STOCHASTIC COMPARISONS IN REVENUE MANAGEMENT UNDER A DISCRETE CHOICE MODEL OF CONSUMER BEHAVIOR

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Abstract   This study examines the monotonicity properties of expected revenue in a revenue management problem with respect to the consumers' choice behavior and the market size. The consumer behavior is described by a general discrete choice model. The firm decides which subset of fare products to offer at each time period. An example shows that the usual stochastic order relation in consumers' preference over the set of fare products is not sufficient for our naive intuition regarding the monotonicity to hold true. We provide sufficient conditions under which our intuition is valid. These conditions identify desirable changes in consumer behavior for the firm.

Keywords: Applied probability, revenue management, stochastic comparison, Markov decision process

1. Introduction

Revenue management (RM) is an important approach of inventory control for perishable products with capacity constraint. Research on RM has an over 40-year history, covering a wide range of topics, such as overbooking, pricing, customer segmentation and demand forecasting, seat inventory control among others. Littlewood's rule (Littlewood [5]) is widely recognized as the origin of the research on RM techniques, which proposes a best rule for solving a single-leg dynamic seat allocation problem with two fare classes. This rule is based on a booking limit and accepts discount fare bookings as long as their revenue value exceeds the expected revenue of future full fare bookings. Since the seminal work by Littlewood, explored are various aspects of the RM problem, including pricing, promotion, overbooking, and long term contract. In particular, Netessine and Shumsky [8] discuss the issue of alliance in the airline industry. Overkill competition bring a limit to so called KAIZEN, the continuous improvement activities for efficient business operation and resource utilization, in all airline companies. Thus, alliance formation is an important strategic issue for the airline companies and the industry as a whole. The literature of RM is huge. Readers are referred to an extensive list of references in Talluri and van Ryzin [12].

RM practice too has a long history. In the early 1970s, BOAC (now British Airways) offered an early bird program providing discounted low fare tickets to passengers booking 21 days prior to departure. In North America, the intensive development of revenue management techniques dates from the launch of Super Saver fares in April of 1977 by American Airlines, see McGill and van Ryzin [7]. The intensive development and implementation of RM techniques in Japan has started since the deregulation of the domestic airline market in 2000, see [9] and Eguchi and Belobaba [2].

In this study, we investigate an RM problem known as a single-leg seat inventory problem. One may think intuitively that the firm's revenue increases when non-price promotions raise

the purchase probability of consumers. We give an example showing that such an intuition is not necessarily true under a discrete choice model of consumer behavior. This example makes us have a second thought on the impact of the change in consumer behavior. To better understand the relationship between the consumer behavior and the firm's revenue, we introduce a concept called business orientedness of consumer market. Business travelers tend to highly value flexibility such as rescheduling and no penalties for cancellation offered by a high-fare product. For airline companies, it is important to segment business travelers and more price-sensitive leisure travelers by various fences such as advanced purchase and booking limit. Such fences cut off business travelers from a low-fare product and prevent price-sensitive travelers from occupying too many seats. Based on the concept of business orientedness of consumer market, we provide sufficient conditions for the monotonicity of expected revenue with respect to the consumer behavior. Also discussed is the monotonicity of the expected revenue with respect to the market size.

Cooper and Gupta [1] investigate stochastic order relations in RM problem. The firm's decision variable is the booking limits in each time period, and the fare class to be offered at each time period is exogenously given. Here, the booking limit is the number of seats available in a period. They demonstrate that using a stylized example with dependent demands, stochastically larger demand can lead to lower expected revenue. They prove that, with the assumption of i.i.d. demands, the expected revenue increases as the demand increases in the sense of increasing concave order. Other interesting instances of violation of intuitively convincing monotonicity properties are reported in Gupta and Cooper [3] and Masuda [6]. Talluri and van Ryzin [11] incorporate consumer choice behavior such as buy-up and buy-down when a set of fare products are offered to consumers. Here, the firm is to decide which subset of fare products to offer at each time period. The probability that a consumer chooses a specific fare product depends on the set of fare products offered. Under this setting, they show that the optimal policy is of simple form in that the firm offers only an "efficient" set of fare products and that the optimal policy is a sequence of "nested" sets. In addition they develop a procedure for estimating model parameters.

The objective of this study is to explore the monotonicity of the expected revenue with respect to the consumer behavior such as the business orientedness in an RM problem under the discrete choice model of consumer behavior as in Talluri and van Ryzin [11]. The spirit of our analysis parallels that of Cooper and Gupta [1] in that the focus of the study is on qualitative aspects rather than quantitative/numerical aspects of RM.

The rest of the paper is organized as follows. The next section provides the model and the main results regarding the monotonicity. Section 3 gives some illustrative numerical examples. Finally, we give some concluding remarks and summarize managerial insights in Section 4.

2. Model and Analysis

2.1. RM under a discrete choice model of consumers behavior

We consider a single-leg seat inventory problem of Talluri and van Ryzin [11]. Let T be the number of time periods in consideration and index t the number of remaining time periods before the departure of the flight. Suppose that a time period is short enough so that in every period at most one customer arrives with probability $\lambda_t > 0$, see Lee and Hersh [4]. Let $N = \{1, 2, \ldots, n\}$ be the finite set of fare products and $r_i > 0$ the fare of product $i \in N$. At the beginning of each period, the firm chooses a subset $S \subset N$ of fare products to offer. Denoted by $P_{it}(S)$ is the probability that the arriving consumer chooses product $i \in S$ in

period t when the set of fare products S is offered. The arriving consumer in period t does not purchase any product with probability $P_{0t}(S)$, so that $P_{0t}(S) + \sum_{i \in S} P_{it}(S) = 1$. For notational convenience, we set $P_{it}(S) = 0$ for $i \notin S, S \subset N$, and $r_0 = 0$. Let C denote the seat capacity of the flight, and x the number of remaining seats.

