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AN ALGEBRAIC CRITERION FOR STRONG STABILITY OF STATIONARY SOLUTIONS OF NONLINEAR PROGRAMS WITH A FINITE NUMBER OF EQUALITY CONSTRAINTS AND AN ABSTRACT CONVEX CONSTRAINT

Toshihiro Matsumoto
Teikyo University of Science & Technology

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Abstract This paper addresses strong stability, in the sense of Kojima, of stationary solutions of nonlinear programs **Pro** with a finite number of equality constraints and one abstract convex constraint defined by the closed convex set K. It intends to extend results of our former paper that treated nonlinear programs **Pro** in a special case that K is the set of nonnegative symmetric matrices S_+. Firstly, it deduces properties of eigenvalues of the Euclidean projector on K. Secondly, it extends the results to programs **Pro** in case that the convex set K satisfies the regular boundary condition that S_+ always satisfies.

Keywords: Nonlinear programming, stationary solution, strong stability, stationary index, Lipschitz continuous map, nonsmooth analysis

1. Introduction

In cited reference [12] Kojima introduced, for the first time, the concept of strong stability of stationary solution of nonlinear programs which have a finite number of equality constraints and finite inequality constraints of C^2 class satisfying the so called Mangasarian-Fromovitz condition; he also gave an algebraic condition which is necessary and sufficient for the stability by means of Jacobian and Hessian matrices. Since then, strong stability for programs of this type has been intensively studied and it is known that various kinds of regularities are equivalent to strong stability for programs of this type [10],[11].

In this paper we investigate strong stability of stationary solution of NPAC, i.e., the following nonlinear programs with an abstract constraint $x \in K$

$$\mathbf{Pro}(f,h) \qquad \left\| \begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in K \\ & h_i(\boldsymbol{x}) = 0 \quad (i = 1, \cdots, \ell) \end{array} \right. \left. \vphantom{\begin{array}{l} a \\ a \\ a \end{array}} \right\},$$

where K is a closed convex set in $\boldsymbol{R}^n$ and $f,\ h_i\ (i = 1, \cdots, \ell)$ are C^2 functions on $\boldsymbol{R}^n$. Then, $\boldsymbol{x} \in K$ is called a stationary solution of program $\mathbf{Pro}(f,h)$ if $-D_{\boldsymbol{x}} f(\boldsymbol{x}) \in \boldsymbol{R} D_{\boldsymbol{x}} h(\boldsymbol{x}) + \sigma(\boldsymbol{x})^T$ holds. Here, $\boldsymbol{R} D_{\boldsymbol{x}} h(\boldsymbol{x})$ denotes the affine space spanned by $\{D_{\boldsymbol{x}} h_i(\boldsymbol{x}) : i = 1, \cdots, \ell\}$, and $\sigma(\boldsymbol{x}) = \{\boldsymbol{v} \in \boldsymbol{R}^n : \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{v} \rangle \leq 0\ (\forall \boldsymbol{y} \in K)\}$ is the normal cone of K at $\boldsymbol{x}$, and $\sigma(\boldsymbol{x})^T = \{\boldsymbol{w} : \boldsymbol{w}^T \in \sigma(\boldsymbol{x})\}$. The stationary solution $\boldsymbol{x}$ is defined to be strongly stable if there exist $\delta > 0$ such that, for any small perturbation (f', h') of (f, h), there exists a unique stationary solution $\boldsymbol{x}(f', h')$ of $\mathbf{Pro}(f', h')$ satisfying $\|\boldsymbol{x}(f', h') - \boldsymbol{x}\| \leq \delta$, and the correspondence $(f', h') \mapsto \boldsymbol{x}(f', h')$ is continuous at (f, h).

Similar programs are treated by Bonnans and Shapiro [2]. Given Banach spaces X, Y and a closed convex subset $\mathcal{K} \subset Y$ and a map $G : X \to Y$, they treat a nonlinear convex program in the form

$$\left\| \begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & G(\boldsymbol{x}) \in \mathcal{K} \end{array} \right\}$$

and show that the second growth condition is equivalent to strong stability when local minimum solutions are considered. However, no criterion of strong stability has yet been found for these programs in general cases.

Matsumoto [17] treated and investigated strong stability of stationary solution of the above program $\mathbf{Pro}(f, h)$ in case that K is the set $S_+(n)$ of $n \times n$ positive semidefinate real symmetric matrices and $f,\ h_i\ (i = 1, \cdots, \ell)$ are functions on the set $S(n)$ of $n \times n$ real symmetric matrices. In [17], by means of Jacobians and Hessians of f and h an algebraic condition equivalent to strong stability for such programs to which he referred as NSDP was deduced under both the linear independence constraint qualification (LICQ) condition defined to those programs and the infiltrative orientation condition. This paper intends to expand the results of [17] for NSDP to that for NPAC.

In section 2,

- we define stationary solutions and strong stability, and we prepare a series of elementary results and facts, and
- under LICQ condition defined to programs $\mathbf{Pro}(f, h)$, we show one theorem that gives an necessary and sufficient condition for strong stability by virtue of one-to-one maps.

In section 3,

- we calculate the generalized Jacobians of $\boldsymbol{x}^+$ and $\boldsymbol{x}^-$, where $\boldsymbol{x}^+$ is the orthogonal projection of $\boldsymbol{x}$ in K, and $\boldsymbol{x}^-$ are defined by $\boldsymbol{x}^- = \boldsymbol{x} - \boldsymbol{x}^+$.

In section 4,

- we derive an algebraic criterion for strong stability under LICQ condition defined to programs $\mathbf{Pro}(f, h)$ and the regular boundary condition of K (that is always satisfied for $K = S_+(n)$) and the infiltrative orientation condition, and
- we define the stationary index of strongly stable stationary solutions after Kojima.

2. Preliminaries

In this section, we define strong stability in the sense of Kojima and we prepare a series of elementary results and facts. For their preparation, we list notations used in this paper:

$\boldsymbol{R}$	:	the field of all real numbers,
$\boldsymbol{R}^n$	:	the space of n dimensional real column vectors,
$S(n)$	:	the set of all $n \times n$ symmetric real matrices,
$S_+(n)$	:	the set of all $n \times n$ positive semidefinite symmetric real matrices,
$\langle \boldsymbol{x}, \boldsymbol{y} \rangle$	:	the standard inner product of $\boldsymbol{x},\ \boldsymbol{y} \in \boldsymbol{R}^n$,
$\boldsymbol{I}_A$	:	the identity map on A for any set A,
$\boldsymbol{I}_r$	:	the $r \times r$ identity matrix, i.e., the identity map on $\boldsymbol{R}^r$,
$\boldsymbol{I}$	:	the identity matrix of an appropriate size,
$\boldsymbol{O}_r$	:	the $r \times r$ zero matrix,
$\boldsymbol{O}$	:	the zero matrix of an appropriate size,

$$
\begin{array}{rcl}
\boldsymbol{X}^T & : & \text{the transposition of the matrix } \boldsymbol{X}, \\
A^T & = & \{\boldsymbol{X}^T : \boldsymbol{X} \in A\} \text{ for a set } A \text{ of matrices}, \\
sgn\ t & = & \begin{cases} 1 & (t > 0) \\ 0 & (t = 0) \\ -1 & (t < 0) \end{cases}, \\
A \setminus B & = & \{x \in A : x \notin B\}, \\
\mathrm{conv}(A) & : & \text{the convex hull of a subset } A \text{ of a vector space } V, \text{ i.e.,} \\
 & & \left\{ \sum_{k=1}^{N} t_k \boldsymbol{a}_k : N = 1, 2, \cdots \text{ and } \boldsymbol{a}_k \in A \text{ and } t_k \geq 0, (k = 1, 2, \cdots, N)\right., \\
 & & \quad \left. \text{with } \sum_{k=1}^{N} t_k = 1 \right\}, \\
\mathrm{int}(A) & : & \text{the interior of a subset } A \text{ of a topological space } X, \\
ex(K) & : & \text{the set of extremal points of a convex set } K, \\
\|\boldsymbol{x}\| & = & \sqrt{\sum_{i=1}^{n} |x_i|^2} \text{ for } \boldsymbol{x} = (x_1, \cdots, x_n) \in \boldsymbol{R}^n, \text{ i.e., the Euclidean norm of } \boldsymbol{R}^n, \\
d(\boldsymbol{x}, K) & = & \inf\{\|\boldsymbol{x} - \boldsymbol{y}\| : \boldsymbol{y} \in K\}, \\
\mathcal{F} & = & \{(f, h) = (f, h_1, \cdots, h_\ell) : f, h_1, \cdots, h_\ell \in C^2(\boldsymbol{R}^n)\}, \\
 & & \text{where } C^2(\boldsymbol{R}^n) \text{ is the set of all functions on } \boldsymbol{R}^n \text{ of } C^2 \text{ class}, \\
F|A & : & \text{the restriction of a map } F \text{ to a subset } A \text{ of the domain where } F \text{ is defined},
\end{array}
$$

The character K denotes a closed convex subset of $\boldsymbol{R}^n$ that is fixed throughout this paper and $\sigma(\boldsymbol{x})$ denotes its normal cone at $\boldsymbol{x} \in K$, i.e., $\sigma(\boldsymbol{x}) = \{\boldsymbol{v} \in \boldsymbol{R}^n : \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{v} \rangle \leq 0\ (\forall \boldsymbol{y} \in K)\}$. The next fact is well known as stated in the inequality (1.8) of [21], but we give its simple direct proof.

Fact 2.1 *For a closed convex subset K of $\boldsymbol{R}^n$, the following (i) and (ii) hold.*

(i) For $\boldsymbol{x} \in \boldsymbol{R}^n$, there exists a unique $\boldsymbol{x}^+ \in K$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}^+\| = d(\boldsymbol{x}, K)$.

(ii) $\|\boldsymbol{x}^+ - \boldsymbol{y}^+\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$ for any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^n$.

