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NUMBERS OF PRIMAL AND DUAL BASES OF NETWORK FLOW AND UNIMODULAR INTEGER PROGRAMS

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Abstract To integer programming, algebraic approaches using Gröbner bases and standard pairs via toric ideals have been studied in recent years. In this paper, we consider a unimodular case, e.g., network flow problems, which enables us to analyze primal and dual problems in an equal setting. By combining existing results in an algebraic approach, we prove a theorem that the maximum number of dual feasible bases is obtained by computing the normalized volume of the convex hull generated from column vectors of a coefficient matrix in the primal standard form. We apply the theorem, partly with Gröbner bases theory, to transportation problems and minimum cost flow problems on acyclic tournament graphs. In consequence, we show new algebraic proofs to the Balinski and Russakoff's result on the dual transportation polytope and Klee and Witzgall's result on the primal transportation polytope. Similarly results for the primal case of acyclic tournament graphs are obtained by using Gelfand, Graev and Postnikov's result for nilpotent hypergeometric functions. We also give a bound of the number of feasible bases for its dual case.

Keywords: Network flow, computational algebra, Gröbner basis, standard pair, normalized volume

1. Introduction

Computational algebraic approaches by Gröbner bases [4] and standard pairs [8] to integer programming reveal various new properties, which can be used to obtain its combinatorial complexity bounds, especially tight ones in the unimodular and graphical cases. This paper investigates the numbers of primal and dual feasible bases of such cases via these computational algebraic methods.

For a linear programming problem whose coefficient matrix A is unimodular, all the feasible basic solutions are integral. The unimodularity are also satisfied in its dual problem, thus these two dualistic pair of problems can be regarded as integer programming problems. Then, computational algebra provides the following type of characterizations.

(1) Dual feasible bases are bases which do not contain any element in the initial ideal of the Gröbner basis for toric ideal of A with respect to a cost vector.
(2) The maximum number of dual feasible bases is equal to a certain volume of a polytope formed by column vectors of A.

Furthermore, from the unimodularity of the dual problem, primal feasible bases can be analyzed similarly. This paper introduces above characterizations about duality, together with their proofs where necessary, to unimodular and network-flow problems. Specifically, we obtain the following results:

- A complete proof of (2) in the unimodular case, which was originally communicated by Ohsugi and Hibi [7], based on an earlier result of the authors [12] in the graphical case.
- Another algebraic proofs of existing results on the numbers of primal [13] and dual [2]

feasible bases of the transportation problem.

- The maximum number of dual feasible bases of the minimum-cost flow problem on an acyclic tournament graph with d vertices is the $(d-1)$-th Catalan number $\frac{1}{d}\binom{2d-2}{d-1}$ by using (2) with a result for hypergeometric functions [6], while a lower bound for the number of primal feasible bases is $\Omega(2^{\lfloor d/6 \rfloor})$.

For the Lawrence lifting $\Lambda(A)$ of A (see [16]), some relations between dual feasible bases and bases of a vector matroid were obtained. As applications of them, the number of dual feasible bases for capacitated minimum cost flow problems on an acyclic tournament graph and that of dual and primal feasible bases for multidimensional transportation problems have been analyzed [10, 11], based on the theorem in this paper.

2. Preliminaries

2.1. Gröbner bases and standard pairs

We concern the family $IP_{A,\boldsymbol{c}}$ of all integer programs

$$IP_{A,\boldsymbol{c}}(\boldsymbol{b}) := minimize\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \mathbb{N}^n\}$$

($A \in \mathbb{Z}^{d\times n}$ and row full rank, $\boldsymbol{b} \in \mathbb{Z}^d$, $\boldsymbol{c} \in \mathbb{R}^n$ and $\mathbb{N}$ is the set of non-negative integers) as $\boldsymbol{b}$ varies in $\mathbb{N}A := \{A\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{N}^n\} \subseteq \mathbb{Z}^d$. We assume that $\boldsymbol{c}$ is generic, i.e., each program in $IP_{A,\boldsymbol{c}}$ has a unique optimal solution. Let $\mathcal{O}_{\boldsymbol{c}} \subseteq \mathbb{N}^n$ denote the set of optimal solutions for an integer program in $IP_{A,\boldsymbol{c}}$, and $\mathcal{N}_{\boldsymbol{c}} := \mathbb{N}^n \setminus \mathcal{O}_{\boldsymbol{c}}$ the set of non-optimal solutions.

To deal with above programs, we introduce two tools of computational algebra, which are *Gröbner bases* and *standard pair decompositions* of toric ideals. The former works as a *test set* for $IP_{A,\boldsymbol{c}}$, which means that we can obtain an optimal solution of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ by transforming any feasible solution with elements of test set. Roughly speaking, we can regard a Gröbner basis as the set $\mathcal{N}_{\boldsymbol{c}}$ of non-optimal solutions. On the other hand, the latter is introduced in order that we cover the set $\mathcal{O}_{\boldsymbol{c}}$ of optimal solutions of integer programs $IP_{A,\boldsymbol{c}}$ with a set of vectors and enumerate its number. Consequently we can say that the two tools correspond to a pair of complement sets in $\mathbb{N}^n$, and in this sense, are dual to each other.

Let k be a field and $k[\boldsymbol{x}] := k[x_1, \ldots, x_n]$ the polynomial ring where $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ be the set of variables. For an exponent vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, we denote $\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$. We define toric ideal I_A of $A \in \mathbb{Z}^{d\times n}$ as a binomial ideal $I_A := \langle \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} | A\boldsymbol{u} = A\boldsymbol{v}, : \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n\rangle$. Every vector $\boldsymbol{u} \in \mathbb{Z}^n$ can be written uniquely as $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$ where $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ are non-negative and have disjoint supports, hence $I_A = \langle \boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} | A\boldsymbol{u} = \boldsymbol{0}\rangle$. For $f \in I_A$ we define the initial term $in_{\boldsymbol{c}}(f)$ of f as the largest term of f by a term order $\succ_{\boldsymbol{c}}$ which is determined with respect to the inner product of an exponent vector and generic $\boldsymbol{c}$. Then we define the initial ideal $in_{\boldsymbol{c}}(I_A)$ as $in_{\boldsymbol{c}}(I_A) := \langle in_{\boldsymbol{c}}(f) | f \in I_A\rangle$. A finite subset $\mathcal{G}_{\boldsymbol{c}} = \{g_1, \ldots, g_s\} \subseteq I_A$ is a Gröbner basis for I_A with respect to $\boldsymbol{c}$ if $in_{\boldsymbol{c}}(I_A) = \langle in_{\boldsymbol{c}}(g_1), \ldots, in_{\boldsymbol{c}}(g_s)\rangle$. By fixing a term order and adding some conditions, Gröbner basis is determined uniquely [16].

We consider to cover the set $\mathcal{O}_{\boldsymbol{c}}$ of optimal solutions by pairs which contain a starting point $\boldsymbol{u}$ and a set of direction σ. For $\boldsymbol{u} \in \mathbb{N}^n$ and an index set $\sigma \subseteq [n] \equiv \{1, \ldots, n\}$, let $(\boldsymbol{u}, \sigma) := \left\{\boldsymbol{u} + \sum_{i\in\sigma} k_i\boldsymbol{e}_i \mid k_i \in \mathbb{N}\right\}$.

