Journal of the Operations Research
Society of Japan
2003, Vol. 46, No. 4, 429-447

2-CYCLIC DESIGN IN AHP

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(Received December 4, 2002; Revised August 11, 2003)

Abstract The simple method called "2-cyclic design" is proposed to select pairs to be compared from the whole set of pairs in AHP, which reduces the comparing number $n(n-1)/2$ for n-object AHP to $2n$. The isomorphism of our designs is investigated, by which we can reduce the range of designs to be considered. The standard errors of our designs are calculated for $n = 5, \ldots, 60$, by which we know the best 2-cyclic designs in this range. Developing the algebra on our designs we can clarify the characteristics of good designs and further propose several conjectures which give a general method to produce the better or the best designs. These conjectures are valid for $n = 5, \ldots, 60$.

Keywords: AHP, paired comparison, logarithmic least square, incomplete information, cyclic design

1. Introduction

The essence of AHP is the paired comparison, but if the number "n" of objects is large the number of comparisons increases with square order of n, so it is often difficult for an evaluator to treat such a large number of comparisons. To overcome this difficulty, short cut methods without paired comparisons such as "absolute method" [2] are proposed. But to neglect paired comparisons is to miss the essence of AHP.

Wang and Takahashi [4] proposed the method to select specific subsets of pairs from the whole set of pairs by graph theoretic considerations, which insists that the strongly regular design based on strongly regular graph is the best selection method of the methods with the same number "m" of pairs to be compared. But the existing of strongly regular graphs is very rare for various values of n (the number of objects) and m (the number of pairs to be compared), and further the construction methods in general are too difficult for practical use.

Here we propose a simple and effective method, which is called **2-cyclic design**, to select the subset S of pairs for the case with $m = 2n$, whose labour to compare is reasonably allowable even if n is rather large.

When the set S of pairs to be compared is not the whole set, there are missing elements in the comparison matrix, which we call **incomplete case** in contrast to the **complete case** with all paired comparisons. There are various analyzing methods to estimate the weights of objects for the incomplete case. We adopt the **logarithmic least square** method (LLS) of them, which can evaluate the estimating error of designs based on various theorems of least square method.

The main results of this paper are to give the standard errors of estimating weights of objects of 2-cyclic designs, which show what is the best 2-cyclic design, for $n = 5, \ldots, 60$, and to conjecture the general method to reveal what is the best 2-cyclic design. The conjectures are valid for all cases $n = 5, \ldots, 60$.

2. 1-cyclic Design

Let $\{0, 1, \cdots, n-1\}$ be the set of n objects (alternatives or criteria) to be evaluated in AHP. We call a set of pairs (i, j) a design, and

$$Q_i = \{(0, i), (1, i+1), \cdots, (n-1, i+(n-1))\} \quad , i = 1, 2, \cdots, n-1 \tag{1}$$

be called **1-cyclic design** (or simply **cycle**) with **initial pair** $(0, i)$, where the numbers i, j in pair symbol(i, j) are calculated in mod n.

For example for $n = 6$,

$$Q_1 = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 0).\}.$$

We can estimate weight $u_i (i = 0, 1, \cdots, 5)$ based on the paired comparison data a_{ij} for $(i, j) \in Q_1$ by LLS method which is stated later. But if we take

$$Q_3 = \{(0, 3), (1, 4), (2, 5), (3, 0), (4, 1), (5, 2)\},$$

we cannot estimate weights of objects by this design.

Note that for pair (i, j) the order of i, j is indifferent, that is, $(i, j) = (j, i)$.

It is clear that for cycle Q_i in n-object AHP

$$Q_1 = Q_{n-1}, Q_2 = Q_{n-2}, \cdots. \tag{2}$$

So we have only to consider 1-cyclic designs

$$\left.\begin{array}{ll} Q_1, Q_2, \cdots, Q_{n/2} & for \ \ even \ \ n \\ Q_1, Q_2, \cdots, Q_{(n-1)/2} & for \ \ odd \ \ n \end{array}\right\} \tag{3}$$

Theorem 1 If n is prime then $Q_1, Q_2, \cdots, Q_{(n-1)/2}$ are all **isomorphic**, that is,

$$Q_1 \approx Q_2 \approx \cdots \approx Q_{(n-1)/2}, \tag{4}$$

and for general n if a and n are relatively prime, that is, a has not common divisors (except 1) with n, then Q_i is isomorphic to Q_{ai}, that is,

$$Q_i \approx Q_{ai}, \tag{5}$$

where ai is calculated in mod n like in formula (2). For two designs P and Q to be isomorphic $P \approx Q$ means the existence of one to one correspondence φ on $\{0, 1, \cdots, n-1\}$ such that

$$Q = \{(\varphi(i), \varphi(j)) \mid (i, j) \in P\}. \tag{6}$$

Proof It is clear that formula (5) includes (4) , so we have only to prove (5).