Proof. We omit the proof of (i) since it can be inferred. We will prove (ii) only. Define $\xi, \zeta, H_{\boldsymbol{x}}, H_{\boldsymbol{y}}$ as

$$
\begin{cases}
\xi & = \boldsymbol{x} - \boldsymbol{x}^+, \\
\zeta & = \boldsymbol{y} - \boldsymbol{y}^+, \\
H_{\boldsymbol{x}} & = \{\boldsymbol{v} \in \boldsymbol{R}^n : \langle \boldsymbol{v} - \boldsymbol{x}^+, \xi \rangle \leq 0\}, \\
H_{\boldsymbol{y}} & = \{\boldsymbol{v} \in \boldsymbol{R}^n : \langle \boldsymbol{v} - \boldsymbol{y}^+, \zeta \rangle \leq 0\}.
\end{cases}
$$

Then, from the definition of $\boldsymbol{x}^+$ and $\boldsymbol{y}^+$ it follows $K \subset H_{\boldsymbol{x}} \bigcap H_{\boldsymbol{y}}$. Therefore,

$$
\begin{cases}
\langle \boldsymbol{y}^+ - \boldsymbol{x}^+, \xi \rangle \leq 0, \\
\langle \boldsymbol{x}^+ - \boldsymbol{y}^+, \zeta \rangle \leq 0.
\end{cases}
$$

Let $F(t) = \|(\boldsymbol{x}^+ + t\xi) - (\boldsymbol{y}^+ + t\zeta)\|^2 = \langle (\boldsymbol{x}^+ + t\xi) - (\boldsymbol{y}^+ + t\zeta), (\boldsymbol{x}^+ + t\xi) - (\boldsymbol{y}^+ + t\zeta) \rangle$ for

$t \in \boldsymbol{R}$. $F(t)$ is a polynomial in t with $deg\ F(t) \leq 2$ and

$$\begin{cases} F(1) &= \|\boldsymbol{x}-\boldsymbol{y}\|^2, \\ F(0) &= \|\boldsymbol{x}^+-\boldsymbol{y}^+\|^2, \\ \dfrac{dF(0)}{dt} &= 2\langle \boldsymbol{x}^+-\boldsymbol{y}^+, \xi-\zeta\rangle \\ &= 2\langle \boldsymbol{x}^+-\boldsymbol{y}^+, \xi\rangle - 2\langle \boldsymbol{x}^+-\boldsymbol{y}^+, \zeta\rangle \geq 0, \\ \dfrac{d^2F(0)}{dt^2} &= 2\|\xi-\zeta\|^2 \geq 0. \end{cases}$$

From these properties, it follows easily that $\|\boldsymbol{x}^+-\boldsymbol{y}^+\| = \sqrt{F(0)} \leq \sqrt{F(1)} = \|\boldsymbol{x}-\boldsymbol{y}\|$. □

$\boldsymbol{x}^+$ stated in Fact 2.1 is actually the orthogonally projected element of $\boldsymbol{x}$ in K. $\boldsymbol{x}^-$ denotes $\boldsymbol{x}-\boldsymbol{x}^+$, i.e., $\boldsymbol{x}^- = \boldsymbol{x}-\boldsymbol{x}^+$. It is readily inferred that both $\boldsymbol{x}^- \in \sigma(\boldsymbol{x}^+)$ and $d(\boldsymbol{x}, K) = \|\boldsymbol{x}^-\|$ hold.

Definition 2.2 Define $\rho^+ : \boldsymbol{R}^n \to \boldsymbol{R}^n$ and $\rho^- : \boldsymbol{R}^n \to \boldsymbol{R}^n$ by $\rho^+(\boldsymbol{x}) = \boldsymbol{x}^+$ and $\rho^-(\boldsymbol{x}) = \boldsymbol{x}^-$.

Definition 2.3 Let $\mathcal{H} = \{(\boldsymbol{y}, \boldsymbol{v}) \in \boldsymbol{R}^n \times \boldsymbol{R}^n : \boldsymbol{y} \in K \text{ and } \boldsymbol{v} \in \sigma(\boldsymbol{y})\}$. We define $\eta : \mathcal{H} \to \boldsymbol{R}^n$, $\rho : \boldsymbol{R}^n \to \mathcal{H}$ by $\eta(\boldsymbol{y}, \boldsymbol{v}) = \boldsymbol{y}+\boldsymbol{v}$ and $\rho(\boldsymbol{x}) = (\rho^+(\boldsymbol{x}), \rho^-(\boldsymbol{x})) = (\boldsymbol{x}^+, \boldsymbol{x}^-)$.

Since both η and ρ are continuous and $\rho^{-1} = \eta$, $\mathcal{H}$ and $\boldsymbol{R}^n$ are homeomorphic by ρ.

Definition 2.4 Let $(f, h) \in \mathcal{F}$. $D_{\boldsymbol{x}}f(\boldsymbol{x})$ and $D_{\boldsymbol{x}}h(\boldsymbol{x})$ denote respectively the Jacobians of $f(\boldsymbol{x})$ and $h(\boldsymbol{x})$. $\boldsymbol{R}D_{\boldsymbol{x}}h(\boldsymbol{x}) = \sum_{i=1}^{\ell} \boldsymbol{R}D_{\boldsymbol{x}}h_i(\boldsymbol{x})$ denotes the affine space spanned by $\{D_{\boldsymbol{x}}h_i(\boldsymbol{x}) : i = 1, \cdots, \ell\}$. Then $\bar{\boldsymbol{x}} \in K$ is called a stationary solution of program $\mathbf{Pro}(f, h)$ if $-D_{\boldsymbol{x}}f(\bar{\boldsymbol{x}}) \in \boldsymbol{R}D_{\boldsymbol{x}}h(\bar{\boldsymbol{x}}) + \sigma(\bar{\boldsymbol{x}})^T$ holds. Also, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{v}}, \bar{\lambda}) \in \mathcal{H} \times \boldsymbol{R}^\ell$ is called a stationary point of program $\mathbf{Pro}(f, h)$ if $D_{\boldsymbol{x}}f(\bar{\boldsymbol{x}}) + \sum_{i=1}^{\ell} \bar{\lambda}_i D_{\boldsymbol{x}}h_i(\bar{\boldsymbol{x}}) + \bar{\boldsymbol{v}}^T = \mathbf{0}$ holds. Identifying $\mathcal{H}$ with $\boldsymbol{R}^n$, $(\bar{\boldsymbol{x}}, \bar{\lambda}) \in \boldsymbol{R}^{n+\ell}$ is also called a stationary point of program $\mathbf{Pro}(f, h)$ if $(\rho(\bar{\boldsymbol{x}}), \bar{\lambda})$ is a stationary point of program $\mathbf{Pro}(f, h)$, i.e., $D_{\boldsymbol{x}}f(\bar{\boldsymbol{x}}^+) + \sum_{i=1}^{\ell} \bar{\lambda}_i D_{\boldsymbol{x}}h_i(\bar{\boldsymbol{x}}^+) + (\bar{\boldsymbol{x}}^-)^T = \mathbf{0}$.

Following are some notations for the remainder of this paper. For $(f, h) \in \mathcal{F}$, we define $L(\cdot, \cdot; f, h) : \boldsymbol{R}^{n+\ell} \to \boldsymbol{R}$, $\psi(\cdot, \cdot; f, h) : \boldsymbol{R}^{n+\ell} \to \boldsymbol{R}^{n+\ell}$, $\Omega \subset \boldsymbol{R}^{n+\ell} \times \mathcal{F}$, $\Xi \subset \boldsymbol{R}^n \times \mathcal{F}$ and $\chi : \Omega \to \Xi$ as follows.

$$\begin{aligned} L(\boldsymbol{x}, \lambda; f, h) &= f(\boldsymbol{x}) + \sum_{i=1}^{\ell} \lambda_i h_i(\boldsymbol{x}), \\ \psi(\boldsymbol{x}, \lambda; f, h) &= (D_{\boldsymbol{x}}L(\boldsymbol{x}^+, \lambda; f, h) + (\boldsymbol{x}^-)^T, D_\lambda L(\boldsymbol{x}^+, \lambda; f, h)) \\ &= (D_{\boldsymbol{x}}f(\boldsymbol{x}^+) + \sum_{i=1}^{\ell} \lambda_i D_{\boldsymbol{x}}h_i(\boldsymbol{x}^+) + (\boldsymbol{x}^-)^T, h(\boldsymbol{x}^+)), \\ \Omega &= \{(\boldsymbol{x}, \lambda, f, h) \in \boldsymbol{R}^{n+\ell} \times \mathcal{F} : (\boldsymbol{x}, \lambda) \text{ be a stationary point of } \mathbf{Pro}(f, h)\} \\ &= \{(\boldsymbol{x}, \lambda, f, h) \in \boldsymbol{R}^{n+\ell} \times \mathcal{F} : \psi(\boldsymbol{x}, \lambda, f, h) = \mathbf{0}\}, \\ &\quad \text{where } \mathbf{0} \text{ denotes the zero vector,} \\ \Xi &= \{(\boldsymbol{x}, f, h) \in \boldsymbol{R}^n \times \mathcal{F} : \boldsymbol{x} \text{ is a stationary solution of } \mathbf{Pro}(f, h)\}, \\ \chi(\boldsymbol{x}, \lambda, f, h) &= (\boldsymbol{x}^+, f, h), \text{ i.e., } \chi : \Omega \to \Xi \text{ is a natural projection.} \end{aligned}$$

Let $\mathcal{M}$ be a C^1 manifold and $\mathcal{N} \subset \mathcal{M}$ be a C^1-submanifold of $\mathcal{M}$ and $\bar{\boldsymbol{x}} \in \mathcal{N}$ and U be a neighborhood of $\bar{\boldsymbol{x}}$ in $\mathcal{M}$. Consider the coordinate system $\boldsymbol{x}$ of $\mathcal{M}$ around $\bar{\boldsymbol{x}}$ and the coordinate system $\boldsymbol{y}$ of $\mathcal{N}$ around $\bar{\boldsymbol{x}}$. Then, the natural immersion $\mathcal{N} \subset \mathcal{M}$ is represented by a unique C^1-map $\boldsymbol{x} = \nu(\boldsymbol{y})$. Let $\boldsymbol{f} : U \to \boldsymbol{R}^n$ be a C^2 map. $D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x})$ and $D^2_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x})$ denote respectively the Jacobian and Hessian of $\boldsymbol{f}(\boldsymbol{x})$. We also use the notation $D_{\boldsymbol{y}}\boldsymbol{f}(\bar{\boldsymbol{x}})$, whose meaning we define to be $D_{\boldsymbol{y}}\boldsymbol{f}(\bar{\boldsymbol{x}}) = D_{\boldsymbol{x}}\boldsymbol{f}(\bar{\boldsymbol{x}})D_{\boldsymbol{y}}\nu(\bar{\boldsymbol{x}})$. For $f \in C^2(\boldsymbol{R}^n)$ and a subset $B \subset \boldsymbol{R}^n$, a norm $\|f\|_B$ is defined by

$$\|f\|_B = \sup\{|f(\boldsymbol{x})|, \|D_{\boldsymbol{x}}f(\boldsymbol{x})\|, \|D^2_{\boldsymbol{x}}f(\boldsymbol{x})\| : \boldsymbol{x} \in B\}.$$

For $(f, h) \in \mathcal{F}$ and a subset $B \subset S(n)$, a norm $\|\cdot\|_B$ is defined by

$$\|(f, h)\|_B = \max\{\|f(\boldsymbol{x})\|_B\ ,\ \|h_i(\boldsymbol{x})\|_B : 1 \leq i \leq \ell\}.$$

We denote by $\mathcal{F}_B$ the space $\mathcal{F}$ with $\|\cdot\|_B$-topology.

In general, given a normed vector space V with its norm $\|\cdot\|$, we define a closed ball and an open ball by $B_\delta(x) = \{y \in V : \|y - x\| \leq \delta\}$ and $\mathrm{int}(B_\delta(x)) = \{y \in V : \|y - x\| < \delta\}$ for $x \in V$ and a positive real number $\delta > 0$.

Definition 2.5 Let $\bar{\boldsymbol{x}} \in \boldsymbol{R}^n$ be a stationary solution of $\mathbf{Pro}(\bar{f}, \bar{h})$. $\bar{\boldsymbol{x}}$ is said to be strongly stable if there exist neighborhoods $U = B_\delta(\bar{\boldsymbol{x}})$ of $\bar{\boldsymbol{x}}$ in $\boldsymbol{R}^n$ and V of $(\bar{f}, \bar{h})$ in $\mathcal{F}_U$ such that the natural projection $pr : \Xi \bigcap (U \times V) \to V$ is bijective and $pr^{-1} : V \to \Xi \bigcap (U \times V)$ is continuous at $(\bar{f}, \bar{h})$.

