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A BAYESIAN SEQUENTIAL BATCH-SIZE DECISION PROBLEM TO MINIMIZE EXPECTED TOTAL COMPLETION TIME ON A SINGLE MACHINE

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Abstract The same kind of n jobs are processed one by one sequentially by a single machine and any number k of them may be processed as a batch. A setup time is necessary before the processing of the first job in a batch. The completion times of all the jobs in a batch are the same as the completion time of the last job in the batch and the processing time of the batch is the sum of a setup time and k times of the processing time of a job. The processing time of the job is known, but the setup time is the random variable which is distributed in the gamma distribution with the parameter whose value is unknown a priori. A conjugate prior distribution for the value is considered. The first batch size is decided by using the prior distribution, the setup time is observed, and then the second batch size is decided by using the posterior distribution revised by using the observed value of the setup time in the first batch. This process is repeated until all the jobs have been processed. The objective is to minimize the expected total completion times. This problem is formulated by using dynamic programming and both the several properties derived from the recursive equations and the critical values for the optimal strategy are derived.

Keywords: Dynamic programming, Bayesian, sequential decision, single machine, setup time, batch size, total completion time

1. Introduction

The batch size decision problem considered in this paper is described as follows: There are n jobs which have to be processed by a single machine sequentially one by one. All the jobs are the same and have the same processing time. Any number of jobs may be processed as a batch, but a setup time is necessary in order to begin the processing of the first job in each batch. The processing time has a predetermined value, but the setup time is the random variable which has some distribution with the parameter whose true value is unknown. We can consider that there is a conjugate prior distribution for the unknown value of the parameter as a prior distribution. All the jobs in each batch has the same completion time that is equal to that of the last job in the batch. For n jobs problem, the batch size k is decided first and then k jobs are processed after observing the value of the setup time X and the posterior distribution is calculated by using X. The processing time of this batch is the sum of the setup time and k times of the processing time. Now, the problem becomes $n-k$ jobs problem and the batch size is decided again by using the revised new prior distribution. The objective is to minimize the expected total completion times of n jobs.

The batch size decision in the scheduling problems have been studied mainly in these twenty years and many papers have been published. When several jobs are processed as a batch, the setup time has an important role in the scheduling problems. If the setup time is much larger than the processing time, the optimal batch size is large, but if the setup time is small enough to neglect, the optimal batch size is 1. The ratio of setup time to processing time is very important in this kind of problem. There are many papers which discussed scheduling problem with the batch size, for example, Santos and Magazine [11], Dobson, Karmarker and Rummel [4], Naddef and Santos [9], Coffman, Yannakakis, Magazine, and Santos [2], Shallcross [12], Potts and Kovalyov [10], and so on. The problems discussed in these papers are deterministic problems. Since the setup is sometimes done by the worker, it is important to consider the setup time not as the predetermined value but as the random variable whose parameter is unknown.

A stochastic scheduling problem with batching was discussed in Koole and Righter [8] whose objective is to minimize the total completion time where the completion time of each batch is the maximum completion time of all the jobs in the batch. This kind of problem occurs in the deterministic cases and there are applications in semiconductor manufacturing and was discussed in Ikura and Gimple [7], Chandru, Lee and Uzsoy [1], Hochbaum and Landy [6], and so on.

The deterministic or the stochastic batch size decision problems whose parameters are known are the interesting problems as the static problems in which batch sizes are determined at once. The stochastic batch size decision problem with at least one unknown parameter is the sequential decision problems with learning. This kind of Bayesian sequential decision was discussed in Hamada and Glazebrook [5] in the single machine scheduling problem. In the scheduling problem with batching, Yanai [13] considered the case that the setup time is distributed in the exponential distribution with an unknown parameter which has the gamma distribution as a prior distribution, formulated it as the recursive equations of the dynamic programming, and solved them directly for the cases that $n = 1$, 2, 3 and 4. He has not solved the cases of 5 or more jobs because of the complication of considering many cases.

In this paper, the case that the processing time is distributed in the gamma distribution with one unknown parameter is not solved directly but is discussed from the different point of view and the several properties which are important in the analysis of the problem are derived. The main results of this paper is to derive the important properties of the critical values which describes the optimal solutions of the recursive equations. The optimal strategy is also derived as the result.

2. Formulation by Dynamic Programming

A sequential batch-size decision problem considered in this paper is described as follows: There are n identical jobs whose processing time is 1 and it is possible to begin the processing of any job at time 0. Since all the jobs have the same processing time, the total completion time, the sum of completion times of n jobs, is indifferent from the sequence of jobs. Once the processing of each job begins, preemption is not allowed until completion. Several jobs are processed as a batch and a setup time is necessary before the processing of the first job of the batch. The length of the setup time is assumed to be the random variable X which is distributed in the gamma distribution with parameter vector (u, s), that is, X has the density function

$$f(x \mid u, s) = \begin{cases} \Gamma(s)^{-1} u^s x^{s-1} e^{-ux}, & if \quad x \geq 0, \\ 0, & if \quad x < 0, \end{cases}$$

where $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$. For example, if we know X is the exponentially distributed random variable with unknown mean, then $s = 1$ and the mean is $1/u$. By the same way, we assume that the true value of s is known, but the true value of u is unknown and it has a conjugate prior distribution with parameter vector (w, α) as a prior distribution. Since the gamma distribution is the conjugate prior distribution of u, the posterior distribution after observing that $X = x$ is the gamma distribution with parameter vector $(w + x, \alpha + s)$ (See, for example, DeGroot [3]). Let $(n; w, \alpha)$ be called the current prior information when there are n jobs remaining and the current prior distribution has the parameter vector (w, α). Also, let $(n - k; w + x, \alpha + s)$ be called the posterior information after deciding a batch size k and observing the length x of the setup time, where $n - k$ jobs are remaining. The batch size is decided sequentially after observing the current prior information. Since the setup time is a random variable, the completion time of each job is also a random variable. Let C_j $(j = 1, 2, \cdots, n)$ be the completion time of j-th job. The completion times of all the jobs in a batch are the same as that of the last job in the batch, that is, $C_i = C_j$ if i-th and j-th jobs are in the same batch. The total completion time is $C_1 + C_2 + \cdots + C_n$, the sum of completion times of n jobs. The objective is to decide sequentially how many jobs are processed as a batch in order to minimize the expected total completion times.