Definition 2.1 *$(\boldsymbol{u}, \sigma)$ is a* standard pair *of $\mathcal{O}_{\boldsymbol{c}}$ if it satisfies the following:*

(i) $\mathrm{supp}(\boldsymbol{u}) \cap \sigma = \emptyset$*, where* $\mathrm{supp}(\boldsymbol{u})$ *is* $\{i \mid u_i \neq 0\}$.

(ii) *Any point represented as $(\boldsymbol{u}, \sigma)$ is contained in $\mathcal{O}_{\boldsymbol{c}}$, i.e. $(\boldsymbol{u}, \sigma) \subseteq \mathcal{O}_{\boldsymbol{c}}$.*

(iii) $(\boldsymbol{u}, \sigma) \not\subset (\boldsymbol{v}, \tau)$ *for any other* $(\boldsymbol{v}, \tau)$ *that satisfies* (i) *and* (ii).

Let $S(\mathcal{O}_{\boldsymbol{c}})$ denote the set of all standard pairs of $\mathcal{O}_{\boldsymbol{c}}$. The cardinality of $S(\mathcal{O}_{\boldsymbol{c}})$ is called the arithmetic degree *of $\mathcal{O}_{\boldsymbol{c}}$ [17].*

At a first glance, standard pairs might look strange, but they have natural meanings in combinatorial optimization. In computational algebra, regular triangulations are used to analyze polytopal structures of standard pairs. Regular triangulations have the same structures with the dual polytope of the linear relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$, and here we introduce regular triangulation in preparation for the algebraic proofs in the next section.

Definition 2.2 *Let $A \in \mathbb{Z}^{d\times n}$ be a matrix with rank d and $\boldsymbol{c}$ be a cost vector. We define the* regular triangulation $\Delta_{\boldsymbol{c}}$ *of* cone(A) *as follows:*

(i) *σ is called a* face *of $\Delta_{\boldsymbol{c}}$ if and only if there exists a vector $\boldsymbol{y} \in \mathbb{R}^d$ such that $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{a}_i = c_i$ $(i \in \sigma)$ and $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{a}_j < c_j$ $(j \notin \sigma)$.*

(ii) *σ is called a* facet *of $\Delta_{\boldsymbol{c}}$ if σ is maximal face of $\Delta_{\boldsymbol{c}}$ with respect to inclusion.*

Faces of $\Delta_{\boldsymbol{c}}$ have a reciprocal inclusive relationship with faces of a dual polyhedron $\{\boldsymbol{y}|\boldsymbol{y}A \le \boldsymbol{c}\}$, more exactly, a feasible basis of the dual problem. Examples of such relationships are as follows:

face of $\Delta_{\boldsymbol{c}}$		dual polyhedron
$\emptyset$	$\Longleftrightarrow$	polytope itself
$\{i \mid i$ is one arbitrary element$\}$	$\Longleftrightarrow$	facets
facets	$\Longleftrightarrow$	vertices

When vertices of conv(A) are in an m-dimensional lattice $L \simeq \mathbb{Z}^m$, we define the *normalized volume* of a facet σ of $\Delta_{\boldsymbol{c}}$ by the volume of σ with normalization such that the volume of the convex hull of $\boldsymbol{0}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_m$ is 1. Here, $\{\boldsymbol{b}_i\}_{1\le i\le m}$ are the basis of the lattice L.

Lemma 2.3 ([16, 17])

(i) *If $\mathcal{O}_{\boldsymbol{c}}$ has $(*, \sigma)$, where $*$ denotes any point in $\mathbb{N}^n$, then σ is a face of $\Delta_{\boldsymbol{c}}$.*

(ii) *$\mathcal{O}_{\boldsymbol{c}}$ has $(\boldsymbol{0}, \sigma)$ as a standard pair if and only if σ is a facet of $\Delta_{\boldsymbol{c}}$.*

(iii) *If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ span an affine hyperplane, the number of standard pairs $(*, \sigma)$ for a facet σ of $\Delta_{\boldsymbol{c}}$ equals the normalized volume of σ.*

In this paper, we consider *unimodular integer programs*, i.e., integer programs whose coefficient matrices are unimodular.

Definition 2.4 *Let $A \in \mathbb{Z}^{d\times n}$ be a matrix with rank d. A is* unimodular *if each nonsingular submatrix of order d has determinant ± 1.*

Proposition 2.5 ([16]) *If A is unimodular, any reduced Gröbner basis of I_A is square-free.*

The best known examples of unimodular matrices are obtained from incidence matrices of directed graphs by deleting one row [14].

If A is unimodular, then all standard pairs are obtained from all facets of $\Delta_{\boldsymbol{c}}$.

Proposition 2.6 ([9]) *If A is unimodular, then $S(\mathcal{O}_{\boldsymbol{c}}) = \{(\boldsymbol{0}, \sigma) \mid \sigma$ is a facet of $\Delta_{\boldsymbol{c}}\}$.*

2.2. Transform of $IP_{A,\boldsymbol{c}}$

In this paper, we also consider the dual problem of the linear relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ which corresponds to some basis B. With non-basis N and basis B, A is represented as $(N\ B)$ and we rewrite the integer program as follows:

$$IP_{(N\ B),\boldsymbol{c}}(\boldsymbol{b}) := minimize\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid N\boldsymbol{x}' + B\boldsymbol{x}'' = \boldsymbol{b},\ \boldsymbol{x} \in \mathbb{N}^n\}$$

Therefore primal and dual integer problems are:

$$P_{(M\ I),\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}}) \quad \begin{array}{ll} \max & (-\widetilde{\boldsymbol{c}})^{\mathrm{T}}\boldsymbol{x}' \\ \text{s.t.} & M\boldsymbol{x}' + I_d\boldsymbol{x}'' = \widetilde{\boldsymbol{b}}, \\ & \boldsymbol{x}', \boldsymbol{x}'' \geq 0 \end{array} \qquad D_{(I\ -M^{\mathrm{T}}),\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}}) \quad \begin{array}{ll} \min & \widetilde{\boldsymbol{b}}^{\mathrm{T}}\boldsymbol{y}'' \\ \text{s.t.} & I_{n-d}\boldsymbol{y}' - M^{\mathrm{T}}\boldsymbol{y}'' = \widetilde{\boldsymbol{c}}, \\ & \boldsymbol{y}', \boldsymbol{y}'' \geq 0 \end{array}$$

where $M = B^{-1}N \in \mathbb{Z}^{d\times(n-d)}$, $\widetilde{\boldsymbol{b}} = (\widetilde{b}_i)_{i\in B} = B^{-1}\boldsymbol{b} \in \mathbb{Z}^d$, $\boldsymbol{x}''$ (resp. $\boldsymbol{x}'$) is a basic (resp. non-basic) variable for $P_{(M\ I),\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}})$, $\boldsymbol{y}'$ (resp. $\boldsymbol{y}''$) is a basic (resp. non-basic) variable for $D_{(I\ -M^{\mathrm{T}}),\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}})$, and $\widetilde{\boldsymbol{c}} = \boldsymbol{c}_N - N^{\mathrm{T}}(B^{-1})^{\mathrm{T}}\boldsymbol{c}_B \in \mathbb{Z}^{n-d}$ is a reduced cost for B. By considering integer programs by such forms, both primal and dual problems come to be expressed in standard forms, hence it enables us to analyze their toric ideals of $(M\ I_d)$ and $(I_{n-d}\ -M^{\mathrm{T}})$ in a dual setting.