It is known that leisure customers tend to make a reservation well in advance while some business travelers book flights just before the time of departure. Thus, we know that the customer behavior in RM is intrinsically inhomogeneous in time. Nevertheless, most of the models in the literature discuss a homogeneous model. This is so because a homogeneous model keeps the notation simple, as stated explicitly in Talluri and van Ryzin [11]. A major premise here is that the analysis based on a homogeneous model can be easily carried over to an inhomogeneous model. As we will see, we need a technical assumption to prove the main result of the paper. This assumption is necessary only for the case of inhomogeneous model, i.e., we cannot readily carry over the analysis of the homogeneous model to the inhomogeneous model. This is the reason why our model incorporates time t explicitly as in $P_{it}(S)$ and λ_t.

The problem of maximizing the firm's expected revenue is formulated as a discrete time Markov decision process as follows. We define the value function $V_t(x)$ as the maximum expected revenue generated from x seats remaining at period t. The optimality equation is written as

$$V_t(x) = \max_{S \subset N} \left\{ \sum_{i \in S} \lambda_t P_{it}(S)(r_i + V_{t-1}(x-1)) + (\lambda_t P_{0t}(S) + 1 - \lambda_t)V_{t-1}(x) \right\} \quad (1)$$

with the boundary conditions $V_t(0) = 0$, $t = 1, 2, \ldots, T$, and $V_0(x) = 0$, $x = 1, 2, \ldots, C$. Optimal policy is a sequence of $S_t(x)$, which is a maximizer of Bellman equation (1).

Talluri and van Ryzin [11] introduce the concept of efficient set. Let $q_t(S) = 1 - P_{0t}(S)$ and $r_t(S) = \sum_{i \in S} P_{it}(S)r_i$. An offer set $S \subset N$ of fare products is said to be *inefficient* at period t if there exist probabilities $\alpha(U), U \subset N$ (including $U = \varnothing$) with $\sum_{U \subset N} \alpha(U) = 1$ such that

$$q_t(S) \geq \sum_{U \subset N} \alpha(U)q_t(U) \quad \text{and} \quad r_t(S) < \sum_{U \subset N} \alpha(U)r_t(U).$$

Otherwise, the set S is said to be *efficient* at period t. Note that by definition $S = \varnothing$ is efficient. Informally speaking, offer set S is inefficient if it is outperformed by a mixture of other offer sets. Talluri and van Ryzin show that the optimal policy for (1) is a sequence of efficient offer sets.

2.2. Monotonicity of the expected revenue

In this section, we investigate the impact of market profile on the expected revenue of the firm. The spirit of our analysis is similar to that of Cooper and Gupta [1]. To be specific, we consider two monotonicity properties. One intuitively expects that the more business oriented the customers are, the higher the expected revenue of the firm is. Also expected is that the larger the market size is, the higher the revenue. Here we shall prove formally these two monotonicity properties. They however may look so intuitive that one can hardly imagine a case when they are violated. We give a simple example showing that the monotonicity involves some twist.

Suppose that the two markets $j = 1, 2$ are identified by the choice probabilities $(P_{it}^j(S) : i \in S \subset N)$. Let $V_t^j(x), j = 1, 2$, be the value function in (1) with $P_{it}(S)$ replaced by $P_{it}^j(S)$. In what follows, a superscript indicates the market profile. Recall that random variable (r.v.) X is *stochastically larger than* r.v. Y, denoted by $X \geq_{st} Y$, if $F_X(x) \leq F_Y(x)$ for all

x where F_X is the distribution function of X. Let $R_t^j(S)$ be a r.v. representing the revenue generated in period t with choice probabilities $(P_{it}^j(S) : i \in S \subset N)$, i.e.,

$$R_t^j(S) = r_i \quad \text{with probability } P_{it}^j(S)$$

We assume with no loss of generality that $0 = r_0 < r_1 \leq r_2 \leq \cdots \leq r_n$. One may naturally wonder if a stochastic order relation:

$$R_t^1(S) \leq_{\text{st}} R_t^2(S) \quad \text{for every } S \subset N \tag{2}$$

is sufficient for $V_t^1(x) \leq V_t^2(x)$. The following stylized and simple example demonstrates that this indeed is not the case.

Example 1 *Let the set of fare products be $N = \{1, 2\}$ with fares $r_1 = 1$ and $r_2 = 2$. Let T be an arbitrary positive integer. For the sake of simplicity, we set $\lambda_t = 1, t = 1, 2, \ldots, T$, and $C = 1$. The choice probabilities in markets 1 and 2 are given as*

$$
\begin{align}
P_{0t}^1(\{1\}) &= P_{0t}^1(\{2\}) = 1, \tag{3} \\
P_{0t}^1(N) &= 1/2, \quad P_{1t}^1(N) = 0, \quad P_{2t}^1(N) = 1/2, \tag{4} \\
P_{0t}^2(\{1\}) &= P_{0t}^2(\{2\}) = 1, \tag{5} \\
P_{0t}^2(N) &= 0, \quad P_{1t}^2(N) = 1/2, \quad P_{2t}^2(N) = 1/2. \tag{6}
\end{align}
$$