Let $G_n(w, \alpha)$ be the minimum expected total completion times when the current prior knowledge is $(n; w, \alpha)$ and the optimal strategy is followed thereafter. Also, let $G_n^k(w, \alpha)$ be the the minimum expected total completion times when the current prior information is $(n; w, \alpha)$, k jobs are processed as a batch, and the optimal strategy is followed thereafter. For any function $f(x)$, let

$$E_X[f(X) \mid u, s] = \int_0^\infty f(x)\Gamma(s)^{-1}u^s x^{s-1}e^{-ux}dx,$$

$$E_U[f(U) \mid w, \alpha] = \int_0^\infty f(u)\Gamma(\alpha)^{-1}w^\alpha u^{\alpha-1}e^{-wu}du,$$

and

$$E[f(X) \mid w, \alpha] = E_U[E_X[f(X) \mid U, s] \mid w, \alpha].$$

Then,

$$E[f(X) \mid w, \alpha] = \int_0^\infty f(x)\{B(\alpha, s)\}^{-1}w^\alpha x^{s-1}(w + x)^{-\alpha-s}dx,$$

where

$$B(\alpha, s) = \Gamma(\alpha)\Gamma(s)/\Gamma(\alpha + s).$$

The problem is formulated by dynamic programming and the recursive equations are derived as follows:

$$G_n(w, \alpha) = \min_{1\le k\le n} G_n^k(w, \alpha) \tag{1}$$

for $n = 1, 2, 3, \cdots$ and

$$G_0(w, \alpha) \equiv 0, \tag{2}$$

where

$$G_n^k(w, \alpha) = E_U[E_X[n(X + k) + G_{n-k}(w + X, \alpha + s) \mid U, s] \mid w, \alpha], \tag{3}$$

that is, (3) is rewritten as follows:

($\mathrm{G}_{n,k}$): $G_n^k(w, \alpha) = nsw(\alpha - 1)^{-1} + nk + E[G_{n-k}(w + X, \alpha + s) \mid w, \alpha]$.

In (3), $n(X+k)$ is the contribution of n jobs to the total completion time in the time interval $(0, X+k)$ and $G_{n-k}(w+X, \alpha+s)$ is also that of $n-k$ jobs in the time interval $(X+k, \infty)$.

From (1), (2), $(\mathrm{G}_{1,1})$, $(\mathrm{G}_{2,1})$ and $(\mathrm{G}_{2,2})$,

$$G_1(w,\alpha) = G_1^1(w,\alpha) = sw(\alpha-1)^{-1}+1, \tag{4}$$

$$G_2(w,\alpha) = \begin{cases} G_2^1(w,\alpha), & if \quad 0 < w < r_{2,1}(\alpha), \\ G_2^2(w,\alpha), & if \quad r_{2,1}(\alpha) \le w, \end{cases} \tag{5}$$

where

$$G_2^1(w,\alpha) = 3sw(\alpha-1)^{-1}+3, \tag{6}$$

$$G_2^2(w,\alpha) = 2sw(\alpha-1)^{-1}+4, \tag{7}$$

and

$$r_{2,1}(\alpha) = (\alpha-1)/s.$$

Also,

$$G_3(w,\alpha) = \begin{cases} G_3^1(w,\alpha), & if \quad 0 < w < r_{3,1}(\alpha), \\ G_3^2(w,\alpha), & if \quad r_{3,1}(\alpha) \le w < r_{3,2}(\alpha), \\ G_3^3(w,\alpha), & if \quad r_{3,2}(\alpha) \le w, \end{cases}$$

where

$$G_3^1(w,\alpha) = \begin{cases} 6sw(\alpha-1)^{-1}+6 \\ \quad + \displaystyle\sum_{i=1}^{s}\left\{-\frac{sr_{2,1}(\alpha+s)}{\alpha+i-2}+1\right\}\frac{\Gamma(\alpha+i-1)}{\Gamma(\alpha)\Gamma(i)}\frac{w^\alpha\{r_{2,1}(\alpha+s)-w\}^{i-1}}{\{r_{2,1}(\alpha+s)\}^{\alpha+i-1}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad if \quad 0 < w < r_{2,1}(\alpha+s), \\ 5sw(\alpha-1)^{-1}+7, \qquad\qquad\qquad\quad if \quad r_{2,1}(\alpha+s) \le w, \end{cases}$$

$$G_3^2(w,\alpha) = 4sw(\alpha-1)^{-1}+7,$$

$$G_3^3(w,\alpha) = 3sw(\alpha-1)^{-1}+9,$$

$$r_{3,2}(\alpha) = 2(\alpha-1)/s,$$

and $r_{3,1}(\alpha)$ is a unique root of the equation

$$-1+2sw(\alpha-1)^{-1}+\sum_{i=1}^{s}\left\{-\frac{sr_{2,1}(\alpha+s)}{\alpha+i-2}+1\right\}\frac{\Gamma(\alpha+i-1)}{\Gamma(\alpha)\Gamma(i)}\frac{w^\alpha\{r_{2,1}(\alpha+s)-w\}^{i-1}}{\{r_{2,1}(\alpha+s)\}^{\alpha+i-1}} = 0$$

of w in the interval $(0, r_{2,1}(\alpha+s))$. The main purpose of this paper is to derive the several properties of the optimal strategy for $n \ge 4$.

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3. Optimal Strategy

In this section, several properties of $G_n^k(w,\alpha)$ for $1 \le k \le n$ and $G_n(w,\alpha)$ for $n \ge 1$ are derived.

Lemma 3.1 *For $w > 0$ and $\alpha > 1$, if $f(x)$ is a continuous and strictly monotone increasing function of x, $E[f(X) \mid w,\alpha]$ is continuous and strictly monotone increasing in w and strictly monotone decreasing in α.*

Proof. For $0 < w_1 < w_2$,

$$E[f(X) \mid w_1, \alpha] - E[f(X) \mid w_2, \alpha]$$
$$= \int_0^\infty f(x)\{B(\alpha, s)\}^{-1}\{w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s} - w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s}\}dx.$$

Let the solution of the equation $w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s} - w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s} = 0$ of x be c, then

$$c = w_2 \left\{1 - \left(\frac{w_1}{w_2}\right)^{\frac{s}{\alpha+s}}\right\} \left\{\left(\frac{w_2}{w_1}\right)^{\frac{\alpha}{\alpha+s}} - 1\right\}^{-1}.$$