3. Maximum Number of Dual Feasible Bases

For a matrix $A \in \mathbb{Z}^{d\times n}$, the *homogenized matrix* $A' \in \mathbb{Z}^{(d+1)\times(n+1)}$ of A is

$$A' := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ & & & & 0 \\ & & A & & \vdots \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ & & & & 0 \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \vdots \\ & & & & 0 \end{pmatrix}.$$

Let $\boldsymbol{a}'_i = \binom{1}{\boldsymbol{a}_i}$ for $1 \leq i \leq n$ and $\boldsymbol{a}'_{n+1}$ be the $(n+1)$-th column vector of A'. We remark that $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_n, \boldsymbol{a}'_{n+1}$ span an affine hyperplane. If column vectors of A themselves span an affine hyperplane, then the normalized volume of $\mathrm{conv}(A')$ is equal to that of $\mathrm{conv}(A)$.

We consider another integer program

$$IP_{A',(\boldsymbol{c},0)}(\boldsymbol{b}, M) := minimize\ \left\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \;\middle|\; A'\binom{\boldsymbol{x}}{x_{n+1}} = \binom{M}{\boldsymbol{b}}, \binom{\boldsymbol{x}}{x_{n+1}} \in \mathbb{N}^{n+1}\right\}$$

for $M \in \mathbb{Z}$, and the family $IP_{A',(\boldsymbol{c},0)}$ of integer programs $IP_{A',(\boldsymbol{c},0)}(\boldsymbol{b}, M)$ as $\binom{M}{\boldsymbol{b}}$ varies in $\{A'\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{N}^{n+1}\}$. If $\boldsymbol{c}$ is generic, $(\boldsymbol{c}, 0)$ is also generic. Let $\mathcal{O}'_{(\boldsymbol{c},0)}$ be the set of all the optimal solutions of all programs in $IP_{A',(\boldsymbol{c},0)}$.

Proposition 3.1 ([17]) *$(\boldsymbol{a}, \sigma) \in S(\mathcal{O}_{\boldsymbol{c}})$ if and only if $\left(\binom{\boldsymbol{a}}{0}, \sigma \cup \{n+1\}\right) \in S(\mathcal{O}'_{(\boldsymbol{c},0)})$.*

As $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_{n+1}$ span an affine hyperplane, the normalized volume of $\mathrm{conv}(A')$ gives the number of standard pairs of $in_{(\boldsymbol{c},k)}(I_{A'})$, which correspond to the maximal faces of $\Delta'_{(\boldsymbol{c},k)}$ by Lemma 2.3 (iii), for any $k \in \mathbb{R}$ such that $(\boldsymbol{c}, k)$ becomes homogeneous. Therefore, using Proposition 3.1, the maximum arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ can be obtained via $\mathrm{conv}(A')$.

Theorem 3.2 ([7]) *If A is unimodular, then there exists a cost vector $\boldsymbol{c}$ such that the number of dual feasible bases for $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ is equal to the normalized volume of* $\mathrm{conv}(A')$.

Proof: For any $\boldsymbol{c}$, the set of standard pairs of $\mathcal{O}_{\boldsymbol{c}}$ is $\{(\boldsymbol{0}, \sigma) \mid \sigma \text{ is a maximal face of } \Delta_{\boldsymbol{c}}\}$, and each $(\boldsymbol{0}, \sigma)$ corresponds to the standard pair $(\boldsymbol{0}, \sigma \cup \{n+1\})$ of $\mathcal{O}'_{(\boldsymbol{c},0)}$. Especially, $\sigma \cup \{n+1\}$ is a facet of $\Delta'_{(\boldsymbol{c},0)}$. Therefore,

$$\begin{aligned} |\mathcal{O}_{\boldsymbol{c}}| &= |\{\sigma \mid (\boldsymbol{0}, \sigma) \in S(\mathcal{O}_{\boldsymbol{c}})\}| \\ &= \left|\left\{\sigma \mid (\boldsymbol{0}, \sigma \cup \{n+1\}) \in S\left(\mathcal{O}'_{(\boldsymbol{c},0)}\right)\right\}\right| \\ &\leq \left|\left\{(*, \tau) \in S\left(\mathcal{O}'_{(\boldsymbol{c},0)}\right) \mid \tau:\ \text{maximal face of } \Delta'_{(\boldsymbol{c},0)}\right\}\right| \\ &= \text{normalized volume of } \mathrm{conv}(A'). \end{aligned}$$

We show there exists a vector $\boldsymbol{c}$ which satisfies above equality. Let $I_A \subset k[x_1, \ldots, x_n]$ and $I_{A'} \subset k[x_1, \ldots, x_n, x_{n+1}]$. Then, $\boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}} x_{n+1}^k \in I_{A'}$ ($\boldsymbol{x}^{\boldsymbol{a}},\ \boldsymbol{x}^{\boldsymbol{b}} \in k[x_1, \ldots, x_n]$) if and only if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i + k$ and $\boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}} \in I_A$. We consider that $\boldsymbol{c} = (1, 1, \ldots, 1)$ and $\succ$ is any term order. For any g in the reduced Gröbner basis $\mathcal{G}$ for $I_{A'}$ with respect to $\succ_{(\boldsymbol{c},0)}$, $in_{\succ_{(\boldsymbol{c},0)}}(g)$ does not contain x_{n+1}, and $in_{\succ_{(\boldsymbol{c},0)}}(g)$ is square-free as $\{in_{\succ_{(\boldsymbol{c},0)}}(g) \mid g \in \mathcal{G}\}$ minimally generates $in_{\succ'_{\boldsymbol{c}}}(I_A)$ where $\succ'$ is the restriction of $\succ$ to $k[x_1, \ldots, x_n]$. Thus, the corresponding triangulation $\Delta'_{\succ(\boldsymbol{c},0)}$ is unimodular [16], and each maximal face for $\Delta'_{\succ(\boldsymbol{c},0)}$ corresponds to a standard pair of $in_{\boldsymbol{c}}(I_A)$ bijectively. Then, the arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ is equal to the number of maximal faces of $\Delta'_{\succ(\boldsymbol{c},0)}$, which is the normalized volume of $\mathrm{conv}(A')$. $\square$

4. Number of Feasible Bases of Several Network Problems

4.1. Maximum number for the transportation problem on $K_{m,n}$

Let A be a matrix obtained by deleting redundant rows of the incidence matrix of the bipartite graph $K_{m,n}$. Then $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ is the transportation problem on $K_{m,n}$:

$$P_{A,\boldsymbol{c}}(\boldsymbol{b}) := \textit{minimize}\ \{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}.$$

We give another proof for the following result by Balinski and Russakoff [2].

