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DUAL ANALYSIS ON HEDGING VAR OF BOND PORTFOLIO USING OPTIONS

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Abstract In this paper, I propose the optimal hedging of bond portfolio VaR using bond options based on dual theory in non-linear optimization and I clarify the relation between the implicit price of bond options in VaR hedging and the price, which is derived by arbitrage pricing theory. Through the dual analysis I provide insight into why out-of-the-money options tend to remain rich and the options on super-long bonds are quite often traded richer than other options. The focus is to investigate the background of these bond market observations from the viewpoint of the managerial decision-making in hedging the VaR of their bond portfolio. As supported by the numerical examples, the optimal hedging strategy derived in the framework quite convincingly explains the reason for the bond market phenomenon even though it does not fully represent the actual bond market.

Keywords: Finance, mathematical modeling, nonlinear programming, optimization, risk management

1. Introduction

Large financial institutions, which trade an enormous amount of financial assets, are exposed to huge market risk and more and more give importance to financial risk management. When financial disasters such as Barings, Orange County, and Metallgesellschaft are highly publicized, the improvement of risk management cannot help but become the top priority in the institutions. Regulators have scrutinized the risk management practice since an early stage. In 1993 the Basel Committee on Banking Supervision give an explicit proposal to use a standardized value at risk model for measuring exposure to fluctuations in interest rates, exchange rates, equity prices, and commodity prices. By January 1996, the Basel Committee finally endorsed the banking industry's use of proprietary value at risk models. Since then not only the banking industry, but also the securities industry and life-damage insurance industry have started to use the VaR model.

In the old days the major tool for managing bond portfolios in financial institutions was the duration risk management model, but today bond portfolios are also managed by the VaR model; as are the other assets. VaR is the maximum potential exposure when some confidence interval and investment horizon are given. The VaR of bond portfolios is usually practiced by the following calculation. First, we assume that the yield change of each individual maturity bond follows the normal distribution and derive the variance-covariance matrix of these yield changes. Second, we multiply the variance-covariance matrix by the duration of these bonds to build the variance-covariance matrix of these price changes. Finally, we multiply the price change based variance-covariance matrix by our bond portfolio positions. Financial institutions manage the risk of their portfolio within their risk limits, which are decided based on their own capitals and managerial strategies. When the risk of bond portfolios (VaR) exceeds the risk limits, they have to reduce their VaR. The simplest way of reducing VaR is to sell some of their bond portfolio.

Recently, we have plenty of preceding research to analyze the most efficient way of using risk capital and the optimal selection of bond portfolio with some constraints in VaR. For example, Dowd [3] insists the importance of capturing risk by the impact of the prospective change on overall value at risk (i.e. the incremental VaR) when we analyze the risk-return trade-off. And in Chow and Kritzman [2], they propose the concept of the risk attribution when we allocate VaR in the selection of our portfolio based on the mean-variance optimization model. We have many other valuable preceding researches. However, the focus of those researches is basically how to optimally sell cash bonds, forwards and futures to reduce VaR.

When we strongly expect a bull market, we will have opportunity loss if we sell cash bonds, forwards and futures just for the purpose of reducing VaR. In this case, investors quite often buy out-of-the-money put options (abbreviated to just options throughout the rest of this paper) to reduce their VaR without losing their profit opportunity. And also, when they expect a bear market, they buy options because the selling of large amounts of cash bonds accelerates the market dive and damages their remaining portfolio. In the business world, hedging VaR using options is quite popular. However, academic research dealing with such hedging activity is scarce and it seems that so far Ahn *et al.* [1] is the only preceding research to do it. However, their research did not consider anything about bond markets.

Thus, in this paper, I propose the optimal hedging strategy using bond options to reduce the VaR of the bond portfolio within the risk limit. The methodology here is different from that of Ahn *et al.* [1] and utilizes the dual theory in non-linear optimization. Regarding the dual theory, Paroush and Prisman [7] adopted the dual theory in linear programming and analyzed the immunization of bond portfolios based on duration risk management. They derived an interesting, insightful and counter-intuitive result that said prioritization on the first duration is not necessarily the most efficient way of immunization.

In this paper, without being limited to the proposal of optimal VaR hedging strategy using bond options, through sensitivity analysis in non-linear optimization, I investigate why out-of-the-money options tend to remain rich; observed as the heavy skew in the implied volatility curve and the options on super-long bonds quite often being traded richer than the other options. Thus, the focus is to investigate the background of these bond market observations from the viewpoint of the managerial decision-making bond in hedging the VaR of their bond portfolio. Most of the bonds traded in the bond market are coupon bonds, but in this analysis we use discount bonds in order to avoid the complicated notation and formula. Because the main focus of this paper is to clarify the relation between the implicit price of bond options in VaR hedging and the usual arbitrage-free bond options price, such a simplification doesn't damage the purpose of this paper.

Initially adopting the 1-factor forward rate model introduced by Ho and Lee [5] as a term structure model of interest rate to describe the dynamics of the discount bond, I derive the main result. Then, the same argument is reexamined based on the 2-factor forward rate model proposed by Heath, Jarrow and Morton [4] to make the result tested in the more actual interest rate dynamics. In some bond markets, such as the Japanese Government Bond market, actual interest rate dynamics are mostly replicated by the 3-factor forward rate model introduced in Miyazaki and Yoshida [6]. However, avoiding the tedious notation and formula, only the cases up to the 2-factor model are examined.

The organization of this paper is as follows. In the next chapter I propose the optimal hedging strategy based on the Ho-Lee model. In Chapter 3, the model is extended to include the actual behavior of management, such as the hesitation to buy rich options in VaR hedging. In Chapter 4, the case that management can use plural kinds of options in hedging is investigated. In Chapter 5, I adopt the 2-factor HJM model and go through the result in Chapter 4 in a more real setting. In Chapter 6, numerical examples are listed. In the last chapter, summary and concluding remarks are added.

2. Optimal Hedging Strategy in Ho-Lee Framework

The Ho-Lee framework presents an arbitrage-free pricing model based on the forward interest rate model of a constant implied volatility across the yield curve. In this chapter we discuss the optimal VaR hedging strategy assuming that all of the options traded in the bond market are fairly priced in the Ho-Lee framework and the implied volatility matches to the historical one in the calculation of the VaR of the bond portfolio.

In the Ho-Lee framework the price and yield dynamics at the time epoch T of the discount bond, whose maturity falls on τ are expressed by

$$P(T,\tau) = \frac{P(0,\tau)}{P(0,T)} \exp\left\{-\frac{\sigma^2}{2}T(\tau-T)\tau - \sigma(\tau-T)\widetilde{B}(T)\right\}, \quad T \in [0,\tau] \quad \text{and}$$

$$Y(T,\tau) = -\frac{1}{\tau-T}\log\frac{P(0,\tau)}{P(0,T)} + \frac{\sigma^2 T\tau}{2} + \sigma\widetilde{B}(T), \quad T \in [0,\tau]$$

where σ is the implied volatility and $\widetilde{B}(T)$ is the standard Brownian motion in the risk-neutral measure.

We define the optimal VaR hedging strategy using options as the selection of a strike price and a hedge ratio to make the option premium minimum reducing VaR within the risk limit. In the selection, as we adopt time horizon T in the calculation of VaR, we restrict our choice to the options whose maturity are T and underlying bonds are τ-year bonds.

When the maturity of the underlying bond and option itself are τ and T respectively and the strike price and implied volatility are K and σ respectively, the put-option premium is given by

$$\widetilde{P} = KP(0,T)\{1-\Phi(d-\sigma(\tau-T)\sqrt{T})\} - P(0,\tau)\{1-\Phi(d)\}$$

where Φ is the distribution function of the standard normal distribution and

$$d = \frac{\log P(0,\tau) - \log KP(0,T)}{\sigma(\tau-T)\sqrt{T}} + \frac{\sigma(\tau-T)\sqrt{T}}{2}.$$

We define the yield volatility of the bond as one standard deviation yield change at the time epoch T. We also define VaR as the yield volatility multiplied by the duration of the bond at the time epoch T; following the actual practice of bond risk management. They are derived as

Yield Volatility	$\sigma\sqrt{T}$,
VaR of the Discount bond	$\sigma(\tau-T)\sqrt{T}$,

respectively.

Throughout this article we capture the loss amount not from the present bond price but from the T forward bond price.

