

A NEW APPROACH OF REVISING UNSTABLE DATA IN ANP BY BAYES THEOREM

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Abstract In the typical type of ANP with a matrix U evaluating alternatives by criteria and a matrix W evaluating criteria by alternatives in the so-called supermatrix S , W is often said to be unstable. Here, we propose a method to revise W into a stable \hat{W} and to calculate the weights of criteria and alternatives at the same time under the revised supermatrix \hat{S} . Our method is formulated as an optimization problem based on Bayes Theorem which T.L. Saaty claimed to be included in ANP scheme. Concurrent Convergence method developed by Kinoshita and Nakanishi also intends to be correct W , but this method includes some contradictions. We prove that our method has no such contradiction. We introduce some eigenvalue problems, which give a lower bound of our optimal value and their special cases coincide with our problem. Furthermore, we clear what perturbations of W preserve weights of criteria and alternatives to be invariant under the concept of inactive alternatives.

Keywords: Decision making, AHP, analytic network process, bayes theorem, fractional programming

1. Introduction

The Analytic Hierarchy Process (AHP) developed by Saaty [6] is a multi-criteria decision method that uses hierarchic structure to represent a decision problem and then provides priorities for alternatives on decision makers' judgments. Saaty extends the hierarchic structure of criteria and alternatives into a network one, and proposes Analytic Network Process (ANP). We consider the simplest type of ANP that is composed of the set of criteria $C = \{C_1, \dots, C_m\}$, the set of alternatives $A = \{A_1, \dots, A_n\}$, the evaluation matrix U of alternatives by criteria and W of criteria by alternatives. This type of ANP has a so-called supermatrix

$$S = \begin{bmatrix} 0 & W \\ U & 0 \end{bmatrix}. \quad (1.1)$$

The main object of ANP is to calculate weights of C_1, \dots, C_m and A_1, \dots, A_n with given values of U and W . Of course decision makers may give two evaluation matrices W and U that include neither interdependence nor reciprocal relationship. This pair of evaluation matrices W and U is often called independent. In the other case there exist some interdependence between W and U , and it is often said that the values of elements of W depend on U . This dependence of w_{ij} is mathematically formulated as

$$w_{ij} = f_{ij}(U)e_{ij} \quad (1.2)$$

where e_{ij} is an error term. If e_{ij} of (1.2) is abnormally large, then W is often called unstable. Kinoshita and Nakanishi assume an interdependence between criteria and alternatives and

then propose Concurrent Convergence Method (CCM) [3] for this type ANP, which revises given values of W by some relations between U and W . It is said to correct the initial values of W by using the information of U . However, it is pointed out by Sekitani and Ueta [10] that their CCM has a contradiction. Further recently Saaty [6, 7] insists that Bayes Theorem is included in the framework of ANP. His idea is briefly summarized as follows; Taking C_1, \dots, C_m as the causes and A_1, \dots, A_n as the outcomes, we can consider U as the set of conditional probabilities of the the outcomes under the occurrences of the causes. So W is to correspond to the formula to calculate aposteriori probabilities in Bayes Theorem. His such idea connecting ANP and Bayes Theorem is ingenious excellent. The object of this paper is to propose the new approach of revising W based on this idea and to show no contradiction of the new one.

The remaining structure of the paper is organized as follows: Section 2 reviews relationships between Bayes theory and ANP with the supermatrix (1.1). The new method of revising unstable W is proposed in Section 3. Section 4 shows no contradiction of the new method and mathematical properties of priorities of alternatives and criteria. With the help of a simple numerical example, Section 5 illustrates the uniqueness and sensitivity of an evaluation weight vector. Section 6 shows how to incorporate some requests of decision makers into the proposed method. A conclusion is summarized in the last section.

2. The Structures of Bayes Theorem and ANP

We will explain the structures of Bayes Theory by the simplest actual example. Consider a group G of human beings (G may be the whole people of U.S.A.). Some of them have a specific disease (say, cancer). Let C_1 be the set of persons of cancer and $C_2 = G \setminus C_1$ be the set of non-cancer ones. Denoting the percent/100 of $C_1(C_2)$ by $p_1(p_2)$, we have $p_1 + p_2 = 1$. Let A_1 be the set of persons who are decided to cancer by the medical checkup. And $A_2 = G \setminus A_1$ is the set of ones decided to have not cancer. Denoting the percent/100 of $A_1(A_2)$ by $q_1(q_2)$, then we have $q_1 + q_2 = 1$. (See Figure 1.) Then we have the following

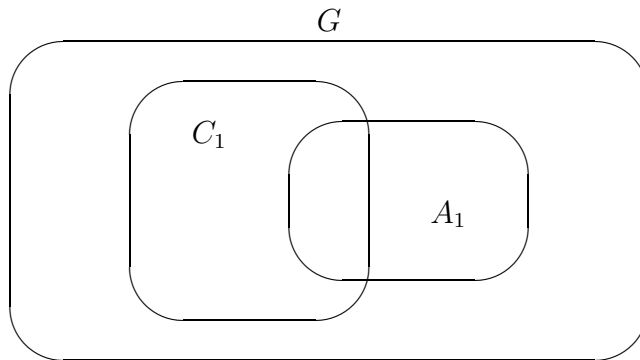


Figure 1: Cancer and medical check

four kinds of conditional probabilities:

$$u_{ji} = \frac{|A_j \cap C_i|}{|C_i|}, i, j = 1, 2, \tag{2.1}$$

where $|X|$ is the number of elements of a set X . For example u_{11} is the probability of a person having cancer who is decided to have cancer by the medical check, and u_{21} is the

probability of a person having cancer who is decided to have not cancer by the check (that is, overlooking probability). All we can know is only the results of the medical check. So we often want to know the (conditional) probability of a person decided to have cancer by the medical check who has really cancer. This is clearly represented as

$$\frac{|A_1 \cap C_1|}{|A_1|} \quad (2.2)$$

by the set theoretical consideration. Bayes Theorem intends to represent (2.2) by p_i, q_j and u_{ij} , then (2.2) is to represent by

$$\frac{u_{11}p_1}{q_1}. \quad (2.3)$$

In Bayes theory the formula like (2.3) is called aposteriori probability. This way of expression is based on their idea taking C_1, C_2 as causes and A_1, A_2 as outcomes. By this way of expression, the aposteriori probability w_{ij} of C_i on the outcome A_j is

$$w_{ij} = \frac{u_{ji}p_i}{q_j}. \quad (2.4)$$

Here, we have clearly $q_j = \sum_i u_{ji}p_i$, so (2.4) is equivalent to

$$w_{ij} = \frac{u_{ji}p_i}{\sum_k u_{jk}p_k}. \quad (2.5)$$

This is the famous Bayes Theorem.

Now we will show how does such probability problem correspond with the evaluation problem like ANP. Again take the simplest actual example of (1.1) type of ANP. Consider two fast food companies A_1 and A_2 , and two evaluation criteria C_1 (quality food) and C_2 (advertisement).

Now assuming that the whole people G of U.S.A can be decomposed into two groups \bar{C}_1 , the set of people supporting C_1 , and \bar{C}_2 , the set of people supporting C_2 . The similar decomposition \bar{A}_1 and \bar{A}_2 is considered. Then evaluating weight $p_i(q_j)$ of $C_i(A_j)$ can be considered to be near percent/100 of $\bar{C}_i(\bar{A}_j)$ in G . Similarly evaluating weight u_{ji} of A_j by C_i can be considered to be near to the percent/100 of \bar{A}_j within \bar{C}_i . Of course such evaluating value is commonly decided by a decision maker's intuition in ANP. But his intuition must be based on the information with above mentioned data. Under these considerations, we have

$$u_{ji} \approx \frac{|\bar{A}_j \cap \bar{C}_i|}{|\bar{C}_i|}. \quad (2.6)$$

If we can accept the validness of (2.6), we must also accept that evaluating weight w_{ij} of C_i by A_j is near to $|\bar{A}_j \cap \bar{C}_i|/|\bar{A}_j|$, that is,

$$w_{ij} \approx \frac{|\bar{A}_j \cap \bar{C}_i|}{|\bar{A}_j|}. \quad (2.7)$$

Formula (2.6) and (2.7) clearly indicate that $U = [u_{ij}]$ and $W = [w_{ij}]$ of the supermatrix (1.1) in ANP can not be independent. Moreover considering $p_i \approx |\bar{C}_i|/|G|$ and $q_i \approx |\bar{A}_j|/|G|$ we have

$$w_{ij} \approx \frac{u_{ji}p_i}{q_j}. \quad (2.8)$$

If (2.8) is valid with exact equality, it completely coincides with Bayes Theorem (2.4). This is a brief explanation of Saaty's claim "ANP includes Bayes Theorem". (Of course the above discussion extremely simplifies the actual situation. For example taking (2.7), you can say that evaluating C_i by A_j must include the policy of the company A_j . Certainly the value of w_{ij} must be based on various factors. But if we want to take every effecting factors, it costs us endless. Theory needs simplifying.)