If we take $\varphi(j) = aj \ (j = 0, 1, \cdots, n-1)$, it is one to one correspondence because a and n are relatively prime. Let

$$Q_i = \{(0, i), (1, i+1), \cdots, (j, i+j), \cdots, (n-1, i+n-1)\}, \tag{7}$$

$$Q_{ai} = \{(0, ai), (1, ai+1), \cdots, (j', ai+j'), \cdots, (n-1, ai+n-1)\}. \tag{8}$$

As the result of transformation of each element in Q_i by φ, we have

$$\varphi(Q_i) = \{(0, ai), (a, a(i+1)), \cdots, (aj, a(i+j), \cdots, (a(n-1), a(i+n-1))\}. \quad (9)$$

There exists exactly one j' for each j such that

$$aj = j' \ (mod \ n), \quad (10)$$

because of relative primality of a and n, and then

$$a(i+j) = ai + aj = ai + j',$$

so $(aj, a(i+j))$ in $\varphi(Q_i)$ coincides with $(j', ai+j')$ in Q_{ai} which means $\varphi(Q_i) = Q_{ai}$ or $Q_i \approx Q_{ai}$. □

Example 1 For $n = 10$, $a = 3$, $i = 2$, we can show that $Q_i = Q_2, Q_{ai} = Q_6$ and $\varphi(Q_i) = 3Q_i$, whose elements are multiples by 3 of elements in Q_i as shown in Table 1. And Q_2 and Q_6 are shown graphically in Fig. 1. □

3. 2-cyclic Design

We call union of two cycles Q_i and Q_j as **2-cyclic design** and denote it as $Q_i + Q_j$. (Symbolically we should denote it as $Q_i \cup Q_j$, but $Q_i + Q_j$ is algebraically convenient because later we claim for Q_i to have the meaning of adjacency matrix.)

Our main object is to find the best 2-cyclic design. To this end, we have to investigate $Q_i + Q_j$ for all combinations of $i, j = 1, \ldots, n/2$(for even n) or $1, \ldots, (n-1)/2$ (for odd n).

But if n is prime we only need to investigate the types of $Q_1 + Q_j (j = 1, \ldots, (n-1)/2)$, because any Q_i is isomorphic to Q_1 by Theorem 1. For general n, there exists a Q_i non-isomorphic to Q_1 if i has a common divisor with n. But for this case Q_i is composed of disconnected subgraphs as shown in Fig.1, which is worse than Q_1 as a design, but $Q_i + Q_j$ can be better than $Q_1 + Q_j$. Treating such special cases in the Appendix 1, we only investigate the type of $Q_1 + Q_j$ for the time being.

Theorem 2 If

$$ij = \pm 1 \ (mod \ n), \quad (11)$$

that is, $ij = 1 (mod \quad n)$ or $ij = -1 (mod \quad n)$, then we have

$$Q_1 + Q_i \approx Q_1 + Q_j. \quad (12)$$

Proof Formula (11) shows that i and j are relatively prime to n. So $\varphi(k) = jk \ (mod \ n) \ (k = 0, \ldots, n-1)$ gives one to one correspondence on $\{0, 1, \cdots, n-1\}$, and we have

$$\varphi(Q_1 + Q_i) = j(Q_1 + Q_i) = Q_j + Q_{ij}. \quad (13)$$

If $ij = 1 \ (mod \ n)$ *then* $\varphi(Q_1 + Q_i) = Q_j + Q_1$ which shows formula (12), and if $ij = -1 \ (mod \ n)$ then $Q_{ij} = Q_{-1} = Q_1$ because of (2). This completes the proof. □

In Table 4 for various values of n the isomorphic relations of 2-cyclic designs are shown. Incidentally we can find j satisfying (11) for given i by Euclidian algorithm [3], which is well known in the field of elementary algebra.