Theorem 4.1 ([2]) *The maximum number of vertices for the dual polyhedron of the transportation problem on $K_{m,n}$ is equal to $\binom{m+n-2}{m-1}$.*

As column vectors of A span an affine hyperplane, the normalized volume of $\mathrm{conv}(A')$ is equal to that of $\mathrm{conv}(A)$. To show the theorem, we show a unimodular triangulation of $\mathrm{conv}(A)$, i.e., a triangulation such that the normalized volume of any facet is 1.

Lemma 4.2 ([16]) *Let $\succ$ be the reverse lexicographic order induced from the variable ordering*

$$x_{1,1} \prec x_{1,2} \prec \cdots \prec x_{1,n} \prec x_{2,1} \prec \cdots \prec x_{m,n}.$$

Then the reduced Gröbner basis of I_A with respect to $\succ$ equals

$$\mathcal{G}_{\succ} = \{\underline{x_{i,l}x_{j,k}} - x_{i,k}x_{j,l} \mid 1 \leq i < j \leq m,\ 1 \leq k < l \leq n\},$$

where underlined term is the initial term.

Corollary 4.3 ([16]) *Let $\boldsymbol{c} \in \mathbb{R}^{mn}$ be a cost vector that satisfies*

$$c_{i,l} + c_{j,k} > c_{i,k} + c_{j,l} \quad \textit{for any } 1 \leq i < j \leq m,\ 1 \leq k < l \leq n.$$

Then $\sigma \subset \{(i,j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a facet of the regular triangulation for $\boldsymbol{c}$ of $\mathrm{conv}(A)$ *if and only if, for any pair (p,q) and (r,s) in σ, $p \leq r$ and $q \leq s$. Furthermore, the normalized volume of σ is* 1.

Let us consider the table of size $m \times n$ as below.

$(1,1)$	$(1,2)$	$\cdots$	$(1,n)$
$(2,1)$	$(2,2)$	$\cdots$	$(2,n)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(m,1)$	$(m,2)$	$\cdots$	(m,n)

Then Corollary 4.3 implies that each path from $(1,1)$ to (m,n) of length $m+n-2$ corresponds to a facet for $\boldsymbol{c}$. Therefore, the normalized volume of $\mathrm{conv}(A)$ is equal to the total number of such paths, which is $\binom{m+n-2}{m-1}$.

4.2. Maximum number for the dual transportation problem on $K_{2,n}$

Now we consider the dual problem of the transportation problem on $K_{2,n}$. Our results in previous sections give another proof for the following result by Klee and Witzgall [13].

Theorem 4.4 ([13]) *The maximum number of vertices for the feasible region of the transportation problem on $K_{2,n}$ is equal to $(n - \lfloor n/2 \rfloor)\binom{n}{\lfloor n/2 \rfloor}$.*

As one constraint of the transportation problem $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ on $K_{2,n}$ is redundant, we can consider the problem $P_{\overline{A},\boldsymbol{c}}(\overline{\boldsymbol{b}})$, which is obtained from $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ by deleting the second constraint.

For the basis $B := \{(1,n),(2,1),(2,2),\ldots,(2,n)\}$, the coefficient matrix of the dictionary of $P_{\overline{A},\boldsymbol{c}}(\overline{\boldsymbol{b}})$ is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and the coefficient matrix D for its dual problem is

$$D := \begin{pmatrix} -1 & -1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Therefore, $\mathrm{conv}(D')$ is a linear transformation of the convex hull of $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{n-1},-\boldsymbol{e}_1,\ldots,-\boldsymbol{e}_{n-1},\mathbf{1},-\mathbf{1},\mathbf{0}\}$, where $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{n-1}$ are unit vectors of $\mathbb{R}^{n-1}$, $\mathbf{1} \in \mathbb{R}^{n-1}$ is the vector all of whose elements are 1, and $\mathbf{0} \in \mathbb{R}^{n-1}$ is the origin. Let P_n denote the convex hull of $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{n-1},-\boldsymbol{e}_1,\ldots,-\boldsymbol{e}_{n-1},\mathbf{1},-\mathbf{1},\mathbf{0}\}$.

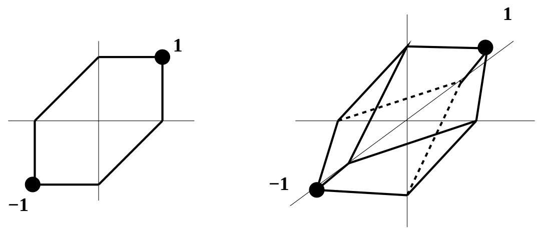

Figure 1: Polytope P_3 and P_4

Let $\boldsymbol{p} = (p_1,\ldots,p_n) \in \mathbb{R}^n$ be a point outside a polytope $P \subset \mathbb{R}^n$, and F be a facet of P whose supporting hyperplane is $a_1x_1 + \cdots + a_nx_n = a_0$. F is *visible* from $\boldsymbol{p}$ if $a_1p_1 + \cdots + a_np_n > a_0$. Then $\mathrm{conv}(P \cup \{\boldsymbol{p}\}) = P \cup \{\mathrm{conv}(F \cup \{\boldsymbol{p}\}) \mid F \text{ is visible from } \boldsymbol{p}\}$ (see [3]).

Proof of Theorem 4.4: We calculate the normalized volume of P_n. We decompose P_n to $V_n := \mathrm{conv}(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{n-1},-\boldsymbol{e}_1,\ldots,-\boldsymbol{e}_{n-1})$ and its outside.

First, we show that the normalized volume of V_n is equal to 2^{n-1} by induction for n. Clearly, the normalized volume of V_2 is 2. As V_n is the convex hull of V_{n-1}, $\boldsymbol{e}_{n-1}$ and $-\boldsymbol{e}_{n-1}$, the normalized volume of V_n is twice that of $\mathrm{conv}(V_{n-1},\boldsymbol{e}_{n-1})$. By the hypothesis of induction, the normalized volume of $\mathrm{conv}(V_{n-1},\boldsymbol{e}_{n-1})$ is 2^{n-2}, and that of V_n is shown to be 2^{n-1}.

We next calculate the normalized volume of outside of V_n. As V_n is an $(n-1)$-dimensional *crosspolytope*, a hyperplane $a_1x_1 + \cdots + a_{n-1}x_{n-1} = a_0$ is a supporting hyperplane of some facet of V_n if and only if $a_0 = 1$ and $|a_i| = 1$ for any $i = 1, \ldots, n-1$ [18]. Let $F(a_1, \ldots, a_{n-1})$ be a facet of V_n whose supporting hyperplane is $a_1x_1 + \cdots + a_{n-1}x_{n-1} = 1$. Then, if $F(a_1, \ldots, a_{n-1})$ is visible from $\mathbf{1}$, $F(-a_1, \ldots, -a_{n-1})$ is visible from $-\mathbf{1}$, and the sets of facets of V_n that are visible from $\mathbf{1}$ and that are visible from $-\mathbf{1}$ are disjoint. Therefore, we need to consider only the facets that are visible from $\mathbf{1}$.