We have stated that there can be a structure of ANP (of (1.1)) such that U and W are not independent. Of course there are many cases where U and W are independently determined. The evaluation structures stated above are based on the same space G . If the evaluation of A_j by C_i and that of C_i by A_j are based on completely different spaces, U and W might be independently determined. So the relationship depends on the structure of evaluations.

Note that we do not claim our method to be applicable to every case of ANP of type (1.1). Our method is applicable to such type of the dependent structure, and is not suitable for the independent type. This is the key to implement our method for actual problems.

3. A New Method of Revising W

In early time Kinoshita and Nakanishi noticed the dependency of U and W and proposed CCM [3]. It is based on some relations between u_{ji}/u_{ki} and w_{ij}/w_{ik} which of course are different from (2.8). Sekitani and Ueta show that CCM includes a fatal contradiction.

Now we will propose our new method which replaces the relations on CCM [3] with (2.8). Let it call Bayes Revising Method (BRM). Of course our method has no contradiction, which is shown later. In BRM it is assumed that U is stable and W is unstable. So it intends to correct the initially given values of W by the informations of U and the relations (2.8) between U and W .

In order to clear BRM we define several symbols as follows:

$$\begin{aligned}
 U &= \begin{bmatrix} u_{11} & \cdots & u_{1m} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nm} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} & : \text{evaluation matrix of alternatives } A_1, \dots, A_n \\
 & & & \text{by criteria } C_1, \dots, C_m. \\
 W &= \begin{bmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{m1} & \cdots & w_{mn} \end{bmatrix} = [\mathbf{w}_1, \dots, \mathbf{w}_n] & : \text{initial value of evaluation matrix of criteria} \\
 & & & \text{by alternatives } (\mathbf{w}_j \text{ is an evaluating vector of} \\
 & & & \text{criteria by } A_j, j = 1, \dots, n.). \\
 \mathbf{p} &= [p_1, \dots, p_m]^\top & : \text{evaluation vector of criteria by an outer fac-} \\
 & & & \text{tor.} \\
 \mathbf{q} &= [q_1, \dots, q_n]^\top & : \text{evaluation vector determined by } \mathbf{q} = U\mathbf{p}.
 \end{aligned}$$

Here we assume as usual ANP

$$\sum_{i=1}^n u_{ij} = 1, \quad \sum_{i=1}^m w_{ij} = 1, \quad w_{ij} \geq 0 \text{ and } u_{ij} \geq 0 \text{ for all } i = 1, \dots, n, j = 1, \dots, m. \quad (3.1)$$

Writing (2.8) by matrix-forms, we have

$$W \approx (\Delta \mathbf{p})U^\top (\Delta \mathbf{q})^{-1}, \quad (3.2)$$

where $\Delta \mathbf{x}$ for vector $\mathbf{x} = [x_1, \dots, x_n]^\top$ is the diagonalization of \mathbf{x} , that is

$$\Delta \mathbf{x} = \begin{bmatrix} x_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & x_n \end{bmatrix}. \quad (3.3)$$

Considering $\mathbf{q} = U\mathbf{p}$, we can write the right hand-side of (3.2) as

$$\mathcal{W}[\mathbf{p}] = (\Delta\mathbf{p}) U^\top (\Delta(U\mathbf{p}))^{-1} \tag{3.4}$$

which is considered to be a transformation of a priori probability into a posteriori probability $\mathcal{W}[\mathbf{p}]$ by Bayes Theorem. Here we call (3.4) Bayes transformation.

Now the principle of BRM is to make Bayes transformation $\mathcal{W}[\mathbf{p}]$ of the convex combination $\mathbf{p} = \sum_{j=1}^n r_j \mathbf{w}_j$ of $\mathbf{w}_1, \dots, \mathbf{w}_n$, to be nearest to W . That is, the principle of BRM is to find

$$\mathbf{p} = \sum_{j=1}^n r_j \mathbf{w}_j, \sum_{j=1}^n r_j = 1 \text{ and } r_j \geq 0 \ (j = 1, \dots, n) \tag{3.5}$$

such that

$$\mathcal{W}[\mathbf{p}] = (\Delta\mathbf{p})U^\top (\Delta(U\mathbf{p}))^{-1} \tag{3.6}$$

is near to W as possible as we can. Then we take $\mathcal{W}[\mathbf{p}]$ as the revised W .

But there are various meanings of "near". For example we can take the minimum L_2 -norm of $(\mathcal{W}[\mathbf{p}] - W)$ as "nearest", which is often formulated as the minimum-norm point problem in mathematical programming (see [12, 15]). Here we take the min-max principle as the nearest; that is, the min-max principle is to minimize

$$\max \left\{ \frac{\hat{w}_{ij}}{w_{ij}}, \frac{w_{ij}}{\hat{w}_{ij}} \mid i = 1, \dots, m \text{ and } j = 1, \dots, n \right\}, \tag{3.7}$$

where

$$\hat{W} = \begin{bmatrix} \hat{w}_{11} & \cdots & \hat{w}_{1n} \\ \vdots & \ddots & \vdots \\ \hat{w}_{m1} & \cdots & \hat{w}_{mn} \end{bmatrix} = \mathcal{W}[\mathbf{p}] \tag{3.8}$$

and \mathbf{p} satisfies (3.5). For all \mathbf{p} and all \hat{W} satisfying (3.5) and (3.8), we have

$$\sum_{i=1}^m \hat{w}_{ij} = 1 \text{ for all } j = 1, \dots, n$$

and

$$\hat{w}_{ij} = \frac{u_{ji}p_i}{\mathbf{u}_j\mathbf{p}} = \frac{u_{ji} \sum_{k=1}^n w_{ik}r_k}{\sum_{k=1}^n (\mathbf{u}_j\mathbf{w}_k)r_k} \text{ for all } i = 1, \dots, m \text{ and all } j = 1, \dots, n, \tag{3.9}$$

where p_i is the i -th component of \mathbf{p} . Thus the actual BRM is essentially to solve

$$\min \max \left\{ \frac{u_{ji} \sum_{k=1}^n w_{ik}r_k}{w_{ij} \sum_{k=1}^n (\mathbf{u}_j\mathbf{w}_k)r_k}, \frac{w_{ij} \sum_{k=1}^n (\mathbf{u}_j\mathbf{w}_k)r_k}{u_{ji} \sum_{k=1}^n w_{ik}r_k} \mid \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix} \right\} \tag{3.10}$$

$$\text{s.t. } \sum_{k=1}^n r_k = 1, r_k \geq 0, k = 1, \dots, n. \tag{3.11}$$

The optimization problem of (3.10) and (3.11) is a typical fractional program and it can be solved by Dinkelbach algorithm [1].

Remark 1 Theoretically the problem to minimize the L_1 -norm or L_2 -norm of $(\mathcal{W}[\mathbf{p}] - W)$ can be considered. But their objective functions include a sum of linear ratios or quadratic ratios. So, it is hard to find an optimal solution exactly and efficiently by research efforts of the recent mathematical programming [2, 5, 13].