Clearly, $R_t^1(S) \leq_{\text{st}} R_t^2(S)$ for every $S \subset N$. Simple algebra shows that

$$
\begin{align}
V_t^1(1) &= 2\left(1 - \left(\frac{1}{2}\right)^t\right), \quad t = 1, 2, \ldots, T, \text{ and} \\
V_t^2(1) &= 1.5, \quad t = 1, 2, \ldots, T.
\end{align}
$$

Thus, $V_t^1(1) > V_t^2(1)$ for $t \geq 3$, violating the plausible monotonicity. The extreme simplicity of this example help us better see through the problem. We note that Equations (3) and (5) imply that the firm has to offer fare products 1 and 2 at the same time to induce a purchase. Thus, in both markets, the firm has no way to "segment" customers while there are two types of customers. Such choice behavior alone does not cause an anomaly. Complications are associated with arriving customers making no purchase and the conditional choice behavior given that a purchase is made, see Equations (4) and (6). In other word, the relation (2) is the stochastic ordering of "per-period revenue", which is crucial when there are a few remaining periods. When the number of remaining periods is large, however, what counts is the seat capacity and the "per-seat revenue". The expected per-seat revenues from markets 1 and 2 are $(r_1 P_{1t}^1(N) + r_2 P_{2t}^1(N)) / (1 - P_{0t}^1(N)) = 2$ and $3/2$, respectively, when the offer set is N. Thus, market 1 generates more revenue than market 2 if there are many remaining time periods. We employ this fact in the definition of business orientedness of a market. More importantly, this example indicates that if promotional efforts shift arriving customers' choice from no-purchase to a low-fare product while for whatever the reason it is difficult to segment business customers and more price-sensitive leisure customers, then the price-sensitive customers may eat up precious resources, lowering the expected revenue.

We note that the violation of monotonicity arises even in a very simple setting like Example 1. The following example shows that the outright simplicity $(\lambda_t = 1, P_{0t}^j(\{1\}) = P_{0t}^j(\{2\}) = 1$ and $C = 1)$ is inessential for the non-monotonicity.

Example 2 *Let the set of fare products be* $N = \{1, 2\}$ *with fares* $r_1 = 1$ *and* $r_2 = 5$. *The arrival probability and the capacity are* $\lambda_t = 0.9$ *and* $C = 10$. *Markets 1 and 2 are characterized by*

$$
\begin{aligned}
P_{0t}^1(\{1\}) &= P_{0t}^2(\{1\}) = 0.8, & P_{1t}^1(\{1\}) &= P_{1t}^2(\{1\}) = 0.2, \\
P_{0t}^1(\{2\}) &= P_{0t}^2(\{2\}) = 0.9, & P_{2t}^1(\{2\}) &= P_{2t}^2(\{2\}) = 0.1, \\
P_{0t}^1(N) &= 0.4, & P_{1t}^1(N) = 0.1, & P_{2t}^1(N) = 0.5, \\
P_{0t}^2(N) &= 0.1, & P_{1t}^2(N) = 0.4, & P_{2t}^2(N) = 0.5.
\end{aligned}
$$

Note that the value function $V_t^j(x)$ *does not depend on* T *so long as* $t \leq T$. *Thus, we suppose that the number* T *of time periods is sufficiently large, so that we can discuss* $V_t^j(x)$ *for large* t. *Figure 1 plots the expected revenue* $V_t^j(C)$ *of this example for* $t = 1, 2, \ldots, 50$. *Clearly* $R_t^1(S) \leq_{st} R_t^2(S)$ *for every* $S \subset N$. *However, Figure 1 shows* $V_t^1(C) > V_t^2(C)$ *for* $t \geq 14$, *violating the plausible monotonicity. Small values of* $P_{1t}^j(\{1\})$ *and* $P_{2t}^j(\{2\})$ *indicate that if there are not enough time periods to sell all the seat inventories, the airline company offers* $S = N$ *to minimize the inventory overage. Another implication of small values of* $P_{1t}^j(\{1\})$ *and* $P_{2t}^j(\{2\})$ *is that the airline company has little ability to segment the market. The offer set* $S = N$ *generates the best per-period revenue of* $\lambda_t \left(P_{1t}^j(N) r_1 + P_{2t}^j(N) r_2 \right)$, *which coincides with the slope of* $V_t^j(C)$ *for small value of* t ($t \leq 10$) *in Figure 1. For this range of* t, *the slope for market 2 is larger than that of market 1, resulting in* $V_t^2(C) > V_t^1(C)$. *For a moderately large value of* t (*for* t *around* 14), *the time constraint is rather relaxed and the seat capacity constraint becomes stringent. For this range of* t, *the offer set* $S = N$ *yields more revenue in market 1 than in market 2 because the offer set* $S = N$ *generates more revenue per seat in market 1 than in market 2. We note that for a very large value of* t, *the airline company tries to maximize the per-seat revenue by offering only the high fare product* ($S = \{2\}$), *so that both* $V_t^1(C)$ *and* $V_t^2(C)$ *approach to* $r_2 C = 50$ *as* t *increases.*