As $w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s} - w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s} > 0$ if $0 < x < c$ and $w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s} - w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s} < 0$ if $c < x$,

$$E[f(X) \mid w_1, \alpha] - E[f(X) \mid w_2, \alpha]$$
$$= \int_0^c f(x)\{B(\alpha, s)\}^{-1}\{w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s} - w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s}\}dx$$
$$+ \int_c^\infty f(x)\{B(\alpha, s)\}^{-1}\{w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s} - w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s}\}dx$$
$$< \int_0^c f(c)\{B(\alpha, s)\}^{-1}\{w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s} - w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s}\}dx$$
$$+ \int_c^\infty f(c)\{B(\alpha, s)\}^{-1}\{w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s} - w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s}\}dx$$
$$= f(c)\int_0^\infty \{B(\alpha, s)\}^{-1} w_1^\alpha x^{s-1}(w_1 + x)^{-\alpha-s}dx - f(c)\int_0^\infty \{B(\alpha, s)\}^{-1} w_2^\alpha x^{s-1}(w_2 + x)^{-\alpha-s}\}dx.$$

Since both integrals of this right hand side are 1, the monotonicity of $E[f(X) \mid w, \alpha]$ is derived. Also for $0 < \alpha_1 < \alpha_2$,

$$E[f(X) \mid w, \alpha_1] - E[f(X) \mid w, \alpha_2]$$
$$= \int_0^\infty f(x)\left[\{B(\alpha_1, s)\}^{-1} w^{\alpha_1} x^{s-1}(w + x)^{-\alpha_1-s} - \{B(\alpha_2, s)\}^{-1} w^{\alpha_2} x^{s-1}(w + x)^{-\alpha_2-s}\right] dx.$$

Let the solution of the equation $\{B(\alpha_1, s)\}^{-1} w^{\alpha_1} x^{s-1}(w+x)^{-\alpha_1-s} - \{B(\alpha_2, s)\}^{-1} w^{\alpha_2} x^{s-1}(w+x)^{-\alpha_2-s} = 0$ of x be c', then

$$c' = \left[\left\{\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_2)}\frac{\Gamma(\alpha_2 + s)}{\Gamma(\alpha_1 + s)}\right\}^{\frac{1}{\alpha_2-\alpha_1}} - 1\right] w.$$

As $\{B(\alpha_1, s)\}^{-1} w^{\alpha_1} x^{s-1}(w + x)^{-\alpha_1-s} - \{B(\alpha_2, s)\}^{-1} w^{\alpha_2} x^{s-1}(w + x)^{-\alpha_2-s} < 0$ if $0 < x < c'$ and $\{B(\alpha_1, s)\}^{-1} w^{\alpha_1} x^{s-1}(w + x)^{-\alpha_1-s} - \{B(\alpha_2, s)\}^{-1} w^{\alpha_2} x^{s-1}(w + x)^{-\alpha_2-s} > 0$ if $c' < x$, $E[f(X) \mid w, \alpha_1] > E[f(X) \mid w, \alpha_2]$ is derived from the monotonicity of $f(x)$ and

$$E[f(X) \mid w, \alpha_1] - E[f(X) \mid w, \alpha_2]$$
$$= \int_0^{c'} f(x)\left[\{B(\alpha_1, s)\}^{-1} w^{\alpha_1} x^{s-1}(w + x)^{-\alpha_1-s} - \{B(\alpha_2, s)\}^{-1} w^{\alpha_2} x^{s-1}(w + x)^{-\alpha_2-s}\right] dx$$
$$+ \int_{c'}^\infty f(x)\left[\{B(\alpha_1, s)\}^{-1} w^{\alpha_1} x^{s-1}(w + x)^{-\alpha_1-s} - \{B(\alpha_2, s)\}^{-1} w^{\alpha_2} x^{s-1}(w + x)^{-\alpha_2-s}\right] dx$$
$$> \int_0^{c'} f(c')\left[\{B(\alpha_1, s)\}^{-1} w^{\alpha_1} x^{s-1}(w + x)^{-\alpha_1-s} - \{B(\alpha_2, s)\}^{-1} w^{\alpha_2} x^{s-1}(w + x)^{-\alpha_2-s}\right] dx$$
$$+ \int_{c'}^\infty f(c')\left[\{B(\alpha_1, s)\}^{-1} w^{\alpha_1} x^{s-1}(w + x)^{-\alpha_1-s} - \{B(\alpha_2, s)\}^{-1} w^{\alpha_2} x^{s-1}(w + x)^{-\alpha_2-s}\right] dx,$$

where this right hand side reduces to 0 and the proof is completed.□

Lemma 3.2 *For $w > 0$ and $\alpha > 1$, $(\mathrm{A}_{n,k})$ for $1 \le k \le n$ holds for $n \ge 2$ and (A_n) holds for $n \ge 1$, where*

$(\mathrm{A}_{n,k})$: *$G_n^k(w,\alpha)$ is continuous and strictly monotone increasing in w and strictly monotone decreasing in α and*

(A_n) : *$G_n(w,\alpha)$ is continuous and strictly monotone increasing in w and strictly monotone decreasing in α.*

Proof. $(\mathrm{A}_{1,1})$ and (A_1) is trivial. For $n \ge 2$, assume that both $(\mathrm{A}_{m,k})$ for $1 \le k \le m$ and (A_m) hold for $1 \le m < n$. For $1 \le k \le n-1$, $(\mathrm{A}_{n,k})$ is derived from $(\mathrm{G}_{n,k})$, (A_{n-k}), and Lemma 3.1. Also, $(\mathrm{A}_{n,n})$ is derived from

$$G_n^n(w,\alpha) = nsw(\alpha-1)^{-1} + n^2$$

and as the result (A_n) holds.□

Five lemmas are derived as follows:

Lemma 3.3 *For $n \ge 2$, $2 \le k \le n$, $1 \le m \le k-1$, $w > 0$ and $\alpha > 1$,*
$(\mathrm{B}_{n,k,m})$: $G_n^k(w,\alpha) - G_{n-m}^{k-m}(w,\alpha) = msw(\alpha-1)^{-1} + m(n+k-m)$.