The next condition is called the Mangasarian-Fromovitz condition, to which we refer as MF condition 2.6.

Condition 2.6

(i) For any $\boldsymbol{x} \in \boldsymbol{R}^n$, $D_{\boldsymbol{x}}h_i(\boldsymbol{x})$ $(1 \leq i \leq \ell)$ are linearly independent.
(ii) For any $\boldsymbol{x} \in K$ with $h(\boldsymbol{x}) = \mathbf{0}$, $\boldsymbol{R}D_{\boldsymbol{x}}h(\boldsymbol{x}) \bigcap \sigma(\boldsymbol{x})^T = \{\mathbf{0}\}$ holds.

In case of $K = S_+(n) \subset S(n) \simeq \boldsymbol{R}^{\frac{n(n+1)}{2}}$, it can be deduced from Lemma 1.2 of the cited reference [16] that $\sigma(\boldsymbol{x})$ is a pointed cone. Therefore, the above MF condition 2.6 is equivalent to the MF condition of [16], and the following proposition can be proved exactly same as Proposition 2.13 of [16].

Proposition 2.7 *Let $\boldsymbol{x} \in K$ be a strongly stable stationary solution of $\mathbf{Pro}(f, h)$. Suppose that neighborhoods $U = B_\delta(\boldsymbol{x})$ of $\boldsymbol{x}$ in $\boldsymbol{R}^n$ and V of (f, h) in $\mathcal{F}_U$ satisfy the condition that the natural projection $pr : \Xi \bigcap (U \times V) \to V$ is bijective and $pr^{-1} : V \to \Xi \bigcap (U \times V)$ is continuous at (f, h). Then pr is a homeomorphism under MF condition 2.6.*

We refer to the next condition as LICQ condition 2.8 since, under the condition, each stationary solution corresponds to a unique stationary point and this condition takes a role in program $\mathbf{Pro}(f, h)$ just as LICQ condition does in the setting of cited reference [12].

Condition 2.8

(i) For any $\boldsymbol{x} \in \boldsymbol{R}^n$, $D_{\boldsymbol{x}}h_i(\boldsymbol{x})$ $(1 \leq i \leq \ell)$ are linearly independent.
(ii) For any $\boldsymbol{x} \in K$ with $h(\boldsymbol{x}) = \mathbf{0}$, $\boldsymbol{R}D_{\boldsymbol{x}}h(\boldsymbol{x}) \bigcap \boldsymbol{R}\sigma(\boldsymbol{x})^T = \{\mathbf{0}\}$.

It is readily inferred that LICQ condition 2.8 implies MF condition 2.6. One can prove the next proposition exactly same as Proposition 2.17 of cited reference [16].

Proposition 2.9 *Under LICQ condition 2.8, for any subset $U \subset \boldsymbol{R}^n$, $\chi : \Omega \bigcap ((\rho^+)^{-1}(U) \times \boldsymbol{R}^\ell \times \mathcal{F}_U) \to \Xi \bigcap (U \times \mathcal{F}_U)$ is a homeomorphism.*

Since any stationary solution $\boldsymbol{x}^+$ corresponds to a unique stationary point $(\boldsymbol{x}, \lambda)$ under LICQ condition 2.8, we can make the following definition.

Definition 2.10 LICQ condition 2.8 we refer to $(\boldsymbol{x}, \lambda)$ as a strongly stable stationary point of $\mathbf{Pro}(f, h)$ if and only if $\boldsymbol{x}^+$ is a strongly stable stationary solution of $\mathbf{Pro}(f, h)$.

Remark 2.11 From Propositions 2.7 and 2.9, we can restate strong stability as follows. Let $(\bar{\boldsymbol{x}}, \bar{\lambda}) \in \boldsymbol{R}^n \times \boldsymbol{R}^\ell$ be a stationary point of $\mathbf{Pro}(\bar{f}, \bar{h})$. Under LICQ condition 2.8, $(\bar{\boldsymbol{x}}, \bar{\lambda})$ is strongly stable if and only if there exist a neighborhood $U = B_{\delta^*}(\bar{\boldsymbol{x}}^+)$ of $\bar{\boldsymbol{x}}^+$ in $\boldsymbol{R}^n$ and V of $(\bar{f}, \bar{h})$ in $\mathcal{F}_U$ such that the natural projection $\pi : \Omega \bigcap ((\rho^+)^{-1}(U) \times \boldsymbol{R}^\ell \times V) \to V$ is a homeomorphism.

We assume LICQ condition 2.8 throughout in the remainder of this document. Exactly same as the proof of Theorem 3.4 of cited reference [17], we can prove the following theorem that gives an equivalent condition for strong stability under LICQ condition 2.8.

Theorem 2.12 *Suppose that LICQ condition 2.8 holds. Let* $(\bar{f}, \bar{h}) \in \mathcal{F}$ *and* $(\bar{\boldsymbol{x}}, \bar{\lambda}) \in \boldsymbol{R}^{n+\ell}$ *be a stationary point of* $\mathbf{Pro}(\bar{f}, \bar{h})$*. Then the following (i) and (ii) are equivalent.*

(i) $(\bar{\boldsymbol{x}}, \bar{\lambda})$ *is strongly stable.*

(ii) There exist neighborhoods $U = B_{\delta^*}(\bar{\boldsymbol{x}}^+)$ *of* $\bar{\boldsymbol{x}}^+$ *in* $\boldsymbol{R}^n$ *and* $W = B_\delta((\bar{\boldsymbol{x}}, \bar{\lambda}))$ *of* $(\bar{\boldsymbol{x}}, \bar{\lambda})$ *with* $W \subset (\rho^+)^{-1}(U) \times \boldsymbol{R}^\ell$ *satisfying the following two conditions.*

  (a) $\bar{\boldsymbol{x}}^+$ *is a unique stationary solution in* U *for* $\mathbf{Pro}(\bar{f}, \bar{h})$*.*

  (b) $V = \{(f, h) \in \mathcal{F} : \psi(\cdot, \cdot; f, h)$ *is one-to-one on* $W\}$ *is a neighborhood of* $(\bar{f}, \bar{h})$ *in* $\mathcal{F}_U$*.*

3. Properties of Generalized Jacobian of $\rho(\boldsymbol{x})$

Next we investigate the structure of the generalized Jacobian $\partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$ of $\rho(\boldsymbol{x}) = (\rho^+(\boldsymbol{x}), \rho^-(\boldsymbol{x}))$ which we consider $\rho : \boldsymbol{R}^n \to \boldsymbol{R}^n \times \boldsymbol{R}^n$.

Definition 3.1 Let V_1 and V_2 be normed vector spaces with their norms denoted by $\|\cdot\|$ and U be an open subset of V_1. Then, a map $\boldsymbol{f} : U \to V_2$ is called Lipschitz continuous with its modulus M if there exists a constant M such that $\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \le M\|\boldsymbol{x} - \boldsymbol{y}\|$ for any $\boldsymbol{x},\ \boldsymbol{y} \in U$.

Since $\rho^+(\boldsymbol{x}) = \boldsymbol{x}^+$ and $\rho^-(\boldsymbol{x}) = \boldsymbol{x}^-$, Fact 2.1 shows that $\rho(\boldsymbol{x}) = (\rho^+(\boldsymbol{x}), \rho^-(\boldsymbol{x}))$ is Lipschitz continuous. Before we state the next definition, we remark that any Lipschitz continuous map is differentiable almost everywhere in the sense of Lebesgue measure by Rademacher's Theorem ([5]).

Definition 3.2 ([3],[9]) Let U be an open set of $\boldsymbol{R}^n$ and $\boldsymbol{f}$ be a Lipschitz continuous map from U to $\boldsymbol{R}^m$. Let $E_{\boldsymbol{f}}$ be the set of all points $\boldsymbol{x} \in U$ where the Jacobian $D_{\boldsymbol{x}}\boldsymbol{f}$ exists. Then, for $\bar{\boldsymbol{x}} \in U$, the generalized Jacobian $\partial_{\boldsymbol{x}}\boldsymbol{f}(\bar{\boldsymbol{x}})$ of $\boldsymbol{f}$ at $\bar{\boldsymbol{x}}$ is defined by

$$\partial_{\boldsymbol{x}}\boldsymbol{f}(\bar{\boldsymbol{x}}) = \mathrm{conv}\{\lim_{k\to\infty} D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}_k) : \boldsymbol{x}_k \in E_{\boldsymbol{f}}\ (k = 1, 2, \cdots) \text{ such that } \lim_{k\to\infty} \boldsymbol{x}_k = \bar{\boldsymbol{x}}\}.$$

In case $m = n$, $\boldsymbol{f}$ is called nonsingular at $\bar{\boldsymbol{x}}$ if rank $\boldsymbol{A} = n$ for any $\boldsymbol{A} \in \partial_{\boldsymbol{x}}\boldsymbol{f}(\bar{\boldsymbol{x}})$, and $\boldsymbol{f}$ is called singular at $\bar{\boldsymbol{x}}$ if $\boldsymbol{f}$ is not nonsingular at $\bar{\boldsymbol{x}}$.

Definition 3.3 $C^{1,1}(\boldsymbol{R}^n)$ denotes the set of the functions on $\boldsymbol{R}^n$ whose derivative $D_{\boldsymbol{x}}f(\boldsymbol{x})$ is Lipschitz continuous on $\boldsymbol{R}^n$. An element $f \in C^{1,1}(\boldsymbol{R})$ is called a $C^{1,1}$ function on $\boldsymbol{R}^n$. $\partial^2_{\boldsymbol{x}}f(\boldsymbol{x})$ denotes $\partial_{\boldsymbol{x}}D_{\boldsymbol{x}}f(\boldsymbol{x})$, i.e., $\partial^2_{\boldsymbol{x}}f(\boldsymbol{x}) = \partial_{\boldsymbol{x}}D_{\boldsymbol{x}}f(\boldsymbol{x})$. It is well known that $\partial^2_{\boldsymbol{x}}f(\boldsymbol{x}) \subset S(n)$ holds for $f \in C^{1,1}(\boldsymbol{R}^n)$ as stated in cited references [7][18].

Definition 3.4 Let $\boldsymbol{x} \in \boldsymbol{R}^n$, U be any subset of $\boldsymbol{R}^n$. We use next notations.

$$\begin{aligned} \sigma^1(\boldsymbol{x}) &= \{\boldsymbol{v} \in \sigma(\boldsymbol{x}) : \|\boldsymbol{v}\| \le 1\} \text{ for } \boldsymbol{x} \in K, \\ \sigma^1(U) &= \bigcup\{\sigma^1(\boldsymbol{x}) : \boldsymbol{x} \in U \textstyle\bigcap K\}. \end{aligned}$$

For any map $\boldsymbol{f} : \boldsymbol{R}^n \to \boldsymbol{R}^m$ and $\bar{\boldsymbol{x}} \in \boldsymbol{R}^n$, we define $D_{\boldsymbol{x}}^T \boldsymbol{f}(\bar{\boldsymbol{x}})$ and $\partial_{\boldsymbol{x}}^T \boldsymbol{f}(\bar{\boldsymbol{x}})$ by $D_{\boldsymbol{x}}^T \boldsymbol{f}(\bar{\boldsymbol{x}}) = (D_{\boldsymbol{x}} \boldsymbol{f}(\bar{\boldsymbol{x}}))^T$ and $\partial_{\boldsymbol{x}}^T \boldsymbol{f}(\bar{\boldsymbol{x}}) = \{\boldsymbol{A}^T : \boldsymbol{A} \in \partial_{\boldsymbol{x}} \boldsymbol{f}(\bar{\boldsymbol{x}})\}$ respectively. We can restate Propositions 2.5.4 and 2.5.7 of cited reference [3] for a closed convex set K as the following facts.