Next we formally define the business orientedness of a market. Let $\mathcal{S}_{1t}$ be the family of all efficient offer sets with respect to market 1 in period t. We say that market 2 is *more business oriented than* market 1 if for every $S \in \mathcal{S}_{1t}$ and $t = 1, 2, \ldots, T$,

$$
E(R_t^1(S) | R_t^1(S) > 0) \leq E(R_t^2(S) | R_t^2(S) > 0). \tag{7}
$$

Note that $E(R_t^j(S) | R_t^j(S) > 0)$ is the conditional expected revenue generated from the offer set S in one period given that a consumer arrives and makes a purchase, which is the expected per-seat revenue. Thus, condition (7) states that the expected per-seat revenue under market 1 is smaller than that under market 2 for every offer set in $\mathcal{S}_{1t}$. In what follows, we impose the following mild technical assumption:

$$
P_{nt}^1(\{n\}) > 0. \tag{8}
$$

That is, if only the highest fare product n is offered in market 1, an arriving customer buys the product with some positive probability. Under this assumption, offer set $S = \{n\}$ dominates offer set $S' \subset N$ with $P_{0t}^1(S') = 1$ (e.g. $S' = \varnothing$) in every period whatever the number x of remaining seats is. Thus, Assumption (8) ensures that $S \subset N$ with $P_{0t}^1(S) = 1$ never arises in the optimal sequence of offer sets $S_t^1(x)$, i.e.,

$$
P_{0t}^1(S_t^1(x)) < 1 \tag{9}
$$

for every t and $x > 0$. We need Property (9) to prove the main result of the paper. Assumption (8) is considered to be mind since the purchase probability $P_{nt}^1(\{n\})$ can be arbitrarily close to zero.
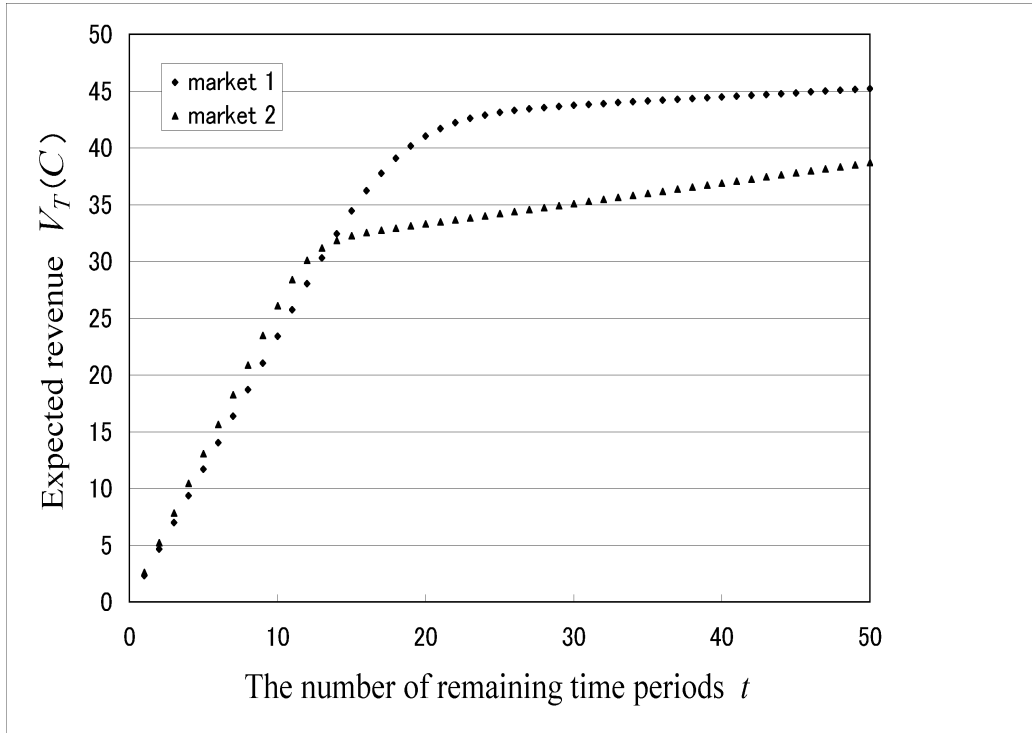
Figure 1: The number of remaining periods vs expected Revenue

It can be shown that, if the arrival and choice probabilities are temporally homogeneous, the assumption (8) is not necessary to ensure (9), see Appendix. The temporal homogeneity of probabilities however is significantly more restrictive than Assumption (8) for practical applications.

The following theorem states that, if the firm can improve both the purchase probability of arriving customers and the conditional choice behavior of customers given that a purchase is made, then firm enjoys a higher expected revenue.

Theorem 3 *Under Assumption (8), if*
(a) $P_{0t}^1(S) \geq P_{0t}^2(S)$, for every $S \in \mathcal{S}_{1t}$ and t, and
(b) market 2 is more business oriented than market 1,
then $V_t^1(x) \leq V_t^2(x)$ for every t and x.

Proof. Let $S_t^1(x)$ be the optimal strategy for (1) corresponding to $V_t^1(x)$, and $\bar{V}_t^2(x)$ the expected revenue resulting from strategy $S_t^1(x)$ under market 2. That is,

$$\bar{V}_t^2(x) \equiv \sum_{i \in S_t^1(x)} \lambda_t P_{it}^2(S_t^1(x))(r_i + \bar{V}_{t-1}^2(x-1)) + (\lambda_t P_{0t}^2(S_t^1(x)) + 1 - \lambda_t)\bar{V}_{t-1}^2(x).$$