The formulation of the optimal VaR hedging strategy is

$$\underset{K,h}{Min}\, \widetilde{P} \cdot h$$

$$g_1(K,h) = (1-h)\cdot\sigma(\tau - T)\sqrt{T} - h\left(K - \frac{P(0,\tau)}{P(0,T)}\right) - c \leq 0 \qquad (1)$$

$$g_2(K,h) = h - 1 \leq 0 \qquad (2)$$

$$g_3(K,h) = -h \leq 0 \qquad (3)$$

$$g_4(K,h) = K - \frac{P(0,\tau)}{P(0,T)} \leq 0 \qquad (4)$$

$$g_5(K,h) = \frac{P(0,\tau)}{P(0,T)} - \sigma(\tau - T)\sqrt{T} - K \leq 0 \qquad (5)$$

where h is a hedge ratio and c is a risk limit. Constraint (1) restricts the VaR of the bond within the risk limit. We recognize the VaR of the hedged portion as the difference between the strike price and the forward price. Constraint (2) restricts the hedge ratio less than 1, while constraint (3) restricts it above 0. Constraint (4) represents our choice of options as the out-of-the-money options. Constraint (5) restricts the strike price to not go below the forward price minus the VaR of the bond because such a far out-of-the-money option is pretty rare in the market.

Theorem 1 The necessary and sufficient condition of optimal VaR hedging is to select the strike price K such that the sensitivity of the options with respect to the strike price matches the amount that the option premium divided by the risk limit minuses the out width of the strike price. The hedge ratio becomes the excess amount of the risk limit divided by the difference between the risk limit and the out width of the strike price.

(*Proof*) The Lagrangean of the previously mentioned mathematical programming is given by

$$L(x,\lambda) = \begin{cases} \widetilde{P}\cdot h + \sum_{i=1}^{5} \lambda_i g_i(x) & \lambda \geq 0, \\ -\infty & \lambda < 0 \end{cases}$$

where $x = (K,h)$ and $\lambda = (\lambda_1, \cdots, \lambda_5)$.

Both of the objective function $f(x)$ that is $\widetilde{P}\cdot h$ and the constraints $g_i(x),\ i = 1,\ldots,5$ are differentiable and the KKT condition that guarantees the necessary condition of the optimality is the existence of $\overline{x} \in R^2$ and $\overline{\lambda} \in R^5$ satisfying

$$\nabla_x L(\overline{x},\overline{\lambda}) = \nabla f(\overline{x}) + \sum_{i=1}^{5} \overline{\lambda}_i \nabla g_i(\overline{x}) = 0,$$

$$\overline{\lambda}_i \geq 0,\ g_i(\overline{x}) \leq 0,\ \overline{\lambda}_i g_i(\overline{x}) = 0,\ i = 1,\ldots,5. \qquad (6)$$

Focusing on the next two equations in the condition (6), we can get

$$\frac{\partial L}{\partial K} = h\cdot\frac{\partial \widetilde{P}}{\partial K} - \lambda_1 h + \lambda_4 - \lambda_5 = 0, \qquad (7)$$

$$\frac{\partial L}{\partial h} = \widetilde{P} - \lambda_1\left(\sigma(\tau - T)\sqrt{T} + K - \frac{P(0,\tau)}{P(0,T)}\right) + \lambda_2 - \lambda_3 = 0. \qquad (8)$$

We put $\overline{\lambda}_i = 0,\ i = 2,\ldots,5$, and choose K and λ_1 as $\overline{K}, \overline{\lambda}_1$; the solution is equation (9).

$$\lambda_1 = \frac{\partial \widetilde{P}}{\partial K} = \frac{\widetilde{P}}{\sigma(\tau - T)\sqrt{T} + K - P(0,\tau)/P(0,T)}. \qquad (9)$$

In this case, $\overline{\lambda}_i g_i(\overline{x}) = 0,\ i = 2, \ldots, 5$ are attained for any $\overline{K}, \overline{h}$. If we put $\overline{\lambda}_1 \neq 0$, $\overline{x}$ has to satisfy $g_1(\overline{x}) = 0$ to attain $\overline{\lambda}_1 g_1(\overline{x}) = 0$.
Thus, we select $\overline{h}$ as follows.

$$\overline{h} = \frac{\sigma(\tau - T)\sqrt{T} - c}{\sigma(\tau - T)\sqrt{T} + \overline{K} - \dfrac{P(0,\tau)}{P(0,T)}}. \tag{10}$$

$\overline{K}$ and $\overline{h}$ are the strike price and the hedge ratio described in Theorem 1 respectively. Furthermore, because all of the constraint functions $g_i(K, h),\ i = 1, \ldots, 5$ are linear functions with respect to K and h, they are convex and concave. To guarantee the convexity of the Lagrangean we have only to show the convexity of $\widetilde{P}(K)$ with respect to K. With the relation as

$$\frac{\phi(d - \sigma(\tau - T)\sqrt{T})}{\phi(d)} = \exp\left(\frac{1}{2}(2d - \sigma(\tau - T)\sqrt{T})\sigma(\tau - T)\sqrt{T}\right) = \frac{P(0,\tau)}{KP(0,T)},$$

$\dfrac{\partial \widetilde{P}}{\partial K}$ and $\dfrac{\partial^2 \widetilde{P}}{\partial K^2}$ are derived as next.

$$\begin{aligned}
\frac{\partial \widetilde{P}}{\partial K} &= P(0,T)\left\{1 - \Phi(d - \sigma(\tau - T)\sqrt{T}) + \frac{\phi(d - \sigma(\tau - T)\sqrt{T})}{\sigma(\tau - T)\sqrt{T}}\right\} - \frac{P(0,\tau)\phi(d)}{\sigma(\tau - T)\sqrt{T}K} \\
&= P(0,T)\left\{1 - \Phi(d - \sigma(\tau - T)\sqrt{T})\right\}, \\
\frac{\partial^2 \widetilde{P}}{\partial K^2} &= P(0,T)\frac{\phi(d - \sigma(\tau - T)\sqrt{T})\sigma(\tau - T)\sqrt{T} - \phi'(d - \sigma(\tau - T)\sqrt{T})}{\sigma^2(\tau - T)^2 TK} \\
&\quad + P(0,\tau)\frac{\phi'(d) + \phi(d)\sigma(\tau - T)\sqrt{T}}{\sigma^2(\tau - T)^2 TK^2} \\
&= \frac{1}{\sigma^2(\tau - T)^2 TK^2}\left\{KP(0,T)\phi(d - \sigma(\tau - T)\sqrt{T})d - P(0,\tau)\phi(d)(d - \sigma(\tau - T)\sqrt{T})\right\} \\
&= \frac{P(0,T)\phi(d - \sigma(\tau - T)\sqrt{T})}{K\sigma(\tau - T)\sqrt{T}} \\
&\geq 0.
\end{aligned}$$

Therefore, the hedging strategy given in Theorem 1 becomes the necessary and sufficient condition of the optimal VaR hedging of the bond. (QED)

Corollary 1 When we regard the right hand side of equation (9), $\dfrac{\widetilde{P}}{\sigma(\tau - T)\sqrt{T} + K - P(0,\tau)/P(0,T)}$, as a function of $K, m(K)$, the minimum of $m(K)$ is attained at $K = \overline{K}$.

(*Proof*) If we put $m'(K) = 0$, we can get equation (9). With equation (9) and the argument in the proof of Theorem 1, $m''(K) = \dfrac{\partial^2 \widetilde{P}}{\partial K^2} \Big/ \dfrac{\partial \widetilde{P}}{\partial K} \geq 0$. (QED)

Corollary 2 In the Ho-Lee framework, we observe the tendency that the smaller the $m(K)$ of an option at the forward price, the smaller the out-ratio and the cost of hedging become. We defined the out-ratio as the percentage of out-width in $\sigma(\tau - T)\sqrt{T}$.

(*Reasoning*) Due to Theorem 1, when λ_1 is small, the hedge cost becomes small. λ_1 matches $\dfrac{\partial \widetilde{P}}{\partial K}$, which becomes small compared to the large out-ratio. Corollary 1 guarantees that λ_1 takes the minimum value $m(\overline{K})$ at $K = \overline{K}$. Thus, the value of $m(K)$ at the forward price K has substantial impact on the value $m(\overline{K})$. Please refer to Figure 1.