$F(a_1, \ldots, a_{n-1})$ is visible from $\mathbf{1}$ if and only if $a_1 + \cdots + a_{n-1} > 1$. We compute the normalized volume of $\mathrm{conv}(F \cup \{\mathbf{1}\})$ for a facet F of P_n that is visible from $\mathbf{1}$. As vertices of P_n lie on the lattice generated by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$, the normalized volume of $\mathrm{conv}(F \cup \{\mathbf{1}\})$ is k if the Euclidean distance between $\mathbf{1}$ and the hyperplane of F is $k/\sqrt{n-1}$. Thus, for a facet $F(a_1, \ldots, a_{n-1})$ of V_n visible from $\mathbf{1}$, the normalized volume of $\mathrm{conv}(F(a_1, \ldots, a_{n-1}) \cup \{\mathbf{1}\})$ is $p-1$ if and only if $|\{i \mid a_i = 1\}| - |\{i \mid a_i = -1\}| = p$. Therefore, for $k = 0, \ldots, \lfloor \frac{n-2}{2} \rfloor$, the number of facets $F(a_1, \ldots, a_{n-1})$ such that the normalized volume of $\mathrm{conv}(F(a_1, \ldots, a_{n-1}) \cup \{\mathbf{1}\}) = n - 2k - 2$ is equal to $\binom{n-1}{k}$, and the normalized volume of P_n is equal to $2^{n-1} + 2\sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-1}{k}(n-2k-2) = (n - \lfloor n/2 \rfloor)\binom{n}{\lfloor n/2 \rfloor}$. □

4.3. Maximum number for the minimum-cost flow problem on the acyclic tournament graph

A matrix A denotes an incidence matrix of a graph. As one constraint of $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ is redundant, we can consider the problem $P_{\overline{A},\boldsymbol{c}}(\overline{\boldsymbol{b}})$, which is obtained from $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ by deleting the last constraint. Then $in_{\boldsymbol{c}}(I_A) = in_{\boldsymbol{c}}(I_{\overline{A}})$, and $\overline{A}$ is row-full rank. In addition, for two dual polyhedra $\{\boldsymbol{y} | A\boldsymbol{y} \leq \boldsymbol{c}\}$ and $\{\overline{\boldsymbol{y}} | \ \overline{\boldsymbol{y}}\overline{A} \leq \boldsymbol{c}\}$ there exists a surjection from $\boldsymbol{y}$ to $\overline{\boldsymbol{y}}$ and both triangulations come to be identical. Let $\Delta_{\boldsymbol{c}}$ denote both triangulations.

As any initial ideal $in_{\boldsymbol{c}}(I_A)$ is generated by square-free monomials(Proposition 2.5) , the standard pairs $S(in_{\boldsymbol{c}}(I_A))$ are $(\mathbf{0}, \sigma)$ where σ ranges among all maximal faces for $\boldsymbol{c}$.

The arcs in the optimum flow of uncapacitated minimum cost flow problems form a forest [1]. Therefore, as the rank of A equals $d-1$, the next proposition is implied by Lemma 2.3, Proposition 2.5, and Proposition 2.6.

Proposition 4.5 $(\boldsymbol{a}, \sigma)$ *is a standard pair of* $in_{\boldsymbol{c}}(I_A)$ *if and only if* $\boldsymbol{x}^{\boldsymbol{a}} = 1$ *and* σ *is a spanning tree of* G_d *such that* $\boldsymbol{x}^\sigma \notin in_{\boldsymbol{c}}(I_A)$.

The results shown in Section 3 indicate that there is a one-to-one correspondence between the standard pairs $(1, *)$,where $*$ means any vector $\subseteq [n]$, of $in_{\boldsymbol{c}}(I_A)$ and the dual feasible bases of $P_{\overline{A},\boldsymbol{c}}(\overline{\boldsymbol{b}})$. Therefore, the Hoşten-Thomas algorithm for the minimum cost flow problem $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ is a variant of the enumeration of dual feasible bases.

The Gröbner bases shown in the previous section give upper and lower bounds for the arithmetic degree (i.e., bounds for the number of vertices of the dual polyhedron). The genericity of $\boldsymbol{c}$ implies that the arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ is equal to or greater than 1.

Theorem 4.6 *The minimum arithmetic degree of* $in_{\boldsymbol{c}}(I_A)$ *in which* $\boldsymbol{c}$ *varies all generic cost vectors is equal to* 1.

Proof: For a cost vector $\boldsymbol{c} = (c_{1,2}, \ldots, c_{1,d}, c_{2,3}, \ldots, c_{d-1,d})$ satisfying $c_{i,j} > c_{i,i+1} + c_{i+1,i+2} \cdots + c_{j-1,j}$ for any $i < j-1$, $in_{\boldsymbol{c}}(I_A) = \langle x_{i,j} \mid j - i > 1 \rangle$. Then $\boldsymbol{x}^{\boldsymbol{a}} \notin in_{\boldsymbol{c}}(I_A)$ if and only if $a_{i,j} = 0$ for any (i,j) such that $j - i > 1$. The set of all such monomials is equal to $(1, \{(1,2), (2,3), \ldots, (d-1,d)\})$. Thus, only this pair is a standard pair of $in_{\boldsymbol{c}}(I_A)$. □

In order to show the upper bound, we use the next result based on the study of hypergeometric systems on unipotent matrices reported by Gelfand et al. [6].

Lemma 4.7 ([6]) *Let A' be the homogenized matrix* (3) *for the incidence matrix A of the acyclic tournament graph with d vertices, and* $\mathrm{conv}(A')$ *be the convex hull of* $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_{n+1}$. *Then, the normalized volume of* $\mathrm{conv}(A')$ *is equal to the* $(d-1)$*-th Catalan number* C_{d-1}.

Therefore, by Theorem 3.2, we obtain the upper bound for the arithmetic degree.

Theorem 4.8 *The maximum arithmetic degree of* $in_{\boldsymbol{c}}(I_A)$ *in which* $\boldsymbol{c}$ *varies all generic cost vectors equals* $C_{d-1} := \frac{1}{d}\binom{2(d-1)}{d-1}$, *which is the* $(d-1)$*-th Catalan number.*

The Catalan number equals $\frac{4^n}{\sqrt{\pi}n^{3/2}}\left(1+O\left(\frac{1}{n}\right)\right)$ (e.g., see [5]). This number is exponential for n.

We show an example of a cost vector which achieves the maximum arithmetic degree in Theorem 4.8.

Theorem 4.9 *For the cost vector satisfying* $c_{i,j}+c_{j,k} > c_{i,k}$ *for any* $i<j<k$ *and* $c_{i,k}+c_{j,l} > c_{i,l}+c_{j,k}$ *for any* $i<j<k<l$, *the arithmetic degree of* $in_{\boldsymbol{c}}(I_A)$ *is the* $(d-1)$*-th Catalan number.*

Proof: $(\mathbf{0}, \sigma)$ is a standard pair of $in_{\boldsymbol{c}}(I_A)$ if and only if σ is a spanning tree of the acyclic tournament graph that satisfies the following two conditions, which is called *standard tree* in [6]:

(a) there are no $1 \leq i < j < k \leq d$ such that both (i,j) and (j,k) are arcs in σ, and
(b) there are no $1 \leq i < j < k < l \leq d$ such that both (i,k) and (j,l) are arcs in σ.