Proof. $(\mathrm{B}_{n,k,m})$ is derived from $(\mathrm{G}_{n,k})$ and $(\mathrm{G}_{n-m,k-m})$.□

Lemma 3.4 *For $n \ge 3$, $2 \le k \le n-1$, $w > 0$ and $\alpha > 1$, $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha)$ satisfies two equations:*

$(\mathrm{C}_{n,k})$: $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) = G_{n-1}^{k-1}(w,\alpha) - G_{n-1}^k(w,\alpha) - 1$, *and*

$(\mathrm{D}_{n,k})$: $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) = -n$
$$+E[G_{n-k}(w+X,\alpha+s) - G_{n-k-1}(w+X,\alpha+s) \mid w,\alpha].$$

Proof. $(\mathrm{C}_{n,k})$ is derived from $(\mathrm{B}_{n,k,1})$ and $(\mathrm{B}_{n,k+1,1})$. Also, $(\mathrm{D}_{n,k})$ is derived from $(\mathrm{G}_{n,k})$ and $(\mathrm{G}_{n,k+1})$.□

Lemma 3.5 *For $n \ge 3$, $1 \le k \le n-1$, $w > 0$ and $\alpha > 1$,*
$(\mathrm{E}_{n,k})$: $G_n^k(w,\alpha) - G_{n-1}^k(w,\alpha) = sw(\alpha-1)^{-1} + k$

$$+E[G_{n-k}(w+X,\alpha+s) - G_{n-k-1}(w+X,\alpha+s) \mid w,\alpha].$$

Proof. $(\mathrm{E}_{n,k})$ is derived from $(\mathrm{G}_{n,k})$ and $(\mathrm{G}_{n-1,k})$.□

Lemma 3.6 *For $n \ge 2$, $w > 0$ and $\alpha > 1$,*

$$sw(\alpha-1)^{-1} + n \le G_n(w,\alpha) - G_{n-1}(w,\alpha) \le nsw(\alpha-1)^{-1} + n.$$

Proof. Let $S(n;w,\alpha)$ be the optimal strategy in the state $(n;w,\alpha)$. Also, let $G_n^{S(n;w,\alpha)}(w,\alpha)$ and $G_{n-1}^{S(n;w,\alpha)}(w,\alpha)$ be the expected total completion times by using the same strategy $S(n;w,\alpha)$ when the current prior informations are $(n;w,\alpha)$ and $(n-1;w,\alpha)$, respectively. Then, $G_n(w,\alpha) = G_n^{S(n;w,\alpha)}(w,\alpha)$ and $G_{n-1}(w,\alpha) \le G_{n-1}^{S(n;w,\alpha)}(w,\alpha)$ are derived. Therefore,

$$G_n(w,\alpha) - G_{n-1}(w,\alpha) \ge G_n^{S(n;w,\alpha)}(w,\alpha) - G_{n-1}^{S(n;w,\alpha)}(w,\alpha) \ge sw(\alpha-1)^{-1} + n,$$

where the last inequality means that the difference, $G_n^{S(n;w,\alpha)}(w,\alpha) - G_{n-1}^{S(n;w,\alpha)}(w,\alpha)$, is the completion time of the n-th job which contains n processing times and at least 1 setup time. Also, let $S(n-1;w,\alpha)$ be the optimal strategy in the state $(n-1;w,\alpha)$, then $G_n(w,\alpha) \le G_n^{S(n-1;w,\alpha)}(w,\alpha)$ and $G_{n-1}(w,\alpha) = G_{n-1}^{S(n-1;w,\alpha)}(w,\alpha)$, from which

$$G_n(w,\alpha) - G_{n-1}(w,\alpha) \le G_n^{S(n-1;w,\alpha)}(w,\alpha) - G_{n-1}^{S(n-1;w,\alpha)}(w,\alpha) \le nsw(\alpha-1)^{-1} + n,$$

where the last inequality means that the difference, $G_n^{S(n-1;w,\alpha)}(w,\alpha) - G_{n-1}^{S(n-1;w,\alpha)}(w,\alpha)$, is the completion time of the n-th job which contains n processing times and at most n setup times. This completes the proof.□

Lemma 3.7 *For $n \ge 2$, $1 \le k \le n-1$, $w > 0$ and $\alpha > 1$,*

$$sw(\alpha-1)^{-1} - k \le G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) \le (n-k)sw(\alpha-1)^{-1} - k. \tag{8}$$

Proof. By taking conditional expectation $E[\cdot \mid w,\alpha]$ of each terms of the inequalities

$$s(w+X)(\alpha+s-1)^{-1} + (n-k) \le G_{n-k}(w+X,\alpha+s) - G_{n-k-1}(w+X,\alpha+s)$$
$$\le (n-k)s(w+X)(\alpha+s-1)^{-1} + (n-k)$$

which is derived from Lemma 3.6 and using

$$E[G_{n-k}(w+X,\alpha+s) - G_{n-k-1}(w+X,\alpha+s) \mid w,\alpha] = G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) + n$$

which is derived from $(\mathrm{D}_{n,k})$,

$$E[s(w+X)(\alpha+s-1)^{-1} \mid w,\alpha] + (n-k) \le G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) + n$$
$$\le E[(n-k)s(w+X)(\alpha+s-1)^{-1} \mid w,\alpha] + (n-k).$$

(8) is derived by using $E[s(w+X)(\alpha+s-1)^{-1} \mid w,\alpha] = sw(\alpha-1)^{-1}$ and subtracting n.□

Now the following theorem is derived.

Theorem 3.1 (P_n) *and* (Q_n) *hold for $n \ge 2$,* (R_n)*,* (S_n)*, and* (T_n) *hold for $n \ge 3$,* (U_n) *holds for $n \ge 4$ with*

$$(\mathrm{U}_3):\ G_3(w,\alpha) - G_2(w,\alpha) = \begin{cases} G_3^1(w,\alpha) - G_2^1(w,\alpha), & if \quad 0 < w < r_{3,1}(\alpha), \\ G_3^2(w,\alpha) - G_2^1(w,\alpha), & if \quad r_{3,1}(\alpha) \le w < r_{2,1}(\alpha), \\ G_3^2(w,\alpha) - G_2^2(w,\alpha), & if \quad r_{2,1}(\alpha) \le w < r_{3,1}(\alpha), \\ G_3^3(w,\alpha) - G_2^2(w,\alpha), & if \quad r_{3,1}(\alpha) \le w, \end{cases}$$

(V_n) *holds for $n \ge 1$, and* (W_n) *holds for $n \ge 2$, where*

(P_n): *for $1 \le k \le n-1$, $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha)$ is continuous and strictly monotone increasing in w and strictly monotone decreasing in α,*

(Q_n): *for $1 \le k \le n-1$, the equation $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) = 0$ of w has a unique root $r_{n,k}(\alpha)$ such that $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) < 0$ if $0 < w < r_{n,k}(\alpha)$, $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) > 0$ if $r_{n,k}(\alpha) < w$, and $r_{n,k}(\alpha) < r_{n,k}(\alpha+1)$,*

(R_n): *for $2 \le k \le n-1$, $r_{n-1,k-1}(\alpha) < r_{n,k}(\alpha)$,*

(S_n): *for $1 \le k \le n-2$, $r_{n,k}(\alpha) \le r_{n-1,k}(\alpha)$,*

$$(\mathrm{T}_n):\ G_n(w,\alpha) = \begin{cases} G_n^1(w,\alpha), & if \quad 0 < w < r_{n,1}(\alpha), \\ G_n^k(w,\alpha), & if \quad r_{n,k-1}(\alpha) \le w < r_{n,k}(\alpha) \quad (2 \le k \le n-1), \\ G_n^n(w,\alpha), & if \quad r_{n,n-1}(\alpha) \le w, \end{cases}$$