Once we had the revised matrix \hat{W} , the analysis of ANP are carried out by the revised supermatrix

$$\hat{S} = \begin{bmatrix} 0 & \hat{W} \\ U & 0 \end{bmatrix}. \tag{3.12}$$

Let the evaluation vector of criteria be \mathbf{x} , and that of alternative be \mathbf{y} , then they are unique solution of

$$\hat{S} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \text{ or } \begin{matrix} U\mathbf{x} = \mathbf{y} \\ \hat{W}\mathbf{y} = \mathbf{x} \end{matrix}. \tag{3.13}$$

That is, $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$ is the principal eigenvector of \hat{S} whose principal eigenvalue is 1 (see the details for [9]). This leads to

$$\hat{W}U\mathbf{x} = \mathbf{x}. \tag{3.14}$$

The solution of (3.14) is \mathbf{p} satisfying (3.9). That is

$$\hat{W}U\mathbf{p} = \mathbf{p}. \tag{3.15}$$

In fact, the i -th component of the left hand-side of (3.15) is

$$\sum_{j=1}^n \sum_{k=1}^m \hat{w}_{ij} u_{jk} p_k = \sum_{j=1}^n \frac{u_{ji} p_i}{\mathbf{u}_j \mathbf{p}} \sum_{k=1}^m u_{jk} p_k = \sum_{j=1}^n \frac{u_{ji} p_i}{\mathbf{u}_j \mathbf{p}} \mathbf{u}_j \mathbf{p} = \sum_{j=1}^n u_{ji} p_i = p_i,$$

which is the i -th component of the right hand-side of (3.15). So we have the following theorem:

Theorem 1 *The evaluation weight vector of criteria of ANP with the supermatrix (3.12) is \mathbf{p}^* of (3.8) and the evaluation weight vector of alternatives is $U\mathbf{p}^*$.*

From Theorem 1 we introduce definitions of the evaluation weight vectors by BRM as follows: For an optimal solution $[r_1^*, \dots, r_n^*]^\top$ of (3.10) and (3.11), an evaluation weight vector of criteria is defined by $\mathbf{p}^* = \sum_{j=1}^n r_j^* \mathbf{w}_j$ and an evaluation weight vector of alternatives is defined by $U\mathbf{p}^* = U \sum_{j=1}^n r_j^* \mathbf{w}_j$. In the sequel, the evaluation weight vector of criteria is denoted by \mathbf{p}^* and that of alternatives is denoted by \mathbf{q}^* .

4. Some Properties of BRM

In this section we introduce some properties of BRM that is to solve the min-max problem of (3.10) and (3.11). Since the evaluation weight vector \mathbf{p}^* of criteria by BRM is an optimal solution of the min-max problem, properties of \mathbf{p}^* are provided by the constraints (3.11) and the objective function (3.10).

We will show that the constraints (3.11) provides an important property of \mathbf{p}^* , which is not obtained by CCM. For the typical mutual evaluation (1.1) Sekitani and Ueta [10] request the following condition for an evaluation weight vector $\mathbf{p} = [p_1, \dots, p_m]^\top$ of criteria:

$$\text{if } w_{1j} > w_{2j} > \dots > w_{mj} \text{ for all } j = 1, \dots, n, \text{ then } p_1 > p_2 > \dots > p_m. \tag{4.1}$$

If an evaluation method provides an evaluation weight vector \mathbf{p} satisfying (4.1) for any supermatrix (1.1), the evaluation method is called no contradiction. CCM has a contradiction for the supermatrix

$$\begin{bmatrix} 0 & 0 & 0.520 & 0.510 & 0.530 \\ 0 & 0 & 0.480 & 0.490 & 0.470 \\ 0.667 & 0.030 & 0 & 0 & 0 \\ 0.250 & 0.968 & 0 & 0 & 0 \\ 0.083 & 0.002 & 0 & 0 & 0 \end{bmatrix} \tag{4.2}$$

and it provides the evaluation weight vector of criteria, $[0.135, 0.865]^\top$ after 5 iterations (see [10] for details). Applying BRM to the supermatrix (4.2), we have an optimal solution of (3.10) and (3.11), $\mathbf{r}^* = [0, 1, 0]^\top$, by using Dinkelbach algorithm. It follows from (3.5) and the definition of the evaluation weight vector \mathbf{p}^* of criteria that

$$\mathbf{p}^* = \sum_{j=1}^n r_j^* \mathbf{w}_j = \mathbf{w}_2 = \begin{bmatrix} 0.51 \\ 0.49 \end{bmatrix}. \tag{4.3}$$

Therefore, the evaluation weight vector of criteria by BRM satisfies (4.1) for the supermatrix (4.2).

For all supermatrices (1.1) with any pair of positive stochastic matrices W and U , BRM always provides an evaluation weight vector of criteria satisfying (4.1) by the following theorem:

Theorem 2 *Let S be a supermatrix (1.1) and let \mathbf{p}^* be an evaluation weight vector of criteria by applying BRM to S . Suppose that W of S satisfies $w_{1j} > w_{2j} > \dots > w_{mj}$ for all $j = 1, \dots, n$, then $p_1^* > p_2^* > \dots > p_m^*$. That is, BRM has no contradiction.*

Proof: Let \mathbf{r}^* be an optimal solution of (3.10) and (3.11), then we have $r_j^* \geq 0$ for all $j = 1, \dots, n$ and there exists an index $l \in \{1, \dots, n\}$ such that $r_l^* > 0$. Since we have $p_i^* = \sum_{j=1}^n w_{ij} r_j^*$ for all $i = 1, \dots, m$ and $(w_{1j} - w_{2j}) r_j^* \geq 0$ for all $j = 1, \dots, n$, it follows from $(w_{1l} - w_{2l}) r_l^* > 0$ that

$$\begin{aligned} p_1^* - p_2^* &= \sum_{j=1}^n w_{1j} r_j^* - \sum_{j=1}^n w_{2j} r_j^* = \sum_{j=1}^n (w_{1j} - w_{2j}) r_j^* \\ &= \sum_{j \neq l} (w_{1j} - w_{2j}) r_j^* + (w_{1l} - w_{2l}) r_l^* \geq (w_{1l} - w_{2l}) r_l^* > 0. \end{aligned}$$

By the same manner, we prove $p_2^* > p_3^* > \dots > p_m^*$. □

Let $C(W)$ be the convex hull of $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$, then the min-max problem of (3.10) and (3.11) is equivalent to

$$\min_{\mathbf{p} \in C(W)} \max \left\{ \frac{u_{ji} p_i}{w_{ij} \mathbf{u}_j \mathbf{p}}, \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\}. \tag{4.4}$$

In order to discuss some properties of \mathbf{p}^* by the objective function (3.10), we relax the feasible region $C(W)$ of the min-max problem (4.4) into the positive orthant $\{\mathbf{p} \mid p_i > 0 \ i = 1, \dots, m\}$ and then consider two optimization problems:

$$\min_{\mathbf{p} > 0} \max \left\{ \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\}. \tag{4.5}$$

and

$$\max_{\mathbf{p} > 0} \min \left\{ \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\}. \tag{4.6}$$

Lemmas 3 and 4 guarantee the existence of optimal solutions of (4.5) and (4.6) and the coincidence between them and principal eigenvectors of some matrices.

Lemma 3 *There exists an optimal solution of (4.5). Let $\bar{\lambda}$ and $\bar{\mathbf{p}}$ be the optimal value of (4.5) and an optimal solution of (4.5), respectively. Let*

$$\frac{w_{ij_i} \mathbf{u}_{j_i} \bar{\mathbf{p}}}{u_{j_i i} \bar{p}_i} = \max \left\{ \frac{w_{ij} \mathbf{u}_j \bar{\mathbf{p}}}{u_{ji} \bar{p}_i} \mid j = 1, \dots, n \right\} \tag{4.7}$$

for all $i = 1, \dots, m$, then $\bar{\mathbf{p}}$ is a positive principal eigenvector of an $m \times m$ matrix

$$\begin{bmatrix} (w_{1j_1}/u_{j_11}) \mathbf{u}_{j_1} \\ \vdots \\ (w_{mj_m}/u_{j_mm}) \mathbf{u}_{j_m} \end{bmatrix}. \quad (4.8)$$

Proof: Let $V^i = \{(w_{ij}/u_{ji})\mathbf{u}_j \mid j = 1, \dots, n\}$ for $i = 1, \dots, m$, then (4.5) is equal to

$$\min_{\mathbf{p} > 0} \max \left\{ \frac{\mathbf{v}^1 \mathbf{p}}{p_1}, \dots, \frac{\mathbf{v}^m \mathbf{p}}{p_m} \mid (\mathbf{v}^1, \dots, \mathbf{v}^m) \in V^1 \times \dots \times V^m \right\}.$$

It follows from Theorem 34 of [8] that the above problem has an optimal solution that is a principal eigenvector of (4.8). □