Fact 3.5 *([3]) Suppose that $D_{\boldsymbol{x}} d(\boldsymbol{x}, K)$ exists and $D_{\boldsymbol{x}} d(\boldsymbol{x}, K) \neq 0$. Then $\boldsymbol{x} \notin K$ and $D_{\boldsymbol{x}}^T d(\boldsymbol{x}, K) = \frac{\boldsymbol{x}^-}{\|\boldsymbol{x}^-\|}$.*

Fact 3.6 *([3]) $\partial_{\boldsymbol{x}}^T d(\boldsymbol{x}, K) = \sigma^1(\boldsymbol{x})$ for $\boldsymbol{x} \in K$.*

We prepare a lemma to prove the proposition below it.

Lemma 3.7 *Suppose that a Lipschitz continuous function f defined on an open set U is differentiable at $\boldsymbol{x} \in U \setminus \mathcal{N}$, where $\mathcal{N}$ has its Lebesgue measure 0, and that there exists a continuous map g on U satisfying $D_{\boldsymbol{x}} f(\boldsymbol{x}) = g(\boldsymbol{x})$, $(\boldsymbol{x} \in U \setminus \mathcal{N})$. Then f is differentiable on U and $D_{\boldsymbol{x}} f(\boldsymbol{x}) = g(\boldsymbol{x})$ holds on U.*

Proof. From the definition of generalized Jacobian, it is directly deduced that $\partial_{\boldsymbol{x}} f(\boldsymbol{x}) = \{g(\boldsymbol{x})\}$. Proposition 2.6.5 of cited reference [3] asserts that for any $\boldsymbol{x}$, $\boldsymbol{w} \in \boldsymbol{R}^n$ there exists $\zeta \in \text{conv}\{g(\boldsymbol{x} + t\boldsymbol{w}) : 0 \leq t \leq 1\}$ satisfying $f(\boldsymbol{x} + \boldsymbol{w}) = f(\boldsymbol{x}) + \zeta \boldsymbol{w}$, which leads to the differentiability of f at $\boldsymbol{x}$ and $D_{\boldsymbol{x}} f(\boldsymbol{x}) = g(\boldsymbol{x})$. $\square$

We can prove the following proposition.

Proposition 3.8 *The following (i) and (ii) hold.*

(i) $\partial_{\boldsymbol{x}}^T d(\boldsymbol{x}, K) = \begin{cases} \left\{\frac{\boldsymbol{x}^-}{\|\boldsymbol{x}^-\|}\right\} & , \ (\boldsymbol{x} \notin K) \\ \sigma^1(\boldsymbol{x}) & , \ (\boldsymbol{x} \in K) \end{cases}.$

(ii) $d(\boldsymbol{x}, K)$ is C^1 except on the boundary $Bd(K) = K \setminus \text{int}(K)$ of K which is open dense in $\boldsymbol{R}^n$, and

$$D_{\boldsymbol{x}}^T d(\boldsymbol{x}, K) = \begin{cases} \frac{\boldsymbol{x}^-}{\|\boldsymbol{x}^-\|} & , \ (\boldsymbol{x} \notin K) \\ \boldsymbol{0} & , \ (\boldsymbol{x} \in \text{int}(K)) \end{cases}.$$

Proof. In case of $\boldsymbol{x} \in K$, part (i) is nothing but Fact 3.6. We will treat the case $\boldsymbol{x} \notin K$ below. Since $d(\boldsymbol{x}, K)$ is Lipschitz continuous with its modulus 1, the set $N = \{\boldsymbol{x} \in \boldsymbol{R}^n : d(\boldsymbol{x}, K)$ is not differentiable at $\boldsymbol{x}\}$ has measure 0 in the sense of Lebesgue by Rademacher's theorem, and therefore $\boldsymbol{R}^n \setminus N$ is a dense subset of $\boldsymbol{R}^n$. Let $\boldsymbol{x} \notin K$. Then it is readily inferred that $d(\boldsymbol{x}+t\boldsymbol{x}^-, K) = (1+t)d(\boldsymbol{x}, K)$, which leads to $\lim_{t \to \infty} \dfrac{d(\boldsymbol{x} + t\boldsymbol{x}^-, K) - d(\boldsymbol{x}, K)}{t} = d(\boldsymbol{x}, K) \neq 0$. This implies that $D_{\boldsymbol{x}} d(\boldsymbol{x}, K) \neq \boldsymbol{0}$, $(\boldsymbol{x} \notin N \bigcup K)$. Hence it follows from Fact 3.5 that $D_{\boldsymbol{x}}^T d(\boldsymbol{x}, K) = \frac{\boldsymbol{x}^-}{\|\boldsymbol{x}^-\|}$, $(\boldsymbol{x} \notin N \bigcup K)$. Therefore we can deduce from Lemma 3.7 that $d(\boldsymbol{x}, K)$ is differentiable on $\boldsymbol{R}^n \setminus K$ and its derivative is $D_{\boldsymbol{x}} d(\boldsymbol{x}, K) = \frac{\boldsymbol{x}^-}{\|\boldsymbol{x}^-\|}$ there. On the other hand, it is readily inferred that $d(\boldsymbol{x}, K) = 0$, $(\boldsymbol{x} \in \text{int}(K))$, which leads to that $d(\boldsymbol{x}, K)$ is differentiable on $\text{int}(K)$ where its derivative is $D_{\boldsymbol{x}} d(\boldsymbol{x}, K) = \boldsymbol{0}$. $\square$

We can prove the following proposition, where $\otimes$ denotes the Kronecker product [6].

Proposition 3.9 *The following (i)-(iv) hold.*

(i) $d(\boldsymbol{x}, K)^2 \in C^{1,1}(\boldsymbol{R}^n)$.

(ii) $\begin{cases} D_{\boldsymbol{x}}^T (\|\boldsymbol{x}\|^2 - d(\boldsymbol{x}, K)^2) = 2\rho^+(\boldsymbol{x}) = 2\boldsymbol{x}^+ \\ D_{\boldsymbol{x}}^T d(\boldsymbol{x}, K)^2 = 2\rho^-(\boldsymbol{x}) = 2\boldsymbol{x}^- \end{cases}.$

(iii) $\begin{cases} \partial_{\boldsymbol{x}} \rho^+(\boldsymbol{x}) \subset S_+(n) \\ \partial_{\boldsymbol{x}} \rho^-(\boldsymbol{x}) \subset S_+(n) \end{cases}.$

(iv) For any $C = (C_+, C_-) \in \partial_{\boldsymbol{x}} \rho(\boldsymbol{x})$, there exists an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^n$ of $\boldsymbol{R}^n$ and λ_i with $0 \leq \lambda_i \leq 1$, $(1 \leq i \leq n)$ such that

$$\begin{aligned} C_+ &= \textstyle\sum_{i=1}^n \lambda_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i, \\ C_- &= \textstyle\sum_{i=1}^n (1 - \lambda_i) \boldsymbol{e}_i \otimes \boldsymbol{e}_i. \end{aligned}$$

Proof. (i)(ii): It is readily inferred from Proposition 3.8 that $\partial_{\boldsymbol{x}} d(\boldsymbol{x}, K)^2 = 2d(\boldsymbol{x}, K)\partial_{\boldsymbol{x}} d(\boldsymbol{x}, K) = \{2\boldsymbol{x}^-\}$. Parts (i) and (ii) follow from this fact.

(iii): Proposition 3.8 shows that $\partial_{\boldsymbol{x}}\rho^-(\boldsymbol{x}) = \partial^2_{\boldsymbol{x}}\frac{1}{2}d(\boldsymbol{x}, K)^2 \subset S(n)$. It is well known that $d(\boldsymbol{x}, K)$ is a convex function as shown by Lemma in p.53 of cited reference [3]. Since both $d(\boldsymbol{x}, K)(\geq 0)$ and the square function $F(t) = t^2 : \boldsymbol{R} \rightarrow \boldsymbol{R}_+ = \{t \in \boldsymbol{R} : t \geq 0\}$ are convex functions and F is increasing on $\boldsymbol{R}_+$, $d(\boldsymbol{x}, K)^2$ is also a convex function. In fact, Let $s,\ t \geq 0$ with $s + t = 1$ and $\boldsymbol{x},\ \boldsymbol{y} \in \boldsymbol{R}^n$. From convexity of $d(\boldsymbol{x}, K)$ the inequality $sd(\boldsymbol{x}, K) + td(\boldsymbol{x}, K) \geq d(s\boldsymbol{x} + t\boldsymbol{y}, K) \geq 0$ follows. Since F is increasing on $\boldsymbol{R}_+$, it is readily inferred that $F(sd(\boldsymbol{x}, K)+td(\boldsymbol{x}, K)) \geq F(d(s\boldsymbol{x}+t\boldsymbol{y}, K))$. By convexity of F, this inequality makes another inequality $sF(d(\boldsymbol{x}, K))+tF(d(\boldsymbol{y}, K)) \geq F(sd(\boldsymbol{x}, K)+td(\boldsymbol{x}, K))$. Combining these two inequalities obtained above, it readily inferred that $sF(d(\boldsymbol{x}, K)) + tF(d(\boldsymbol{y}, K)) \geq F(d(s\boldsymbol{x} + t\boldsymbol{y}, K))$.

So it is readily inferred that $\partial_{\boldsymbol{x}}\rho^-(\boldsymbol{x}) = \frac{1}{2}\partial^2_{\boldsymbol{x}} d(\boldsymbol{x}, K)^2 \subset S_+(n)$. The assertion that $\partial_{\boldsymbol{x}}\rho^+(\boldsymbol{x}) \subset S_+(n)$ follows from part (iv).

(iv): It is readily inferred that $0 \leq \|C_-\boldsymbol{v}\| \leq \|\boldsymbol{v}\|$ for any $\boldsymbol{v} \in \boldsymbol{R}^n$ since $\rho^-(\boldsymbol{x})$ is a Lipschitz continuous with its modulus 1. Therefore any eigenvalue μ of $C_- \in \partial_{\boldsymbol{x}}\rho^-(\boldsymbol{x})$ satisfies the inequality $|\mu| \leq 1$. With $\partial_{\boldsymbol{x}}\rho^-(\boldsymbol{x}) \subset S_+(n)$ of part (iii), it follows that $0 \leq \mu \leq 1$. From (ii) of this proposition it follows that $C_+ + C_- = \boldsymbol{I}_n$ holds for $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\boldsymbol{x})$. Therefore C_+ and C_- commute to each other, and as a result of it (iv) holds with $\lambda_i = 1 - \mu_i \geq 0$. □

Definition 3.10 Let $\boldsymbol{A} \in S(n)$. Then we denote by $V(\boldsymbol{A}; > 0)$ (respectively, $V(\boldsymbol{A}; \geq 0)$, $V(\boldsymbol{A}; < 0)$, $V(\boldsymbol{A}; \leq 0)$, $V(\boldsymbol{A}; = 0)$) the space spanned by the eigenvectors of $\boldsymbol{A}$ whose eigenvalues are positive (resp. nonnegative, negative, nonpositive, zero). By inspection, $\boldsymbol{R}^n = V(\boldsymbol{A}; > 0) \oplus V(\boldsymbol{A}; = 0) \oplus V(\boldsymbol{A}; < 0)$ holds.

