar{x})\})$, the above result reduces to the following Corollary.

Corollary 3.3.*(a) Let $f, g : R^n \to R^n$ be continuously differentiable and $\Phi(x)$ be the penalized Fischer-Burmeister function and $\Psi(x) = \frac{1}{2}||\Phi||^2$. Let $\bar{x}$ be a stationary point of Ψ such that $\nabla g(\bar{x})$ is nonsingular and $\nabla f(\bar{x}) \nabla g(\bar{x})^{-1}$ is $\mathbf{P_0}$-matrix. Then $\bar{x}$ is a solution of GCP(f, g).*

(b) Let $f : R^n \to R^n$ be continuously differentiable and $\Phi(x)$ be the penalized Fischer-Burmeister function and $\Psi(x) = \frac{1}{2}||\Phi||^2$. Let $\bar{x}$ be a stationary point of Ψ and $\nabla f(\bar{x})$ is $\mathbf{P_0}$-matrix. Then $\bar{x}$ is a solution of NCP(f).

If $g(x) = x$ in part(a) Corollary 3.3, GCP(f, g) reduces to NCP(f) and part(a) result reduces to part(b).

Concluding Remarks This paper is considered as a continuation of [34]. In this paper, we give the sufficient conditions on the functions f and g so that we can guarantee that stationary points of the merit function solve the GCP(f, g). For continuously differentiable functions, the nonsingularity of $T_g = \{\nabla g\}$ is very important in an algorithmic point of view and studying the error bounds for GCP(f, g), please see examples in [1, 20, 38–41] where $T_g = \{\nabla g\}$ consists of nonsingular matrices.

We considered a generalized complementarity problem corresponding to H-differentiable functions, with an associated GCP function Φ and a merit function $\Psi(x) = \frac{1}{2}||\Phi||^2$, while in [35] the author considered the merit function $\Psi(\bar{x}) := \sum_{i=1}^n \Phi_i(\bar{x})$ where the associated GCP function Φ need to be nonnegative and have certain properties.

In this article, we showed under certain semimonotone (E_0), strictly semimonotone (E), and regularity conditions the global/local minimum or a stationary point of Ψ is a solution of GCP(f, g). For generalized complementarity problem based on the penalized Fischer-Burmeister function, our results give various results for generalized complementarity problem when the underlying functions are continuously differentiable (locally Lipschitzian, semismooth, and directionally differentiable) functions. For example, we have the following:

- When f and g are C^1 in which case $T_f(\bar{x}) = \{\nabla f(\bar{x})\}$ and $T_g(\bar{x}) = \{\nabla g(\bar{x})\}$, our results will be true when the underlying functions are C^1.
- When f is C^1 and $g(x) = x$ (in which case we can let $T_f(\bar{x}) = \{\nabla f(\bar{x})\})$, GCP$(f, g)$ reduces to nonlinear complementarity problem NCP(f) and the results of this paper will be valid for NCP(f).
- In view of Example 3, if f is locally Lipschitzian with $T_f(\bar{x}) = \partial f(\bar{x})$ and $g(x) = x$, our results will be applicable to NCP(f) when the underlying data are locally Lipschitzian.

To the best of our knowledge, solving GCP(f, g) on the basis of penalized Fischer-Burmeister function seems to be new.

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References

[1] R. Andreani, A. Friedlander, and S.A. Santos: On the resolution of the generalized nonlinear complementarity problem. *SIAM Journal on Optimization*, **12** (2001), 303–321.

[2] B. Chen, X. Chen, and C. Kanzow: A penalized Fischer-Burmeister NCP-function. *Mathematical Programming*, **88** (2000), 211–216.

[3] F.H. Clarke: *Optimization and Nonsmooth Analysis* (Wiley, New York, 1983) (reprinted by SIAM, Philadelphia, PA, 1990).

[4] R.W. Cottle, J.-S. Pang, and R.E. Stone: *The Linear Complementarity Problem* (Academic Press, Boston, 1992).

[5] F. Facchinei and C. Kanzow: On unconstrained and constrained stationary points of the implicit Lagrangian. *Journal of Optimization Theory and Applications*, **92** (1997), 99–115.

[6] F. Facchinei and J. Soares: A new merit function for nonlinear complementarity problems and related algorithm. *SIAM Journal on Optimization*, **7** (1997), 225–247.

[7] F. Facchinei, A. Fischer, and C. Kanzow: Regularity properties of a semismooth reformulation of variational inequalities. *SIAM Journal on Optimization*, **8** (1998), 850–869.

[8] F. Facchinei and J.-S. Pang: *Finite-Dimensional Variational Inequalities and Complementarity Problems* (Vol. I/Vol.II, Springer Series in Operations Research, Springer-Verlag, New York, 2003).

[9] M.C. Ferris and D. Ralph: Projected gradient methods for nonlinear complementarity problems via normal maps. In D.Z. Du, L.Qi, and R.S. Womersley (eds.): *Recent Advances in Nonsmooth Optimization* (World Scientific Publishers, Singapore, Republic of Singapore 1995), 57–87.

[10] M.C. Ferris and J.-S. Pang: Engineering and economic applications of complementarity problems. *SIAM Review*, **39** (1997), 669–713.

[11] A. Fischer: Solution of monotone complementarity problems with locally Lipschitzian functions. *Mathematical Programming*, **76** (1997), 513–532.

[12] M.S. Gowda: A Note on H-differentiable functions. Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, Maryland, November, 1998.