Lemma 4 *There exists an optimal solution of (4.6). Let $\underline{\lambda}$ and $\underline{\mathbf{p}}$ be the optimal value of (4.6) and an optimal solution of (4.6), respectively. Let*

$$\frac{w_{ij} \mathbf{u}_{ji} \underline{\mathbf{p}}}{u_{ji} \underline{p}_i} = \min \left\{ \frac{w_{ij} \mathbf{u}_{j} \underline{\mathbf{p}}}{u_{ji} \underline{p}_i} \mid j = 1, \dots, n \right\}$$

for all $i = 1, \dots, m$, then $\underline{\mathbf{p}}$ is a positive principal eigenvector of an $m \times m$ matrix

$$\begin{bmatrix} (w_{1j_1}/u_{j_11}) \mathbf{u}_{j_1} \\ \vdots \\ (w_{mj_m}/u_{j_mm}) \mathbf{u}_{j_m} \end{bmatrix}. \quad (4.9)$$

Proof: By the same manner as the proof of Lemma 3, this assertion directly follows (see also Theorem 30 of [8]). □

Both the optimal value of (4.5) and that of (4.6) correspond to lower bounds of the optimal value of the min-max problem (4.4) as follows:

Lemma 5 *Let $\bar{\lambda}$ and $\underline{\lambda}$ be the optimal value of (4.5) and (4.6), respectively, then*

$$\text{the optimal value of (4.4)} \geq \max \left\{ \frac{1}{\underline{\lambda}}, \bar{\lambda} \right\}. \quad (4.10)$$

Proof: Since $w_j > 0$ for all $j = 1, \dots, n$, we have $C(W) \subseteq \{\mathbf{p} \mid \mathbf{p} > 0\}$. This implies from $\underline{\lambda} > 0$ that

$$\begin{aligned} & \min_{\mathbf{p} \in C(W)} \max \left\{ \frac{u_{ji} p_i}{w_{ij} \mathbf{u}_j \mathbf{p}}, \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} \\ & \geq \min_{\mathbf{p} > 0} \max \left\{ \frac{u_{ji} p_i}{w_{ij} \mathbf{u}_j \mathbf{p}}, \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} \\ & = \min_{\mathbf{p} > 0} \max \left\{ \max \left\{ \frac{u_{ji} p_i}{w_{ij} \mathbf{u}_j \mathbf{p}} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\}, \max \left\{ \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} \right\} \\ & \geq \max \left\{ \min_{\mathbf{p} > 0} \max \left\{ \frac{u_{ji} p_i}{w_{ij} \mathbf{u}_j \mathbf{p}} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\}, \min_{\mathbf{p} > 0} \max \left\{ \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} \right\} \\ & = \max \left\{ \left(\max_{\mathbf{p} > 0} \min \left\{ \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} \right)^{-1}, \min_{\mathbf{p} > 0} \max \left\{ \frac{w_{ij} \mathbf{u}_j \mathbf{p}}{u_{ji} p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} \right\} \\ & = \max \left\{ \frac{1}{\underline{\lambda}}, \bar{\lambda} \right\}. \quad \square \end{aligned}$$

When the optimal value of (4.4) is equal to that of (4.5), an evaluation weight vector \mathbf{p}^* of criteria by BRM has the following properties:

Lemma 6 *Let $\bar{\lambda}$ and $\bar{\mathbf{p}}$ be the optimal value of (4.5) and an optimal solution of (4.5), respectively, and suppose that $\bar{\lambda}$ is equal to the optimal value of (4.4), then any optimal solution of (4.4) is $(\bar{\mathbf{p}}/\sum_{i=1}^m \bar{p}_i)$.*

Proof: Suppose that there exists an optimal solution of $\mathbf{p}^* \neq \bar{\mathbf{p}}/\sum_{i=1}^m \bar{p}_i$ of (4.4), then it follows from $\sum_{i=1}^m p_i^* = 1$, Frobenius' Theorem (e.g., Theorem 9 of [11]) and Lemma 3 that

$$\bar{\lambda} < \max \left\{ \frac{(w_{1j_1}/u_{j_1}) \mathbf{u}_{j_1} \mathbf{p}^*}{p_1^*}, \dots, \frac{(w_{mj_m}/u_{j_m}) \mathbf{u}_{j_m} \mathbf{p}^*}{p_m^*} \right\},$$

where j_i is defined by (4.7) for all $i = 1, \dots, m$. This means that $\bar{\lambda}$ is less than the optimal value of (4.4), which is a contradiction. Hence, any optimal solution (4.4) is $\bar{\mathbf{p}}/\sum_{i=1}^m \bar{p}_i$. \square

Theorem 7 means that an evaluation weight vector \mathbf{p}^* of criteria by BRM is an eigenvector of some matrix when the optimal value of (4.4) is equal to that of (4.5).

Theorem 7 *Suppose that the optimal value of (4.5) is equal to the optimal value of (4.4), then an optimal solution of (4.4) is a positive principal eigenvector of the matrix (4.8).*

Proof: This theorem follows from lemmas 3 and 6. \square

In the other case, that is, when the optimal value of (4.4) is a reciprocal of that of (4.6), an evaluation weight vector \mathbf{p}^* of criteria by BRM is an eigenvector of some matrix by the following lemma and theorem:

Lemma 8 *Let $\underline{\lambda}$ and $\underline{\mathbf{p}}$ be the optimal value of (4.6) and an optimal solution of (4.6), respectively, and suppose that $\underline{\lambda}^{-1}$ is equal to the optimal value of (4.4), then any optimal solution of (4.4) is $(\underline{\mathbf{p}}/\sum_{i=1}^m \underline{p}_i)$.*

Proof: In the similar way to the proof of Lemma 6, we prove it. \square

Theorem 9 *Suppose that the optimal value of (4.6) is a reciprocal of the optimal value of (4.4), then an optimal solution of (4.4) is a positive principal eigenvector of the matrix (4.9).*

Proof: This theorem follows from lemmas 4 and 8. \square

We show the existence of W that provides no gap between the optimal value of (4.4) and that of (4.5).

Theorem 10 *Suppose that there exists a positive vector $\bar{\mathbf{p}}$ such that $W = \mathcal{W}[\bar{\mathbf{p}}]$ and $\sum_{i=1}^m \bar{p}_i = 1$, then the optimal value of (4.4) is equal to that of (4.5), and both an optimal solution of (4.5) and that of (4.4) are $\bar{\mathbf{p}}$.*

Proof: It follows from $W = \mathcal{W}[\bar{\mathbf{p}}]$ that $w_{ji} = (u_{ij}\bar{p}_i)/(\mathbf{u}_j \bar{\mathbf{p}})$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. Hence, we have

$$\frac{(w_{ji}/u_{ij}) \mathbf{u}_j \bar{\mathbf{p}}}{\bar{p}_i} = 1 \quad \text{for all } i = 1, \dots, m \text{ and all } j = 1, \dots, n.$$

From Frobenius' Theorem (e.g., Corollary 4 of [11]) every positive vector $\mathbf{p} \notin \{\mu \bar{\mathbf{p}} \mid \mu > 0\}$ satisfies

$$\min \left\{ \frac{(w_{ji}/u_{ij}) \mathbf{u}_j \mathbf{p}}{p_i} \mid i = 1, \dots, m \right\} < 1 < \max \left\{ \frac{(w_{ji}/u_{ij}) \mathbf{u}_j \mathbf{p}}{p_i} \mid i = 1, \dots, m \right\}$$

for all $j = 1, \dots, n$. Hence, for every positive vector $\mathbf{p} \neq \bar{\mathbf{p}}$ with $\sum_{i=1}^m p_i = 1$, we have

$$\begin{aligned} \max \left\{ \frac{u_{ji}\bar{p}_i}{w_{ij}\mathbf{u}_j\bar{\mathbf{p}}}, \frac{w_{ij}\mathbf{u}_j\bar{\mathbf{p}}}{u_{ji}\bar{p}_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} &= 1 < \max \left\{ \frac{u_{ji}p_i}{w_{ij}\mathbf{u}_j\mathbf{p}}, \frac{w_{ij}\mathbf{u}_j\mathbf{p}}{u_{ji}p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} \\ \text{and } \max \left\{ \frac{w_{ij}\mathbf{u}_j\bar{\mathbf{p}}}{u_{ji}\bar{p}_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\} &= 1 < \max \left\{ \frac{w_{ij}\mathbf{u}_j\mathbf{p}}{u_{ji}p_i} \mid \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array} \right\}. \end{aligned}$$

It follows from Theorem 1 that there exists an positive vector \mathbf{y} such that $W\mathbf{y} = \bar{\mathbf{p}}$ and $\sum_{j=1}^n y_j = 1$. This means that $\bar{\mathbf{p}} \in C(W)$. Therefore, both the optimal value of (4.4) and that of (4.5) are 1 and their optimal solution are $\bar{\mathbf{p}}$. \square

Theorem 10 also implies that BRM provides a unique evaluation weight vector \mathbf{p}^* satisfying $W = \mathcal{W}[\mathbf{p}^*]$ if W is given by the Bayes Theorem (2.5).