The next lemma is proved directly.

Lemma 3.11 *Let $\bar{\boldsymbol{x}} \in \boldsymbol{R}^n$. Then, the following (i)-(v) hold.*

(i) $C_+,\ C_- \in S_+(n)$ *for any* $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$.

(ii) C_+ *and* C_- *are simultaneously diagonalized for any* $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$.

(iii) C_+ *and* C_- *are commutative, i.e.,* $C_+C_- = C_-C_+$ *for any* $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$.

(iv) $C_+ + C_- = \boldsymbol{I}_n$ *holds for any* $C \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$.

(v) $\boldsymbol{R}^n = V(C_+; > 0) \oplus V(C_+; = 0)$ *holds for any* $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$.

We refer to the following condition as the regular boundary condition 3.12 of K at $\boldsymbol{x}^+$. Lemma 4.15 of cited reference [17] showed that this condition is always fulfilled in case $K = S_+(n)$.

Condition 3.12 $V(C_+; = 0) \subset \boldsymbol{R}\sigma(\boldsymbol{x}^+)$ for $C_+ \in \partial_{\boldsymbol{x}}\rho^+(\boldsymbol{x})$.

Condition 3.12 does not hold in general as shown in Remark 3.13 below.

Remark 3.13 (a) In general, it is not true that $V(C_+; = 0) \subset \boldsymbol{R}\sigma(\boldsymbol{x}^+)$ for $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\boldsymbol{x})$. For example, $K = \{(x, y) \in \boldsymbol{R}^2 : y \geq 1 + |x|^{\frac{3}{2}}\}$ and $\bar{\boldsymbol{x}} = \boldsymbol{0} = (0, 0)$. Then $\bar{\boldsymbol{x}}^+ = (0, 1)$ and $\boldsymbol{R}\sigma(\bar{\boldsymbol{x}}^+) = \boldsymbol{R}$. Let $\rho^+(\boldsymbol{x}) = (u, v)$. When $\boldsymbol{x} = (x, y)$ is sufficiently near $\boldsymbol{0} = (0, 0)$, $\boldsymbol{x}^+ = (u, v)$ are related to $\boldsymbol{x} = (x, y)$ by the equations $y = -(sgn\ x)\frac{2}{3}|u|^{-\frac{1}{2}}(x-u)+v$ and $v = 1 + |u|^{\frac{3}{2}}$. From these equations, $t = |u|^{\frac{1}{2}}$ satisfies the equation $t^4 + (sgn\ x)\frac{2}{3}t^2 + (1-y)t-(sgn\ x)\frac{2}{3}x = 0$. Therefore, it can be easily deduced that $\frac{\partial t}{\partial x} = \frac{2(sgn\ x)}{12t^3+4(sgn\ x)t+3(1-y)}$ and $\frac{\partial t}{\partial y} = \frac{3t}{12t^3+4(sgn\ x)t+3(1-y)}$. Since $u = (sgn\ x)t^2$ and $v = 1 + t^3$, we have

$$D_{\boldsymbol{x}}\rho^+(\boldsymbol{x}) = \begin{pmatrix} \frac{4t}{12t^3+4(sgn\ x)t+3(1-y)} & \frac{6(sgn\ x)t^2}{12t^3+4(sgn\ x)t+3(1-y)} \\ \frac{6(sgn\ x)t^2}{12t^3+4(sgn\ x)t+3(1-y)} & \frac{9t^3}{12t^3+4(sgn\ x)t+3(1-y)} \end{pmatrix}.$$

On account of $\lim_{\boldsymbol{x}\to\boldsymbol{0}} D_{\boldsymbol{x}}\rho^+(\boldsymbol{x}) = \boldsymbol{O}$, we conclude that $\partial_{\boldsymbol{x}}\rho^+(\boldsymbol{0}) = \{\boldsymbol{O}\}$ and that $V(C_+; = 0) = \boldsymbol{R}^2$ and $\boldsymbol{R}\sigma(\bar{\boldsymbol{x}}^+) = \{0\} \times \boldsymbol{R}$. This implies that $V(C_+; = 0) \not\subset \boldsymbol{R}\sigma(\bar{\boldsymbol{x}}^+)$ for $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\boldsymbol{0})$.

(b) Let $\bar{\boldsymbol{x}} \in \boldsymbol{R}^n$ and $\bar{\boldsymbol{x}}^+ \in K$ and $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\boldsymbol{x})$. Since the fact that $V(C_+; = 0) \subset \boldsymbol{R}\sigma(\boldsymbol{x}^+)$ is equivalent to the fact that $V(C_+; > 0) \supset \sigma(\bar{\boldsymbol{x}})^\perp$, it follows from (a) that $V(C_+; > 0) \supset \sigma(\bar{\boldsymbol{x}})^\perp$ does not holds in general for $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$.

We can prove the following lemma exactly same as Lemma 5.6 of cited reference [17].

Lemma 3.14 *Under the regular boundary condition 3.12,* rank $D_{\boldsymbol{x}}h(\boldsymbol{x})C_+ = \ell$ *for any* $C_+ \in \partial_{\boldsymbol{x}}\rho^+(\boldsymbol{x})$.

4. Algebraic Criterion for Strong Stability by Generalized Jacobian of $\psi(\cdot, \cdot; f, h)$

In this section we investigate strong stability under LICQ condition 2.8 and the regular boundary condition 3.12. We can deduce an algebraic criterion for stability when a condition, to which we will refer as the infiltrative orientation condition, holds for the stationary solution considered, and can also define the stationary index of strongly stable stationary solutions under the same conditions. Although methods and techniques of this section are almost same as those in the section 5 of our former paper ([17]), we explain them again for readability.

Definition 4.1 Let $\boldsymbol{A}$ be an $n \times n$ real matrix whose eigenvalues are all real. We denote the number of positive (zero, negative) eigenvalues of $\boldsymbol{A}$ by $posi(\boldsymbol{A})$ (resp. $zero(\boldsymbol{A})$, $nega(\boldsymbol{A})$). We define $Type(\boldsymbol{A}) = (posi(\boldsymbol{A}), zero(\boldsymbol{A}), nega(\boldsymbol{A}))$.

The next fact can be proved without difficulties. This fact is considered a direct reason on which the stationary index can be well defined in Definition 4.10.

Fact 4.2 *Let U be an open set of $\boldsymbol{R}^n$ and $\boldsymbol{f}$ be a Lipschitz continuous map from U to $\boldsymbol{R}^n$ and $\bar{\boldsymbol{x}} \in U$. Then, the following (i) and (ii) are equivalent.*

(i) $\boldsymbol{f}$ is nonsingular at $\bar{\boldsymbol{x}}$.

(ii) sgn $\det\boldsymbol{A}$ *is nonzero and constant for $\boldsymbol{A} \in \partial_{\boldsymbol{x}}\boldsymbol{f}(\bar{\boldsymbol{x}})$.*

Moreover, in case that all eigenvalues of $\boldsymbol{A}$ are real for any $\boldsymbol{A} \in \partial_{\boldsymbol{x}}\boldsymbol{f}(\bar{\boldsymbol{x}})$, the above (i), (ii) and the following (iii) are equivalent.

(iii) $Type(\boldsymbol{A}) = (posi(\boldsymbol{A}), zero(\boldsymbol{A}), nega(\boldsymbol{A}))$ is constant and $zero(\boldsymbol{A}) = 0$ for $\boldsymbol{A} \in \partial_{\boldsymbol{x}}\boldsymbol{f}(\bar{\boldsymbol{x}})$.

Let $C = (C_+, C_-) \in \partial\rho(\boldsymbol{X})$. Since C_+ and C_- commute to each other, $V(C_+; > 0)$ and $V(C_+; = 0)$ are invariant spaces with respect to C_+ and C_-. Therefore we restrict C_+ and C_- to $V(C_+; > 0)$ and $V(C_+; = 0)$. We denote by C_{++} and C_{+-} the restrictions of C_+ to the spaces $V(C_+; > 0)$ and $V(C_+; = 0)$ respectively, and by C_{-+} and C_{--} the restrictions of C_- to the spaces $V(C_-; > 0)$ and $V(C_-; = 0)$ respectively, i.e.,

$$\begin{aligned}
C_{++} &= C_+|V(C_+; > 0) : V(C_+; > 0) \to V(C_+; > 0),\\
C_{+-} &= C_+|V(C_+; = 0) : V(C_+; = 0) \to V(C_+; = 0),\\
C_{-+} &= C_-|V(C_-; > 0) : V(C_-; > 0) \to V(C_-; > 0),\\
C_{--} &= C_-|V(C_-; = 0) : V(C_-; = 0) \to V(C_-; = 0).
\end{aligned}$$

For the remainder of this paper we treat the case $(\bar{\boldsymbol{x}}, \bar{\lambda}) \in \boldsymbol{R}^n \times \boldsymbol{R}^\ell$ is a stationary point of $\mathbf{Pro}(\bar{f}, \bar{h})$ and $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$. Subspaces $W_1(C_+, \bar{h})$ and $W_2(C_+, \bar{h})$ of $\boldsymbol{R}^n$ are defined by

$$\begin{aligned} W_1(C_+, \bar{h}) &= V(C_+; > 0) \bigcap T_{\bar{\boldsymbol{x}}^+}\mathcal{N}(\bar{h}), \\ W_2(C_+, \bar{h}) &= V(C_+; > 0) \bigcap (W_1(C_+, \bar{h})^\perp) \\ &= \{\boldsymbol{w}_2 \in V(C_+; > 0) : \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle = 0, (\forall \boldsymbol{w}_1 \in W_1(C_+, \bar{h})\}. \end{aligned}$$

It is readily inferred that

$$\begin{aligned} V(C_+; > 0) &= W_1(C_+, \bar{h}) \oplus W_2(C_+, \bar{h}) \quad \text{and,} \\ \boldsymbol{R}^n &= V(C_+; > 0) \oplus V(C_+; = 0) \\ &= W_1(C_+, \bar{h}) \oplus W_2(C_+, \bar{h}) \oplus V(C_+; = 0). \end{aligned}$$

Let $\boldsymbol{y}_{11}$ and $\boldsymbol{y}_{12}$ be linear coordinate systems of $W_1(C_+, \bar{h})$ and $W_2(C_+, \bar{h})$, respectively, and therefore $\boldsymbol{y}_1 = (\boldsymbol{y}_{11}, \boldsymbol{y}_{12})$ is a linear coordinate system of $V(C_+; > 0)$. Let $\boldsymbol{y}_2$ be a linear coordinate system of $V(C_+; = 0)$. Then $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2) = (\boldsymbol{y}_{11}, \boldsymbol{y}_{12}, \boldsymbol{y}_2)$ is a linear coordinate system of $\boldsymbol{R}^n$. We list these notations as follows.