If $i^2 = \pm 1 \ (mod \ n)$, then it holds that

$$\varphi(Q_1 + Q_i) = Q_i + Q_{i^2} = Q_i + Q_1, \tag{14}$$

where φ is the correspondence produced by iQ_j, that is, $\varphi(Q_j) = iQ_j$.
This means $Q_1 + Q_i$ is transformed to itself by φ, so $Q_1 + Q_i$ is called **self-isomorphic** design. If n is prime then self-isomorphic designs exist if and only if $n-1$ is a multiple of 4. Table 4 shows that almost all self isomorphic designs are the best cyclic designs (with the lowest standard error).

4. Statistical Reliability Analysis of 2-cyclic Design

As for statistical reliability analysis of our design, we almost go to along that of [4]. Let a_{ij} be the value of paired comparisons of object i to j in 2-cyclic design $Q_1 + Q_k$, then we assume the following model;

$$\begin{aligned} &a_{ij} = \tfrac{u_i}{u_j} e_{ij} \ , \ \ i < j \ , \ \ (i,j) \in Q_1 + Q_k, \\ &a_{ii} = 1, \end{aligned} \tag{15}$$

where u_i is weight of object i $(i = 0, \ldots, n-1)$ and $e_{ij} > 0$ is the error. Taking logarithm (the base can be arbitrary) we have

$$\begin{aligned} &\dot{a_{ij}} = \dot{u_i} - \dot{u_j} + \dot{e_{ij}}, \ \ i < j, \ \ (i,j) \in Q_1 + Q_k, \\ &(\dot{a_{ij}} = \log a_{ij}, \ \ \dot{u_i} = \log u_j, \cdots). \end{aligned} \tag{16}$$

By minimizing the sum of squares of $\dot{e_{ij}}$, we have LLS estimate $\hat{u_i}$ of u_i. We assume that $\dot{e_{ij}}$'s are independent random variables with zero-expectation and common variance σ^2, that is,

$$E[\dot{e_{ij}}] = 0 \ , \ \ V[\dot{e_{ij}}] = \sigma^2. \tag{17}$$

Since $u_0, \cdots, u_{n-1}$ are arbitrary by a constant multiple we can assume $u_0 u_1 \cdots u_{n-1} = 1$, then

$$\dot{u}_0 + \dot{u}_1 + \cdots + \dot{u}_{n-1} = 0. \tag{18}$$

So our LLS must be done under the condition of (18).

Example 2 For $Q_1 + Q_2 (n = 6)$, we write down (16) and (18) in Table 2 neglecting the error term. This is often called **data table**. □

As for the general 2-cyclic design the format of data table is similar to Table 2. We denote the coefficient matrix of the right hand side of data table by X, and column vector of the left hand side by $\dot{\boldsymbol{a}}$, and $\dot{\boldsymbol{u}}^T = [\dot{u}_0, \dot{u}_1, \cdots, \dot{u}_{n-1}]$, then the normal equation of least square method is

$$(X^T X)\dot{\boldsymbol{u}} = X^T \dot{\boldsymbol{a}}. \tag{19}$$

And i-th element $\hat{\dot{u_i}}$ of the solution $\hat{\dot{\boldsymbol{u}}}$ of (19) gives the LLS estimate of $\dot{u}_i$ $(i = 0, \ldots, n-1)$.

The coefficient matrix $X^T X$ of the normal equation plays an important role and is called **information matrix** M, that is,

$$M = X^T X. \tag{20}$$

We have a graph corresponding to a design as shown in Fig.1. Let the point-to-point **adjacency matrix** (whose (i, j) and (j, i) elements are 1 if edge (i, j) exists and zero otherwise) of the graph be called **matrix of the design**. Let N be the matrix of a design, and if the corresponding graph to the design is a **regular graph** whose points have the common degree d (the number of edges connected with a point), then the information matrix M of the design is represented as

$$M = dI + J - N, \tag{21}$$

where I is unit matrix and J is all-1 matrix.The important formula (21) is stated and proved in [1]. Also note that the values of the diagonal elements m^{ii} of the inverse of M in (21) are independent of i [1]. Now the variance $V[\hat{u}_i]$ of the solution $\hat{u}_i$ of (19), or its square root $\sqrt{V[\hat{u}_i]}$, is the measure of error of estimate $\hat{u}_i$. In our case $V[\hat{u}_i]$ is independent of i and is represented by