It is obvious that $\bar{V}_t^2(x) \leq V_t^2(x)$. We prove the theorem by showing $V_t^1(x) \leq \bar{V}_t^2(x)$

inductively. For $t = 1$, we have

$$
\begin{aligned}
V_1^1(x) &= \sum_{i \in S_1^1(x)} \lambda_t P_{it}^1(S_1^1(x)) r_i \\
&= \left(1 - P_{0t}^1(S_1^1(x))\right) \sum_{i \in S_1^1(x)} \lambda_t \frac{P_{it}^1(S_1^1(x))}{1 - P_{0t}^1(S_1^1(x))} r_i \\
&\leq \left(1 - P_{0t}^2(S_1^1(x))\right) \sum_{i \in S_1^1(x)} \lambda_t \frac{P_{it}^2(S_1^1(x))}{1 - P_{0t}^2(S_1^1(x))} r_i \\
&= \sum_{i \in S_1^1(x)} \lambda_t P_{it}^2(S_1^1(x)) r_i \\
&= \bar{V}_1^2(x)
\end{aligned}
$$

where the inequality follows from $S_t^1(x) \in \mathcal{S}_{1t}$ and (a), and the denominator is non-zero because of (8), see (9). As an induction hypothesis, suppose that $V_{t-1}^1(x) \leq \bar{V}_{t-1}^2(x)$ for every x. The optimality of $S_t^1(x)$ implies $V_t^1(x) \geq V_{t-1}^1(x)$. Thus,

$$
\sum_{i \in S_t^1(x)} \lambda_t P_{it}^1\left(S_t^1(x)\right)\left(r_i + V_{t-1}^1(x-1)\right) + \lambda_t \left(P_{0t}^1\left(S_t^1(x)\right) - 1\right) V_{t-1}^1(x) \geq 0. \tag{10}
$$

Invoking (9) and (a), we multiply Equation (10) by

$$
\frac{1 - P_0^2(S_t^1(x))}{1 - P_0^1(S_t^1(x))} - 1 \geq 0.
$$

Then we obtain

$$
\frac{\sum_{i \in S_t^1(x)} \lambda_t P_{it}^1\left(S_t^1(x)\right)\left(r_i + V_{t-1}^1(x-1)\right)\left(1 - P_{0t}^2\left(S_t^1(x)\right)\right)}{1 - P_{0t}^1(S_t^1(x))}
$$
$$
- \sum_{i \in S_t^1(x)} \lambda_t P_{it}^1\left(S_t^1(x)\right)\left(r_i + V_{t-1}^1(x-1)\right)
$$
$$
- \lambda_t \left(\left(1 - P_{0t}^2(S_t^1(x))\right) - \left(1 - P_{0t}^1(S_t^1(x))\right)\right) V_{t-1}^1(x) \geq 0
$$

Adding this to the right hand side of

$$
V_t^1(x) = \sum_{i \in S_t^1(x)} \lambda_t P_{it}^1(S_t^1(x))(r_i + V_{t-1}^1(x-1)) + (1 + \lambda_t \left(P_{0t}^1(S_t^1(x)) - 1\right)) V_{t-1}^1(x),
$$

we obtain

$$
\begin{aligned}
V_t^1(x) &\leq \sum_{i \in S_t^1(x)} \lambda_t \left(1 - P_{0t}^2(S_t^1(x))\right) \frac{P_{it}^1(S_t^1(x))}{1 - P_{0t}^1(S_t^1(x))}(r_i + V_{t-1}^1(x-1)) \\
&\quad + (1 + \lambda_t(P_{0t}^2(S_t^1(x)) - 1)) V_{t-1}^1(x)
\end{aligned}
$$

Applying the business orientedness condition (b) and the induction hypothesis, one sees

$$
\begin{aligned}
V_t^1(x) &\leq \sum_{i \in S_t^1(x)} \lambda_t P_{it}^2(S_t^1(x))(r_i + V_{t-1}^1(x-1)) + (1 + \lambda_t \left(P_{0t}^2(S_t^1(x)) - 1\right)) V_{t-1}^1(x) \\
&\leq \sum_{i \in S_t^1(x)} \lambda_t P_{it}^2(S_t^1(x))(r_i + \bar{V}_{t-1}^2(x-1)) + (1 + \lambda_t \left(P_{0t}^2(S_t^1(x)) - 1\right)) \bar{V}_{t-1}^2(x) \\
&= \bar{V}_t^2(x).
\end{aligned}
$$

Hence, the statement holds true. ∎

We note that business orientedness is related to standard stochastic order relations. For r.v. Z and event A, we denote by $[Z \mid A]$ a r.v. having a distribution function equal to the conditional distribution of Z given A. The business orientedness condition is implied by the following conditional stochastic order:

$$[R^1(S) \mid R^1(S) > 0] \leq_{\text{st}} [R^2(S) \mid R^2(S) > 0] \quad \text{for all } S \in \mathcal{S}_{1t}, \tag{11}$$

or equivalently $\sum_{i=m}^n P_{it}^1(S)/\left(1 - P_{0t}^1(S)\right) \leq \sum_{i=m}^n P_{it}^2(S)/(1 - P_{0t}^2(S))$ for all $S \in \mathcal{S}_{1t}, t = 1, 2, \ldots, T$, and $m = 1, 2, \ldots, n$. A sufficient condition for (a) and (b) in Theorem 3 is phrased by means of likelihood ratio order. For discrete r.v.s X and Y with probability functions f and g, X is said to be *larger than Y in the likelihood ratio order*, denoted by $X \geq_{\text{lr}} Y$, if $f(t)/g(t)$ is non-decreasing on the relevant region. It is known that $X \geq_{\text{lr}} Y$ iff $[X \mid a \leq X \leq b] \geq_{\text{st}} [Y \mid a \leq Y \leq b]$ whenever $a < b$. Thus, the likelihood ratio order

$$R_t^1(S) \leq_{\text{lr}} R_t^2(S) \quad \text{for all } S \in \mathcal{S}_{1t} \text{ and } t \tag{12}$$

implies (a) and (b) in Theorem 3. We note that the usual stochastic order (2) is neither stronger nor weaker than (a) and (b) in Theorem 3. Also note that the conditional stochastic order (11) with (a) in Theorem 3 implies the usual stochastic order (2). For the definitions of various types of stochastic orders and their properties, readers are referred to Shaked and Shanthikumar [10].