The number of such spanning trees is the $(d-1)$-th Catalan number (e.g., see [6, 15]). □

4.4. Lower bound for the dual minimum-cost flow problem on the acyclic tournament graph

Similar to Section 4.3, we assume here that $\widetilde{\boldsymbol{b}}$ is generic. As any initial ideal $in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$ is generated by square-free monomials, any standard pair in $S(in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})}))$ is of the form $(1, *)$. Moreover, the support of each optimal solution of $D_{(I\ -M^{\mathrm{T}}),\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}})$ does not include a cutset.

Proposition 4.10 *For a cost vector* $\widetilde{\boldsymbol{b}}$ *such that the linear system* $(M\ I)\boldsymbol{x} = \widetilde{\boldsymbol{b}}_B$ *has a non-negative solution,* $I_{(I\ -M^{\mathrm{T}})}$ *has a reduced Gröbner basis with respect to* $\widetilde{\boldsymbol{b}}$.

Proposition 4.11 $(\boldsymbol{a}, \sigma)$ *is a standard pair of* $in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$ *if and only if* $\boldsymbol{x}^{\boldsymbol{a}} = 1$ *and* σ *is a co-tree of* G_d *such that* $\boldsymbol{x}^{\sigma} \notin in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$.

Theorem 4.12 *For any* $\widetilde{\boldsymbol{b}}$ *that satisfies the condition in Proposition 4.10, there exists* $S \subset \{1, \ldots, d-1\}$ *with* $|S| \geq \lfloor (d-1)/6 \rfloor$ *such that, for any* $\sigma \subseteq S$, *there exists a spanning tree* T_σ *of* G_d *which satisfies the following:*

(A) T_σ *contains the arc set* $\{(i, i+1) \mid i \in S \backslash \sigma\}$ *and does not contain any arc in* $\{(j, j+1) \mid j \in \sigma\}$
(B) $(1, \overline{T_\sigma})$ *is a standard pair of* $in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$, *where* $\overline{T_\sigma} := E \backslash T_\sigma$ *is a co-tree of* T_σ.

In particular, as $T_\sigma \neq T_\tau$ *for any* $\sigma, \tau \subseteq S$ $(\sigma \neq \tau)$, $in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$ *has at least* $\Omega(2^{\lfloor d/6 \rfloor})$ *standard pairs for any generic* $\boldsymbol{b}$ *that satisfies the condition in Proposition 4.10.*

Proof: We divide $\{1, \ldots, d-1\}$ into the following four subsets.

$$M_0 := \{i \in \{1, \ldots, d-1\} \mid x_{i,i+1} \in in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})\}$$
$$M_k := \{i \in \{1, \ldots, d-1\} \mid i \notin M_0,\ i \equiv k-1 \pmod 3\} \text{ for } k = 1, 2, 3.$$

Lemma 4.13 $|M_0| \le \lceil (d-1)/2 \rceil$.

Proof of Lemma 4.13: We consider a cutset D which corresponds to (V^+, V^-) such that f_D contains $x_{i,j}$ as a term of degree 1. Without loss of generality, we set $i \in V^+$. On the assumption that $j - i > 1$, for any k $(i < k < j)$, if $k \in V^+$ then f_D contains $x_{k,j}$ and $x_{i,j}$ in the same term, otherwise f_D contains $x_{i,k}$ and $x_{i,j}$ in the same term, which contradicts that $x_{i,j}$ is a term of f_D of degree 1. Thus, $j = i+1$. In addition, $k \in V^-$ for any $k < i$ and $k \in V^+$ for any $k > i+1$. Therefore, $V^+ = \{i, i+2, i+3, \dots, d\}$ and $V^- = \{1, \dots, i-1, i+1\}$.

We consider that $in_{\widetilde{\boldsymbol{b}}}(f) = x_{i,i+1}$ for some $f \in I_{(I\ -M^{\mathrm{T}})}$. If $x_{i-1,i} \in in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$, then f can be reduced by the binomial corresponding to the cutset between $\{i-1\ , i+1, \dots, d\}$ and $\{1, \dots, i-2, i\}$ to

$$f' \ := \ x_{i,i+1} - \left\{ \left(\prod_{k \le i-2} x_{k,i} \right) \left(\prod_{k \ge i+2} x_{i+1,k} \right) \left(\prod_{\substack{k \le i-1, \\ l \ge i+2}} x_{k,l} \right) \left(\prod_{k \le i-2} x_{k,i-1} \right) \left(\prod_{k \ge i+1} x_{i,k} \right) \left(\prod_{\substack{k \le i-2, \\ l \ge i+1}} x_{k,l} \right) \right\},$$

and its initial term is $x_{i,i+1}$. As both terms of this binomial contain $x_{i,i+1}$, this implies that $in_{\widetilde{\boldsymbol{b}}}(f'/x_{i,i+1}) = 1$. As $\widetilde{\boldsymbol{b}}$ defines a term order by Proposition 4.10, this is a contradiction. $\square$

Thus, at least one of $M_1,\ M_2,\ M_3$ has at least $\lfloor (d-1)/6 \rfloor$ elements. Let S be one such M_i $(i = 1, 2, 3)$. For any $\sigma := \{i_1 > i_2 > \cdots > i_r\} \subseteq S$, we construct the desired spanning trees $T_\emptyset,\ T_{\{i_1\}}, T_{\{i_1, i_2\}},\ \dots,\ T_\sigma$ inductively.

· **Initial step:**

Let $T_\emptyset := \{(1,2), (2,3), \dots, (d-1, d)\}$. Clearly, $T_\emptyset$ is a spanning tree. As the reduced Gröbner basis corresponds to a subset of cutsets, the initial term of any element of the reduced Gröbner basis contains a variable $x_{i,i+1}$ for some i. Thus, $\boldsymbol{x}^{\overline{T_\emptyset}} \notin in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$.

· **Induction step:**

Let $T_{\sigma \setminus \{i_r\}}$ be the desired spanning tree for $\sigma \setminus \{i_r\}$. We define two edge sets

$$T^1 := \left\{ T_{\sigma \setminus \{i_r\}} \setminus \{(i_r, i_r+1)\} \right\} \cup \{(i_r, i_r+2)\},$$
$$T^2 := \{T^1 \setminus \{(i_r+1, i_r+2)\}\} \cup \{(i_r-1, i_r+1)\}.$$

Then, both T^1 and T^2 are spanning trees and satisfy the condition (A). We show here that either T^1 or T^2 satisfies the condition (B).

(a) The case where T^1 satisfies the condition (ii).

T^1 is the desired spanning tree T_σ.

(b) The case where T^1 does not satisfy the condition (ii).

In this case, $\boldsymbol{x}^{\overline{T^1}} \in in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$. Let $\mathcal{G}$ be the reduced Gröbner basis for $I_{(I\ -M^{\mathrm{T}})}$ with respect to $\widetilde{\boldsymbol{b}}$. Then, $\boldsymbol{x}^{\overline{T^1}}$ can be reduced by some binomial $g \in \mathcal{G}$, and such g is of the following form (See Figure 3).