$$V[\hat{u}_i] = \{m^{1,1} - (\sum_{j=0}^{n-1} m^{1,j})^2\}\sigma^2, \tag{22}$$

where $m^{i,j} = (M^{-1})_{ij}((i, j)$ element of $M^{-1})$. (22) shows that $V[\hat{u}_i]$ is independent of data a_{ij} and only depends on M, that is, the design structure. Note that for the model without constraint (18) we have simply $V[\hat{u}_i] = m^{1,1}\sigma^2$ but we need the second term within $\{m^{1,1} - (\sum_{j=0}^{n-1} m^{1,j})^2\}$ of (22) because of (18). The proof of (22) is shown in the Appendix 2. We have calculated these values $= \sqrt{V[\hat{u}_i]}/\sigma$ for all 2-cyclic designs of the type of $Q_1 + Q_i$ for $n = 5, \ldots, 60$ in Table 5. The value, standard error $\sqrt{V[\hat{u}_i]}$ divided by σ, shows the statistical reliability measure of the estimated weight $\hat{u}_i$. The best designs (with the smallest value of $\sqrt{V[\hat{u}_i]}/\sigma$) are marked by $*$. Further the **standard error** $\sqrt{V[\hat{u}_i]}$ for the complete case is

$$\sqrt{V[\hat{u}_i]} = \frac{\sqrt{n-1}}{n}\sigma, \tag{23}$$

which are added in Table 5. We can compare the standard error of the best 2-cyclic design with that of complete case for each n. We have (23) by the following way; from (21) we have $M = nI$ and $M^{-1} = \frac{1}{n}I$, so from (22) we have $V[\hat{u}_i] = (\frac{1}{n} - (\frac{1}{n})^2)\sigma^2$ which leads us to (23).

We pick up several values of n and compare these two in Table 3.

Considering the overwhelming reduction of labors of 2-cyclic designs, we can say that the standard errors of the best 2-cyclic designs are not so worse than those of the complete cases for smaller values of n. For larger values of n, the difference becomes manifest, where we are ready to propose 3-cyclic designs.

Note that isomorphic designs (shown in Table 4) naturally have the same value of the standard error.

Table 5 certainly shows what is the best 2-cyclic design for $n = 5, \ldots, 60$, but further we would like to know what characteristics of the matrix $Q_1 + Q_i$ give the good 2-cyclic designs. In order to investigate these characteristics we study the algebraic structure in the next section.

5. Algebra on Cyclic Designs

So far such symbols as Q_i and $Q_i + Q_j$ represent designs, that is, sets of pairs of objects, but later on let us make these symbols to represent the (adjacent) matrices of the corresponding designs.

Then, $Q_i + Q_j$ is the sum (as matrices) of Q_i and Q_j, and further we have

$$Q_i = P_i + P_{-i}, \tag{24}$$

where P_i is **cyclic matrix** whose $(j, i+j)$ element is 1 $(j = 0, \ldots, n-1)$ and other elements are zeros, and P_{-i} is defined by the same way in *mod* $n-$calculation. Further we have

$$P_i P_j = P_{i+j}. \tag{25}$$

The following relations are easily proved by matrix calculations and relation (24) and (25).

$$\begin{array}{l} Q_i = Q_{-i} \\ Q_i^2 = 2I + Q_{2i} \\ Q_i Q_j = Q_j Q_i = Q_{i+j} + Q_{i-j} \end{array} \tag{26}$$

Further note that considering (2) and (3) we have

$$Q_l = Q_k \;\; if \;\; l = \pm k \;\; (mod \;\; n). \tag{27}$$

Note that we have

$$\begin{array}{l} I + Q_1 + Q_2 + \cdots + Q_{(n-1)/2} = J, \;\; for \;\; odd \;\; n, \\ I + Q_1 + Q_2 + \cdots + \frac{1}{2} Q_{n/2} = J, \;\; for \;\; even \;\; n, \end{array}$$