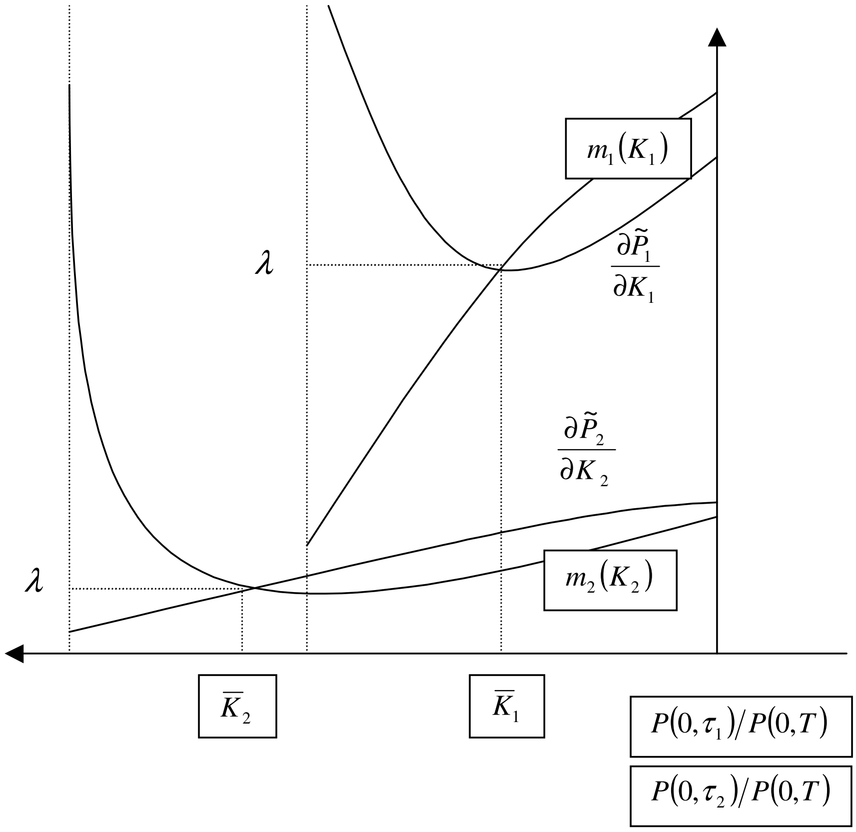

Figure 1: K gives impacts on the value $m(\overline{K})$

3. Disincentive in Hedging

In Theorem 1 we discuss the optimal selection of options in VaR hedging without considering any disincentive to buy rich options compared to the historical yield volatility. However, in reality, a portfolio manager hesitates to buy an option, whose implied volatility is much higher than the historical one σ in the VaR calculation. The portfolio manager has disincentive to buy a rich option even though he knows that it is the best one for the VaR hedging. To incorporate such a portfolio manager's behavior into the mathematical programming model in Chapter 2, we introduce disincentive $\gamma(\widetilde{P} - \widetilde{P}^H)$ (where γ is a positive number expressing the disincentive coefficient and $\widetilde{P}^H$ is the option premium derived using the historical volatility σ) into the objective function of the model. The optimal hedging strategy in this case is different from that of Theorem 1 and is stated in Theorem 2.

Theorem 2 In the parameter set σ, τ, T of Theorem 1, if we incorporate the disincentive of the portfolio manager into the hedging scheme, the strike price of the option in the optimal hedging becomes higher than that in Theorem 1, while the hedge ratio becomes smaller. The hedging cost also becomes higher.

(*Proof*) Because the constraints (1)$\sim$(5) are exactly the same as in Theorem 1, for the purpose of minimal hedging cost, the objective function of Theorem 1 clearly provides better strategy than the objective function that concentrates on the pareto optimality of minimizing the combination of the hedging cost and the disincentive. The hedge ratio is adversely related to the strike price K. This is clear once we prove the statement regarding the strike price K. Thus, we focus on proving it. We show how the KKT condition is modified in Theorem 2.

Equation (7) becomes

$$\frac{\partial L}{\partial K} = (h-\gamma)\frac{\partial \widetilde{P}}{\partial K} + \gamma\frac{\partial \widetilde{P}^H}{\partial K} - \lambda_1 h + \lambda_4 - \lambda_5 = 0. \tag{11}$$

Thus, equation (9) becomes

$$\lambda_1 = \frac{(h-\gamma)\dfrac{\partial \widetilde{P}}{\partial K} + \gamma\dfrac{\partial \widetilde{P}^H}{\partial K}}{h} = \frac{\widetilde{P}}{\sigma^H(\tau - T)\sqrt{T} + K - P(0,\tau)/P(0,T)}. \tag{12}$$

Contrary to the case in Theorem 1, we cannot derive K using only equation (12). Instead, we insert equation (10)

$$h = \frac{\sigma^H(\tau - T)\sqrt{T} - c}{\sigma^H(\tau - T)\sqrt{T} + \overline{K} - P(0,\tau)/P(0,T)}$$

into equation (12) and derive equation (13), which has only the variable K.

$$\begin{aligned}&(\sigma^H(\tau - T)\sqrt{T} + K - P(0,\tau)/P(0,T))\frac{\partial \widetilde{P}}{\partial K} - \widetilde{P} \\ &= \frac{\gamma\left(\dfrac{\partial \widetilde{P}}{\partial K} - \dfrac{\partial \widetilde{P}^H}{\partial K}\right)(\sigma^H(\tau - T)\sqrt{T} + K - P(0,\tau)/P(0,T))^2}{\sigma^H(\tau - T)\sqrt{T} - c}.\end{aligned} \tag{13}$$

The K giving the optimal VaR hedging strategy in Theorem 1 is the solution $\overline{K}$ of equation (13) in the case that the right hand side of the equation is 0. The right hand side of equation (13) is positive because $\gamma > 0$, $\dfrac{\partial \widetilde{P}}{\partial K} - \dfrac{\partial \widetilde{P}^H}{\partial K} > 0$ and $\sigma^H(\tau - T)\sqrt{T} - c > 0$. We compare the scale of $\overline{\overline{K}}$, (the solution of equation (13)) and $\overline{K}$. Differentiating the left hand side of equation (13) with respect to K, we get

$$(\sigma^H(\tau - T)\sqrt{T} + K - P(0,\tau)/P(0,T))\frac{\partial^2 \widetilde{P}}{\partial K^2} \geq 0.$$

$\dfrac{\partial^2 \widetilde{P}}{\partial K^2}$ is positive due to the convexity with respect to K. Thus, the left hand side of equation (13) is an increasing function of K. Because the right hand side of equation (13) is some positive, we denote it as ξ. We also denote the right hand side of equation (13) as $f(K)$ observing it as a function of K. $\overline{\overline{K}}$ (the K value of the crossing point of the graph $y = f(K)$ and $y = \xi$) should be larger than $\overline{K}$ (the K value of the crossing point of the graph $y = f(K)$ and $y = 0$). (QED)

Remark 1 The result of Theorem 2 may be interpreted as follows. When a portfolio manager feels strong pressure to protect his portfolio in the downward bias of the market, the disincentive coefficient becomes small and the large out-ratio option is selected in optimal hedging. Therefore, the out-of-the-money option is loved more than the at-the-money option by most of the portfolio managers and we observe the skew phenomenon such that the implied volatility of the out-of-the-money option is much larger than that of the at-the-money option. On the other hand, when the bond market is stable, the disincentive coefficient γ becomes large and the small out-ratio or at-the-money option is selected in optimal VaR hedging. In this case, the skew phenomenon is seldom observed and the magnitude is quite small if it appears.

4. In the Case That We May Use the Option, Whose Underlying Bond Is Not the One We Have To Hedge

Another extension of Theorem 1 is to consider the framework that we may use the option, whose underlying bond is not the one we have to hedge. For example, in order to hedge a 10-year bond, the framework in Theorem 1 allows us to use only 10-year bond options, while the framework in this chapter provide us with the opportunity to use a variety of options other than the 10-year bond options.