(i) $g^{(1)}_{(p)}$, which corresponds to the cutset for (V^+, V^-), $V^+ = \{p, p+1, \dots, i_r, i_r+2, i_r+3, \dots, d\}$ and $V^- = \{1, 2, \dots, p-1, i_r+1\}$ for some $p \le i_r$, and its initial term is a product of variables corresponding to arcs from V^+ to V^-, or

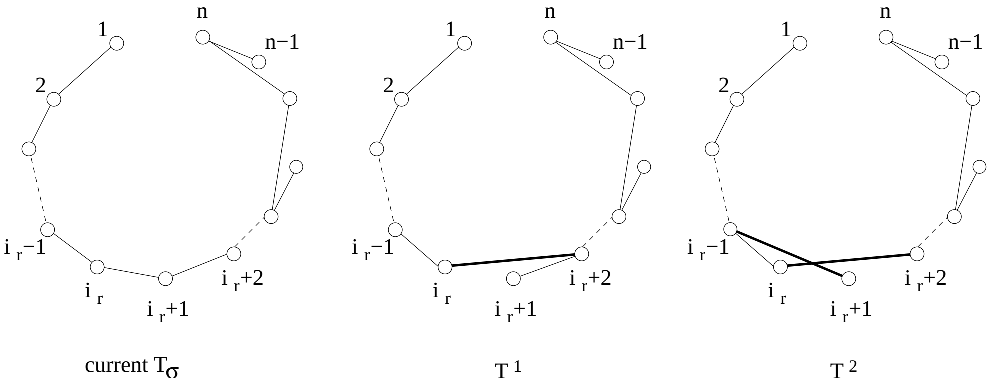

Figure 2: Two trees T^1 (middle) and T^2 (right) for the current spanning tree (left)

(ii) (The case of $r > 1$) $g^{(2)}_{(p,t)}$, which corresponds to the cutset for (V^+, V^-), $V^- = \{1, 2, \ldots, p-1, i_r+1, i_{q(1)}+1, \ldots, i_{q(t)}+1\}$ and $V^+ = V \setminus V^-$ for $1 \leq {}^{\exists}q(t) < \cdots < {}^{\exists}q(1) < r$ such that $(i_{q(k)}+1, i_{q(k)}+2) \in T_{\sigma\setminus\{i_r\}}$ for $k = 1, \ldots, t$ and $1 \leq {}^{\exists}p \leq i_r$, and its initial term is a product of variables corresponding to arcs from V^+ to V^-.

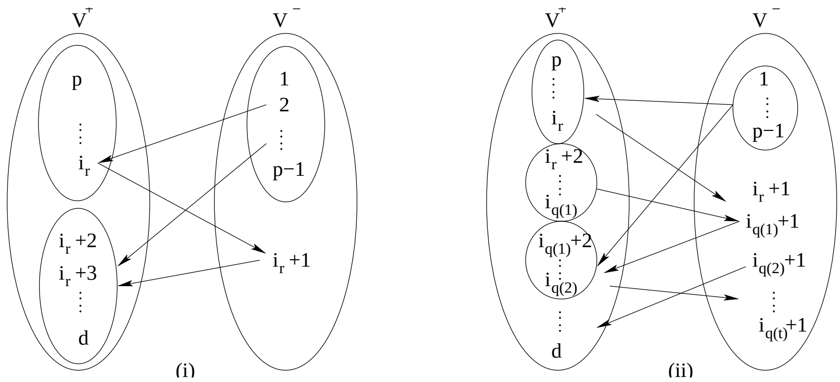

Figure 3: Cutsets corresponding to binomials $g^{(1)}_{(p)}$ (left) and $g^{(2)}_{(p,t)}$ (right)

Lemma 4.14 $g^{(1)}_{(p)} \in \mathcal{G}$ *for some* p *and* $\boldsymbol{x}^{\overline{T^1}}$ *can be reduced by* $g^{(1)}_{(1)}$*, i.e., the initial term of* $g^{(1)}_{(1)}$ *corresponds to the set of arcs* $\{(k, i_r+1) : k \leq i_r\}$.

Proof of Lemma 4.14: The case of $r = 1$ is trivial.

We suppose that $r > 1$ and $\boldsymbol{x}^{\overline{T^1}}$ cannot be reduced by any $g^{(1)}_{(p)}$. Then, $\boldsymbol{x}^{\overline{T^1}}$ can be reduced by some $g^{(2)}_{(p,t)}$, which is an element of $\mathcal{G}$, and $\boldsymbol{x}^{\overline{T^1}}$ can also be reduced by $g^{(2)}_{(1,t)}$ (otherwise, $g^{(2)}_{(p,t)}$ is reduced by $g^{(2)}_{(1,t)}$ and $g^{(2)}_{(p,t)}$ cannot be an element of $\mathcal{G}$).

Suppose that $\boldsymbol{x}^{\overline{T^1}}$ can be reduced by $g^{(2)}_{(1,t)}$ with $t = 1$. Let m_1 be the monomial obtained by reducing $\boldsymbol{x}^{\overline{T^1}}$ by $g^{(2)}_{(1,t)}$. Then m_1 can be reduced to the monomial m_2 by $g^{(1)}_{(1)}$ (the initial term of $g^{(1)}_{(1)}$ is a product of variables corresponding to arcs from V^- to V^+ by assumption).

For a binomial $f_D \in I_{(I\ -M^{\mathrm{T}})}$, which corresponds to the cutset D for (V^+_D, V^-_D) such that $V^-_D = \{i_{q(1)}+1\}$ and $V^+_D = V \setminus V^-_D$, $in_{\tilde{\boldsymbol{b}}}(f_D)$ corresponds to arcs from V^-_D to V^+_D (otherwise, $\boldsymbol{x}^{\overline{T_{\sigma\setminus\{i_r\}}}}$ can be reduced by f_D, which contradicts the assumption of the induction). Then, m_2 can be reduced by f_D, and the resulting monomial is $\boldsymbol{x}^{\overline{T^1}}$, which contradicts the definition of term order by $\tilde{\boldsymbol{b}}$.

Table 1: Divided and multiplied variables while reducing by $g^{(2)}_{(1,1)}$ and $g^{(1)}_{(1)}$

reduce by $g^{(2)}_{(1,1)}$		*reduce by* $g^{(1)}_{(1)}$	
divided variables	multiplied variables	divided variables	multiplied variables
$\{x_{k,i_r+1} : k \le i_r\}$, $\{x_{k,i_{q(1)}+1} :$ $k \le i_{q(1)},\ k \ne i_r + 1\}$	$\{x_{i_r+1,l} : l \ge i_r + 2,$ $l \ne i_{q(1)} + 1\}$, $\{x_{i_{q(1)}+1,l} :\ l \ge i_{q(1)} + 2\}$	$\{x_{i_r+1,l} :$ $l \ge i_r + 2\}$	$\{x_{k,i_r+1} : k \le i_r\}$

Similarly, in the case that in which $\boldsymbol{x}^{\overline{T1}}$ can be reduced by $g^{(2)}_{(1,t)}$ for some $t > 1$, using $f_D \in I_{(I\ -M^{\mathrm{T}})}$, which corresponds to the cutset D for (V^+_D, V^-_D) such that $V^-_D = \{i_{q(1)} + 1, i_{q(2)} + 1, \ldots, i_{q(t)} + 1\}$, and $V^+_D = V \setminus V^-_D$, we can show a contradiction. Thus, there exists some p such that $g^{(1)}_{(p)} \in \mathcal{G}$.