(U_n): $G_n(w,\alpha) - G_{n-1}(w,\alpha)$

$$= \begin{cases} G_n^1(w,\alpha) - G_{n-1}^1(w,\alpha), & if \quad 0 < w < r_{n,1}(\alpha), \\ G_n^2(w,\alpha) - G_{n-1}^1(w,\alpha), & if \quad r_{n,1}(\alpha) \le w < r_{n-1,1}(\alpha), \\ G_n^k(w,\alpha) - G_{n-1}^k(w,\alpha), & if \quad r_{n-1,k-1}(\alpha) \le w < r_{n,k}(\alpha), \\ G_n^{k+1}(w,\alpha) - G_{n-1}^k(w,\alpha), & if \quad r_{n,k}(\alpha) \le w < r_{n-1,k}(\alpha), \\ & \qquad (2 \le k \le n-2) \\ G_n^{n-1}(w,\alpha) - G_{n-1}^{n-1}(w,\alpha), & if \quad r_{n-1,n-2}(\alpha) \le w < r_{n,n-1}(\alpha), \\ G_n^n(w,\alpha) - G_{n-1}^{n-1}(w,\alpha), & if \quad r_{n,n-1}(\alpha) \le w, \end{cases}$$

(V_n): $G_n(w,\alpha) - G_{n-1}(w,\alpha)$ *is strictly monotone increasing in* w *and strictly monotone decreasing in* α*,*

(W_n): $G_n(w,\alpha) - G_{n-1}(w,\alpha) \ge G_{n-1}(w,\alpha) - G_{n-2}(w,\alpha) + 1$.

Proof. (P_2), (Q_2), (V_1), (V_2), and (W_2) are easily derived from the explicit formula of (2), (4), (5), (6) and (7) given in Section 2. (P_3) is derived from Lemma 3.1, ($\mathrm{D}_{3,1}$), ($\mathrm{D}_{3,2}$),

$$G_2(w+x,\alpha+s) = \begin{cases} 3s(w+x)(\alpha+s-1)^{-1}+3, & if \quad 0 < w+x < r_{2,1}(\alpha+s), \\ 2s(w+x)(\alpha+s-1)^{-1}+4, & if \quad r_{2,1}(\alpha+s) \le w+x, \end{cases}$$

$$G_1(w+x,\alpha+s) = s(w+x)(\alpha+s-1)^{-1}+1$$

and

$$G_0(w+x,\alpha+s) = 0.$$

For $1 \le k \le 2$, it is derived from Lemma 3.7 that $G_3^k(w,\alpha) - G_3^{k+1}(w,\alpha) < 0$ if $0 < w < (\alpha-1)k/\{(3-k)s\}$ and $G_3^k(w,\alpha) - G_3^{k+1}(w,\alpha) > 0$ if $(\alpha-1)k/s < w$. As $G_3^k(w,\alpha) - G_3^{k+1}(w,\alpha)$ is continuous and strictly monotone increasing in w, the equation $G_3^k(w,\alpha) - G_3^{k+1}(w,\alpha) = 0$ of w has a unique root $r_{3,k}(\alpha)$ in the closed interval $[(\alpha-1)k/\{(3-k)s\}, (\alpha-1)k/s]$, and $r_{3,k}(\alpha) < r_{3,k}(\alpha+1)$ is derived from the property that $G_3^k(w,\alpha) - G_3^{k+1}(w,\alpha)$ is strictly monotone decreasing in α, from which (Q_3) is derived. (R_3) is directly derived from $r_{2,1}(\alpha) = (\alpha-1)/s$ and $r_{3,2}(\alpha) = 2(\alpha-1)/s$. By considering the case of $0 < w < r_{2,1}(\alpha+s)$, the inequality $G_3^1(w,\alpha) - G_3^2(w,\alpha) > sw(\alpha-1)^{-1} - 1$ is easily derived and $G_3^1(r_{2,1}(\alpha),\alpha) - G_3^2(r_{2,1}(\alpha),\alpha) > 0$ holds, which means $r_{3,1}(\alpha) < r_{2,1}(\alpha)$ and (S_3) is derived. (T_3) is the immediate consequence of $r_{3,1}(\alpha) < r_{2,1}(\alpha) < r_{3,2}(\alpha)$ and (Q_3), and as the results (U_3) is derived from (5) and (T_3). As

$$G_3(w,\alpha) - G_2(w,\alpha) = \begin{cases} E[G_2(w+X,\alpha+s) \mid w,\alpha], & if \quad 0 < w < r_{3,1}(\alpha), \\ sw(\alpha-1)^{-1}+4, & if \quad r_{3,1}(\alpha) \le w \le r_{2,1}(\alpha), \\ 2sw(\alpha-1)^{-1}+3, & if \quad r_{2,1}(\alpha) < w < r_{3,2}(\alpha), \\ sw(\alpha-1)^{-1}+5, & if \quad r_{3,2}(\alpha) \le w, \end{cases} \tag{9}$$

and $G_2(w+x,\alpha+s)$ is strictly increasing in x, (V_3) is derived by using Lemma 3.1. Also

$$G_2(w+X,\alpha+s) \ge 2s(w+X)(\alpha+s-1)^{-1}+3$$

and

$$G_2(w,\alpha) - G_1(w,\alpha) + 1 = \begin{cases} 2sw(\alpha-1)^{-1}+3, & if \quad 0 < w < r_{2,1}(\alpha), \\ sw(\alpha-1)^{-1}+4, & if \quad r_{2,1}(\alpha) \le w, \end{cases}$$

with (9) means $G_3(w,\alpha) - G_2(w,\alpha) \ge G_2(w,\alpha) - G_1(w,\alpha) + 1$ and (W_3) holds. For $n \ge 4$, suppose that (P_m), (Q_m), (R_m), (S_m), (T_m), (U_m), (V_m), and (W_m) hold for $3 \le m \le n-1$. From ($\mathrm{D}_{n,k}$) and the monotone increasing property of $G_{n-k}(w+x,\alpha+s) - G_{n-k-1}(w+x,\alpha+s)$ in x obtained in (V_{n-k}) for $1 \le k \le n-1$, (P_n) is proved in the same way as the proof of