In this case the mathematical programming is formulated as follows.

$$\underset{K_1,K_2,h_1,h_2}{Min}\ \widetilde{P}_1 h_1 + \widetilde{P}_2 h_2$$

$$g_1(K_1,K_2,h_1,h_2) = (1-h_1)\sigma(\tau_1 - T)\sqrt{T} - h_1\left(K_1 - \frac{P(0,\tau_1)}{P(0,T)}\right) - h_2\left(K_2 - \left(\frac{P(0,\tau_2)}{P(0,T)} - \sigma(\tau_2 - T)\sqrt{T}\right)\right) - c \le 0 \tag{14}$$

$$g_{2,1}(K_1,h_1) = h_1 - 1 \le 0, \qquad g_{2,2}(K_2,h_2) = h_2 - 1 \le 0 \tag{15}$$

$$g_{3,1}(K_1,h_1) = -h_1 \le 0, \qquad g_{3,2}(K_2,h_2) = -h_2 \le 0 \tag{16}$$

$$g_{4,1}(K_1,h_1) = K_1 - \frac{P(0,\tau_1)}{P(0,T)} \le 0, \qquad g_{4,2}(K_2,h_2) = K_2 - \frac{P(0,\tau_2)}{P(0,T)} \le 0 \tag{17}$$

$$g_{5,1}(K_1,h_1) = \frac{P(0,\tau_1)}{P(0,T)} - \sigma(\tau_1 - T)\sqrt{T} - K_1 \le 0,$$

$$g_{5,2}(K_2,h_2) = \frac{P(0,\tau_2)}{P(0,T)} - \sigma(\tau_2 - T)\sqrt{T} - K_2 \le 0. \tag{18}$$

The above notations are the same as we used in (1) through (5) where the suffix 1, 2 in the equation represents the option 1 and the option 2. When the price of the τ_1-maturity bond decreases from its forward price $\frac{P(0,\tau_1)}{P(0,T)}$ by its VaR ($\sigma(\tau_1 - T)\sqrt{T}$) at time epoch T, the price of the τ_2-maturity bond decreases from its forward price $\frac{P(0,\tau_2)}{P(0,T)}$ by its VaR ($\sigma(\tau_2 - T)\sqrt{T}$) because the Ho-Lee model assumes a constant volatility across different maturities. This indicates the price of the τ_2-maturity bond is $\frac{P(0,\tau_2)}{P(0,T)} - \sigma(\tau_2 - T)\sqrt{T}$ at time epoch T. Buying the τ_2-maturity bond at a market price of $\frac{P(0,\tau_2)}{P(0,T)} - \sigma(\tau_2 - T)\sqrt{T}$ and selling it at price K_2 with the execution of the option on the τ_2-maturity bond, we obtain a profit of $K_2 - \left(\frac{P(0,\tau_2)}{P(0,T)} - \sigma(\tau_2 - T)\sqrt{T}\right)$. The profit multiplied by the option amount h_2 contributes to the risk reduction and appears the third part of equation (14). Constraints (15)$\sim$(18) are provided in an identical fashion to the constraints (2)$\sim$ (5).

Theorem 3 When we have two kinds of options in the VaR hedging of a bond, we first decide the K of each option for optimal hedging assuming that either option may be used and secondly choose the option which gives the smaller λ evaluated at each previously decided strike price. If λ of both options is identical, we may use either or both of the options in the optimal VaR hedging of the bond. When we use one of the options, the optimal hedge ratio is the one described in Theorem 1. If we use both of the options, the decrease of one unit of option 1 deserves the increase of $\dfrac{\tilde{P}_2}{\tilde{P}_1}$ units of option 2.

(*Proof*) The Lagrangean of the above mathematical programming is

$$L(x,\lambda) = \begin{cases} \tilde{P}_1 h_1 + \tilde{P}_2 h_2 + \lambda_1 g_1(x) + \sum_{j=1}^{2}\sum_{i=2}^{5} \lambda_{i,j} g_{i,j}(x) & \lambda \geq 0, \\ -\infty & \lambda < 0 \end{cases}$$

where $x = (K_1, K_2, h_1, h_2)$ and $\lambda = (\lambda_1, \lambda_{2,1}, \ldots, \lambda_{5,2})$.
The objective function $f(x)$ that is $\tilde{P}_1 h_1 + \tilde{P}_2 h_2$ and constraints $g_1(x), g_{i,j}(x),\ i = 2, \ldots, 5, j = 1, 2$ are differentiable and the KKT condition is the existence of $\overline{x} \in R^4, \overline{\lambda} \in R^9$ which satisfy

$$\nabla_x L(\overline{x}, \overline{\lambda}) = \nabla f(\overline{x}) + \overline{\lambda}_1 \nabla g_1(\overline{x}) + \sum_{j=1}^{2}\sum_{i=2}^{5} \overline{\lambda}_{i,j} \nabla g_{i,j}(\overline{x}) = 0,\ \overline{\lambda}_1 \geq 0,\ \overline{\lambda}_{i,j} \geq 0,$$
$$g_1(\overline{x}) \leq 0,\ g_{i,j}(\overline{x}) \leq 0,\ \overline{\lambda}_1 g_1(\overline{x}) = 0,\ \overline{\lambda}_{i,j} g_{i,j}(\overline{x}) = 0,\ i = 2, \ldots, 5,\ j = 1, 2. \tag{19}$$

Let's focus on the following two equations of constraint (19)

$$\frac{\partial L}{\partial K_1} = h_1 \cdot \frac{\partial \tilde{P}_1}{\partial K_1} - \lambda_1 h_1 + \lambda_{4,1} - \lambda_{5,1} = 0, \tag{20-1}$$
$$\frac{\partial L}{\partial h_1} = \tilde{P}_1 - \lambda_1 \left(\sigma(\tau_1 - T)\sqrt{T} + K_1 - \frac{P(0,\tau_1)}{P(0,T)} \right) + \lambda_{2,1} - \lambda_{3,1} = 0 \tag{21-1}$$

and consider the case $h_1 \neq 0$ and $h_2 = 0$ in (20-1), (21-1).
Putting $\lambda_{4,1} = \lambda_{5,1} = \lambda_{2,1} = \lambda_{3,1} = 0$, we get $\overline{K}_1, \overline{\lambda}_1$ as a solution of equation (22-1).

$$\lambda_1 = \frac{\partial \tilde{P}_1}{\partial K_1} = \frac{\tilde{P}_1}{\sigma(\tau_1 - T)\sqrt{T} + K_1 - P(0,\tau_1)/P(0,T)}. \tag{22-1}$$

Because we set $\overline{\lambda}_{i,1} = 0, i = 2, \ldots, 5, \overline{\lambda}_{i,1} g_{i,1}(\overline{x}) = 0, i = 2, \ldots, 5$ are satisfied for any $\overline{K}_1, \overline{h}_1$. In the case of $\overline{\lambda}_1 \neq 0$, we have only to select $\overline{h}_1$ as below by solving $g_1(\overline{x}) = 0$ to satisfy $\overline{\lambda}_1 g_1(\overline{x}) = 0$.

$$\overline{h}_1 = \frac{\sigma(\tau_1 - T)\sqrt{T} - c}{\sigma(\tau_1 - T)\sqrt{T} + \overline{K}_1 - \dfrac{P(0,\tau_1)}{P(0,T)}}. \tag{23-1}$$

In a parallel fashion, we get similar equations for option 2.

$$\frac{\partial L}{\partial K_2} = h_2 \cdot \frac{\partial \tilde{P}_2}{\partial K_2} - \lambda_1 h_2 + \lambda_{4,2} - \lambda_{5,2} = 0, \tag{20-2}$$
$$\frac{\partial L}{\partial h_2} = \tilde{P}_2 - \lambda_1 \left(\sigma(\tau_2 - T)\sqrt{T} + K_2 - \frac{P(0,\tau_2)}{P(0,T)} \right) + \lambda_{2,2} - \lambda_{3,2} = 0. \tag{21-2}$$

Putting $h_2 \neq 0, \lambda_{4,2} = \lambda_{5,2} = \lambda_{2,2} = \lambda_{3,2} = 0$, we get $\overline{K}_2, \overline{\lambda}_1$ as a solution of equation (20-2), (21-2).

$$\lambda_1 = \frac{\partial \widetilde{P}_2}{\partial K_2} = \frac{\widetilde{P}_2}{\sigma(\tau_2 - T)\sqrt{T} + K_2 - P(0,\tau_2)/P(0,T)}. \tag{22-2}$$

Because we set $\overline{\lambda}_{i,2} = 0$, $i = 2, \ldots, 5$, $\overline{\lambda}_{i,2} g_{i,2}(\overline{x}) = 0$, $i = 2, \ldots, 5$ are satisfied for any $\overline{K}_2$, $\overline{h}_2$. In the case of $\overline{\lambda}_1 \neq 0$, putting $h_1 = 0$, we have only to select $\overline{h}_2$ as below by solving $g_1(\overline{x}) = 0$ to obtain $\overline{\lambda}_1 g_1(\overline{x}) = 0$.

$$\overline{h}_2 = \frac{\sigma(\tau_1 - T)\sqrt{T} - c}{\sigma(\tau_2 - T)\sqrt{T} + \overline{K}_2 - \dfrac{P(0,\tau_2)}{P(0,T)}}. \tag{23-2}$$

It is important for us to examine the case of $h_1 \neq 0$ and $h_2 \neq 0$ carefully. In this case, putting $\lambda_{4,1} = \lambda_{4,2} = \lambda_{5,1} = \lambda_{5,2} = \lambda_{2,1} = \lambda_{2,2} = \lambda_{3,1} = \lambda_{3,2} = 0$ into equations (20-1), (20-2), (21-1) and (21-2), $\overline{\lambda}_1$ has to satisfy both of the following equations simultaneously.