$$\left\{ \begin{array}{ccl} \boldsymbol{y}_{11} & : & \text{a linear coordinate system of } W_1(C_+, \bar{h}), \\ \boldsymbol{y}_{12} & : & \text{a linear coordinate system of } W_2(C_+, \bar{h}), \\ \boldsymbol{y}_1 = (\boldsymbol{y}_{11}, \boldsymbol{y}_{12}) & : & \text{a linear coordinate system of } V(C_+; > 0), \\ \boldsymbol{y}_2 & : & \text{a linear coordinate system of } V(C_+; = 0), \\ \boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2) = (\boldsymbol{y}_{11}, \boldsymbol{y}_{12}, \boldsymbol{y}_2) & : & \text{a linear coordinate system of } \boldsymbol{R}^n. \end{array} \right.$$

With respect to the linear coordinate system $\boldsymbol{y}_1 = (\boldsymbol{y}_{11}, \boldsymbol{y}_{12})$ of $V(C_+; > 0) = W_1(C_+, \bar{h}) \oplus W_2(C_+, \bar{h})$, we represent

$$\left\{ \begin{array}{lcl} C_{++} & = & \begin{pmatrix} C_{++11} & C_{++12} \\ C_{++21} & C_{++22} \end{pmatrix}, \\ C_{-+} & = & \begin{pmatrix} C_{-+11} & C_{-+12} \\ C_{-+21} & C_{-+22} \end{pmatrix}, \\ C_{-+}C_{++}^{-1} & = & \begin{pmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{pmatrix}. \end{array} \right.$$

By identification of $T_{\bar{\boldsymbol{x}}^+}\boldsymbol{R}^n = \boldsymbol{R}^n$, we assume that $\boldsymbol{x} = \bar{\boldsymbol{x}}^+ + \boldsymbol{y}$ is a coordinate system of $\boldsymbol{R}^n$ around $\bar{\boldsymbol{x}}^+$. In the remainder of this paper, we use the following notations of coordinate systems around $\bar{\boldsymbol{x}}^+$

$$\left\{ \begin{array}{lcl} \boldsymbol{x}_{11} & = & \bar{\boldsymbol{x}}^+ + \boldsymbol{y}_{11} \\ \boldsymbol{x}_{12} & = & \bar{\boldsymbol{x}}^+ + \boldsymbol{y}_{12} \\ \boldsymbol{x}_1 & = & (\boldsymbol{x}_{11}, \boldsymbol{x}_{12}) = \bar{\boldsymbol{x}}^+ + \boldsymbol{y}_1 \\ \boldsymbol{x}_2 & = & \bar{\boldsymbol{x}}^+ + \boldsymbol{y}_2 \end{array} \right.$$

and the notations of derivatives

$$\left\{ \begin{array}{lcl} \boldsymbol{T} & = & \boldsymbol{T}(C; \bar{\boldsymbol{x}}, \bar{\lambda}, \bar{f}, \bar{h}) \\ & = & D^2_{\boldsymbol{x}_1} L(\bar{\boldsymbol{x}}^+, \bar{\lambda}; \bar{f}, \bar{h}) + C_{-+}C_{++}^{-1} \\ \boldsymbol{T}_{ij} & = & \boldsymbol{T}_{ij}(C; \bar{\boldsymbol{x}}, \bar{\lambda}, \bar{f}, \bar{h}) \\ & = & D_{\boldsymbol{x}_{1i}} D_{\boldsymbol{x}_{1j}} L(\bar{\boldsymbol{x}}^+, \bar{\lambda}; \bar{f}, \bar{h}) + \boldsymbol{M}_{ij}, \ (\forall i, \forall j = 1, 2) \end{array} \right. .$$

Remark 4.3 (i) It follows from $C_{+-} = \boldsymbol{O}$ that $C_{--} = \boldsymbol{I}_{V(C_+;>0)}$. Then we can write C_+ and C_- as

$$\begin{cases} C_+ &= \begin{pmatrix} C_{++} & \boldsymbol{O} \\ \boldsymbol{O} & C_{+-} \end{pmatrix} = \begin{pmatrix} C_{++} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}, \\ C_- &= \begin{pmatrix} C_{-+} & \boldsymbol{O} \\ \boldsymbol{O} & C_{--} \end{pmatrix} = \begin{pmatrix} C_{-+} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} \end{cases}$$

and we have

$$\begin{cases} D^2_{\boldsymbol{x}} L(\boldsymbol{x}^+, \lambda; f, h) &= \begin{pmatrix} D^2_{\boldsymbol{x}_1} L(\boldsymbol{x}^+, \lambda; f, h) & D_{\boldsymbol{x}_1} D_{\boldsymbol{x}_2} L(\boldsymbol{x}^+, \lambda; f, h) \\ D_{\boldsymbol{x}_2} D_{\boldsymbol{x}_1} L(\boldsymbol{x}^+, \lambda; f, h) & D^2_{\boldsymbol{x}_2} L(\boldsymbol{x}^+, \lambda; f, h) \end{pmatrix}, \\ D^2_{\boldsymbol{x}} L(\boldsymbol{x}^+, \lambda; f, h) C_+ + C_- &= \begin{pmatrix} D^2_{\boldsymbol{x}_1} L(\boldsymbol{x}^+, \lambda; f, h) C_{++} + C_{-+} & \boldsymbol{O} \\ D_{\boldsymbol{x}_2} D_{\boldsymbol{x}_1} L(\boldsymbol{x}^+, \lambda; f, h) C_{++} & \boldsymbol{I} \end{pmatrix}, \\ D_{\boldsymbol{x}} h(\boldsymbol{x}^+) C_+ &= \begin{pmatrix} D_{\boldsymbol{x}_1} h(\boldsymbol{x}^+) C_{++} & \boldsymbol{O} \end{pmatrix}. \end{cases}$$

(ii) By chain rule of generalized Jacobian [11] we have

$$\partial_{(\boldsymbol{x},\lambda)} \psi(\boldsymbol{x}, \lambda; f, h) = \left\{ \begin{pmatrix} D^2_{\boldsymbol{x}} L(\boldsymbol{x}^+, \lambda; f, h) C_+ + C_- & (D_{\boldsymbol{x}} h(\boldsymbol{x}^+))^T \\ D_{\boldsymbol{x}} h(\boldsymbol{x}^+) C_+ & \boldsymbol{O} \end{pmatrix} : C \in \partial_{\boldsymbol{x}} \rho(\boldsymbol{x}) \right\}.$$

(iii) Let $\boldsymbol{A} \in \partial_{(\boldsymbol{x},\lambda)} \psi(\boldsymbol{x}, \lambda; f, h)$ and represent $\boldsymbol{A}$ by $C \in \partial_{\boldsymbol{x}} \rho(\boldsymbol{x})$ as

$$\begin{aligned} \boldsymbol{A} &= \begin{pmatrix} D^2_{\boldsymbol{x}} L(\boldsymbol{x}^+, \lambda; f, h) C_+ + C_- & (D_{\boldsymbol{x}} h(\boldsymbol{x}^+))^T \\ D_{\boldsymbol{x}} h(\boldsymbol{x}^+) C_+ & \boldsymbol{O} \end{pmatrix} \\ &= \begin{pmatrix} D^2_{\boldsymbol{x}_1} L(\boldsymbol{x}^+, \lambda; f, h) C_{++} + C_{-+} & \boldsymbol{O} & (D_{\boldsymbol{x}_1} h(\boldsymbol{x}^+))^T \\ D_{\boldsymbol{x}_2} D_{\boldsymbol{x}_1} L(\boldsymbol{x}^+, \lambda; f, h) C_{++} & \boldsymbol{I} & (D_{\boldsymbol{x}_2} h(\boldsymbol{x}^+))^T \\ D_{\boldsymbol{x}_1} h(\boldsymbol{x}^+) C_{++} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}. \end{aligned}$$

Therefore,

$$sgn \det \boldsymbol{A} = \begin{cases} (-1)^\ell sgn \det \boldsymbol{T}_{11}(C; \boldsymbol{x}, \lambda, f, h) &, \ (\text{ rank } D_{\boldsymbol{x}_1} h(\boldsymbol{x}) C_{++} = \ell \ ) \\ 0 &, \ (\text{ rank } D_{\boldsymbol{x}_1} h(\boldsymbol{x}) C_{++} < \ell \ ) \end{cases}.$$

Similarly as in cited reference [17] we can prove directly the next proposition which implies that ψ has an advantageous property that Kojima function does not have. The following proposition asserts that we can apply Fact 4.2 to $\partial_{(\boldsymbol{x},\lambda)} \psi(\boldsymbol{x}, \lambda; f, h)$.

Proposition 4.4 *([17]) All eigenvalues of $\boldsymbol{A}$ are real for any $\boldsymbol{A} \in \partial_{(\boldsymbol{x},\lambda)} \psi(\boldsymbol{x}, \lambda; f, h)$.*

Definition 4.5 ([20]) Let U be an open subset of $\boldsymbol{R}^n$ and $F : U \to \boldsymbol{R}^n$ be a continuous map with $\bar{x} \in U$. Take $\delta > 0$ such that $B_\delta(\bar{x}) \subset U$. From the homology theory there exists a canonical isomorphism $H_n(B_\delta(\bar{x}), B_\delta(\bar{x}); \boldsymbol{Z}) \setminus \{\bar{x}\}; \boldsymbol{Z}) \simeq \boldsymbol{Z}$, where $\boldsymbol{Z}$ denotes the ring of integers. The theory asserts that F induces the morphism of homology groups: $F_* : \boldsymbol{Z} \simeq H_n(B_\delta(\bar{x}; \boldsymbol{Z})) \to H_n(B_\delta(F(\bar{x}); \boldsymbol{Z})) \simeq \boldsymbol{Z}$, and that this morphism F_* is independent on the choice of $\delta > 0$. Then, the Brouwer's degree $deg(\bar{x}; F)$ of the map F around $\bar{x}$ is defined as $deg(\bar{x}; F) = F_*(1) \in \boldsymbol{Z}$.

Remark 4.6 ([18]) We use the following properties of $deg(\cdot;\cdot)$.

(1) When F is a local homeomorphism around $\bar{x}$, $deg(\bar{x}; F) = F_*(1) = \pm 1$ since $F_* : \boldsymbol{Z} \to \boldsymbol{Z}$ is an isomorphism of the abelian group $\boldsymbol{Z}$. For example, when F is one-to-one around $\bar{x}$, $deg(\bar{x}; F) = \pm 1$ holds since F is a local homeomorphism around $\bar{x}$ by the Brouwer's invariance theorem of domain.

(2) *Homotopy property:* Let $I = \{t \in \boldsymbol{R} : 0 \le t \le 1\}$ and $F_t : U \times I \to \boldsymbol{R}^n; (x, t) \mapsto F_t(x)$ be continuous and F_t be one-to-one, $(\forall t \in I)$. Then $deg(\bar{x}; F_0) = deg(\bar{x}; F_1)$ holds.