Lemma 3.1. From Lemma 3.7, for $1 \leq k \leq n-1$ $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) < 0$ if $0 < w < (\alpha-1)k\{(n-k)s\}^{-1}$ and $G_n^k(w,\alpha)-G_n^{k+1}(w,\alpha) > 0$ if $(\alpha-1)k/s < w$, which with (P_n) means the existence of the unique root $r_{n,k}(\alpha)$ of the equation $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) = 0$ of w. Also, $r_{n,k}(\alpha) < r_{n,k}(\alpha+1)$ is derived from the property that $G_n^k(w,\alpha)-G_n^{k+1}(w,\alpha)$ is strictly monotone decreasing in α, and therefore (Q_n) is derived. (R_n) is derived from the continuity and monotone increasing properties of $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha)$ in w, the definition of $r_{n,k}(\alpha)$, and $G_n^k(r_{n-1,k-1}(\alpha),\alpha) - G_n^{k+1}(r_{n-1,k-1}(\alpha),\alpha) = -1$ which is obtained by the definition of $r_{n-1,k-1}(\alpha)$ and $(\mathrm{C}_{n,k})$. (S_n) is derived from the definitions of $r_{n-1,k}(\alpha)$ and $r_{n,k}(\alpha)$ and the inequality $G_n^k(w,\alpha) - G_n^{k+1}(w,\alpha) \geq G_{n-1}^k(w,\alpha) - G_{n-1}^{k+1}(w,\alpha)$ which is obtained from $(\mathrm{D}_{n,k})$ and (W_{n-k}). (T_n) is derived from the definitions of $r_{n,k}(\alpha)$ for $1 \leq k \leq n-1$ and the inequalities

$$r_{n,1}(\alpha) < r_{n,2}(\alpha) < r_{n,3}(\alpha) < \cdots < r_{n,n-2}(\alpha) < r_{n,n-1}(\alpha)$$

obtained by (R_n) and (S_n). (U_n) is derived from (T_n) and (T_{n-1}) by using (R_n) and (S_n). To prove (V_n), the continuous and monotone properties of $G_n^k(w,\alpha) - G_{n-1}^k(w,\alpha)$ for $1 \leq k \leq n-1$ in w and α are derived from $(\mathrm{E}_{n,k})$ with Lemma 3.1 and those of $G_n^{k+1}(w,\alpha) - G_{n-1}^k(w,\alpha)$ for $1 \leq k \leq n-1$ are obtained from $(\mathrm{B}_{n,k+1,1})$. To prove (W_n), $r_{n-1,k}(\alpha)$ for $1 \leq k \leq n-2$ in (U_n) and (U_{n-1}) have the important roles in both $G_n(w,\alpha) - G_{n-1}(w,\alpha)$ and $G_{n-1}(w,\alpha) - G_{n-2}(w,\alpha)$. When $0 < w < r_{n-1,1}(\alpha)$, two cases, $0 < w < r_{n,1}(\alpha)$ and $r_{n,1}(\alpha) \leq w < r_{n-1,1}(\alpha)$, have to be considered. In the case of $0 < w < r_{n,1}(\alpha)$,

$$\begin{aligned}&\{G_n(w,\alpha) - G_{n-1}(w,\alpha)\} - \{G_{n-1}(w,\alpha) - G_{n-2}(w,\alpha)\}\\ &= \{G_n^1(w,\alpha) - G_{n-1}^1(w,\alpha)\} - \{G_{n-1}^1(w,\alpha) - G_{n-2}^1(w,\alpha)\}\end{aligned}$$

from which (W_n) is derived by using $(\mathrm{E}_{n,1})$, $(\mathrm{E}_{n-1,1})$ and (W_{n-1}). In the case of $r_{n,1}(\alpha) \leq w < r_{n-1,1}(\alpha)$, (W_n) is derived from

$$\begin{aligned}&\{G_n(w,\alpha) - G_{n-1}(w,\alpha)\} - \{G_{n-1}(w,\alpha) - G_{n-2}(w,\alpha)\}\\ &= \{G_n^2(w,\alpha) - G_{n-1}^1(w,\alpha)\} - \{G_{n-1}^1(w,\alpha) - G_{n-2}^1(w,\alpha)\}\\ &\geq \{G_n^2(w,\alpha) - G_{n-1}^2(w,\alpha)\} - \{G_{n-1}^1(w,\alpha) - G_{n-2}^1(w,\alpha)\}\end{aligned}$$

by using $(\mathrm{E}_{n,2})$ and $(\mathrm{E}_{n-1,1})$. When $r_{n-1,k-1}(\alpha) \leq w < r_{n-1,k}(\alpha)$ for $2 \leq k \leq n-2$, three cases, $r_{n-1,k-1}(\alpha) \leq w < \min\{r_{n,k}(\alpha), r_{n-2,k-1}(\alpha)\}$, $\min\{r_{n,k}(\alpha), r_{n-2,k-1}(\alpha)\} \leq w < \max\{r_{n,k}(\alpha), r_{n-2,k-1}(\alpha)\}$, and $\max\{r_{n,k}(\alpha), r_{n-2,k-1}(\alpha)\} \leq w < r_{n-1,k}(\alpha)$, have to be considered if $r_{n,k}(\alpha) \neq r_{n-2,k-1}(\alpha)$ and the first and third cases have to be considered if $r_{n,k}(\alpha) = r_{n-2,k-1}(\alpha)$. In the case of $r_{n-1,k-1}(\alpha) \leq w < \min\{r_{n,k}(\alpha), r_{n-2,k-1}(\alpha)\}$, (W_n) is derived from

$$\begin{aligned}&\{G_n(w,\alpha) - G_{n-1}(w,\alpha)\} - \{G_{n-1}(w,\alpha) - G_{n-2}(w,\alpha)\}\\ &= \{G_n^k(w,\alpha) - G_{n-1}^k(w,\alpha)\} - \{G_{n-1}^k(w,\alpha) - G_{n-2}^{k-1}(w,\alpha)\}\\ &\geq \{G_n^k(w,\alpha) - G_{n-1}^{k-1}(w,\alpha)\} - \{G_{n-1}^k(w,\alpha) - G_{n-2}^{k-1}(w,\alpha)\}\end{aligned}$$

by using $(\mathrm{B}_{n,k,1})$ and $(\mathrm{B}_{n-1,k,1})$. In the case of $\max\{r_{n,k}(\alpha), r_{n-2,k-1}(\alpha)\} \leq w < r_{n-1,k}(\alpha)$, (W_n) is derived from $(\mathrm{E}_{n,k+1})$, $(\mathrm{E}_{n-1,k})$ and

$$\begin{aligned}&\{G_n(w,\alpha) - G_{n-1}(w,\alpha)\} - \{G_{n-1}(w,\alpha) - G_{n-2}(w,\alpha)\}\\ &= \{G_n^{k+1}(w,\alpha) - G_{n-1}^k(w,\alpha)\} - \{G_{n-1}^k(w,\alpha) - G_{n-2}^k(w,\alpha)\}\\ &\geq \{G_n^{k+1}(w,\alpha) - G_{n-1}^{k+1}(w,\alpha)\} - \{G_{n-1}^k(w,\alpha) - G_{n-2}^k(w,\alpha)\}.\end{aligned}$$