5. Uniqueness and Sensitivity of Evaluation Weight Vector

For practical use of BRM, we discuss the uniqueness of an optimal solution of (4.1) and sensitivity of its optimal solution, with the help of a numerical example [14],

$$U = \begin{bmatrix} 1/3 & 0.3 \\ 1/6 & 0.6 \\ 1/2 & 0.1 \end{bmatrix} \tag{5.1}$$

$$W = \begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.3 & 0.6 & 0.8 \end{bmatrix}. \tag{5.2}$$

The min-max problem (3.10) and (3.11) defined by U of (5.1) and W of (5.2) has an optimal solution

$$\mathbf{r}^* = [r_1^*, r_2^*, r_3^*]^\top = [0.0, 0.586, 0.414]^\top \tag{5.3}$$

and the optimal value $\lambda^* = 3.4948$. An optimal solution of the min-max problem (4.4) is

$$\mathbf{p}^* = W\mathbf{r}^* = \begin{bmatrix} 0.317 \\ 0.683 \end{bmatrix}. \tag{5.4}$$

From (3.9) we have

$$\hat{W} = \mathcal{W}[\mathbf{p}^*] = \begin{bmatrix} 0.340 & 0.114 & 0.699 \\ 0.660 & 0.886 & 0.301 \end{bmatrix}. \tag{5.5}$$

For two matrices, W of (5.2) and \hat{W} of (5.5), we have

$$\max \left\{ \frac{w_{ij}}{\hat{w}_{ij}}, \frac{\hat{w}_{ij}}{w_{ij}} \mid i = 1, 2, j = 1, 2, 3 \right\} = \frac{w_{12}}{\hat{w}_{12}} = \frac{\hat{w}_{13}}{w_{13}} = \lambda^* = 3.4948, \tag{5.6}$$

where λ^* is the optimal value of (4.4). From \mathbf{p}^* of (5.4) we have

$$\mathbf{q}^* = U\mathbf{p}^* = \begin{bmatrix} 0.310 \\ 0.463 \\ 0.227 \end{bmatrix}. \tag{5.7}$$

As stated above, an optimal solution \mathbf{p}^* of the min-max problem (4.4) is given as an evaluation weight vector of criteria by BRM. Is \mathbf{p}^* unique? To check the existence of alternative evaluation weight vector of criteria by BRM other than \mathbf{p}^* , we solve

$$\begin{aligned} \max \quad & \sum_{i=1}^m |p_i - p_i^*| \\ \text{s.t.} \quad & u_{ji}p_i - \lambda^* w_{ij} \mathbf{u}_j \mathbf{p} \leq 0 \text{ for all } i = 1, \dots, m \text{ and } j = 1, \dots, n \\ & w_{ij} \mathbf{u}_j \mathbf{p} - \lambda^* u_{ji} p_i \leq 0 \text{ for all } i = 1, \dots, m \text{ and } j = 1, \dots, n \\ & \mathbf{p} \in C(W), \end{aligned} \quad (5.8)$$

where λ^* is the optimal value of the min-max problem of (3.10) and (3.11). If the optimal value of (5.8) is 0, an evaluation weight vector of criteria by BRM is unique. Otherwise there exist multiple evaluation weight vectors of criteria by BRM, whose set is

$$\left\{ \mathbf{p} \left| \begin{array}{l} u_{ji}p_i - \lambda^* w_{ij} \mathbf{u}_j \mathbf{p} \leq 0 \text{ for all } i = 1, \dots, m \text{ and } j = 1, \dots, n \\ w_{ij} \mathbf{u}_j \mathbf{p} - \lambda^* u_{ji} p_i \leq 0 \text{ for all } i = 1, \dots, m \text{ and } j = 1, \dots, n \\ \mathbf{p} \in C(W) \end{array} \right. \right\}. \quad (5.9)$$

The problem (5.8) defined by U of (5.1), W of (5.2), \mathbf{p}^* of (5.4) and $\lambda^* = 3.4948$ has the optimal value 0. Hence, an optimal solution \mathbf{p}^* of the min-max problem (4.4) is unique. Applying BRM to the supermatrix (1.1) with U of (5.1) and W of (5.2), \hat{W} is (5.5) and an evaluation weight vector of criteria is \mathbf{p}^* of (5.4). Moreover, an evaluation weight vector of alternates is \mathbf{q}^* of (5.7).

Since judgment of criteria by alternative may be unstable, it is desirable that \hat{W} given by \mathbf{p}^* does not fluctuate widely with small changes in W . So we illustrate how sensitive the evaluation weight vector \mathbf{p}^* of criteria is to slight changes in W of (5.2). A way of perturbation of W is to change only one column of W and then to fix other columns. This is called a single column-perturbation of W . Replacing the first column $[0.7, 0.3]^\top$ of (5.2) with $\mathbf{w}_1 = [w_{11}, w_{21}]^\top$, we consider

$$W(w_{11}) = \begin{bmatrix} w_{11} & 0.4 & 0.2 \\ w_{21} & 0.6 & 0.8 \end{bmatrix} \quad (5.10)$$

as a perturbation of W , where $w_{11} + w_{21} = 1$, $w_{11} > 0$ and $w_{21} > 0$.

Setting $w_{11} = 0.5, 0.55, \dots, 0.9$, we generate 9 perturbations of (5.2), $W(0.5), W(0.55), \dots, W(0.9)$. For each of $w_{11} = 0.5, \dots, 0.9$ we apply BRM to the supermatrix (1.1) with U of (5.1) and $W(w_{11})$, and then solve (5.8) in order to test the uniqueness of an optimal solution \mathbf{p}^* of (4.4). Here, the optimal value of (5.8) is denoted by σ^* . These computational results are summarized in Table 1 that documents the optimal value λ^* , an evaluation weight vector \mathbf{p}^* of criteria, an evaluation weight vector $\mathbf{q}^* = U\mathbf{p}^*$ of alternatives, a revising matrix $\hat{W} = \mathcal{W}[\mathbf{p}^*]$ and the optimal value σ^* of (5.8) for each of $w_{11} = 0.5, \dots, 0.9$.

Since we have $\sigma^* = 0$ for every $w_{11} = 0.5, \dots, 0.9$, an evaluation weight vector \mathbf{p}^* of criteria by BRM is unique. The evaluation weight vector \mathbf{p}^* of criteria is invariant to $w_{11} = 0.5, \dots, 0.8$ and then $\mathcal{W}[\mathbf{p}^*]$ is so. Therefore, the evaluation weight vector \mathbf{p}^* of criteria and the revising matrix $\mathcal{W}[\mathbf{p}^*]$ do not fluctuate with changes of $w_{11} = 0.7 \pm 0.05$ and 0.7 ± 0.1 .