$$h_1\left(\frac{\partial \widetilde{P}_1}{\partial K_1} - \lambda_1\right) = 0, \quad h_2\left(\frac{\partial \widetilde{P}_2}{\partial K_2} - \lambda_1\right) = 0, \quad \widetilde{P}_1 - \lambda_1\left(\sigma(\tau_1 - T)\sqrt{T} + K_1 - \frac{P(0,\tau_1)}{P(0,T)}\right) = 0$$

and $$\widetilde{P}_2 - \lambda_1\left(\sigma(\tau_2 - T)\sqrt{T} + K_2 - \frac{P(0,\tau_2)}{P(0,T)}\right) = 0.$$

Due to $h_1 \neq 0$ and $h_2 \neq 0$, the above equations are reduced to the next equation.

$$\begin{aligned}\lambda_1 = \frac{\partial \widetilde{P}_1}{\partial K_1} = \frac{\partial \widetilde{P}_2}{\partial K_2} &= \frac{\widetilde{P}_1}{\sigma(\tau_1 - T)\sqrt{T} + K_1 - P(0,\tau_1)/P(0,T)} \\ &= \frac{\widetilde{P}_2}{\sigma(\tau_2 - T)\sqrt{T} + K_2 - P(0,\tau_2)/P(0,T)}.\end{aligned} \tag{22-3}$$

The above equation indicates that the $\overline{\lambda}_1$ derived from equation (22-1) and derived from equation (22-2) should match. Adversely, only one of the options is used in the optimal hedge when $\dfrac{\partial \widetilde{P}_2}{\partial K_2} = \dfrac{\partial \widetilde{P}_1}{\partial K_1}$ is not satisfied. The $\overline{\lambda}_1$ is the dual price of the option and means that how much the hedging cost increases if the VaR risk limit is tighten by a unit. Thus, in the case of $\lambda_1 = \dfrac{\partial \widetilde{P}_2}{\partial K_2} < \dfrac{\partial \widetilde{P}_1}{\partial K_1}$ as an example, only option 2 is used in the optimal hedge. Actually, in order to satisfy $h_1\left(\dfrac{\partial \widetilde{P}_1}{\partial K_1} - \lambda_1\right) = 0$, h_1 should be zero. This is the case of $h_1 = 0$ and $h_2 \neq 0$. Based on the above argument, for the optimal hedge, we first decide K_1 by solving the equation $\dfrac{\partial \widetilde{P}_1}{\partial K_1} = \dfrac{\widetilde{P}_1}{\sigma(\tau_1 - T)\sqrt{T} + K_1 - P(0,\tau_1)/P(0,T)}$ and decide K_2 by solving the equation $\dfrac{\partial \widetilde{P}_2}{\partial K_2} = \dfrac{\widetilde{P}_2}{\sigma(\tau_2 - T)\sqrt{T} + K_2 - P(0,\tau_2)/P(0,T)}$.
Second, adopting the K_1 and K_2, we compare $\dfrac{\partial \widetilde{P}_1}{\partial K_1}$ and $\dfrac{\partial \widetilde{P}_2}{\partial K_2}$.
We have only to use the option that gives us the smaller value. When the values are the same, we may use both of the options and it is simply a case of $h_1 \neq 0$ and $h_2 \neq 0$. In this

case, the optimal hedge amounts of the options have to satisfy the following equation due to the constraint (14).

$$h_1\left(\sigma(\tau_1-T)\sqrt{T}+K_1-\frac{P(0,\tau_1)}{P(0,T)}\right)+h_2\left(K_2-\left(\frac{P(0,\tau_2)}{P(0,T)}-\sigma(\tau_2-T)\sqrt{T}\right)\right)$$
$$=\sigma(\tau_1-T)\sqrt{T}-c.$$

The equation is reduced to the next equation by utilizing equation (22-3).

$$h_1+h_2\frac{\widetilde{P}_2}{\widetilde{P}_1}=\frac{\sigma(\tau_1-T)\sqrt{T}-c}{\left(\sigma(\tau_1-T)\sqrt{T}+K_1-\dfrac{P(0,\tau_1)}{P(0,T)}\right)}.$$

And a decrease of one unit of option 1 deserves the increase of $\dfrac{\widetilde{P}_2}{\widetilde{P}_1}$ units of option 2.

Corollary 3 We observe the tendency that an option written on a longer maturity bond makes the hedging cost lower than one written on a shorter maturity bond in the market conditions of a high interest rate and steep yield curve. When these conditions are constant, the higher the volatility, the higher the hedging cost.

(*Proof*) According to Theorem 3, when we have two kinds of options in the VaR hedging of a bond, we have only to choose the option that gives us the smaller λ. Due to Corollary 2, the scale of $m(K)$ evaluated at the forward price has substantial importance on the size of λ.
The at-the-money option premium is

$$\widetilde{P}(0,\tau)=P(0,\tau)\left\{\Phi\left(\frac{\sigma(\tau-T)}{2}\right)-\Phi\left(-\frac{\sigma(\tau-T)}{2}\right)\right\}.$$

Taylor expansion of a standard normal density function $\dfrac{1}{\sqrt{2\pi}}\exp\left(-\dfrac{x^2}{2}\right)$ around zero is $\dfrac{1}{\sqrt{2\pi}}\exp\left(1-\dfrac{x^2}{2}\right)$. And due to the following approximation

$$\Phi\left(\frac{\sigma(\tau-T)}{2}\right)-\Phi\left(-\frac{\sigma(\tau-T)}{2}\right)\approx\frac{1}{\sqrt{2\pi}}\int_{-\frac{\sigma(\tau-T)}{2}}^{\frac{\sigma(\tau-T)}{2}}\left(1-\frac{x^2}{2}\right)dx$$
$$=\frac{1}{\sqrt{2\pi}}\left(\sigma(\tau-T)-\frac{\{\sigma(\tau-T)\}^3}{24}\right)=\frac{1}{\sqrt{2\pi}}\sigma(\tau-T).$$

$\widetilde{P}(0,\tau)$ is approximated as next in ordinal scale of σ,τ,T.

$$\widetilde{P}(0,\tau)\approx P(0,\tau)\frac{1}{\sqrt{2\pi}}\sigma(\tau-T).$$

Thus, the scale of $m(K)$ evaluated at the forward price becomes approximately $\dfrac{P(0,\tau)}{\sqrt{2\pi T}}$. This indicates that $m(K)$ of the option written on the longer maturity bond evaluated at the forward price becomes small when the interest rate level is high and the yield curve is steep. The latter half of the Corollary 3 is trivial. (QED)

5. Optimal Hedging Strategy in the 2-Factor HJM Framework

The result of Corollary 3 tells us that an option written on a longer maturity bond is almost always the one we had better use when the hedging the VaR of the bond. However, when we hedge our 10-year bond, it is counter-intuitive that we had better use the option written on the 20-year bond when the implied volatility is the same as the historical volatility. Therefore, adopting the 2-factor HJM model (which captures actual interest rate dynamics more precisely) we reexamine the sector preference of options stated in Theorem 3 in a more realistic setting.

2-factor HJM Model $\quad df(t,T) = \alpha(t,T)dt + \sigma_1 dB_1(t) + \sigma_2 e^{-\lambda(T-t)/2} dB_2(t)$

In the same manner as we did in the case of the Ho-Lee model, in the 2-factor HJM model the yield volatility of the discount bond and its VaR at time epoch T are calculated as

τ_1-year Yield Volatility $\quad Y\sigma_{HJM2}(\tau_1) = \sqrt{\sigma_1^2 T + \frac{\sigma_2^2}{2\lambda^3\tau_1^2}(1-e^{-\lambda\tau_1})^2(1-e^{-2\lambda T})}$

VaR of the Discount Bond $\quad P\sigma_{HJM2}(\tau_1) = (\tau_1 - T)\sqrt{\sigma_1^2 T + \frac{\sigma_2^2}{2\lambda^3\tau_1^2}(1-e^{-\lambda\tau_1})^2(1-e^{-2\lambda T})}$

In the hedging of this VaR, when we use the option written on the τ_1-year bond, the hedging scheme is the same as the one in Theorem 1 except for the magnitude of the volatility. However, the hedging scheme in using the option written on the τ_2-year bond is different from the one in the Ho-Lee framework in Theorem 3; not only the yield volatility but also the direction of yield change are the same in both the τ_1-year bond and the τ_2-year bond. In the 2-factor HJM framework, the volatility differs sector by sector and the correlation of yield change among sectors becomes less than 1. The correlation of the yield change between the τ_1-year bond and the τ_2-year bond is given by the following lemmas.