In the case of $\min\{r_{n,k}(\alpha), r_{n-2,k-1}(\alpha)\} \leq w < \max\{r_{n,k}(\alpha), r_{n-2,k-1}(\alpha)\}$, two cases, $r_{n,k}(\alpha) \leq r_{n-2,k-1}(\alpha)$ and $r_{n,k}(\alpha) > r_{n-2,k-1}(\alpha)$, have to be considered. (W_n) is derived in the former case by using $(\mathrm{B}_{n,k+1,1})$, $(\mathrm{B}_{n-1,k,1})$ and

$$\begin{aligned}&\{G_n(w,\alpha)-G_{n-1}(w,\alpha)\}-\{G_{n-1}(w,\alpha)-G_{n-2}(w,\alpha)\}\\=&\{G_n^{k+1}(w,\alpha)-G_{n-1}^{k}(w,\alpha)\}-\{G_{n-1}^{k}(w,\alpha)-G_{n-2}^{k-1}(w,\alpha)\}\end{aligned}$$

and it is also derived in the latter case by using $(\mathrm{E}_{n,k})$, $(\mathrm{E}_{n-1,k})$, (W_{n-k}) and

$$\begin{aligned}&\{G_n(w,\alpha)-G_{n-1}(w,\alpha)\}-\{G_{n-1}(w,\alpha)-G_{n-2}(w,\alpha)\}\\=&\{G_n^{k}(w,\alpha)-G_{n-1}^{k}(w,\alpha)\}-\{G_{n-1}^{k}(w,\alpha)-G_{n-2}^{k}(w,\alpha)\}.\end{aligned}$$

When $r_{n-1,n-2}(\alpha)\le w$, two cases, $r_{n-1,n-2}(\alpha)\le w<r_{n,n-1}(\alpha)$ and $r_{n,n-1}(\alpha)<w$, have to be considered and (W_n) is derived from

$$\begin{aligned}&\{G_n(w,\alpha)-G_{n-1}(w,\alpha)\}-\{G_{n-1}(w,\alpha)-G_{n-2}(w,\alpha)\}\\=&\{G_n^{n-1}(w,\alpha)-G_{n-1}^{n-1}(w,\alpha)\}-\{G_{n-1}^{n-1}(w,\alpha)-G_{n-2}^{n-2}(w,\alpha)\}\\\ge&\{G_n^{n-1}(w,\alpha)-G_{n-1}^{n-2}(w,\alpha)\}-\{G_{n-1}^{n-1}(w,\alpha)-G_{n-2}^{n-2}(w,\alpha)\},\end{aligned}$$

$(\mathrm{B}_{n,n-1,1})$ and $(\mathrm{B}_{n-1,n-1,1})$ in the former case and it is also derived from

$$\begin{aligned}&\{G_n(w,\alpha)-G_{n-1}(w,\alpha)\}-\{G_{n-1}(w,\alpha)-G_{n-2}(w,\alpha)\}\\=&\{G_n^{n}(w,\alpha)-G_{n-1}^{n-1}(w,\alpha)\}-\{G_{n-1}^{n-1}(w,\alpha)-G_{n-2}^{n-2}(w,\alpha)\},\end{aligned}$$

$(\mathrm{B}_{n,n,1})$ and $(\mathrm{B}_{n-1,n-1,1})$ in the later case. This completes the proofs of both (W_n) and this theorem.□

From this theorem, the optimal strategy for the state (n,w,α) is described as follows:

(i) In the case of $n=1$, perform 1 job as a batch.

(ii) In the case of $n=2$, perform 1 job as a batch if $0<w<r_{2,1}(\alpha)$ and perform 2 jobs as a batch if $r_{2,1}(\alpha)\le w$.

(iii) In the case of $n\ge 3$, perform 1 job as a batch if $0<w<r_{n,1}(\alpha)$, perform k jobs as a batch if $r_{n,k-1}(\alpha)\le r_{n,k}(\alpha)$ for $2\le k\le n-1$, and perform n jobs as a batch if $r_{n,n-1}(\alpha)\le w$.

If there are some jobs remaining after performing k jobs and observing $X=x$, then the new state becomes $(n-k;w+x,\alpha+s)$, where $1\le k\le n-1$. The optimal strategy is applied for this new state again. This process is repeated until there is no job remaining.

Example 1. In the case of $n=5$, $w=w_0$, and $\alpha=\alpha_0$, the initial state is $(5,w_0,\alpha_0)$.

(I) If $0<w_0<r_{5,1}(\alpha_0)$, let the first batch size be 1 and observe $X=x_1$. There are 4 jobs remaining and the new state becomes $(4,w_0+x_1,\alpha_0+s)$. We have to consider four cases, (i), (ii), (iii), and (iv):

(i) If $0<w_0+x_1<r_{4,1}(\alpha_0+s)$, let the second batch size be 1 and observe $X=x_2$. The new state becomes $(3,w_0+x_1+x_2,\alpha_0+2s)$ and we have to consider 3 cases (a), (b), and (c):

(a) If $0<w_0+x_1+x_2<r_{3,1}(\alpha_0+2s)$, let the third batch size be 1, observe $X=x_3$, and the new state becomes $(2,w_0+x_1+x_2+x_3,\alpha_0+3s)$. If $0<w_0+x_1+x_2+x_3<r_{2,1}(\alpha_0+3s)$, let the forth batch size be 1 and observe $X=x_4$, and let the fifth batch size be 1, observe $X=x_5$, and stop. Otherwise, let the forth batch size be 2, observe $X=x_4$, and stop.

(b) If $r_{3,1}(\alpha_0+2s)\le w_0+x_1+x_2<r_{3,2}(\alpha_0+2s)$, let the third batch size be 2, observe $X=x_3$, and the new state becomes $(1,w_0+x_1+x_2+x_3,\alpha_0+3s)$. Then, the forth batch size be 1, observe $X=x_4$, and stop.