In order to discuss invariance of \mathbf{p}^* to the single column-perturbation analytically, we introduce a definition of an inactive alternative. For two stochastic positive matrices W and

Table 1: Perturbations of W and BRM

w_{11}	\mathbf{p}^*	\mathbf{q}^*	λ^*	$\mathcal{W}[\mathbf{p}^*]$	σ^*
0.5					
\vdots	$[0.317, 0.683]^\top$	$[0.310, 0.463, 0.227]^\top$	3.495	$\begin{bmatrix} 0.340 & 0.114 & 0.699 \\ 0.660 & 0.886 & 0.301 \end{bmatrix}$	0
0.8					
0.85	$[0.399, 0.601]^\top$	$[0.313, 0.427, 0.259]^\top$	3.841	$\begin{bmatrix} 0.424 & 0.156 & 0.768 \\ 0.576 & 0.844 & 0.232 \end{bmatrix}$	0
0.9	$[0.498, 0.502]^\top$	$[0.316, 0.385, 0.299]^\top$	4.763	$\begin{bmatrix} 0.524 & 0.216 & 0.832 \\ 0.476 & 0.784 & 0.168 \end{bmatrix}$	0

U of (1.1), an alternative j is called inactive if the optimal value λ^* of (4.4) and any optimal solution \mathbf{p}^* of (4.4) satisfies

$$\lambda^* > \max \left\{ \frac{u_{ji}p_i^*}{w_{ij}\mathbf{u}_j\mathbf{p}^*}, \frac{w_{ij}\mathbf{u}_j\mathbf{p}^*}{u_{ji}p_i^*} \mid i = 1, \dots, m \right\} \quad (5.11)$$

$$\text{and } \mathbf{p}^* \in C(\mathbf{w}_1, \dots, \mathbf{w}_{j-1}, \mathbf{w}_{j+1}, \dots, \mathbf{w}_n), \quad (5.12)$$

where $C(\mathbf{w}_1, \dots, \mathbf{w}_{j-1}, \mathbf{w}_{j+1}, \dots, \mathbf{w}_n)$ is the convex hull of $\{\mathbf{w}_1, \dots, \mathbf{w}_{j-1}, \mathbf{w}_{j+1}, \dots, \mathbf{w}_n\}$. An inactive alternative has the following property for the single column-perturbation of W :

Lemma 11 *Let λ^* and \mathbf{p}^* be the optimal value of (4.4) and the optimal solution of (4.4), respectively and suppose that an alternative j is inactive. Suppose that $\tilde{\mathbf{w}}_j$ is a positive vector with $\sum_{i=1}^m \tilde{w}_{ij} = 1$ and consider $\tilde{W} = [\mathbf{w}_1, \dots, \mathbf{w}_{j-1}, \tilde{\mathbf{w}}_j, \mathbf{w}_{j+1}, \dots, \mathbf{w}_n]$ as a single column-perturbation of W , then \mathbf{p}^* is also a feasible solution of (4.4) with \tilde{W} . If*

$$\lambda^* \geq \max \left\{ \frac{\tilde{w}_{ij}\mathbf{u}_j\mathbf{p}^*}{u_{ji}p_i^*}, \frac{u_{ji}p_i^*}{\tilde{w}_{ij}\mathbf{u}_j\mathbf{p}^*} \mid i = 1, \dots, m \right\}, \quad (5.13)$$

then \mathbf{p}^* attains the same objective function value of (4.4) with \tilde{W} as λ^* , that is

$$\lambda^* = \max \left\{ \begin{array}{l} \max \left\{ \frac{u_{li}p_i^*}{w_{il}\mathbf{u}_l\mathbf{p}^*}, \frac{w_{il}\mathbf{u}_l\mathbf{p}^*}{u_{li}p_i^*} \mid i = 1, \dots, m \right\}, \\ \max \left\{ \frac{u_{ji}p_i^*}{\tilde{w}_{ij}\mathbf{u}_j\mathbf{p}^*}, \frac{\tilde{w}_{ij}\mathbf{u}_j\mathbf{p}^*}{u_{ji}p_i^*} \mid i = 1, \dots, m \right\} \end{array} \right\}. \quad (5.14)$$

Proof: Since the alternative j is inactive, it follows from (5.12) that

$$\mathbf{p}^* \in C(\mathbf{w}_1, \dots, \mathbf{w}_{j-1}, \mathbf{w}_{j+1}, \dots, \mathbf{w}_n) \subseteq C(\mathbf{w}_1, \dots, \mathbf{w}_{j-1}, \tilde{\mathbf{w}}_j, \mathbf{w}_{j+1}, \dots, \mathbf{w}_n) = C(\tilde{W}).$$

Therefore, \mathbf{p}^* is a feasible solution of (4.4) with \tilde{W} .

Since λ^* and \mathbf{p}^* are the optimal value of (4.4) and its optimal solution, respectively, it follows from (5.11) and (5.13) that

$$\begin{aligned} \lambda^* &= \max \left\{ \frac{u_{li}p_i^*}{w_{il}\mathbf{u}_l\mathbf{p}^*}, \frac{w_{il}\mathbf{u}_l\mathbf{p}^*}{u_{li}p_i^*} \mid \begin{array}{l} i = 1, \dots, m \\ l = 1, \dots, n \end{array} \right\} = \max \left\{ \frac{u_{li}p_i^*}{w_{il}\mathbf{u}_l\mathbf{p}^*}, \frac{w_{il}\mathbf{u}_l\mathbf{p}^*}{u_{li}p_i^*} \mid \begin{array}{l} i = 1, \dots, m \\ l \neq j \end{array} \right\} \\ &= \max \left\{ \max \left\{ \frac{u_{li}p_i^*}{w_{il}\mathbf{u}_l\mathbf{p}^*}, \frac{w_{il}\mathbf{u}_l\mathbf{p}^*}{u_{li}p_i^*} \mid \begin{array}{l} i = 1, \dots, m \\ l \neq j \end{array} \right\}, \max \left\{ \frac{u_{ji}p_i^*}{\tilde{w}_{ij}\mathbf{u}_j\mathbf{p}^*}, \frac{\tilde{w}_{ij}\mathbf{u}_j\mathbf{p}^*}{u_{ji}p_i^*} \mid i = 1, \dots, m \right\} \right\}. \quad \square \end{aligned}$$

Let $\tilde{W} = [\mathbf{w}_1, \dots, \mathbf{w}_{j-1}, \tilde{\mathbf{w}}_j, \mathbf{w}_{j+1}, \dots, \mathbf{w}_n]$, then the min-max problem (4.4) with \tilde{W} is formulated as

$$\min_{\mathbf{p} \in C(\tilde{W})} \max \left\{ \begin{array}{l} \max \left\{ \frac{u_{li}p_i}{w_{li}\mathbf{u}_l\mathbf{p}}, \frac{w_{li}\mathbf{u}_l\mathbf{p}}{u_{li}p_i} \mid \begin{array}{l} i = 1, \dots, m \\ l \neq j \end{array} \right\}, \\ \max \left\{ \frac{u_{ji}p_i}{\tilde{w}_{ij}\mathbf{u}_j\mathbf{p}}, \frac{\tilde{w}_{ij}\mathbf{u}_j\mathbf{p}}{u_{ji}p_i} \mid i = 1, \dots, m \right\} \end{array} \right\}. \quad (5.15)$$

Lemma 11 means that an optimal solution \mathbf{p}^* of the min-max problem (4.4) is a feasible solution of (5.15) when the alternative j is inactive and that the objective function value attained by \mathbf{p}^* is invariant to the inactive alternative j 's vector $\tilde{\mathbf{w}}_j$ satisfying (5.13). Therefore, if the alternative j is inactive and the optimal value λ^* of (4.4) is also the optimal value of (5.15), then the optimal solution \mathbf{p}^* of (4.4) is also that of (5.15).

By the numerical results of (5.1) and (5.2), we confirm the invariance of the evaluation weight vector \mathbf{p}^* of criteria to $\tilde{\mathbf{w}}_j$ of an inactive alternative j . Since the min-max problem (4.4) with W of (5.2) has the unique optimal solution \mathbf{p}^* of (5.4) and the optimal value $\lambda^* = 3.495$, it follows from $\mathbf{p}^* = 0.586\mathbf{w}_2 + 0.414\mathbf{w}_3$ that

$$\max \left\{ \frac{u_{1i}p_i^*}{w_{1i}\mathbf{u}_1\mathbf{p}^*}, \frac{w_{1i}\mathbf{u}_1\mathbf{p}^*}{u_{1i}p_i^*} \mid i = 1, 2 \right\} = \max \{0.486, 2.057, 2.199, 0.455\} < 3.495 = \lambda^*$$

and $\mathbf{p}^* \in C(\mathbf{w}_2, \mathbf{w}_3)$.

