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POLYHEDRA AND OPTIMIZATION RELATED TO A WEAK ABSOLUTE MAJORIZATION ORDERING

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Abstract A vector $x \in \mathbf{R}^n$ is weakly k-majorized by a vector $q \in \mathbf{R}^k$ if the sum of r largest components of x is less than or equal to the sum of r largest components of q for $r = 1, 2, \cdots, k$ and $k \leq n$. In this paper we extend the components of x to their absolute values in the above description and generalize some results in [2] and [3] by G. Dahl and F. Margot.

Keywords: Majorization, bisubmodular functions, polyhedra

1. Introduction

For $p, q \in \mathbf{R}^n$ we say that p is *weakly sub-majorized* by q if $\sum_{j=1}^{r} p_{[j]} \leq \sum_{j=1}^{r} q_{[j]}$ for $r = 1, 2, \cdots, n$. Here $p_{[j]}$ denotes the jth largest component of p. If $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j$ also holds, p is *majorized* by q and we write $p \prec q$. Furthermore, for integers k, n with $k \leq n$ and $x \in \mathbf{R}^n$, $q \in \mathbf{R}^k$, we say that x is *weakly k-majorized* by q and write $x \prec_k q$ if

$$\sum_{j=1}^{r} x_{[j]} \leq \sum_{j=1}^{r} q_{[j]} \text{ for } r = 1, 2, \cdots, k. \tag{1.1}$$

Majorization is a concept appearing in several branches of mathematics and applied mathematics as indicated in [2]. Here, we extend the components of x in (1.1) to their absolute values as follows:

$$\sum_{j=1}^{r} |x_{[j]}| \leq \sum_{j=1}^{r} q_{[j]} \text{ for } r = 1, 2, \cdots, k. \tag{1.2}$$

We say that x is *weakly absolutely k-majorized* by q and write $x_{\text{abs}} \prec_k q$. In the following we investigate properties induced by (1.2) and generalize some results in [2] and [3].

Hereafter, we assume that *majorant* $q \in \mathbf{R}^k$ satisfies

$$q_1 \geq q_2 \geq \cdots \geq q_k \geq 0. \tag{1.3}$$

2. Polyhedra Induced by Absolute Majorization

Let

$$P(q; k) := \{x \in \mathbf{R}^n | x \prec_k q\}, \tag{2.1}$$

and denote $\mathbf{N}_t := \{1, 2, \cdots, t\}$. Define $x(X) := \sum_{j \in X} x_j$. Then we have [2]

$$P(q; k) = \{x \in \mathbf{R}^n | x(X) \leq q(\mathbf{N}_r) \text{ for all } X \subseteq \mathbf{N}_n \text{ with } r = |X| \leq k\}, \tag{2.2}$$

so $P(q;k)$ is a polyhedron.

In the case of absolute majorization, we first denote by $3^{\mathbf{N}_n}$ the set of all ordered pairs of disjoint subsets of $\mathbf{N}_n$, i.e., $3^{\mathbf{N}_n} := \{(X,Y) \mid X, Y \subseteq N_n, X \cap Y = \emptyset\}$. Similarly, let

$$P_S(q;k) := \{x \in \mathbf{R}^n | x_{\text{abs}} \prec_k q\}. \tag{2.3}$$

Then we have

$$P_S(q;k) = \{x \in \mathbf{R}^n |\, x(X) - x(Y) \leq q(\mathbf{N}_r) \text{ for all } (X,Y) \in 3^{\mathbf{N}_n} \text{ with } r = |X \cup Y| \leq k\}. \tag{2.4}$$

Hence $P_S(q;k)$ is also a polyhedron, or more precisely, a polytope. Note here that the absolute-value notation disappears, and instead we have a signed linear form description.

Define the set function $f : 3^{\mathbf{N}_n} \to \mathbf{R}$ by

$$f(X,Y) = \sum_{i=1}^{r} q_i \quad \text{for all } (X,Y) \in 3^{\mathbf{N}_n} \text{ where } r = |X \cup Y|. \tag{2.5}$$

We call a function $f : 3^{\mathbf{N}_n} \to \mathbf{R}$ *bisubmodular* if f satisfies

$$f(X_1,Y_1)+f(X_2,Y_2) \geq f(X_1 \cap X_2, Y_1 \cap Y_2) + f((X_1 \cup X_2)-(Y_1 \cup Y_2),(Y_1 \cup Y_2)-(X_1 \cup X_2)). \tag{2.6}$$

for any $(X_1,Y_1),(X_2,Y_2) \in 3^{\mathbf{N}_n}$.