It is worth mentioning that the conditional stochastic order (11) and the likelihood ratio order (12) are far stronger than the business orientedness condition and the sufficient condition in Theorem 3, respectively.

We next turn to the monotonicity of the expected revenue with respect to the effective market size. Technically speaking, this turns out to be a simple consequence of Theorem 3.

Corollary 4 *Let $V_t^j(x), j = 1, 2$, be the value function in (1) with λ_t replaced by λ_t^j. If $\lambda_t^1 \leq \lambda_t^2$, then $V_t^1(x) \leq V_t^2(x)$ for every t and x.*

Proof. Set $\lambda_t = 1$ and $P_{it}^j(S), S \subset N$, as

$$P_{0t}^j(S) = \lambda_t^j P_{0t}(S) + 1 - \lambda_t^j,$$
$$P_{it}^j(S) = \lambda_t^j P_{it}(S), \qquad i \neq 0.$$

Consider two markets $j = 1, 2$, with arrival probability $\lambda_t = 1$ and the choice probabilities $\left(P_{it}^j(S)\right)$. From the construction of λ_t and $P_{it}^j(S)$, the consumer behavior under market $\left(\lambda_t^j, (P_{it}(S))\right)$ is probabilistically identical to that under $\left(\lambda_t, \left(P_{it}^j(S)\right)\right)$ so that these two markets result in the same expected revenue. Furthermore, $\lambda_t^1 \leq \lambda_t^2$ implies $P_{0t}^2(S) \leq P_{0t}^1(S)$. Also, one sees $P_{it}^1(S)/\left(1 - P_{0t}^1(S)\right) = P_{it}^2(S)/(1 - P_{0t}^2(S))$ for every $i \neq 0$, which implies

$$\frac{\sum_{i=1}^n r_i P_{it}^1(S)}{1 - P_{0t}^1(S)} = \frac{\sum_{i=1}^n r_i P_{it}^2(S)}{1 - P_{0t}^2(S)}.$$

Thus, the market $(\lambda_t, (P_{it}^2(S)))$ is more business oriented than the market $(\lambda_t, (P_{it}^1(S)))$. The statement now follows from Theorem 3. ∎

Theorem 3 states that, all else alike, if the choice probabilities are more business oriented, then the Markov decision process will have a greater value function at all states and times. By combining Theorem 3 with Corollary 4, one can extend the applicability of Theorem 3.

Table 1: Choice probabilities $P_{it}(S)$

	S	$P_{0t}(S)$	$P_{1t}(S)$	$P_{2t}(S)$	$P_{3t}(S)$	$\frac{\sum r_i P_{it}(S)}{1-P_{0t}(S)}$
	$\varnothing$	1	0	0	0	—
	$\{1\}$	0.4	0.6	0	0	100.00
	$\{2\}$	0.5	0	0.5	0	150.00
$0 \leq t \leq 200$	$\{3\}$	0.6	0	0	0.4	250.00
	$\{1,2\}$	0.1	0.55	0.35	0	119.44
	$\{1,3\}$	0.2	0.55	0	0.25	146.88
	$\{2,3\}$	0.3	0	0.5	0.2	178.57
	$\{1,2,3\}$	0	0.5	0.35	0.15	140.00
	$\varnothing$	1	0	0	0	—
	$\{1\}$	0.4	0.6	0	0	100.00
	$\{2\}$	0.55	0	0.45	0	150.00
$201 \leq t \leq 1000$	$\{3\}$	0.7	0	0	0.3	250.00
	$\{1,2\}$	0.15	0.55	0.3	0	117.65
	$\{1,3\}$	0.3	0.55	0	0.15	132.14
	$\{2,3\}$	0.45	0	0.45	0.1	168.18
	$\{1,2,3\}$	0.15	0.5	0.3	0.05	126.47

3. Numerical Examples

In this section, we give some numerical examples to illustrate the results in the previous section. Let the set of fare products be $N = \{1,2,3\}$ with fares $r_1 = 100, r_2 = 150$ and $r_3 = 250$. First, we examine the impact of the market size, represented by the arrival probability $\lambda_t = \lambda$, on the expected revenue. The consumers' choice behavior is described by time-inhomogeneous probabilities $P_{it}(S)$ in Table 11. The last column of Table 11 provides the conditional expected revenue given that a consumer makes a purchase. The time horizon is divided into two time intervals $0 \leq t \leq 200$ and $201 \leq t \leq T = 1000$. There are more business oriented customers arriving per time period in the time interval $0 \leq t \leq 200$ (just before the departure) than in the interval $201 \leq t \leq 1000$, so that the probability of selling an expensive ticket is higher in the former interval than the latter. Figure 2 plots the expected revenue $V_t(x)$ as a function of arrival probability λ for $t = 1000$, and the remaining number of seats $x = 5, 10, 15$, and 20. For λ close to zero, the arriving customers are more precious than the remaining seats, so the optimal policy is to offer $S = \{1,2,3\}$ having the highest per-period revenue $\sum r_i P_{it}(S) = 140$ and 107.5 for $0 \leq t \leq 200$ and $201 \leq t \leq 1000$, respectively. Thus, the slope of $V_t(x)$ for $\lambda \approx 0$ is $140 \times 200 + 107.5 \times 800 = 114000$. As Corollary 4 states, the expected revenue $V_t(x)$ increases with the market size λ.