Lemma 1 The correlation of the yield change between the τ_1-year bond and the τ_2-year bond is

$$\rho(\sigma_1,\sigma_2,\lambda,\tau_1,\tau_2,T) = \frac{\sigma_1^2 T + \frac{\sigma_2^2}{2\lambda^3\tau_1\tau_2}(1-e^{-\lambda\tau_1})(1-e^{-\lambda\tau_2})(1-e^{-2\lambda T})}{\sqrt{\sigma_1^2 T + \frac{\sigma_2^2}{2\lambda^3\tau_2^2}(1-e^{-\lambda\tau_2})^2(1-e^{-2\lambda T})}\sqrt{\sigma_1^2 T + \frac{\sigma_2^2}{2\lambda^3\tau_1^2}(1-e^{-\lambda\tau_1})^2(1-e^{-2\lambda T})}}.$$

(*Proof*) Computing the next expectation, we can get the Lemma 1.

$$\rho(\sigma_1,\sigma_2,\lambda,\tau_1,\tau_2,T) = \frac{Cov\left[\sigma_1\widetilde{B}_1(T) + \frac{\sigma_2}{\lambda\tau_1}e^{-\lambda T}(1-e^{-\lambda\tau_1})\int_0^T e^{\lambda\nu}d\widetilde{B}_2(T), \sigma_1\widetilde{B}_1(T) + \frac{\sigma_2}{\lambda\tau_2}e^{-\lambda T}(1-e^{-\lambda\tau_2})\int_0^T e^{\lambda\nu}d\widetilde{B}_2(T)\right]}{\sqrt{E\left[\left(\sigma_1\widetilde{B}_1(T) + \frac{\sigma_2}{\lambda\tau_1}e^{-\lambda T}(1-e^{-\lambda\tau_1})\int_0^T e^{\lambda\nu}d\widetilde{B}_2(T)\right)^2\right] E\left[\left(\sigma_1\widetilde{B}_1(T) + \frac{\sigma_2}{\lambda\tau_2}e^{-\lambda T}(1-e^{-\lambda\tau_2})\int_0^T e^{\lambda\nu}d\widetilde{B}_2(T)\right)^2\right]}}.$$

(QED)

Lemma 2 When σ_1 and σ_2 are the same scale, the correlation becomes small, otherwise the correlation becomes closer to 1. When λ is given, the larger the distance between the maturity τ_1 and the maturity τ_2, the lower the correlation. When the distance is given, the larger λ, the lower the correlation.

(*Proof*) In the case of $\sigma_1 \gg \sigma_2$ or $\sigma_1 \ll \sigma_2, \rho \approx 1$ is clear. And also in $\sigma_1 \approx \sigma_2$, it is obvious that ρ becomes small. Based on Lemma 1, we examine the behavior of the correlation when we make τ_2 larger, fixing τ_1 as $\tau_1 \leq \tau_2$.
Because $\sigma_1^2 T$ is fixed in both of the numerator and the denominator, we have only to compare

$$\frac{\sigma_2^2}{2\lambda^3\tau_1\tau_2}2(1-e^{-\lambda\tau_1})(1-e^{-\lambda\tau_2})(1-e^{-2\lambda T})$$

and

$$\frac{\sigma_2^2}{2\lambda^3}(1-e^{-2\lambda T})\left\{\left(\frac{1-e^{-\lambda\tau_2}}{\tau_2}\right)^2+\left(\frac{1-e^{-\lambda\tau_1}}{\tau_1}\right)^2\right\}.$$

Because

$$\frac{\frac{\sigma_2^2}{2\lambda^3\tau_1\tau_2}2(1-e^{-\lambda\tau_1})(1-e^{-\lambda\tau_2})(1-e^{-2\lambda T})}{\frac{\sigma_2^2}{2\lambda^3}(1-e^{-2\lambda T})\left\{\left(\frac{1-e^{-\lambda\tau_2}}{\tau_2}\right)^2+\left(\frac{1-e^{-\lambda\tau_1}}{\tau_1}\right)^2\right\}}$$

$$=\frac{2\left(\frac{1-e^{-\lambda\tau_2}}{\tau_2}\right)\left(\frac{1-e^{-\lambda\tau_1}}{\tau_1}\right)}{\left(\frac{1-e^{-\lambda\tau_2}}{\tau_2}\right)^2+\left(\frac{1-e^{-\lambda\tau_1}}{\tau_1}\right)^2}=\frac{2\frac{\left(\frac{1-e^{-\lambda\tau_2}}{\tau_2}\right)}{\left(\frac{1-e^{-\lambda\tau_1}}{\tau_1}\right)}}{1+\frac{\left(\frac{1-e^{-\lambda\tau_2}}{\tau_2}\right)}{\left(\frac{1-e^{-\lambda\tau_1}}{\tau_1}\right)}}$$

when we fix τ_1 at some level, the larger τ_2 $(>\tau_1)$ becomes, the smaller $\frac{\left(\frac{1-e^{-\lambda\tau_2}}{\tau_2}\right)}{\left(\frac{1-e^{-\lambda\tau_1}}{\tau_1}\right)}$ becomes.

(QED)

The hedging scheme in the 2-factor HJM framework is shown in Figure 2. The distinctive part of the mathematical programming in this scheme is constraint (14). In inequality (14), when we hedge the τ_1-year bond, an option written on either the τ_1-year bond or the τ_2-year bond gives us an identical hedge effect because the correlation is 1. However, in this scheme, as the idea in Figure 2 indicates, the hedge effect decreases from 1 to ρ. Because the correlation between the τ_1-year bond yield and the τ_2-year bond yield is ρ, the τ_2-year bond yield goes up not by $Y\sigma_{HJM2}(\tau_2)$, but by $\rho Y\sigma_{HJM2}(\tau_2)$ when the τ_1-year bond yield goes up $Y\sigma_{HJM2}(\tau_1)$. This decreases the profit of the put-option on the τ_2-year bond.

Thus, the inequality (14) is modified as follows.

$$\begin{aligned}g_1(K_1,K_2,h_1,h_2)=(1-h_1)P\sigma_{HJM2}(\tau_1)-h_1\left(K_1-\frac{P(0,\tau_1)}{P(0,T)}\right)\\-h_2\left(K_2-\left(\frac{P(0,\tau_2)}{P(0,T)}-\rho P\sigma_{HJM2}(\tau_2)\right)\right)-c\leq 0.\end{aligned}\tag{24}$$

Other modifications in this scheme are summarized below.

(1) The option premium $\widetilde{P}$ in the objective function based on the Ho-Lee framework is changed to $\widetilde{\widetilde{P}}$ based on the 2-factor HJM model.

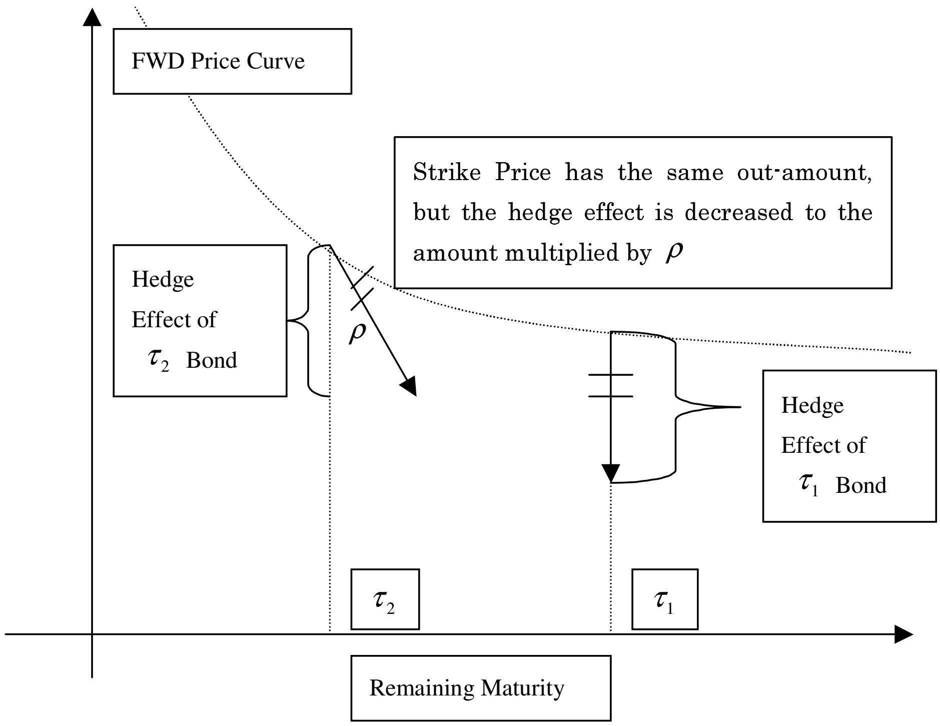

Figure 2: Hedging scheme in the 2-factor HJM framework

(2) The VaR, $\sigma(\tau_1 - T)\sqrt{T}$ and $\sigma(\tau_2 - T)\sqrt{T}$ in constraints (17) and (18) are modified to $P\sigma_{HJM2}(\tau_1)$ and $P\sigma_{HJM2}(\tau_2)$ respectively.