and incidentally $Q_{n/2} = 2P_{n/2}$. Note that for any 2-cyclic design $N = Q_i + Q_j$, any polynomial of N is represented by a linear combination of $I, Q_1, Q_2, \ldots$ for the sake of (26). And N has two terms with unity coefficients, so the sum of coefficients is two. Generally the sum of coefficients of $(a_1 Q_{j_1} + a_2 Q_{j_2} + \ldots + a_k Q_{j_k})(Q_i + Q_j)$ is clearly $4(a_1 + a_2 + \ldots + a_k)$, because we have $a_\alpha Q_{j_\alpha}(Q_i + Q_j) = a_\alpha(Q_{j_\alpha + i} + Q_{j_\alpha - i} + Q_{j_\alpha + j} + Q_{j_\alpha - j})$ except $j_\alpha = i$. If $j_\alpha = i$, we have $Q_{j_\alpha + i} Q_i = 2I + Q_{2i}$, but we can write $2I = Q_0$, so let us count the coefficient as unity. Then without exception, the sum of coefficients of $N^2 = N(Q_i + Q_j)$ is 4 times of that of N, that is 2×4. Similarly the sum of coefficient of N^3 is $2 \times 4^2, \ldots$. Let N be the matrix of a graph then (i, j) element of N^2 is the number of paths from point i to j with length 2, that is, if (i, j) element of N^2 has a positive value, then point i and j are connected with 2-steps. Of course (i, j) element of N itself represents the connection of 1-step. So we can say that $N + N^2$ have the more positive elements (the less zero elements), the connectivity of the graph for $N + N^2$ becomes the stronger. The same logic goes along

$$N + N^2 + N^3 \;\; and \;\; N + N^2 + N^3 + N^4 \cdots.$$

Along above discussions we introduce

$$Cr = I + \frac{1}{2} N + \frac{1}{2 \times 4} N^2 + \frac{1}{2 \times 4^2} N^3 + \cdots + \frac{1}{2 \times 4^{r-1}} N^r, \tag{28}$$

and let Cr be called as **r-th order connectivity matrix**, where the weight of each (28) is the inverse of the sum of coefficients (of $Q_0, Q_1, Q_2, \ldots$) of its term and by this the contribution of each term to C_r becomes even, and unit matrix I is included for consistence of further calculation.

Example 3 For $n = 15$, we have seven possible 1-cyclic designs

$$Q_1, Q_2, \cdots, Q_7 (\text{see formula } (3)).$$

We consider two 2-cyclic designs $Q_1 + Q_2$ and $Q_1 + Q_4$.
We have

$$\begin{aligned}(Q_1+Q_2)^2 &= Q_1^2 + 2Q_1Q_2 + Q_2^2 \\ &= 2I + Q_2 + 2(Q_1+Q_3) + 2I + Q_4 \\ &= 4I + 2Q_1 + Q_2 + 2Q_3 + Q_4, \\ (Q_1+Q_4)^2 &= Q_1^2 + 2Q_1Q_4 + Q_4^2 \\ &= 2I + Q_2 + 2(Q_3+Q_5) + 2I + Q_7,\end{aligned}$$

by formula (25) and (26).
Their 2-nd connectivity matrices are

$$\begin{aligned} C_2 \ of \ (Q_1+Q_2) &= I + \frac{(Q_1+Q_2)}{2} + \frac{(Q_1+Q_2)^2}{8} \\ &= \frac{3}{2}I + \frac{6Q_1 + 5Q_2 + 2Q_3 + 8Q_4}{8}, \qquad (29) \\ C_2 \ of \ (Q_1+Q_4) &= I + \frac{(Q_1+Q_4)}{2} + \frac{(Q_1+Q_4)^2}{8} \\ &= \frac{3}{2}I + \frac{Q_1}{2} + \frac{Q_2}{8} + \frac{Q_3}{4} + \frac{Q_4}{2} + \frac{Q_5}{4} + \frac{Q_7}{8}. \qquad (30)\end{aligned}$$

Comparing (29) and (30), we can say that connectivity of $Q_1 + Q_4$ is stronger than that of $Q_1 + Q_2$, by which we can say that $Q_1 + Q_4$ is better than $Q_1 + Q_2$. This is also certified by the following values of standard errors,

$$The \ standard \ error \ of \ Q_1 + Q_2 = 0.5794,$$
$$The \ standard \ error \ of \ Q_1 + Q_4 = 0.5085,$$

shown in Table 5.