(3) Let F is a local homeomorphism around $\bar{x}$, Then $deg(x;F)$ is locally constant as the function of x. In fact, suppose $F : U \to \boldsymbol{R}^n$. Let $\delta > 0$ satisfy $B_{2\delta}(\bar{x}) \subset U$. Define $G : B_\delta(\bar{x}) \times B_\delta(\mathbf{0}) \to \boldsymbol{R}^n$ by $G(x,w) = F(x+w)$. Since $G_w = G(\cdot,w) : B_\delta(\bar{x}) \to \boldsymbol{R}^n$, $(x \mapsto G(x,w))$ is one-to-one with a parameter $w \in B_\delta(\mathbf{0})$, it is readily inferred from (2) that $deg(\bar{x};F) = deg(\bar{x};G_{\mathbf{o}}) = deg(\bar{x};G_w) = deg(\bar{x}+w;F)$ holds for $w \in B_\delta(\mathbf{0})$.

(4) Suppose that F is differentiable around $\bar{x}$. Then, $deg(\bar{x};F) = sgn\ \det D_xF(\bar{x})$ holds.

Throughout the remainder of this paper, we suppose that the regular boundary condition 3.12 holds for K. Under this condition, we could calculate

$$(*) \quad sgn\ \det \begin{pmatrix} D^2_{\boldsymbol{x}}L(\boldsymbol{x}^+,\lambda;f,h)C_+ + C_- & (D_{\boldsymbol{x}}h(\boldsymbol{x}^+))^T \\ D_{\boldsymbol{x}}h(\boldsymbol{x}^+)C_+ & \boldsymbol{O} \end{pmatrix} = (-1)^\ell sgn\ \det \boldsymbol{T}_{11}(C;\boldsymbol{x},\lambda,f,h).$$

We can deduce a necessary condition for strong stability in the following proposition, where we denote $deg(\boldsymbol{x},\lambda;f,h) = deg((\boldsymbol{x},\lambda);\psi(\cdot,\cdot;f,h))$.

Proposition 4.7 *Suppose that LICQ condition 2.8 holds. Let $(\bar{\boldsymbol{x}},\bar{\lambda})$ be a stationary point for* $\mathbf{Pro}(\bar{f},\bar{h})$ *and that $(\bar{\boldsymbol{x}},\bar{\lambda})$ is a strongly stable stationary point for* $\mathbf{Pro}(\bar{f},\bar{h})$. *Then $sgn\ \det(\boldsymbol{A}) = deg(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})$ for any $\boldsymbol{A} \in ex(\partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h}))$.*

Proof. Theorem 2.12 asserts that there exist neighborhoods $U = B_{\delta^*}(\bar{\boldsymbol{x}}^+)$ of $\bar{\boldsymbol{x}}^+ \in K$ and $W = B_\delta((\bar{\boldsymbol{x}},\bar{\lambda}))$ of $(\bar{\boldsymbol{x}},\bar{\lambda})$ with $W \subset (\rho^+)^{-1}(U) \times \boldsymbol{R}^\ell$ such that $V = \{(f,h) \in \mathcal{F} : \psi(\cdot,\cdot;f,h)$ is one-to-one on $W\}$ is a neighborhood of $(\bar{f},\bar{h})$ in $\mathcal{F}_U$. Therefore, from Remark 4.6, we may assume that

$$s = deg(\boldsymbol{x},\lambda;f,h) \text{ is nonzero and constant for } (\boldsymbol{X},\lambda,f,h) \in W \times V. \tag{4.1}$$

We can deduce a contradiction against value s of degree through exactly the same procedure used by Kojima in cited reference [12]. Let $s = deg(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})$ and $\bar{s} = \begin{cases} -1 & , \quad (s=1) \\ \ \ 1 & , \quad (s=-1) \end{cases}$.
Suppose that there exists an element $\boldsymbol{A} \in ex(\partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h}))$ such that $sgn\ \det(\boldsymbol{A}) = t$ with $t \neq s$, i.e., $t = 0$ or $t = \bar{s}$. Represent $\boldsymbol{A}$ as
$\boldsymbol{A} = \begin{pmatrix} D^2_{\boldsymbol{x}}L(\bar{\boldsymbol{x}}^+,\bar{\lambda};\bar{f},\bar{h})C_+ + C_- & (D_{\boldsymbol{x}}\bar{h}(\bar{\boldsymbol{x}}^+))^T \\ D_{\boldsymbol{x}}\bar{h}(\bar{\boldsymbol{x}}^+)C_+ & \boldsymbol{O} \end{pmatrix}$ for some $C \in \partial_{(\boldsymbol{x},\lambda)}\rho(\bar{\boldsymbol{x}})$. Then $C = (C_+,C_-) \in ex(\partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}}))$ follows from $\boldsymbol{A} \in ex(\partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h}))$. Calculation (*) implies that $sgn\ \det\boldsymbol{A} = (-1)^\ell sgn\ \det\boldsymbol{T}_{11}$ holds. Therefore, $sgn\ \det\ \boldsymbol{T}_{11} = (-1)^\ell t$. It is readily inferred from definitions of $\boldsymbol{x}_{11}$ and $\boldsymbol{x}_{12}$ that $D_{\boldsymbol{x}_{11}}\bar{h}(\bar{\boldsymbol{x}}^+) = \boldsymbol{O}$ and that $D_{\boldsymbol{x}_{12}}\bar{h}(\bar{\boldsymbol{x}}^+)$ is a nonsingular matrix of degree ℓ. Without difficulties, it can be proved that there exist $\epsilon_0 > 0$ and $\boldsymbol{B}_{11} \in End_{\boldsymbol{R}}(W_1) = W_1 \otimes W_1$ such that $\boldsymbol{T}_{11}(\epsilon) = \boldsymbol{T}_{11} + \epsilon\boldsymbol{B}_{11}$ satisfies that $sgn\ \det\boldsymbol{T}_{11}(\epsilon) = (-1)^\ell\bar{s}$ for any $0 < \forall\epsilon < \epsilon_0$. Let $f_\epsilon(\boldsymbol{x}) = \bar{f}(\boldsymbol{x}) + \epsilon\boldsymbol{x}_{11}^T\boldsymbol{B}_{11}\boldsymbol{x}_{11}$. Simple calculation shows that $\boldsymbol{A}(\epsilon) = \begin{pmatrix} D^2_{\boldsymbol{x}}L(\bar{\boldsymbol{x}}^+,\bar{\lambda};f_\epsilon,\bar{h})C_+ + C_- & (D_{\boldsymbol{x}}\bar{h}(\bar{\boldsymbol{x}}^+))^T \\ D_{\boldsymbol{x}}\bar{h}(\bar{\boldsymbol{x}}^+)C_+ & \boldsymbol{O} \end{pmatrix} \in \partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};f_\epsilon,\bar{h})$.
It is readily inferred that

$$\begin{cases} \boldsymbol{T}_{11} & = \ D^2_{\boldsymbol{x}_{11}}L(\bar{\boldsymbol{x}}^+,\bar{\lambda};\bar{f},\bar{h}) + \boldsymbol{M}_{11}, \\ \boldsymbol{T}_{11}(\epsilon) & = \ D^2_{\boldsymbol{x}_{11}}L(\bar{\boldsymbol{x}}^+,\bar{\lambda};f_\epsilon,\bar{h}) + \boldsymbol{M}_{11}. \end{cases}$$

Therefore, from calculation (*) we can deduce that $sgn\ \det\boldsymbol{A}(\epsilon) = (-1)^\ell sgn\ \det\boldsymbol{T}_{11}(\epsilon) = \bar{s} \neq 0$. Since $C = (C_+,C_-)$ is an extremal element of $\partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$, there exists a sequence $\boldsymbol{x}^{(k)}$ $(k = 1,2,\cdots)$ such that $\lim_{k\to\infty}\boldsymbol{x}^{(k)} = \bar{\boldsymbol{x}}$ and $\lim_{k\to\infty}D_{\boldsymbol{x}}\rho(\boldsymbol{x}^{(k)}) = C$.
Since $\lim_{k\to\infty}D_{\boldsymbol{X}}\psi(\boldsymbol{x}^{(k)},\bar{\lambda};f_\epsilon,\bar{h}) = \boldsymbol{A}(\epsilon)$ and $sgn\ \det\boldsymbol{A}(\epsilon) = \bar{s} \neq 0$, we have

$\lim_{k\to\infty} sgn\ \det D_{\boldsymbol{X}}\psi(\boldsymbol{x}^{(k)},\bar{\lambda};f_\epsilon,\bar{h}) = \bar{s}$. Especially, for large k, we may assume that $sgn\ \det D_{\boldsymbol{x}}\psi(\boldsymbol{x}^{(k)},\bar{\lambda};f_\epsilon,\bar{h}) = \bar{s}$, which implies that $deg(\boldsymbol{x}^{(k)},\bar{\lambda};f_\epsilon,\bar{h}) = \bar{s}$ by Remark 4.6. This result contradicts (4.1). □

We introduce the following condition to which we would refer as the infiltrative orientation condition. This condition always holds for classical programs NLP as showed in Theorem 3.1 of cited reference [9]. However it does not hold for NSDP as shown in Remark 5.13 of cited reference [17], and so it does not for NPAC in general.

Condition 4.8 If sgn det $\boldsymbol{A} = s \neq 0$ for any $\boldsymbol{A} \in ex(\partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h}))$, then sgn det $\boldsymbol{A} = s$ for any $\boldsymbol{A} \in \partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})$.

If K satisfies the regular boundary condition 3.12 and the infiltrative orientation condition 4.8 holds for $\mathbf{Pro}(f,h)$, the following theorem proposes an algebraic criterion for strong stability in terms of Jacobian $D_{\boldsymbol{x}}h(\bar{\boldsymbol{x}}^+)$ and Hessian $D^2_{\boldsymbol{x}}L(\bar{\boldsymbol{x}}^+,\bar{\lambda};\bar{f},\bar{h})$. We denote $deg(\boldsymbol{x},\lambda;f,h) = deg((\boldsymbol{x},\lambda);\psi(\cdot,\cdot;f,h))$ in its proof.

Theorem 4.9 *Suppose that LICQ condition 2.8 holds. Let $(\bar{\boldsymbol{x}},\bar{\lambda})$ be a stationary point for $\mathbf{Pro}(\bar{f},\bar{h})$, and suppose that the regular boundary condition 3.12 of K hold at $\bar{\boldsymbol{x}}^+$. Then*

(1) the following (i)-(iv) are equivalent.

(i) $\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})$ is nonsingular at $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(ii) sgn det $\boldsymbol{A}$ is nonzero and constant for any $\boldsymbol{A} \in \partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})$.

(iii) $Type(\boldsymbol{A})$ is constant and $zero(\boldsymbol{A}) = 0$ for any $\boldsymbol{A} \in \partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})$.

(iv) sgn det $\boldsymbol{T}_{11}(C;\bar{\boldsymbol{x}},\bar{\lambda},\bar{f},\bar{h})$ is nonzero and constant for any $C \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$.

(2) Any of (i)-(iv) implies the following

(v) $\bar{\boldsymbol{x}}^+$ is a strongly stable stationary solution for $\mathbf{Pro}(\bar{f},\bar{h})$.

(3) (i)-(v) are equivalent if the infiltrative orientation condition 4.8 holds for $\mathbf{Pro}(\bar{f},\bar{h})$ at $(\bar{\boldsymbol{x}},\bar{\lambda})$.