Using the call-option premium, which is equation (40) in Heath, Jarrow and Morton [4] with our notation and the put-call parity formula, we can get the put-option premium based on the 2-factor HJM model as follows.

$$\begin{aligned}
\tilde{\tilde{P}} &= KP(0,T)\{1-\Phi(d-q)\} - P(0,\tau)\{1-\Phi(d)\},\\
d &= \frac{\ln P(0,\tau) - \ln KP(0,T)}{q} + \frac{q}{2},\\
q^2 &= \sigma_1^2(\tau-T)^2T + \frac{4\sigma_2^2}{\lambda^3}\left(e^{-\frac{\lambda\tau}{2}} - e^{-\frac{\lambda T}{2}}\right)^2\left(e^{\lambda T}-1\right).
\end{aligned}$$

Theorem 4 The dual price of the option given in the equations (22-1) and (22-2) in Theorem 3 is modified to the one given by equation (25-1) and (25-2) based on the 2-factor HJM framework.

$$\lambda_1 = \frac{\partial\,\tilde{\tilde{P}}_1}{\partial K_1} = \frac{\tilde{\tilde{P}}_1}{P\sigma_{HJM2}(\tau_1) + K_1 - P(0,\tau_1)/P(0,T)}, \tag{25-1}$$

$$\lambda_1 = \frac{\partial\,\tilde{\tilde{P}}_2}{\partial K_2} = \frac{\tilde{\tilde{P}}_2}{P\sigma_{HJM2}(\tau_2) + K_1 - P(0,\tau_2)/P(0,T)}. \tag{25-2}$$

In equation (25-1), we don't observe the correlation ρ in the denominator. Thus, the other conditions are the same; λ_1 in (25-1) becomes smaller than that in (25-2) and the magnitude follows Lemma 2.

(*Proof*) Considering the above mentioned modifications, we have only to repeat the same argument in the proof of Theorem 3. (QED)

Remark 2 Based on Theorem 4, in this case, we can also observe the tendency we insisted in Corollary 3. According to the numerical examples provided in the next chapter, the difference in the yield volatilities has a bigger impact on the hedging cost than the scale of the correlation. When the interest rate level is low and the yield curve is flat or downward, the option written on the τ_1-year bond becomes more the effective hedging instrument than the one written on the τ_2-year bond. This result differs from that in Chapter 4.

Table 1: Scenario of yield curve and the optimal hedging (Ho-Lee framework)

Scenario	Yield Level	Yield Shape	1Yr	5Yr	7Yr	10Yr	20Yr
1	High	Upward	0.02	0.04	0.045	0.05	0.055
2	High	Flat	0.05	0.05	0.05	0.05	0.05
3	High	Downward	0.08	0.06	0.055	0.05	0.045
4	Low	Upward	0.01	0.02	0.0225	0.025	0.0275
5	Low	Flat	0.025	0.025	0.025	0.025	0.025
6	Low	Downward	0.04	0.03	0.0275	0.025	0.0225

Scenario	Important Values	5Yr	7Yr	10Yr	20Yr
1	lambda	2.7E−01	2.1E−01	1.2E−01	8.4E−04
	Hedge Ratio	5.0E−01	4.1E−01	3.9E−01	7.8E−01
	Out-Ratio	5.0E−01	5.9E−01	7.1E−01	9.3E−01
	Hedging Cost	1.4E−03	1.1E−03	6.0E−04	4.2E−06
2	lambda	2.6E−01	2.0E−01	1.3E−01	4.3E−03
	Hedge Ratio	5.1E−01	4.1E−01	3.7E−01	5.8E−01
	Out-Ratio	5.1E−01	6.0E−01	7.0E−01	9.1E−01
	Hedging Cost	1.3E−03	1.0E−03	6.4E−04	2.1E−05
3	lambda	2.4E−01	2.0E−01	1.4E−01	1.4E−02
	Hedge Ratio	5.3E−01	4.1E−01	3.5E−01	4.3E−01
	Out-Ratio	5.3E−01	5.9E−01	6.8E−01	8.8E−01
	Hedging Cost	1.2E−03	9.7E−04	6.9E−04	7.1E−05
4	lambda	3.3E−01	2.9E−01	2.4E−01	8.4E−02
	Hedge Ratio	4.2E−01	3.2E−01	2.6E−01	2.4E−01
	Out-Ratio	4.1E−01	4.8E−01	5.7E−01	7.8E−01
	Out-Ratio	4.2E−01	4.7E−01	5.5E−01	7.4E−01
	Hedging Cost	1.6E−03	1.4E−03	1.2E−03	5.4E−04
5	lambda	3.2E−01	2.9E−01	2.4E−01	1.1E−01
	Hedge Ratio	4.3E−01	3.2E−01	2.5E−01	2.1E−01
	Out-Ratio	4.2E−01	4.7E−01	5.5E−01	7.4E−01
	Hedging Cost	1.6E−03	1.4E−03	1.2E−03	5.4E−04
6	lambda	3.1E−01	2.8E−01	2.5E−01	1.4E−01
	Hedge Ratio	4.4E−01	3.2E−01	2.4E−01	1.8E−01
	Out-Ratio	4.3E−01	4.8E−01	5.4E−01	7.1E−01
	Hedging Cost	1.5E−03	1.4E−03	1.2E−03	6.8E−04

6. Numerical Examples

In this numerical example, we assume that the portfolio manager initially has a 10-year bond and it is hedged by one of the options written on 5-year, 7-year, 10-year and 20-year bonds. The λ, the hedge ratio h, the out-ratio and the hedging cost in the optimal hedging using the option written on each bond are shown in various bond market conditions such as the interest rate level, the shape of the yield curve and the shape of the yield volatility. The purpose of this chapter is to confirm the argument of the previous chapter through numerical examples.

(1) Ho-Lee framework

As an example, we assume that the 10-year yield volatility is 50BP (1BP=0.01%) and the risk limit of VaR of the 10-year bond is 4% annually. In each scenario, the λ, the hedge ratio h, the out-ratio and the hedging cost in the optimal hedging are summarized in Table 1. In any scenario in Table 1, hedging by using the option written on the longer maturity bond has a smaller dual price λ and hedging cost and has a larger out-ratio. Comparing the result in scenario 2 with the result in the scenario 5, in the case of a high interest rate level, the option written on the longer maturity bond becomes by far the better hedging tool than the one written on the shorter maturity bond. The difference between the result of the scenario 1 and that of scenario 3 or the difference between the result of the scenario 4 and that of scenario 6 indicates that whatever the interest rate level is, the option written on the longer maturity bond becomes superior to the one written on the shorter bond when the yield curve is upward sloping. These results support Corollary 3.

(2) 2-factor HJM framework

As we did in the case of the Ho-Lee framework, we assume that the 10-year yield volatility is 50BP and the risk limit of VaR of the 10-year bond is 4% annually. As a 2-factor HJM framework specific parameter set, we choose (I) $\sigma_1 = 0.0035, \sigma_2 = 0.001, \lambda = -0.2$ (The shape of the yield volatility is upward sloping.) and (II) $\sigma_1 = 0.0042, \sigma_2 = 0.041, \lambda = 1$ (The shape of the yield volatility is downward sloping.). In Table 2, the yield volatility of each maturity bond and the correlation between the 10-year yield change and other maturity yield change in both (I) and (II) are provided. In both of the parameter sets, the correlations between the 10-year bond and 5-year, and 7-year bonds are higher than 0.94, while the correlation between the 10-year bond and the 20-year bond in case (I) is 0.85 and is a little below 0.97 in case (II). In each of scenario, the λ, the hedge ratio h, the out-ratio and the hedging cost for optimal hedging are summarized in Table 3. Comparing the result in scenario 2 and scenario 3 with the result in scenario 5 and scenario 6 (as is same as in the case of the Ho-Lee framework), in the case of a high interest rate level, the option written on the longer maturity bond becomes by far the better hedging tool than the one written on the shorter maturity bond. In the parameter set (I), the option written not on

Table 2: Yield volatility and the correlation with 10Yr yield (2-factor HJM framework)