For the precise measurement of connectivity the variance of coefficient of $Q_0(= 2I), Q_1, Q_2, \cdots$ is suitable which is denoted by $|C_r|$. As for (29) and (30) the coefficients and the variances are as follows.

	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	$variance\|C_2\|$
$Q_1 + Q_2$	3/4	6/8	5/8	1/4	1/8	0	0	0	1.496
$Q_1 + Q_4$	3/4	1/2	1/8	1/4	1/2	1/4	0	1/8	1.121

(31)

□

Example 4 For $n = 17$ the possible cycles are $Q_1, Q_2, \cdots, Q_8$. Let us compare $Q_1 + Q_4$ and $Q_1 + Q_5$.
By the same way as Example 3, we have the following coefficients of 2-nd order connectivity of both designs.

	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	$variance\|C_2\|$
$Q_1 + Q_4$	3/4	1/2	1/8	2/8	1/2	2/8	0	0	1/8	1.142
$Q_1 + Q_5$	3/4	1/2	1/8	0	2/8	1/2	2/8	1/8	0	1.142

(32)

Both have the same 2-nd order connectivity. But the slight difference of standard errors (0.5210 of $Q_1 + Q_4$ and 0.5222 of $Q_1 + Q_5$) is revealed by the 3-rd order C_3 as follows;

	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	$variance\|C_3\|$
$Q_1 + Q_4$	3/4	27/32	7/32	9/32	27/32	9/32	3/32	3/32	7/32	1.903
$Q_1 + Q_5$	3/4	25/32	5/32	4/32	8/32	25/32	11/32	7/32	3/32	2.111

(33)

□

6. Characteristics of the Goodness of Designs

Already we have Tables 5 and 6 so we are able to know the best 2-cyclic design for $n = 5, \ldots, 60$. But the characteristics of good designs is useful to know the best design for $n > 60$, and to extend our 2-design theory to 3-cyclic, 4-cyclic design theories.

First we define **strength** r of a 2-cyclic design. A 2-cyclic design has strength r if there are no duplications among terms Q_k in C_r .For a general **2-cyclic design** $N = Q_i + Q_j$ we have

$$\begin{aligned} 2C_1 &= 2I + Q_i + Q_j, \\ 2C_2 &= 2I + Q_i + Q_j + (Q_i + Q_j)^2/4 \\ &= 3I + Q_i + Q_j + \frac{1}{4}Q_{2i} + \frac{1}{4}Q_{2j} + \frac{1}{2}Q_{i+j} + \frac{1}{2}Q_{i-j}. \end{aligned}$$

So we have terms

$$I \ , \ Q_i \ , \ Q_j \ , \ Q_{2i} \ , \ Q_{2j} \ , \ Q_{i+j} \ , \ Q_{i-j} \tag{34}$$

in C_2. If i , j , $2i$, $2j$, $i+j$, $i-j$ are different from each other in the meaning of (27), then no duplications occur among terms in (34). For $n = 17$, design $Q_1 + Q_4$ have strength 2, because the set of terms becomes

$$I \ , \ Q_1 \ , \ Q_2 \ , \ Q_4 \ , \ Q_8 \ , \ Q_5 \ , \ Q_3, \tag{35}$$

which has no duplications. But for $Q_1 + Q_6$ we have $2i = 1 - i$, so (34) becomes

$$I \ , \ Q_1 \ , \ Q_2 \ , \ Q_6 \ , \ Q_5 \ , \ Q_7 \ , \ Q_5, \tag{36}$$

So $Q_1 + Q_6$ has not strength 2.

If 2-cyclic design N has strength r then it has also strength $r-1, r-2, \cdots$. And it is easily seen that if 2-cyclic designs N and N' both have strength r then their coefficient patterns in C_r (as seen in Examples 3 and 4) are the same, so they have same connectivity$|C_r|$(the variance of coefficients).

Further we present the following Conjectures 1 through 3.

Conjecture 1 If for some r, N has strength r and N' has not, then N is better design than N' (or the standard error of N is smaller than that of N'). □

For example as shown in (35) and (36), $Q_1 + Q_4$ has strength 2 and $Q_1 + Q_6$ has not, and the standard error of $Q_1 + Q_4$ is 0.5210 and that of $Q_1 + Q_6$ is 0.5353.