Proof. (1): Equivalence between (i), (ii) and (iii) is readily deduced by Fact 4.2 and Proposition 4.4; equivalence between (ii) and (iv) is directly deduced from the relation sgn det $\boldsymbol{A} = (-1)^\ell sgn$ det $\boldsymbol{T}_{11}(C;\bar{\boldsymbol{x}},\bar{\lambda},\bar{\boldsymbol{f}},\bar{h})$.

(2): The implication from (i) to (v) is clear from the Implicit Function Theorem Theorem 2.1 of cited reference [9].

(3): We have only to prove the implication (v)⇒(i).

(v)⇒(i): Suppose that $\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})$ is singular at $(\bar{\boldsymbol{x}},\bar{\lambda})$. Then, Fact 4.2 asserts that either the following statement (a) or (b) holds.

(a) There exists an element $\boldsymbol{A} \in \partial_{\boldsymbol{x}}\psi(\cdot,\cdot;\bar{f},\bar{h})$ such that sgn $\det\boldsymbol{A} = 0$.

(b) There exists elements $\boldsymbol{A},\ \boldsymbol{B} \in \partial_{\boldsymbol{x}}\psi(\cdot,\cdot;\bar{f},\bar{h})$ such that sgn $\det\boldsymbol{A} = 1$ and sgn $\det\boldsymbol{B} = -1$.

Since case (b) is readily reduced to case (a) by virtue of convexity of $\partial_{\boldsymbol{x}}\psi(\cdot,\cdot;\bar{f},\bar{h})$, we treat case (a) only, i.e., sgn det $\boldsymbol{A} = 0$. Theorem 2.12 asserts that there exist neighborhoods $U = B_{\delta^*}(\bar{\boldsymbol{x}}^+)$ of $\bar{\boldsymbol{x}}^+$ in $S(n)$ and $W = B_\delta((\bar{\boldsymbol{x}},\bar{\lambda}))$ of $(\bar{\boldsymbol{x}},\bar{\lambda})$ with $W \subset (\rho^+)^{-1}(U) \times \boldsymbol{R}^\ell$ such that $V = \{(f,h) \in \mathcal{F} : \psi(\cdot,\cdot;f,h)$ is one-to-one on $W\}$ is a neighborhood of $(\bar{f},\bar{h})$ in $\mathcal{F}_U$. Therefore, from Remark 4.6 and for the simplicity, we may assume that $deg(\boldsymbol{x},\lambda;f,h) = 1$ for any $(\boldsymbol{x},\lambda,f,h) \in W \times V$. From Proposition 4.7 it follows that sgn $\det(\boldsymbol{A}) = 1$ for any $\boldsymbol{A} \in ex(\partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h}))$. By the infiltrative orientation condition 4.8, we may assume

$$\det\ \boldsymbol{A}' \geq 0 \text{ for any } (\boldsymbol{x},\lambda;f,h) \in W \times V \text{ and any } \boldsymbol{A}' \in \partial_{(\boldsymbol{x},\lambda)}\psi(\boldsymbol{x},\lambda;f,h). \tag{4.2}$$

We will deduce a contradiction against (4.2) in the below. Represent $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{pmatrix} D^2_{\boldsymbol{x}}L(\bar{\boldsymbol{x}}^+,\bar{\lambda};\bar{f},\bar{h})C_+ + C_- & (D_{\boldsymbol{x}}\bar{h}(\bar{\boldsymbol{x}}^+))^T \\ D_{\boldsymbol{x}}\bar{h}(\bar{\boldsymbol{x}}^+)C_+ & \boldsymbol{O} \end{pmatrix}$$

with $C = (C_+, C_-) \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$. We use the same symbols $\boldsymbol{T}_{11}$, $\boldsymbol{M}_{11}$, $W_1 = W_1(C_+, \bar{h})$, and $W_2 = W_2(C_+, \bar{h})$ as in Definition 4.4. Since we have the relation $sgn \det \boldsymbol{A} = (-1)^\ell sgn \det \boldsymbol{T}_{11}$, we can deduce $\det \boldsymbol{T}_{11} = 0$. It is readily inferred from definitions of $\boldsymbol{x}_{11}$ and $\boldsymbol{x}_{12}$ that $D_{\boldsymbol{x}_{11}}\bar{h}(\bar{\boldsymbol{x}}^+) = \mathbf{0}$ and that $D_{\boldsymbol{x}_{12}}\bar{h}(\bar{\boldsymbol{x}}^+)$ is a nonsingular matrix of degree ℓ. Without difficulties, it can be proved that there exist $\epsilon_0 > 0$ and $\boldsymbol{B}_{11} \in \mathcal{M}(W_1) = End_{\boldsymbol{R}}(W_1) = W_1 \otimes W_1$ such that $\boldsymbol{T}_{11}(\epsilon) = \boldsymbol{T}_{11} + \epsilon \boldsymbol{B}_{11}$ satisfies that $sgn \det \boldsymbol{T}_{11}(\epsilon) = -(-1)^\ell$ for any $0 < \forall\epsilon < \epsilon_0$. Let $f_\epsilon(\boldsymbol{x}) = \bar{f}(\boldsymbol{x}) + \epsilon \boldsymbol{x}_{11}^T \boldsymbol{B}_{11} \boldsymbol{x}_{11}$. Simple calculation shows that

$$\begin{cases} \boldsymbol{A}(\epsilon) & = \begin{pmatrix} D_{\boldsymbol{x}}^2 L(\bar{\boldsymbol{x}}^+, \bar{\lambda}; f_\epsilon, \bar{h})C_+ + C_- & (D_{\boldsymbol{x}}\bar{h}(\bar{\boldsymbol{x}}^+))^T \\ D_{\boldsymbol{x}}\bar{h}(\bar{\boldsymbol{x}}^+)C_+ & \boldsymbol{O} \end{pmatrix} \in \partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; f_\epsilon, \bar{h}), \\ sgn \det \boldsymbol{A}(\epsilon) & = (-1)^\ell sgn \det \boldsymbol{T}_{11}(\epsilon) = -1 \text{ for any } 0 < \forall\epsilon < \epsilon_0. \end{cases}$$

This result contradicts (4.2). □

Definition 4.10 Under the same conditions of Theorem 4.9 we can define the stationary index after Kojima ([12]). For a stationary solution $\bar{\boldsymbol{x}}^+$ of $\mathbf{Pro}(\bar{f}, \bar{h})$ that is associated with its stationary point $(\bar{\boldsymbol{x}}, \bar{\lambda})$, we can define the stationary index $s.index(\bar{\boldsymbol{x}}^+; \bar{f}, \bar{h})$ by $s.index(\bar{\boldsymbol{x}}^+; \bar{f}, \bar{h}) = nega(\boldsymbol{T}_{11}(C; \bar{\boldsymbol{x}}, \bar{\lambda}, \bar{f}, \bar{h}))$. We remark that this definition is independent of choice of $C \in \partial_{\boldsymbol{x}}\rho(\bar{\boldsymbol{x}})$; therefore, it is also independent of $\boldsymbol{A} \in \partial_{(\boldsymbol{x},\lambda)}\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$ from Fact 4.2. This stationary index is an important invariant because it is a nonlinear version of Morse index and characterizes the local behavior of f on $\{\boldsymbol{x} \in K : h(\boldsymbol{x}) = \mathbf{0}\}$.

5. Conclusions

We investigated strong stability, in the sense of Kojima, of stationary solutions of nonlinear programs $\mathbf{Pro}(f, h)$. Firstly we have made clear the structure of $\partial_{\boldsymbol{x}}\rho(\boldsymbol{x})$ in section 3. Secondly in section 4 we have proved that an algebraic criterion for strong stability exists under LICQ condition 2.8 and the regular boundary condition 3.12 of K if the infiltrative orientation condition 4.8 holds for the stationary point $(\bar{\boldsymbol{x}}, \bar{\lambda})$, and defined the stationary index under the same conditions.

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References

[1] D. Bertsekas: *Convex Analysis and Optimization* (Athena Scientific, Massachusetts, 2003).

[2] J. Bonnans and A. Shapiro: *Perturbation Analysis of Optimization Problems* (Springer, New York, 2000).

[3] F. Clarke: *Optimization and Nonsmooth Analysis* (Wiley Interscience Publication, New York, 1969).

[4] F. Facchinei and J. Pang: *Finite-Dimensional Variational Inequalities and Complementarity Problems* (Springer, New York, 2003).

[5] H. Federer: *Geometric Measure Theory* (Springer, Berlin-Heidelberg-New York, 1969)

[6] A. Graham: *Kronecker Products and Matrix Calculus with Applications* (J. Wiley and Sons, New York, 1981).

[7] J. Hiriart-Urruty, J. Strodiot, and V. Nguyen: Generalized Hessian matrix and second-oder optimality conditions for problems with $C^{1,1}$ data. *Applied Mathematics and Optimization*, **11** (1984), 43–56.

[8] B. Iversen: *Cohomology of Sheaves* (Springer-Verlag, New York, 1986).

[9] H.Th. Jongen, D. Klatte, and K. Tammer: Implicit functions and sensitivity of stationary points. *Mathematical Programming*, **49** (1990), 123–138.

[10] D. Klatte and B. Kummer: Strong stability in nonlinear programming revised. *Journal of Australian Mathematical Society, Ser.B*, **40** (1999), 336–352.

[11] D. Klatte and B. Kummer: *Nonsmooth Equations in Optimization* (Kluwer Academic Publishers, Dordrecht, 2002).

[12] M. Kojima: Strongly stable stationary solutions in nonlinear programs. In S. M. Robinson (ed.): *Analysis and Computation of Fixed Points* (Academic Press, New York, 1980), 93–138.

[13] B. Kummer: Lipschitzian inverse functions, directional derivatives and application in $C^{1,1}$ optimization. *Journal of Optimization Theory and Applications*, **70** (1991), 559–580.

[14] B. Kummer: An implicit-function theorem for $C^{0,1}$-equations and parametric $C^{1,1}$-optimization. *Journal of Mathematical Analysis and Applications*, **158** (1991), 35–46.

[15] A. Levy: Solution sensitivity from general principles. *SIAM Journal on Control and Optimization*, **40** (2001), 1–38.

[16] T. Matsumoto: On the stability of stationary solutions of nonlinear positive semidefinite programs. *Journal of Operations Research Society of Japan*, **46** (2003), 22–34.

[17] T. Matsumoto: An algebraic condition equivalent to strong stability of stationary solution of nonlinear positive semidefinite programs, to appear in *SIAM Journal on Optimization.*

[18] J. Ortega and W. Rheinboldt: *Iterative Solutions of Nonlinear Equations in Several Variables* (Academic Press, New York, 1970).

[19] R. Rockafellar and R. Wets: *Variational Analysis* (Springer-Verlag, Berlin, 1998).

[20] E. Spanier: *Algebraic Topology* (McGraw-Hill, New York, 1966).

[21] E. Zarantonello: Projections on convex sets in Hilbert space and spectral theory. In E. Zarantonello (ed.): *Contributions to Nonlinear Functional Analysis* (Academic Press, New York, 1971), 237-424.

Toshihiro Matsumoto
Department of Media and Information Systems
Faculty of Science & Engineering
Teikyo University of Science & Technology
2525 Yatsuzawa, Uenohara-shi,
Yamanashi 409-0193, Japan
E-mail: `matsu@ntu.ac.jp`