Second, we illustrate the monotonicity of the expected revenue with respect to the business orientedness. We consider three market profiles. The choice probabilities in market 1 are described in Table 1, and those in markets 2 and 3 are as in Tables 2 and 3. Note that market 1 (3) is least (most) business oriented and $P_{0t}^1(S) \geq P_{0t}^2(S) \geq P_{0t}^3(S)$ for all $S \subset N$, so that Theorem 3 is applicable. We set the seat capacity to be $C = 10$ and the arrival probabilities

$$\lambda_t^j = \begin{cases} 0.06, & 0 \leq t \leq 200 \\ 0.04, & 201 \leq t \leq T = 1000 \end{cases}$$

for $j = 1,2,3$. Figure 3 plots the expected revenue $V_t(x)$ for each market. One sees that
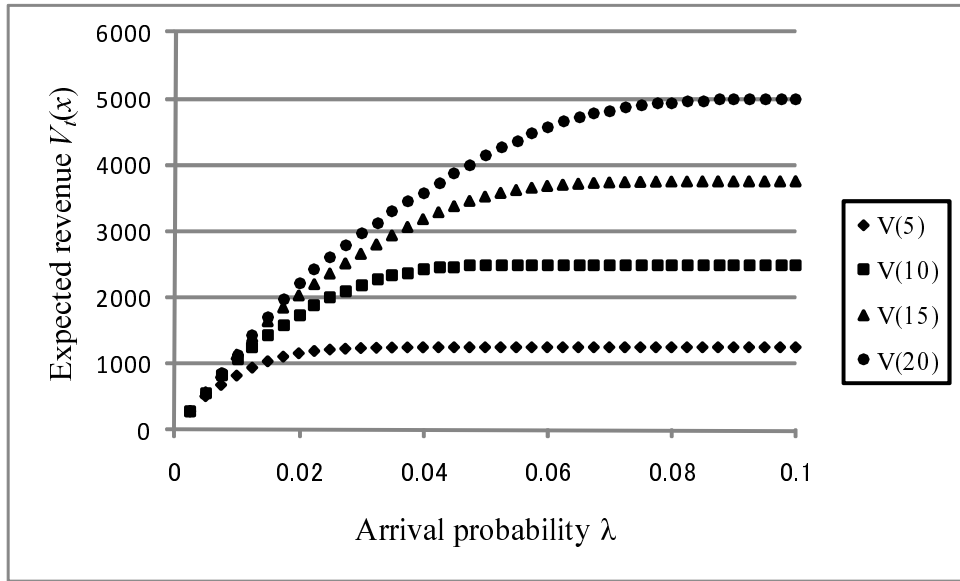
Figure 2: Arrival probability vs expected revenue

the firm earns the highest revenue from market 3 as predicted in Theorem 3. The value function $V_t(x)$ has a kink at $t = 200$ as one may expect.
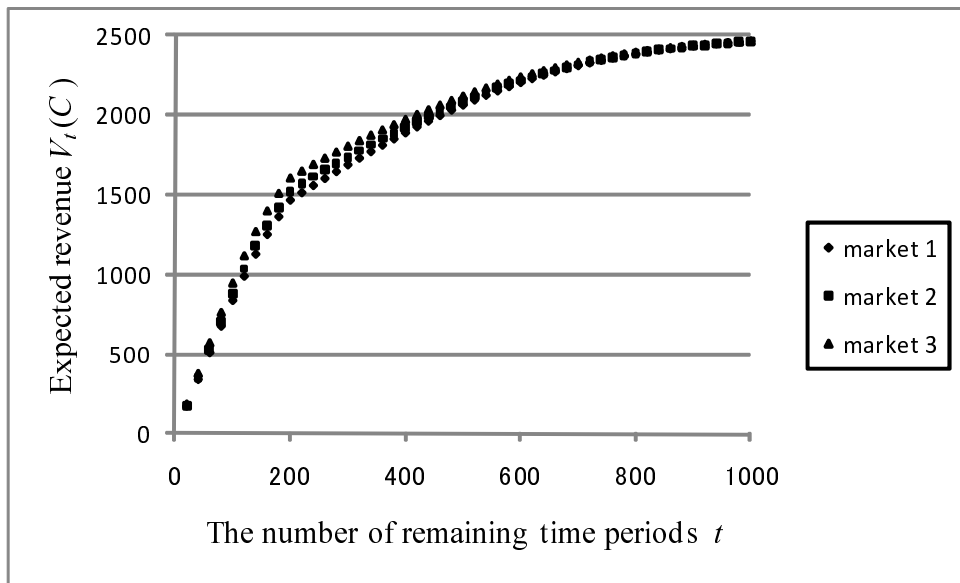

Figure 3: The number of remaining periods vs expected revenue

4. Conclusions

This paper studies the monotonicity properties of expected revenue in an RM problem with respect to consumer behavior under a discrete choice model. We give a simple and stylized example that violates a plausible monotonicity property, and provide a sufficient condition for the monotonicity. The sufficient condition is implied by some conditions involving well known stochastic order relations such as the conditional stochastic order and the likelihood ratio order. These conditions however are far stronger than the sufficient condition given in

Table 2: Choice probabilities $P_{it}^2(S)$ for market 2

	S	$P_{0t}^2(S)$	$P_{1t}^2(S)$	$P_{2t}^2(S)$	$P_{3t}^2(S)$	$\frac{\sum r_i P_{it}^2(S)}{1-P_{0t}^2(S)}$
	$\varnothing$	1	0	0	0	–
	$\{1\}$	0.4	0.6	0	0	100.00
	$\{2\}$	0.5	0	0.5	0	150.00
$0 \le t \le 200$	$\{3\}$	0.6	0	0	0.4	250.00
	$\{1,2\}$	0.1	0.55	0.35	0	119.44
	$\{1,3\}$	0.2	0.5	0	0.3	156.25
	$\{2,3\}$	0.3	0	0.45	0.25	185.71
	$\{1,2,3\}$	0	0.45	0.35	0.2	147.50
	$\varnothing$	1	0	0	0	–
	$\{1\}$	0.4	0.6	0	0	100.00
	$\{2\}$	0.55	0	0.45	0	150.00
$201 \le t \le 1000$	$\{3\}$	0.7	0	0	0.3	250.00
	$\{1,2\}$	0.15	0.55	0.3	0	117.65
	$\{1,3\}$	0.3	0.5	0	0.2	142.86
	$\{2,3\}$	0.45	0	0.4	0.15	177.27
	$\{1,2,3\}$	0.15	0.45	0.3	0.1	135.29

the main theorem of the paper.