Proposition 2.1 *The function f defined by* (2.5) *is a bisubmodular function.*

(Proof) For any $(X_1,Y_1),(X_2,Y_2) \in 3^{\mathbf{N}_n}$, we show that the function satisfies bisubmodular inequality (2.6).

Rewrite the inequality (2.6) as

$$f(X_1,Y_1)-f(X_1 \cap X_2, Y_1 \cap Y_2) \geq f((X_1 \cup X_2)-(Y_1 \cup Y_2),(Y_1 \cup Y_2)-(X_1 \cup X_2))-f(X_2,Y_2). \tag{2.7}$$

Let $s = |X_1 \cap X_2|+|Y_1 \cap Y_2|$, $t = |X_2|+|Y_2|$, and $l_1 = |X_1 \cup Y_1 - X_2 \cup Y_2|$, $l_2 = |X_1 \cap Y_2|+|X_2 \cap Y_1|$. By the definition of function $f(X,Y)$ in (2.5), the left-hand side of (2.7) is

$$\sum_{i=s+1}^{s+l_1+l_2} q_i, \tag{2.8}$$

and the right-hand side of (2.7) is

$$\begin{cases} \displaystyle\sum_{i=t+1}^{t+l_1-l_2} q_i & (\text{if } l_1 \geq l_2) \\ \displaystyle\sum_{i=t+l_2-l_1+1}^{t} -q_i & (\text{if } l_1 < l_2) \end{cases} \tag{2.9}$$

Since $t \geq s$, it follows from the assumption (1.3) that the left-hand side of (2.7) is, indeed, greater than or equal to the right-hand side of (2.7) (see Figure 1). Here note that $|L_1' \cup L_1''| = l_1$ and $|L_2' \cup L_2''| = l_2$. □

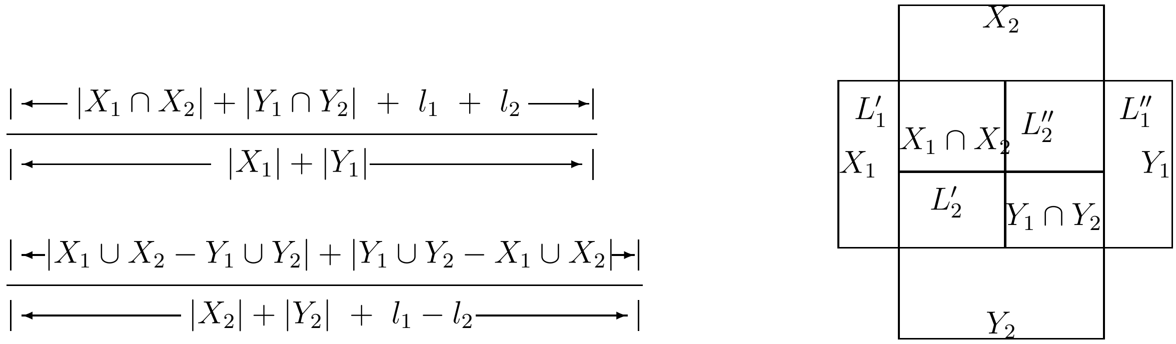

Figure 1: Lengths of sets and Venn diagram of Proposition 2.1

From the above proposition, we have a diameter description of $P_S(q;n)$.

Let P be a convex polytope. The *diameter* of P, denoted by $\delta(P)$, is the smallest number $\delta(P)$ such that any two vertices in P can be connected by a path with at most $\delta(P)$ edges. Let ϕ be a linear form defined on P. Suppose that any v can be connected to v^* by a nonincreasing (nondecreasing) path induced by ϕ with at most $\delta(P,\phi)$ edges, where v^* minimizes (maximizes) ϕv. The *monotonic diameter* of P is defined as

$$\delta^*(P) = \max\{\delta(P,\phi) \mid \phi \text{ is a linear form on } P\}. \tag{2.10}$$

Now we have the following result ([5], [1]).

Proposition 2.2 *If all the components of q take different values, the diameter and the monotonic diameter of $P_S(q,n)$ are n^2.*

(Outline of the proof): ¿From the proof of 2.1, we can easily see that different components of q guarantee strict inequalities of bisubmodular function. Therefore, each vertex of $P_S(q;n)$ can be characterized by a signed *chain*, a list of signed elements, or a special signed Hasse diagram [1]. Then we traverse to adjacent vertices along the edges of $P_S(q;n)$ by two types of operations.