The number of terms (except unit matrix I) in C_r is easily known to be $r(r+1)$. Let m be the number of actual cycles Q_1,Q_2,$\cdots$,Q_m ($m = (n-1)/2$ for odd n and $=n/2 - 1$ for even n), then if $m < r(r+1)$ then some duplications must occur, so any 2-cyclic designs have not strength r. So only in the case

$$m \geq r(r+1) \tag{37}$$

they can have strength r.

Example 5 For $n = 13$, $m = 6$, $r = 2$, (37) is valid with equality and $Q_1 + Q_5$ generates the following terms in C_2 of (34).

$$I \ , Q_1 \ , Q_2 \ , Q_5 \ , Q_3 \ , Q_6 \ , Q_4$$

□

Example 6 For $n = 26, m = 12, r = 3$, (37) is also valid with equality. Actually $Q_1 + Q_{10}$ generates in the following terms C_3.

$$\begin{aligned} I \ , Q_1 \ , Q_i \ , Q_2 \ , Q_{2i} \ , Q_3 \ , Q_{3i} \ , Q_{1+i} \ , Q_{1-i} \ , Q_{1+2i} \ , Q_{1-2i} \ , Q_{2+i} \ , Q_{2-i} \\ = I, Q_1 \ , Q_{10} \ , Q_2 \ , Q_6 \ , Q_3 \ , Q_4 \ , Q_{11} \ , Q_9 \ , Q_5 \ , Q_7 \ , Q_{12} \ , Q_8 \end{aligned}$$

□

Conjecture 2 If (37) is valid with equality, and N has strength r then it is the best 2-cyclic design. □

So far we discuss the goodness of design through the strength which gives the simple method only to check duplications. By this we can cover fairly large case to find better designs, but to find the best design we must use the connectivity, that is, the variance $|C_r|$ of coefficients of terms in C_r ((31), (32)$\cdots$), except the rare case stated in Conjecture 2.

Conjecture 3 Except the case of holding equality of (37), the best 2-cyclic design has strength r for the largest r with $m > r(r+1)$, and if there are several such 2-cyclic designs, the best design has the least value of connectivity $|C_{r+1}|$ among them. □

Example 7 For $n = 15, m = 7$, the largest r with $7 > r(r+1)$ is $r = 2$. There are two designs $Q_1 + Q_4$ and $Q_1 + Q_6$ with strength 2. The coefficients in C_3 of both designs and their variances $|C_3|$ are

	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	$variance\|C_3\|$
$Q_1 + Q_4$	3/4	25/32	7/32	10/32	25/32	8/32	6/32	7/32	1.883
$Q_1 + Q_6$	3/4	25/32	7/32	6/32	6/32	8/32	25/32	11/32	1.891

so Q_1+Q_4 is the best by our Conjecture 3. Actually the standard error of Q_1+Q_4 is 0.5085 and that of $Q_1 + Q_6$ is 0.5093. □

Through these conjectures we can efficiently find the better or the best 2-cyclic designs for the given value of n (even if we have not Table 5 at hand).

For $6 \leq m < 12$, first we find the set S_2 whose designs have strength 2. $N \in S_2$ is better (than designs $\notin S_2$) designs (the best design for $m = 6$). Further calculate $|C_3|$ of $N \in S_2$, then N having the smallest $|C_3|$ is the best design.

For $12 \leq$ m < 20, first find the set S_3 whose designs have strength 3. $N \in S_3$ is better (than designs $\notin S_3$) or the best for $m = 12$. Further calculate $|C_4|$ of $N \in S_3$ then N having the smallest $|C_4|$ is the best.

The same way goes for $20 \leq$ m < 30, $30 \leq$ m $< 42 \cdots$.

7. Conclusions and Further Researches

1. We proposed 2-cyclic designs which give the simple and efficient method to select pairs from the whole set of pairs of n objects, which reduce the number $n(n-1)/2$ of paired comparisons to $m = 2n$.
2. The properties of isomorphism relations of 1-cyclic and 2-cyclic designs were investigated. It contributes to reduce the possible range of 2-cyclic designs to be investigated. In Table 4 all possible isomorphism relations in 2-cyclic designs are shown.
3. The analyzing and error estimating methods of 2-cyclic designs are stated and the standard errors of all possible 2-cyclic designs for $n = 5, \ldots, 60$ are shown in Table 5 and Table 6, by which we can find the best 2-cyclic design for each value of $n = 5, \ldots, 60$.
4. Developing the algebra on cyclic design matrices we introduce the connectivity or strength of designs which reveal characteristics of good designs. We propose several conjectures based on these concepts, which give very simple way to find the better or the best 2-cyclic designs for general values of n.
5. For larger n, the set of pairs to be investigated in 2-cyclic designs is too small compared with the whole set. We would like to extend our methods and theories to 3-cyclic or 4-cyclic designs. This is left to the future researches.
6. Mathematical proofs of 3 conjectures stated in §6 are also left to the future research.