Therefore, the alternative 1 is inactive for U of (5.1) and W (0.7). Let $\tilde{W} = \begin{bmatrix} \tilde{w}_{11} & 0.4 & 0.2 \\ \tilde{w}_{21} & 0.6 & 0.8 \end{bmatrix}$, then the min-max problem (5.15) corresponds to (4.4) defined by W of (5.2) that includes a perturbation factor $\tilde{\mathbf{w}}_1$, and $\lambda^* = 3.495$ is the objective function value of (5.15) at \mathbf{p}^* of (5.4) when $\tilde{\mathbf{w}}_1$ satisfies (5.13), that is

$$\frac{1}{\lambda^*} \leq \frac{\tilde{w}_{i1}\mathbf{u}_1\mathbf{p}^*}{u_{1i}p_i^*} \leq \lambda^* \quad (5.16)$$

for $i = 1, 2$. The inequalities (5.16) implies from $\mathbf{p}^* = [0.317, 0.683]^\top$, $\mathbf{u}_1 = [1/3, 0.3]$, $\lambda^* = 3.495$ and $\tilde{w}_{11} + \tilde{w}_{21} = 1$ that $0.097 \leq \tilde{w}_{11} \leq 0.811$. Since $\tilde{w}_{11} \in [0.097, 0.811]$ for $\tilde{w}_{11} = 0.5, 0.55, \dots, 0.8$ and $\lambda^* = 3.495$ is the optimal value of (5.15) with $\tilde{w}_{11} = 0.5, 0.55, \dots, 0.8$, the evaluation weight vector \mathbf{p}^* of criteria is invariant to $\tilde{w}_{11} = 0.5, 0.55, \dots, 0.8$.

Under the equivalence between the optimal value of (4.4) and that of (5.15), Lemma 11 guarantees the invariance of the evaluation weight vector of criteria. However, the following theorem does not need the equivalence between them:

Theorem 12 *Let λ^* and \mathbf{p}^* be the optimal value of (4.4) and the optimal solution of (4.4), respectively. Let $\bar{\lambda}$ and $\underline{\lambda}$ be the optimal value of (4.5) and that of (4.6), respectively. Suppose that $\lambda^* = \max \{ \bar{\lambda}^{-1}, \underline{\lambda} \}$ and that an alternative j is inactive. If a positive vector $\tilde{\mathbf{w}}_j$ satisfies $\sum_{i=1}^m \tilde{w}_{ij} = 1$ and (5.13), then \mathbf{p}^* is also an optimal solution of (5.15).*

Proof: Let $\tilde{\lambda}$ and $\tilde{\mathbf{p}}$ be the optimal value of (5.15) and an optimal solution of (5.15), respectively. Since $\tilde{\mathbf{w}}_j$ satisfies (5.13), it follows from Lemma 11 that $\lambda^* \geq \tilde{\lambda}$.

Assume that $\lambda^* > \tilde{\lambda}$, then we have

$$\tilde{\mathbf{p}} \neq \mathbf{p}^* \text{ and } \sum_{i=1}^m \tilde{p}_i = \sum_{i=1}^m p_i^* = 1. \quad (5.17)$$

Consider the case $\lambda^* = \max\{\underline{\lambda}^{-1}, \bar{\lambda}\} = \bar{\lambda}$, then it follows from Lemmas 3 and 6 that there exist m indices j_1, \dots, j_m such that \mathbf{p}^* is a positive principal eigenvector of (4.8). Since the alternative j is inactive and the principal eigenvalue of (4.8) is $\lambda^* = \bar{\lambda}$, it follows from (5.11) that $j \notin \{j_1, \dots, j_m\}$. This implies from (5.17) and Frobenius' Theorem that

$$\begin{aligned} \lambda^* = \bar{\lambda} &< \max \left\{ \frac{w_{1j_1} \mathbf{u}_{j_1} \tilde{\mathbf{p}}}{u_{j_1 1} \tilde{p}_1}, \dots, \frac{w_{mj_m} \mathbf{u}_{j_m} \tilde{\mathbf{p}}}{u_{j_m m} \tilde{p}_m} \right\} \\ &\leq \max \left\{ \max \left\{ \frac{u_{li} \tilde{p}_i}{w_{il} \mathbf{u}_i \tilde{\mathbf{p}}}, \frac{w_{il} \mathbf{u}_i \tilde{\mathbf{p}}}{u_{li} \tilde{p}_i} \mid i = 1, \dots, m \right\}, \right. \\ &\quad \left. \max \left\{ \frac{u_{ji} \tilde{p}_i}{\tilde{w}_{ij} \mathbf{u}_j \tilde{\mathbf{p}}}, \frac{\tilde{w}_{ij} \mathbf{u}_j \tilde{\mathbf{p}}}{u_{ji} \tilde{p}_i} \mid i = 1, \dots, m \right\} \right\} = \tilde{\lambda}. \end{aligned}$$

This is contradiction for $\lambda^* > \tilde{\lambda}$. For the other case, that is $\lambda^* = \max\{\underline{\lambda}^{-1}, \bar{\lambda}\} = \underline{\lambda}^{-1}$, we lead to contradiction by the same manner. Therefore, we have $\lambda^* = \tilde{\lambda}$ and it follows from Lemmas 6 and 8 that $\mathbf{p}^* = \tilde{\mathbf{p}}$. □

Consider the min-max problem (4.4) defined by U of (5.1) and $W(0.85)$ of (5.10), then the optimal value λ^* of (4.4) is equal to the optimal value $\underline{\lambda}$ of (4.6) and the optimal solution \mathbf{p}^* of (4.4) is a principal eigenvector of

$$\begin{bmatrix} (w_{13}/u_{31}) \mathbf{u}_3 \\ (w_{21}/u_{12}) \mathbf{u}_1 \end{bmatrix}. \tag{5.18}$$

In fact, by applying the Dinkelbach algorithm [1] or the coloring algorithm [8] to the max-min problem (4.6) defined by U of (5.1) and $W(0.85)$ of (5.10), we have its optimal value $\underline{\lambda} = 0.2603$ and its optimal solution $\underline{\mathbf{p}} = [0.399, 0.601]^\top$. Since

$$\frac{w_{13} \mathbf{u}_3 \underline{\mathbf{p}}}{u_{31} \underline{p}_1} = \min \left\{ \frac{w_{1j} \mathbf{u}_j \underline{\mathbf{p}}}{u_{j1} \underline{p}_1} \mid j = 1, 2, 3 \right\} \text{ and } \frac{w_{21} \mathbf{u}_1 \underline{\mathbf{p}}}{u_{12} \underline{p}_2} = \min \left\{ \frac{w_{2j} \mathbf{u}_j \underline{\mathbf{p}}}{u_{j2} \underline{p}_2} \mid j = 1, 2, 3 \right\},$$

it follows from Lemma 4 that $\underline{\mathbf{p}}$ is a principal eigenvector of the matrix (5.18) such as

$$\begin{bmatrix} \left(\frac{0.2}{1/2}\right) 1/2 & \left(\frac{0.2}{1/2}\right) 0.1 \\ \left(\frac{0.15}{0.3}\right) 1/3 & \left(\frac{0.15}{0.3}\right) 0.3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{25} \\ \frac{1}{6} & \frac{3}{20} \end{bmatrix},$$

whose principal eigenvalue is $\underline{\lambda} = 0.2603$. Since the optimal value $\lambda^* = 3.841$ of (4.4) is equal to $\underline{\lambda}^{-1} = 0.2603^{-1} = 3.841$, it follows from Theorem 9 that $\underline{\mathbf{p}}$ is an optimal solution of (4.4).

From (5.1), (5.2) and $\mathbf{p}^* = [0.399, 0.601]^\top$ we have

$$\begin{aligned} \max \left\{ \frac{u_{2i} p_i^*}{w_{i2} \mathbf{u}_1 \mathbf{p}^*}, \frac{w_{i2} \mathbf{u}_2 \mathbf{p}^*}{u_{2i} p_i^*} \mid i = 1, 2 \right\} &= \max \{2.567, 0.390, 0.710, 0.409\} < 3.841 = \lambda^* \\ \text{and } \mathbf{p}^* &= 0.306 \mathbf{w}_1 + 0.694 \mathbf{w}_3 \in C(\mathbf{w}_1, \mathbf{w}_3). \end{aligned}$$

Hence, the alternative 2 is inactive for U of (5.1) and $W(0.85)$ of (5.10). Let $\tilde{W} = \begin{bmatrix} 0.85 & \tilde{w}_{12} & 0.2 \\ 0.15 & \tilde{w}_{22} & 0.8 \end{bmatrix}$, then the min-max problem (5.15) corresponds to (4.4) defined by $W(0.85)$

including a perturbation factor \tilde{w}_2 , then \mathbf{p}^* is an optimal solution of (5.15) when \tilde{w}_2 satisfies (5.13), that is

$$\frac{1}{\lambda^*} \leq \frac{\tilde{w}_{i2}\mathbf{u}_2\mathbf{p}^*}{u_{2i}p_i^*} \leq \lambda^* \tag{5.19}$$

for $i = 1, 2$. The inequalities (5.19) implies from $\mathbf{p}^* = [0.399, 0.601]^\top$, $\mathbf{u}_2 = [1/6, 0.6]$, $\lambda^* = 3.841$ and $\tilde{w}_{12} + \tilde{w}_{22} = 1$ that $0.0405 \leq \tilde{w}_{12} \leq 0.599$. Therefore, for U of (5.1) and \tilde{W} , the evaluation weight vector \mathbf{p}^* of criteria is invariant to $\tilde{w}_{12} \in [0.0405, 0.599]$.