(1) Exchange the pair of two adjacent elements of a chain.

(2) Change the sign of an element at one fixed terminal of a chain.

Hence, what we have to do is counting the number of operations of types (1) and (2). Without loss of generality, we suppose that a maximally distant pair of vertices are characterized by following two chains.

$$(-x_1,\ -x_2,\ \cdots,\ -x_n), \tag{2.11}$$
$$(x_1,\ x_2,\ \cdots,\ x_n). \tag{2.12}$$

Hence, the total number of operations to obtain from one chain to another is

$$1 + (2\times 2 - 1) + \cdots + (2i-1) + \cdots + (2n-1) = n^2. \tag{2.13}$$

The monotonic property can be obtained from the definition of the bisubmodularity inequality. □

3. Optimization Problems and Integrality

Let $c \in \mathbf{R}^n$ and consider the following optimization problem,

$$\max\{c^T x \mid x_{\text{abs}} \prec_k q\}. \tag{3.1}$$

We know from (2.4) that the above problem is equivalent to

$$\max\{c^{T}x \mid x \in P_S(q;k)\}, \tag{3.2}$$

a linear programming (LP). And it is clear that when $k = n$, i.e., in the bisubmodular case, the above LP can be solved by greedy algorithm [4].

In the following, we generalize some results in [3] and [2] for $k < n$.

First, for $g \in \mathbf{R}^k$, we define *tail average* of g by $\bar{g}_{s:k} := 1/(k-s+1)\sum_{i=s}^{k}|g_i|$.

Suppose that $c \in \mathbf{R}^k$ satisfies $|c_1| \geq |c_2| \geq \cdots \geq |c_{k-1}|$ (note that c_k may be arbitrary). Then there is an $m \in \{1, 2, \cdots, k\}$ such that [2]

$$\bar{c}_{1:k} \geq \cdots \geq \bar{c}_{m:k} \leq \bar{c}_{m+1:k} \leq \cdots \leq \bar{c}_{k:k} = |c_k|. \tag{3.3}$$

For $0 \leq s \leq k-1$ and $c \in \mathbf{R}^n$ with $|c_1| \geq |c_2| \geq \cdots \geq |c_n|$, we define *signed sth q-average* $w^s \in \mathbf{R}^n$ by

$$w^s := ((-1)^{i_1}q_1, \cdots, (-1)^{i_s}q_s, (-1)^{i_{s+1}}\bar{q}_{s+1:k}, \cdots, (-1)^{i_n}\bar{q}_{s+1,k}), \tag{3.4}$$

where $i_l = 0$ or 1 for $l = 1, 2, \cdots, n$, and define $w_c^s \in \mathbf{R}^n$, *signed sth q-average related to c* by

$$w_c^s := (sign(c_1)q_1, \cdots, sign(c_s)q_s, sign(c_{s+1})\bar{q}_{s+1:k}, \cdots, sign(c_n)\bar{q}_{s+1,k}). \tag{3.5}$$

Theorem 3.1 *The optimal solution of Problem* (3.2) *can be obtained as a permutation of* w_c^s*, where* $s = m - 1$.

(Proof) To prove the optimality, we only consider the case when for $i = 1, 2, \cdots, n$, (1) $x_i \geq 0$ if $c_i \geq 0$ and (2) $x_i < 0$ if $c_i < 0$. Otherwise, reversing the signs of components of x would increase the value of $c^{\top}x$ without violating the constraints $x_{abs} \prec_k q$. Our proof is only based on the form of $c^T x$, we may suppose $c_i \geq 0$ and $x_i \geq 0$ for $i = 1, 2, \cdots, n$. Otherwise, let $c_i x_i = (-c_i)(-x_i)$ and omit the signs before them for convenience.

For the second part of the proof, see appendix. □

Conversely, by the same arguments as the proof of Theorem 3.1 and by an appropriate choice of c (see Theorem 5 of [3]), we can prove that each w^s ($s = 0, 1, \cdots, k-1$) is a unique optimal solution of (3.2) on $P_S(q;k)$, and therefore a vertex of $P_S(q;k)$. Combining it with Theorem 3.1, we have the following result.

Theorem 3.2 *The set of vertices of* $P_S(q;k)$ *is precisely the set of vectors that can be obtained by permutations of* w^s, $s = 0, 1, \cdots, k-1$.