(c) If $r_{3,2}(\alpha_0+2s) \leq w_0+x_1+x_2$, let the third batch size be 3, observe $X = x_3$, and stop.

(ii) If $r_{4,1}(\alpha_0+s) \leq w_0+x_1 < r_{4,2}(\alpha_0+s)$, let the second batch size be 2, observe $X = x_2$, and the new state becomes $(2, w_0+x_1+x_2, \alpha_0+2s)$. Now, we have to consider two cases, (a) and (b):

(a) If $0 < w_0+x_1+x_2 < r_{2,1}(\alpha_0+2s)$, let the third batch size be 1, observe $X = x_3$, and the new state becomes $(1, w_0+x_1+x_2+x_3, \alpha_0+3s)$. Then, let the forth batch size be 1, observe $X = x_4$ and stop.

(b) Otherwise, let the third batch size be 2, observe $X = x_3$, and stop.

(iii) If $r_{4,2}(\alpha_0+s) \leq w_0+x_1 < r_{4,3}(\alpha_0+s)$, let the second batch size be 3, observe $X = x_2$, and the new state becomes $(1, w_0+x_1+x_2, \alpha_0+2s)$. Let the third batch size be 1, observe $X = x_3$, and stop.

(iv) If $r_{4,3}(\alpha_0+s) \leq w_0+x_1$, let the second batch size be 4 and observe $X = x_2$. There is no job remaining and stop.

(II) If $r_{5,1}(\alpha_0) \leq w_0 < r_{5,2}(\alpha_0)$, let the first batch size be 2 and observe $X = x_1$. There are 3 jobs remaining and the new state is $(3, w_0+x_1, \alpha_0+s)$. We have to consider three cases (i), (ii) and (iii):

(i) If $0 < w_0+x_1 < r_{3,1}(\alpha_0+s)$, let the second batch size be 1, observe $X = x_2$, and the new state becomes $(2, w_0+x_1+x_2, \alpha_0+2s)$. Furthermore, if $0 < w_0+x_1+x_2 < r_{2,1}(\alpha_0+2s)$, let the third batch size be 1 and observe $X = x_3$, then let the forth batch sizes be 1, observe $X = x_4$, and stop. Otherwise, let the third batch size be 2, observe $X = x_3$, and stop.

(ii) If $r_{3,1}(\alpha_0+s) \leq w_0+x_1 < r_{3,2}(\alpha_0+s)$, let the second batch size be 2, observe $X = x_2$, and the new state becomes $(1, w_0+x_1+x_2, \alpha_0+2s)$. Let the third batch size be 1, observe $X = x_3$, and stop.

(iii) If $r_{3,2}(\alpha_0+s) \leq w_0+x_1$, let the second batch size be 3, observe $X = x_2$, and stop.

(III) If $r_{5,2}(\alpha_0) \leq w_0 < r_{5,3}(\alpha_0)$, let the first batch size be 3 and observe $X = x_1$. There are 2 jobs remaining and the new state is $(2, w_0+x_1, \alpha_0+s)$. We have to consider two cases: (i) $0 < w_0+x_1 < r_{2,1}(\alpha_0+s)$ and (ii) $r_{2,1}(\alpha_0+s) \leq w_0+x_1$. In the former case, let the second batch size be 1 and observe $X = x_2$, then let the third batch size be 1, observe $X = x_3$, and stop. In the later case, let the second batch size be 2, observe $X = x_2$, and stop.

(IV) If $r_{5,3}(\alpha_0) \leq w_0 < r_{5,4}(\alpha_0)$, let the first batch size be 4 and observe $X = x_1$. There is 1 job remaining and the new state is $(1, w_0+x_1, \alpha_0+s)$. Let the second batch size be 1, observe $X = x_2$, and stop.

(V) If $r_{5,4}(\alpha_0) \leq w_0$, let the first batch size be 5 and observe $X = x_1$. There is no job remaining and stop.□

Example 2. Let $n = 7$, $s = 2$, $w = w_0$, and $\alpha = 2$, that is, we consider the case that 7 jobs have to be processed by single machine and the setup time is distributed in the gamma distribution with parameters $(u, 2)$, where the true value of u is unknown and we assume that u has the gamma distribution with parameters $(w_0, 2)$ as the prior distribution. The processing times of all the jobs are assumed to be 1. Let C_j $(j = 1, 2, \cdots, 7)$ be the completion time of j-th job. In this case, if $r_{7,1}(2) \leq w_0 < r_{7,2}(2)$, we decide that 2 jobs have to be processed as the first batch. After observing the setup time $X_1 = x_1$ of the first batch composed of the first and the second jobs, we have $C_1 = C_2 = x_1 + 2$. Now, the posterior distribution of u is the gamma distribution with parameters $(w_0+x_1, 4)$ and the number of remaining jobs is 5. From Theorem 3.1, $r_{7,1}(2) \leq r_{6,1}(2) \leq r_{5,1}(2) < r_{5,1}(3) < r_{5,1}(4)$, and we assume that $w_0+x_1 < r_{5,1}(4)$. As the optimal second batch size for the state $(5; w_0+x_1, 4)$

is 1, the second batch is composed of only one job and we observe the setup time $X_2 = x_2$. The completion time of the third job is $C_3 = (x_1 + 2) + (x_2 + 1) = x_1 + x_2 + 3$. The new state is $(4; w_0 + x_1 + x_2, 6)$. From Theorem 3.1, we have $r_{5,1}(4) \le r_{4,1}(4) \le r_{4,1}(5) \le r_{4,1}(6)$, and if $w_0 + x_1 + x_2 < r_{4,1}(6)$, the optimal third batch size is 1. After observing the setup time $X_3 = x_3$, $C_4 = x_1 + x_2 + x_3 + 4$. For the new state $(3; w_0 + x_1 + x_2 + x_3, 8)$, if $r_{3,2}(8) < w_0 + x_1 + x_2 + x_3$, the optimal forth batch size is 3 and after observing the forth setup time $X_4 = x_4$, all the jobs have been completed. The completion time of the last three jobs are the same and the value is $C_5 = C_6 = C_7 = x_1 + x_2 + x_3 + x_4 + 7$. The total completion time for 7 jobs is

$$\begin{aligned}\sum_{j=1}^{7} C_j &= 2(x_1+2) + (x_1+x_2+3) + (x_1+x_2+x_3+4) + 3(x_1+x_2+x_3+x_4+7) \\ &= 7(x_1+2) + 5(x_2+1) + 4(x_3+1) + 3(x_4+3) \\ &= 7x_1 + 5x_2 + 4x_3 + 3x_4 + 32.\end{aligned}$$

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