In the type (1.1) of ANP, the number n of alternatives is often greater than that of criteria, m . Every point $\mathbf{p} \in C(W)$ can be given as a convex combination of at most m vectors of $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$. Note that the set of optimal solutions of (4.4) is (5.9), we see that at most m inequality constraints of (5.10) are held on equality at almost optimal solutions of (4.4). Since $u_{ji}p_i - \lambda^*w_{ij}\mathbf{u}_j\mathbf{p} = 0$ is equivalent to $w_{ij}\mathbf{u}_j\mathbf{p}/u_{ji}p_i = \lambda^*$, there exists an inactive alternative if $n > 2m$. Here, we consider an assumption that every index j of (5.11) satisfies (5.12). The assumption is valid in all numerical examples as stated above. Under the assumption there exists an inactive alternative if $n > m$. Therefore, the evaluation weight vector by BRM may often be invariant to small perturbation of judgments from one alternative to criteria.

6. Incorporating Requirements of Decision Makers into BRM

BRM satisfies the requirement of (4.1) and changes the evaluation matrix W into the revised one \hat{W} . However, all judgments of criteria from alternatives by decision makers are not always satisfied with the revised matrix \hat{W} . As stated in the previous section, the first column of W of (5.2) is

$$w_{11} = 0.7 \geq 0.3 = w_{21}. \tag{6.1}$$

This means that decision makers with the viewpoint of the alternative 1 prefer the criterion 1 to the criterion 2. The first column of \hat{W} of (5.5) is

$$\hat{w}_{11} = 0.34 \leq 0.66 = \hat{w}_{21} \tag{6.2}$$

and the preference order among criteria by (6.2) is against that by (6.1). Hence, the revised evaluation matrix \hat{W} might neglect some judgments of the decision makers that are quantified to W . There may exist a requirement of the decision makers that order of criteria by some columns of W are invariant to a revision from W to \hat{W} . Furthermore, they may require the revision of W to keep all values of some columns of W invariant.

These requirements of decision makers can be dealt with BRM by adding constraints into the optimization problem (3.10) and (3.11). The coincidence between order among $\{w_{1j}, w_{2j}, \dots, w_{mj}\}$ and that among $\{\hat{w}_{1j}, \hat{w}_{2j}, \dots, \hat{w}_{mj}\}$ is called the j -th column order invariance condition. The equivalence between $\{w_{1j}, w_{2j}, \dots, w_{mj}\}$ and $\{\hat{w}_{1j}, \hat{w}_{2j}, \dots, \hat{w}_{mj}\}$ is called the j -th column invariance condition. Without loss of generality, we consider $w_{1j} \geq w_{2j} \geq \dots \geq w_{mj}$ for some j . The j -th column order invariance condition is formulated as $(w_{ij} - w_{i+1j})(\hat{w}_{ij} - \hat{w}_{i+1j}) \geq 0$ for all $i = 1, \dots, m - 1$. From (3.9) we have $\hat{w}_{ij} - \hat{w}_{i+1j} = u_{ji}p_i/\mathbf{u}_j\mathbf{p} - u_{j(i+1)}p_{i+1}/\mathbf{u}_j\mathbf{p} = (u_{ji}p_i - u_{j(i+1)}p_{i+1})/\mathbf{u}_j\mathbf{p}$ and hence, it follows from $\mathbf{u}_j\mathbf{p} > 0$ that $(w_{ij} - w_{i+1j})(\hat{w}_{ij} - \hat{w}_{i+1j}) \geq 0$ is equivalent to $(w_{ij} - w_{i+1j})(u_{ji}p_i - u_{j(i+1)}p_{i+1}) \geq 0$. Therefore, the j -th column order invariance condition is equivalently reduced to

$$\begin{aligned} (w_{1j} - w_{2j})(u_{j1}p_1 - u_{j2}p_2) &\geq 0, \\ &\vdots \\ (w_{m-1j} - w_{mj})(u_{j(m-1)}p_{m-1} - u_{jm}p_m) &\geq 0. \end{aligned} \tag{6.3}$$

From (3.9), the j -th column invariance condition is formulated as

$$w_{ij}\mathbf{u}_j\mathbf{p} - u_{ji}p_i = 0 \quad \text{for all } i = 1, \dots, m. \quad (6.4)$$

By introducing linear constraints (6.3), (6.4) and $\mathbf{p} = \sum_{j=1}^n r_j\mathbf{w}_j$ into the optimization problem (3.10) and (3.11), BRM can deal with two invariance requirements of decision makers and it satisfies (4.1). Since three additional constraints (6.3), (6.4) and $\mathbf{p} = \sum_{j=1}^n r_j\mathbf{w}_j$ are linear functions of \mathbf{r} , Dinkelbach algorithm can be applied to the optimization problem (3.10) and (3.11) with the additional constraints.

With the supermatrix (1.1) consisting of the pair of (5.1) and (5.2), we illustrate BRM with invariance requirements of decision makers. Suppose that decision makers require the first column order invariance condition. The requirement of decision makers is formulated as

$$0 \leq (w_{11} - w_{21})(u_{11}p_1 - u_{12}p_2). \quad (6.5)$$

We add (6.5) and $\mathbf{p} = \sum_{j=1}^3 r_j\mathbf{w}_j$ into constraints (3.11) of the optimization problem. Solving the optimization problem with additional constraints by Dinkelbach algorithm, we have the optimal function value 4.4, its optimal solution $\mathbf{r}^* = [0.912, 0, 0.088]^\top$ and

$$\begin{aligned} \mathbf{p}^* &= [0.474, 0.526]^\top, \\ \mathbf{q}^* &= [0.316, 0.395, 0.289]^\top, \\ \mathcal{W}[\mathbf{p}^*] &= \begin{bmatrix} 0.5 & 0.2 & 0.818 \\ 0.5 & 0.8 & 0.182 \end{bmatrix}. \end{aligned} \quad (6.6)$$

Since the first column of (6.6) implies $\hat{w}_{11} \geq \hat{w}_{21}$, the revised evaluation matrix \hat{W} is satisfied with the requirement of decision makers. (In order to obtain $\hat{w}_{11} > \hat{w}_{21}$, it is sufficient to replace the left hand-side value 0 of (6.5) with a small positive number ϵ and then to solve the optimization problem with additional constraints.)

7. Conclusion

For the typical ANP with the supermatrix (1.1), we develop a method of revising the unstable W into the stable \hat{W} . The idea of our method is based on Bayes Theorem stated in Section 2. Section 3 formulates it as a fractional programming problem (3.10), (3.11) or (4.4), which is solved by Dinkelbach algorithm. Theorem 1 guarantees that our method, at the same time, gives the solution of the revised ANP with \hat{W} .

In Section 4 Theorem 2 shows that our method has no contradictions appear in CCM [10], and we introduce some eigenvalue problems which give the lower bound of the optimal value λ^* of (4.4) by Lemma 11. Further Theorems 7 and 9 show that if λ^* becomes equal to this bound, the solution of (4.4) coincides with that of this eigenvalue problem.

Section 5 illustrates several numerical examples for (4.4) and further introduces the concept of inactive alternatives, (5.11) and (5.12). Lemma 11 clarifies the range within which the evaluation vector \mathbf{w}_j by the inactive alternative j can fluctuate without changing the optimal value. Further Theorem 12 shows that this range is explicitly represented in the case for λ^* that is equal to the lower bound. We can not prove the uniqueness of solutions of our problem, but all our examples are found to have uniqueness by checking the program (5.8). In addition to advantage of the sensitivity analysis, Section 6 shows flexibility of BRM modeling.

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