Scenario	Yield Level	Yield Shape	Volatility Shape	Yield Volatility 1Yr	5Yr	7Yr	10Yr	20Yr	Correlation with 10Yr 1Yr	5Yr	7Yr	20Yr
1	High	Upward	(I)Upward	3.7E−03	4.0E−03	4.3E−03	5.0E−03	1.5E−02	0.90	0.96	0.98	0.85
	High	Upward	(II)Downward	1.8E−02	6.8E−03	5.7E−03	5.0E−03	4.4E−03	0.73	0.94	0.99	0.97
2	High	Flat	(I)Upward	3.7E−03	4.0E−03	4.3E−03	5.0E−03	1.5E−02	0.90	0.96	0.98	0.85
	High	Flat	(II)Downward	1.8E−02	6.8E−03	5.7E−03	5.0E−03	4.4E−03	0.73	0.94	0.99	0.97
3	High	Downward	(I)Upward	3.7E−03	4.0E−03	4.3E−03	5.0E−03	1.5E−02	0.90	0.96	0.98	0.85
	High	Downward	(II)Downward	1.8E−02	6.8E−03	5.7E−03	5.0E−03	4.4E−03	0.73	0.94	0.99	0.97
4	Low	Upward	(I)Upward	3.7E−03	4.0E−03	4.3E−03	5.0E−03	1.5E−02	0.90	0.96	0.98	0.85
	Low	Upward	(II)Downward	1.8E−02	6.8E−03	5.7E−03	5.0E−03	4.4E−03	0.73	0.94	0.99	0.97
5	Low	Flat	(I)Upward	3.7E−03	4.0E−03	4.3E−03	5.0E−03	1.5E−02	0.90	0.96	0.98	0.85
	Low	Flat	(II)Downward	1.8E−02	6.8E−03	5.7E−03	5.0E−03	4.4E−03	0.73	0.94	0.99	0.97
6	Low	Downward	(I)Upward	3.7E−03	4.0E−03	4.3E−03	5.0E−03	1.5E−02	0.90	0.96	0.98	0.85
	Low	Downward	(II)Downward	1.8E−02	6.8E−03	5.7E−03	5.0E−03	4.4E−03	0.73	0.94	0.99	0.97

Table 3: Lambda, hedge ratio, out-ratio and hedging cost in the optimal hedging (2-factor HJM framework)

Scenario	Volatility Shape	Important Values	5Yr	7Yr	10Yr	20Yr
1	(I)Upward	lambda	1.5E−01	9.6E−02	4.2E−02	3.5E−03
		Hedge Ratio	7.6E−01	6.3E−01	5.9E−01	6.8E−01
		Out-Ratio	6.4E−01	7.3E−01	8.2E−01	7.8E−01
		Hedgng Cost	7.3E−04	4.7E−04	2.0E−04	1.7E−05
1	(II)Downward	lambda	9.6E−01	6.5E−01	3.8E−01	3.4E−03
		Hedge Ratio	2.6E−01	1.7E−01	1.6E−01	6.3E−01
		Out-Ratio	0.0E+00	0.0E+00	3.2E−01	8.8E−01
		Hedgng Cost	5.0E−03	3.2E−03	1.9E−03	1.7E−05
2	(I)Upward	lambda	1.4E−01	9.2E−02	4.7E−02	1.2E−02
		Hedge Ratio	7.9E−01	6.3E−01	5.6E−01	5.0E−01
		Out-Ratio	6.5E−01	7.3E−01	8.1E−01	7.5E−01
		Hedgng Cost	6.6E−04	4.4E−04	2.3E−04	5.9E−05
2	(II)Downward	lambda	9.2E−01	6.3E−01	3.9E−01	1.2E−02
		Hedge Ratio	2.6E−01	1.7E−01	1.5E−01	4.6E−01
		Out-Ratio	0.0E+00	0.0E+00	2.8E−01	8.5E−01
		Hedgng Cost	4.8E−03	3.1E−03	1.9E−03	6.0E−05
3	(I)Upward	lambda	1.2E−01	8.8E−02	5.4E−02	3.1E−02
		Hedge Ratio	8.2E−01	6.4E−01	5.2E−01	3.8E−01
		Out-Ratio	6.6E−01	7.3E−01	7.9E−01	7.2E−01
		Hedgng Cost	6.0E−04	4.2E−04	2.6E−04	1.5E−04
3	(II)Downward	lambda	8.7E−01	6.1E−01	3.9E−01	3.1E−02
		Hedge Ratio	2.6E−01	1.7E−01	1.4E−01	3.5E−01
		Out-Ratio	0.0E+00	0.0E+00	2.4E−01	8.2E−01
		Hedgng Cost	4.5E−03	3.0E−03	1.9E−03	1.5E−04
4	(I)Upward	lambda	2.0E−01	1.6E−01	1.2E−01	1.3E−01
		Hedge Ratio	6.5E−01	4.8E−01	3.7E−01	2.1E−01
		Out-Ratio	5.8E−01	6.5E−01	7.1E−01	6.2E−01
		Hedgng Cost	9.4E−04	7.7E−04	5.8E−04	6.2E−04
4	(II)Downward	lambda	1.1E+00	7.6E−01	5.2E−01	1.3E−01
		Hedge Ratio	2.6E−01	1.7E−01	1.1E−01	1.9E−01
		Out-Ratio	0.0E+00	0.0E+00	0.0E+00	6.9E−01
		Hedgng Cost	5.5E−03	3.8E−03	2.5E−03	6.4E−04
5	(I)Upward	lambda	1.9E−01	1.6E−01	1.3E−01	1.6E−01
		Hedge Ratio	6.6E−01	4.8E−01	3.6E−01	1.9E−01
		Out-Ratio	5.9E−01	6.5E−01	7.0E−01	5.8E−01
		Hedgng Cost	9.0E−04	7.5E−04	6.0E−04	7.5E−04
5	(II)Downward	lambda	1.0E+00	7.5E−01	5.2E−01	1.6E−01
		Hedge Ratio	2.6E−01	1.7E−01	1.1E−01	1.7E−01
		Out-Ratio	0.0E+00	0.0E+00	0.0E+00	6.6E−01
		Hedgng Cost	5.4E−03	3.7E−03	2.5E−03	7.9E−04
6	(I)Upward	lambda	1.8E−01	1.5E−01	1.3E−01	1.9E−01
		Hedge Ratio	6.7E−01	4.9E−01	3.6E−01	1.7E−01
		Out-Ratio	6.0E−01	6.5E−01	7.0E−01	5.5E−01
		Hedgng Cost	8.7E−04	7.4E−04	6.2E−04	9.0E−04
6	(II)Downward	lambda	1.0E+00	7.3E−01	5.2E−01	1.9E−01
		Hedge Ratio	2.6E−01	1.7E−01	1.1E−01	1.5E−01
		Out-Ratio	0.0E+00	0.0E+00	0.0E+00	6.2E−01
		Hedgng Cost	5.3E−03	3.7E−03	2.5E−03	9.4E−04

the 20-year bond but the 10-year bond is selected for optimal VaR hedging of the 10-year bond in scenario 4, scenario 5 and scenario 6. Being different from the result in the Ho-Lee framework, in the high interest rate condition, even though the out-ratio is small, the option written on the longer maturity bond sometimes becomes the optimal hedging tool and the relation between the out-ratio and λ (or the cost of hedging) is not maintained. Of course, as we stated in Theorem 3, the option which gives us the smallest λ becomes the optimal hedging tool. As is same as in the Ho-Lee framework, whatever the interest rate level is, the option written on the longer maturity bond becomes superior to the one written on the shorter bond when the yield curve is upward sloping.

7. Summary and Concluding Remarks

Analyzing the optimal VaR hedging of a bond portfolio through the dual theory of non−linear optimization, I found out that (1) the optimal hedge tends to be obtained by choosing the option whose strike price has the biggest out-ratio, (2) the longer the maturity of the underlying bond becomes, the bigger the out-ratio tends to become in the optimal hedge in an arbitrage-free pricing framework using the Ho-Lee term structure model of interest rate. Thanks to the result, from the view point of the hedging decision making, we can explain why out-of-the-money options tend to remain rich and the options on super-long bonds are quite often traded richer than the other options. The result in (1) is the same as that stated in Ahn *et al.* [1], which analyzed the optimal VaR hedge by using options based on a different approach.

When the above argument is reexamined in a more realistic setting based on the 2-factor HJM model (which can capture the difference of yield volatility sector by sector and the correlation among sectors), we are able to observe the results against those from the Ho-Lee model in conditions such as a low interest rate, a flat or downward term structure of interest rate and upward volatility structure. Overall, in most of the conditions, the results obtained from the Ho-Lee model are maintained and the analysis in this paper shows some implications to clarify the relation between common market observation and managerial decisions in optimal hedging of the VaR of bond portfolios.

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