By Theorem 3.1, the simple (greedy) algorithm for solving problem (3.1) is similar to that described in [2] and its time complexity is $O(n^2)$. The difference is that we compute s by taking absolute values of components of c and we take minus components of w^s when the corresponding components of c are less than zero.

Now we consider the following integer programming problem:

$$\max\{c^{T}x \mid x_{\rm abs} \prec_k q, x \text{ is integral}\}. \tag{3.6}$$

We represent the integer hull of $P_S(q;k)$ by

$$Q_S(q;k) := \text{conv}\{x \in \mathbf{R}^n \mid x_{\rm abs} \prec_k q, x \text{ is integral}\}. \tag{3.7}$$

We assume that q is an integer vector, otherwise round down each component of q without changing $Q_S(q;k)$. Note here that $Q_S(q;k)$ is full-dimensional if $q \neq 0$ (together with the assumption that q is integral and non-negative).

In [3] and [2], a complete description of vertices and facets of polyhedra $Q(q;k) = \text{conv}\{x \in R^n | x \prec_k q, x \text{ is integral}\}$ are provided. By the symmetry, these results can be generalized to polytope $Q_S(q;k)$ without much modifications. We summarize two main results here.

Let $\alpha \in [q_k, \bar{q}_{1,k}]$ and define $s(\alpha) = \max\{0 \le s < k \,|\, \bar{q}_{s+1,k} \ge \alpha\}$ and $\Delta(\alpha) = \sum_{i=s+1}^{k} q_i - (k - s(\alpha) - 1)\alpha$. We define the vector $x(\alpha) \in \mathbf{R}^n$ by

$$x(\alpha) := ((-1)^{i_1} q_1, \cdots, (-1)^{i_s} q_{s(\alpha)}, (-1)^{i_{s+1}} \Delta(\alpha), (-1)^{i_{s+2}} \alpha, \cdots, (-1)^{i_n} \alpha), \tag{3.8}$$

for $i_l = 0$ or 1 and $l = 1, 2, \cdots, n$, where $\alpha \le \bar{q}_{1,k}$ is a round down or round up of $\bar{q}_{s+1,k}$ and $s = 0, 1, \cdots, k-1$. And we say $x(\alpha)$ a *signed rounded q-average.*

Theorem 3.3 *The set of vertices of $Q_S(q;k)$ is precisely the set of vectors that can be obtained by permutations of signed rounded q-averages.*

Let s and t be integers with $0 \le s < k < t \le n$, and put $\delta^s := \sum_{i=s+1}^{k} q_i - (k-s)\lfloor \bar{q}_{s+1:k} \rfloor = \sum_{i=s+1}^{k} q_i \bmod (k-s)$. Then, define $\alpha_0^{s,t} := (t-k)/(k-s-\delta^s) \sum_{i=1}^{s} q_i + \sum_{i=1}^{k} q_i + (t-k)\lfloor \bar{q}_{s+1:k} \rfloor$. We call

$$(t - s - \delta^s)/(k - s - \delta^s)(x(X_1) - x(Y_1)) + x(X_2) - x(Y_2) \le \alpha_0^{s,t} \tag{3.9}$$

a *signed q-average inequality* if $\bar{q}_{s+1:k}$ is fractional, where X_1, Y_1, X_2, Y_2 are pairwise disjoint, $|X_1 \cup Y_1| = s$ and $|X_2 \cup Y_2| = t - s$.

We call

$$x(X) - x(Y) \le \sum_{i=1}^{r} q_i \qquad \text{for} \quad (X, Y) \in 3^{\mathbf{N}_n}, \ \ |X \cup Y| = r \tag{3.10}$$

a *signed set size inequality* if r=1 or $q_1 > q_r$.

Theorem 3.4 *A complete and non-redundant facet description of $Q_S(q;k)$ is given by signed set size inequalities and signed q-average inequalities.*

4. Appendix

For the second part of the proof of Theorem 3.1 we first assume $c_1 \ge c_2 \ge \cdots \ge c_n$. Let $x \ge 0$ be a feasible solution. Without loss of generality, we may assume that $x_1 \ge \cdots \ge x_k = \cdots = x_n$. Let $d_i = c_i$ for $i = 1, 2, \cdots, k-1$, and $d_k = \sum_{j=k}^{n} c_j$. Let $x^* = w_c^s$, as indicated in the theorem.