[13] M.S. Gowda: Inverse and implicit function theorems for H-differentiable and semismooth functions. *Optimization Methods and Software*, **19** (2004), 443–461.

[14] M.S. Gowda and G. Ravindran: Algebraic univalence theorems for nonsmooth functions. *Journal of Mathematical Analysis and Applications*, **252** (2000), 917–935.

[15] P.T. Harker and J.-S. Pang: Finite dimension variational inequality and nonlinear

complementarity problems: A survey of theory, algorithms and applications. *Mathematical programming*, **48** (1990), 161–220.

[16] D.H. Hyer, G. Isac, and T.M. Rassias: *Topics in Nonlinear Analysis and Applications* (World scientific Publishing Company, Singapore, 1997).

[17] G. Isac: *Complementarity Problems* (Lecture Notes in Mathematics 1528, Springer Verlag, Berlin, Germany, 1992).

[18] H. Jiang, M. Fukushima, L. Qi, and D. Sun: A trust region method for solving generalized complementarity problems. *SIAM Journal on Optimization*, **8** (1998), 140–157.

[19] C. Kanzow and H. Kleinmichel: A new class of semismooth Newton-type methods for nonlinear complementarity problems. *Computational Optimization and Applications*, **11** (1998), 227–251.

[20] C. Kanzow and M. Fukushima: Equivalence of the generalized complementarity problem to differentiable unconstrained minimization. *Journal of Optimization Theory and Applications*, **90** (1996), 581–603.

[21] T.D. Luca, F. Facchinei, and C. Kanzow: A semismooth equation approach to the solution of nonlinear complementarity problems. *Mathematical Programming*, **75** (1996), 407–439.

[22] R. Mifflin: Semismooth and semiconvex functions in constrained optimization. *SIAM Journal on Control and Optimization*, **15** (1977), 952–972.

[23] J.J. Moré: Global methods for nonlinear complementarity problems. *Mathematics of Operations Research*, **21** (1996), 589-614.

[24] M.A. Noor: Quasi-complementarity problem. *Journal of Mathematical Analysis and Applications*, **130** (1988), 344–353.

[25] J.S. Pang: The implicit complementarity problem. In O.L. Mangasarian, R. R. Meyer, and S. M. Robinson(eds.): *Nonlinear Programming* (Academic Press, New York 1981), 487–518.

[26] L. Qi: Convergence analysis of some algorithms for solving nonsmooth equations. *Mathematics of Operations Research*, **18** (1993), 227–244.

[27] L. Qi: C-differentiability, C-differential operators and generalized Newton methods. Research Report, School of Mathematics, The University of New South Wales, Sydney, New South Wales 2052, Australia, January, 1996.

[28] L. Qi and J. Sun: A nonsmooth version of Newton's method. *Mathematical Programming*, **58** (1993), 353–367.

[29] R.T. Rockafellar and R.J.-B. Wets: *Variational Analysis* (Grundlehren der Mathematischen Wissenschaften, 317, Springer-Verlag, Berlin, 1998).

[30] Y. Song, M.S. Gowda, and G. Ravindran: On characterizations of $\mathbf{P}$- and $\mathbf{P}_0$- properties in nonsmooth functions. *Mathematics of Operations Research*, **25** (2000), 400–408.

[31] Y. Song, M.S. Gowda, and G. Ravindran: On some properties of $\mathbf{P}$-matrix sets. *Linear Algebra and its Applications*, **290** (1999), 246–273.

[32] M.A. Tawhid: On the local uniqueness of solutions of variational inequalities under H-differentiability. *Journal of Optimization Theory and Applications*, **113** (2002), 149–164.

[33] M.A. Tawhid: Unconstrained methods for nonsmooth nonlinear complementarity problems. *Advanced Modeling and Optimization*, **12** (2010), 20–35.

[34] M.A. Tawhid: Nonsmooth generalized complementarity as unconstrained optimization. *Journal of Industrial and Management Optimization*, **6** (2010), 411–423.

[35] M.A. Tawhid: An application of H-differentiability to nonnegative and unrestricted generalized complementarity problems. *Computational Optimization and Applications*, **39** (2008), 51–74.

[36] M.A. Tawhid and J.L. Goffin: On minimizing some merit functions for nonlinear complementarity problems under H-differentiability. *Journal of Optimization Theory and Applications*, **139** (2008), 127–140.

[37] M.A. Tawhid and M.S. Gowda: On two applications of H-differentiability to optimization and complementarity problems. *Computational Optimization and Applications*, **17** (2000), 279–299.

[38] P. Tseng, N. Yamashita, and M. Fukushima: Equivalence of complementarity problems to differentiable minimization: A unified approach. *SIAM Journal on Optimization*, **6** (1996), 446–460.

[39] X. Zhang, H. Jiang, and Y. Wang: A smoothing Newton-type method for generalized nonlinear complementarity problem. *Journal of Computational and Applied Mathematics*, **212** (2008), 75–85.

[40] X. Zhang, F. Ma, and Y. Wang: A Newton-type algorithm for generalized linear complementarity problem over a polyhedral cone. *Applied Mathematics and Computation*, **169** (2005), 388–401.

[41] Y. Wang, X. Zhang, and L. Qi: Unconstrained optimization reformulation of the generalized nonlinear complementarity problem and related method. *Optimization*, **54** (2005), 563–577.

Mohamed Aly Tawhid
Department of Mathematics and Statistics
Thompson Rivers University
900 McGill Road, PO Box 3010, Kamloops,
BC V2C 5N3, Canada
E-mail: `mtawhid@tru.ca`