Summarized below are the resulting managerial insights regarding promotional efforts that affect the consumer behavior.

1. It is likely that the promotional efforts raise the purchase probability of arriving customers. If the promotional efforts shift their choice from no-purchase to a low-fare product while it is difficult to segment business customers and more price-sensitive leisure customers, then the promotional efforts could be damaging to the firm. Related concepts of importance are per-period revenue vs per-seat revenue, as indicated in Example 1, and Example 2.

2. If the promotion improves the conditional choice behavior of customers given that a purchase is made, then it is beneficial to the firm, see Theorem 3.

3. The improvement of the arrival probability raises the expected revenue, see Corollary 4.

Acknowledgement

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Table 3: Choice probabilities $P_{it}^3(S)$ for market 3

	S	$P_{0t}^3(S)$	$P_{1t}^3(S)$	$P_{2t}^3(S)$	$P_{3t}^3(S)$	$\frac{\sum r_i P_{it}^3(S)}{1-P_{0t}^3(S)}$
	$\varnothing$	1	0	0	0	—
	$\{1\}$	0.4	0.6	0	0	100.00
	$\{2\}$	0.5	0	0.5	0	150.00
$0 \le t \le 200$	$\{3\}$	0.6	0	0	0.4	250.00
	$\{1,2\}$	0.1	0.5	0.4	0	122.22
	$\{1,3\}$	0.2	0.45	0	0.35	165.63
	$\{2,3\}$	0.3	0	0.4	0.3	192.86
	$\{1,2,3\}$	0	0.4	0.3	0.3	160.00
	$\varnothing$	1	0	0	0	—
	$\{1\}$	0.4	0.6	0	0	100.00
	$\{2\}$	0.55	0	0.45	0	150.00
$201 \le t \le 1000$	$\{3\}$	0.7	0	0	0.3	250.00
	$\{1,2\}$	0.15	0.5	0.35	0	120.59
	$\{1,3\}$	0.3	0.45	0	0.25	153.57
	$\{2,3\}$	0.45	0	0.35	0.2	186.36
	$\{1,2,3\}$	0.15	0.4	0.25	0.2	150.00

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Appendix

To prove the main result of the paper, we need Property (9), which holds true under the technical assumption (8). When the probabilities $P_{it}(S)$ and λ_t do not depend on time t, however, Assumption (8) is not necessary to show the main result of the paper, which we prove in this appendix. Hereafter we drop the index t from $P_{it}(S)$, λ_t and $\mathcal{S}_{1t}$.

Lemma 5 *(a) If $\lambda < 1$, the optimal solution $(S_t(x))$ for (1) satisfies $P_0(S_t(x)) < 1$ for all t and $x > 0$. (b) The expected revenue $V_t(x)$ in (1) is continuous with respect to λ.*

Proof. Suppose that $P_0(S_\ell(y)) = 1$ for some ℓ and y. Consider a problem with $T = \ell$ and $C = y$. Let $(S'_t(\cdot))_{t=1}^\ell$ be a new strategy defined by

$$S'_t(x) = \begin{cases} S_{t-1}(x) & \text{if } 2 \leq t \leq \ell, \\ R & \text{if } t = 1 \end{cases}$$

where $R \subset N$ is an offer set generating a positive expected revenue, i.e., $\sum_{i \in R} P_i(R) r_i > 0$. We note that the offer set $S_\ell(y)$ generates no revenue in period ℓ. Since the customers' behavior $(P_i(S))$ is temporally homogeneous, the expected revenue generated from $(S_t(\cdot))$ in periods $t = 1, 2, \cdots, \ell-1$, is equal to that generated from $(S'_t(\cdot))$ in periods $t = 2, 3, \cdots, \ell$. Furthermore, the condition $\lambda < 1$ implies that under any strategy the number of remaining seats at period 1 is positive with some positive probability. Thus, the offer set R yields a strictly positive expected revenue in period 1. Hence, the expected revenue arising from $(S'_t(\cdot))_{t=1}^\ell$ is larger than that arising from $(S_t(\cdot))_{t=1}^\ell$, which is a contradiction. Part (b) can be proved by induction with respect to t. ∎

We now see that when the probabilities are homogeneous in time, the main result of the paper holds true without the technical assumption (8).

Proposition 6 *In the absence of (8), if*
(a) $P_0^1(S) \geq P_0^2(S)$, for every $S \in \mathcal{S}_1$, and
(b) market 2 is more business oriented than market 1,
then $V_t^1(x) \leq V_t^2(x)$ for every t and x.

Proof. Note that $\lambda < 1$ implies Property (9) from Lemma 5 (a). Thus, following the argument identical to the proof of Theorem 3, we see that if $\lambda < 1$, (a) and (b) hold true. This together with Lemma 5 (b) implies the inequalities in (a) and (b) are valid even when $\lambda = 1$. ∎

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