If $\boldsymbol{x}^{\overline{T1}}$ cannot be reduced by $g^{(1)}_{(1)}$, i.e., the initial term of $g^{(1)}_{(1)}$ corresponds to the set of arcs $\{(i_r + 1, l) :\ l \ge i_r + 2\}$, then $g^{(1)}_{(p)}$ can be reduced by $g^{(1)}_{(1)}$, which contradicts that $g^{(1)}_{(p)}$ is an element of reduced Gröbner basis $\mathcal{G}$. Thus, the second statement follows. □

If $\boldsymbol{x}^{\overline{T1}} \in in_{\tilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$, then $\boldsymbol{x}^{\overline{T2}}$ cannot be reduced by any binomial in $\mathcal{G}$. If $\boldsymbol{x}^{\overline{T2}}$ can be reduced by some $g \in \mathcal{G}$, then g is of the following form.

(i) the binomial $g^{(1)}_{(i_r)}$, and its initial term is x_{i_r,i_r+1},

(ii) any binomial that corresponds to the cutset for (V^+, V^-) such that $i_r + 1 \in V^+$ and $1, 2, \ldots, i_r,\ i_r + 2 \in V^-$, and its initial term is a product of variables corresponding to arcs from V^+ to V^-, or

(iii) (The case of $r > 1$) $g^{(2)}_{(i_r,t)}$, and its initial term is a product of variables corresponding to arcs from V^+ to V^-.

If case (i) occurs, the initial term of $g^{(1)}_{(i_r)}$ is x_{i_r,i_r+1}, which contradicts $i_r \notin M_0$. On the other hand, a binomial of type (ii) can be reduced by $g^{(1)}_{(1)}$ by the above lemma, and cannot be contained in $\mathcal{G}$.

Let us consider that case (iii) occurs. If $\boldsymbol{x}^{\overline{T2}}$ can be reduced by $g^{(2)}_{(i_r,t)}$ with $t = 1$, then the monomial to which $\boldsymbol{x}^{\overline{T2}}$ is reduced by $g^{(2)}_{(i_r,1)}$ can be reduced by a binomial $f_D \in I_{(I\ -M^{\mathrm{T}})}$, for the cutset D which corresponds to (V^+_D, V^-_D) where $V^+_D = \{1, 2, \ldots, i_r - 1, i_r + 1\}$ and $V^-_D = V \setminus V^+_D$, to some monomial m (the initial term of f_D is a product of variables corresponding to arcs from V^+_D to V^-_D since $i_r \notin M_0$).

Table 2: Divided and multiplied variables while reducing by $g^{(2)}_{(i_r,1)}$ and f_D

reduce by $g^{(2)}_{(i_r,1)}$		*reduce by* f_D	
divided variables	multiplied variables	divided variables	multiplied variable
$x_{i_r,i_r+1}, x_{i_r,i_{q(1)}+1}$, $x_{i_r+2,i_{q(1)}+1}$, $x_{i_r+3,i_{q(1)}+1}$, $\ldots, x_{i_{q(1)},i_{q(1)}+1}$	$\{x_{k,i_r} :\ k \le i_r - 1\}$, $\{x_{k,l} : k \le i_r + 1,\ k \ne i_r,$ $l \ge i_r + 2, l \ne i_{q(1)} + 1\}$, $\{x_{i_{q(1)}+1,l} :\ l \ge i_{q(1)} + 2\}$	$\{x_{k,i_r} : k \le i_r - 1\}$, $\{x_{k,l} : k \le i_r + 1,$ $k \ne i_r, l \ge i_r + 2\}$	x_{i_r,i_r+1}

For a binomial $f_{D'} \in I_{(I\ -M^{\mathrm{T}})}$, which corresponds to the cutset D' for $(V^+_{D'}, V^-_{D'})$ such that $V^-_{D'} = \{i_{q(1)} + 1\}$ and $V^+_{D'} = V \setminus V^-_{D'}$, $in_{\tilde{\boldsymbol{b}}}(f_{D'})$ corresponds to arcs from $V^-_{D'}$ to $V^+_{D'}$ (otherwise,

$\boldsymbol{x}^{\overline{T_\sigma \setminus \{i_r\}}}$ can be reduced by $f_{D'}$, which contradicts the assumption of the induction). Then, m can be reduced by $f_{D'}$, and the resulting monomial is $\boldsymbol{x}^{\overline{T2}}$, which contradicts the definition of a term order by $\boldsymbol{b}$.

Similarly, in the case in which $\boldsymbol{x}^{\overline{T2}}$ can be reduced by $g^{(2)}_{(i_r,t)}$ for some $t > 1$, using the same f_D and $f_{D'} \in I_{(I\ -M^{\mathrm{T}})}$, which corresponds to the cutset D' for $(V^+_{D'}, V^-_{D'})$ such that $V^-_{D'} = \{i_{q(1)}+1, i_{q(2)}+1, \ldots, i_{q(t)}+1\}$, and $V^+_{D'} = V \setminus V^-_{D'}$, we can show a contradiction.

Therefore, $\boldsymbol{x}^{\overline{T2}} \notin in_{\tilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$, and T^2 is the desired spanning tree T_σ. □

5. Concluding Remarks

In this paper, we proposed a unified approach to count the maximum number of dual feasible bases by computing a normalized volume of the convex hull of column vectors generated by a homogenized matrix. Then we applied the approach to the maximum number of dual and primal feasible bases of network flow problems and obtained following results:

Table 3: The number of dual and primal feasible bases

	primal problem (dual feasible bases)	dual problem (primal feasible bases)
transportation problem for $K_{m,n}$	$\binom{m+n-2}{m-1}$	$(n - \lfloor n/2 \rfloor)\binom{n}{\lfloor n/2 \rfloor} \cdots (m = 2)$
minimum cost flow problem for d vertices acyclic tournament graph	Catalan number C_{d-1}	$\Omega(2^{\lfloor d/6 \rfloor}) \cdots (*)$

We gave new algebraic proofs to existing results on transportation problems, moreover the maximum numbers of dual feasible bases on minimum cost flow problem were newly obtained. However, for primal feasible bases on minimum cost flow problem $(*)$, we showed only an exponential lower bound. Furthermore, other interesting optimization problems may be analyzed through computing its number of feasible bases by algebraic methods. With these points, following open problems are left:

- Compute the normalized volume of primal polyhedra (i.e. a tight bound of the number of primal feasible bases) for minimum cost flow problem on acyclic tournament graph.
- Characterize the number of feasible bases of general integer programs by the approach using normalized volume.

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