$$\begin{aligned}\sum_{i=1}^{n} c_i(x_i^* - x_i) &= \sum_{i=1}^{k} d_i(x_i^* - x_i) = \sum_{i=1}^{s} d_i(q_i - x_i) + \bar{q}_{s+1:k} \sum_{i=s+1}^{k} d_i - \sum_{i=s+1}^{k} d_i x_i \\ &= \sum_{i=1}^{s} d_i(q_i - x_i) + (k-s)\bar{d}_{s+1:k}\bar{q}_{s+1:k} - \sum_{i=s+1}^{k} d_i x_i,\end{aligned} \tag{4.1}$$

where,

$$
\begin{aligned}
&-\sum_{i=s+1}^{k} d_i x_i = -\sum_{i=s+1}^{k-1}\Bigg(\sum_{j=i}^{k} d_j - \sum_{j=i+1}^{k} d_j\Bigg)x_i - d_k x_k \\
&= -\sum_{i=s+1}^{k-1}\big((k-i+1)\bar{d}_{i:k} - (k-i)\bar{d}_{i+1:k}\big)x_i - d_k x_k \\
&= -\Bigg(\sum_{i=s}^{k-2}(k-i)\bar{d}_{i+1:k}x_{i+1} - \sum_{i=s+1}^{k-1}(k-i)\bar{d}_{i+1:k}x_i\Bigg) - d_k x_k \\
&= \sum_{i=s+1}^{k-2}(k-i)\bar{d}_{i+1:k}(x_i - x_{i+1}) - (k-s)\bar{d}_{s+1:k}x_{s+1} + \bar{d}_{k:k}x_{k-1} - d_k x_k. \qquad (4.2)
\end{aligned}
$$

Note that $x_i - x_{i+1} \geq 0$. From the definition of s we obtain

$$
\begin{aligned}
&\sum_{i=s+1}^{k-2}(k-i)\bar{d}_{i+1:k}(x_i - x_{i+1}) \\
&\geq \bar{d}_{s+1:k}\sum_{i=s+1}^{k-2}(k-i)(x_i - x_{i+1}) \\
&= \bar{d}_{s+1:k}\sum_{i=s+1}^{k-2}(k-i)x_i - \bar{d}_{s+1:k}\sum_{i=s+2}^{k-1}(k-i+1)x_i \\
&= -\bar{d}_{s+1:k}\sum_{i=s+2}^{k-2}x_i + (k-s-1)\bar{d}_{s+1:k}x_{s+1} - 2\bar{d}_{s+1:k}x_{k-1} \\
&= -\bar{d}_{s+1:k}\sum_{i=s+1}^{k-1}x_i + (k-s)\bar{d}_{s+1:k}x_{s+1} - \bar{d}_{s+1:k}x_{k-1}. \qquad (4.3)
\end{aligned}
$$

Now, the last whole row of equation (4.2) (after deleting $(k-s)\bar{d}_{s+1:k}x_{s+1} - (k-s)\bar{d}_{s+1:k}x_{s+1})$) is greater than

$$
\begin{aligned}
&-\bar{d}_{s+1:k}\sum_{i=s+1}^{k-1}x_i - \bar{d}_{s+1:k}x_{k-1} + \bar{d}_{k:k}x_{k-1} - d_k x_k \\
&= -\bar{d}_{s+1:k}\sum_{i=s+1}^{k}x_i + \bar{d}_{s+1:k}x_k - \bar{d}_{s+1:k}x_{k-1} + \bar{d}_{k:k}x_{k-1} - d_k x_k \\
&= -\bar{d}_{s+1:k}\sum_{i=s+1}^{k}x_i + (d_k - \bar{d}_{s+1:k})(x_{k-1} - x_k) \\
&\geq -\bar{d}_{s+1:k}\sum_{i=s+1}^{k}x_i. \qquad (4.4)
\end{aligned}
$$

Finally, since $q_i - x_i \geq 0$ for all $i = 1, 2, \cdots, s$, we have

$$
\sum_{i=1}^{s} d_i(q_i - x_i) \geq \bar{d}_{s+1:k}\sum_{i=1}^{s}(q_i - x_i). \qquad (4.5)
$$

Summarizing the above equations and inequalities, we have

$$\sum_{i=1}^{k} d_i(x_i^* - x_i) \geq \bar{d}_{s+1:k}\Big(\sum_{i=1}^{k} q_i - \sum_{i=1}^{k} x_i\Big) \geq 0. \tag{4.6}$$

□

Note that this is a direct proof of Theorem 2 in [2].

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