Acknowlegment

This research was originated through discussion with staffs of Human Resource Research Institute (HRR) about actual personnel affairs. We thank them for offering such interesting problems. We also thank the referees for their useful comments.

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Appendix 1 Standard errors of $Q_i + Q_j$ type for non-prime n

Here we investigate standard errors of $Q_i + Q_j$ for non-prime n. First if i or j is relatively prime to n then $Q_i + Q_j \approx Q_1 + Q_k$, so we have only to consider the case for the i and j to have common divisor($\neq 1$) with n. Further if i and j have common divisor ($\neq 1$) then $Q_i + Q_j$ is disconnected. Besides $Q_{n/2}$ (for even n) is omitted, because it takes duplicate pairs so it is clearly a bad design.

Considering above, we calculate the standard errors of the case for the i and j to have common divisors ($\neq 1$) and i and j are relatively prime, and omit $Q_{n/2}$ (for even n), and show them in Table 6. Note that for example for the case of $n = 14$ or 16 etc. There is no $Q_i + Q_j$ with above conditions; such n is omitted in Table 6.

The cases where a standard error in Table 6 is better than that in Table 5 are only;

$Q_2 + Q_3$ (better than $Q_1 + Q_3, n = 12$)
$Q_3 + Q_4$ (better than $Q_1 + Q_5, n = 24$)
$Q_3 + Q_5$ (better than $Q_1 + Q_5, n = 30$)
$Q_4 + Q_5$ (better than $Q_1 + Q_7, n = 40$)

Our conjectures are still valid for these cases. For example we have the following coefficients of C_2 for $Q_2 + Q_3$ and $Q_1 + Q_3 (n = 12)$

	I	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
$Q_2 + Q_3$	3/2	1/4	1/2	1/2	1/8	1/4	1/8
$Q_1 + Q_3$	3/2	1/2	3/8	1/2	1/4	0	1/8

and it is clear that $Q_2 + Q_3$ has the smaller variance than that of $Q_1 + Q_3$.

Appendix 2

By the formula shown in [1] we have $[V[\hat{u}_i, \hat{u}_j]]/\sigma^2 = M^{-1} X_0^T X_0 M^{-1}$ where $V[\hat{u}_i, \hat{u}_j]$ is the variance and covariance matrix of $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{n-1}$. Let denote the original data table matrix by X_0, that is

$$X = \begin{bmatrix} X_0 \\ e^T \end{bmatrix}$$

where e is all 1 column vector.

Let $X_0 = [x_{ij}]$, $M = [m_{ij}]$, $M^{-1} = [m^{ij}]$, $W = M^{-1} X_0^T = [w_{ij}]$
Then we have

$$V[\hat{u}_i] \;/\sigma^2 = \textstyle\sum_j {w_{ij}}^2 = \sum_j \sum_k m^{ik} x_{kj} \sum_l m^{il} x_{lj}$$

$$= \textstyle\sum_k \sum_l m^{ik} m^{il} \sum_j x_{kj} x_{lj} = \sum_k \sum_l m^{ik} m^{il} (m_{kl} - 1)$$

$$= \textstyle\sum_k \sum_l m^{ik} m^{il} m_{kl} - \sum_k \sum_l m^{ik} m^{il}$$

$$= \textstyle\sum_k \delta_{ik} m^{ik} - (\sum_k m^{ik})^2 \qquad (\delta_{ik} \text{ is Kronecker's delta})$$

$$= m^{ii} - \left(\textstyle\sum_k m^{ik}\right)^2$$

which proves (31).

Table 1: Isomorphic design

Q_i ‖ Q_2	Q_{ai} ‖ Q_6	$\varphi(Q_i)$ ‖ $3Q_i$
02	06	06
13	17	39
24	28	62
35	39	95
46	40	28
57	51	51
68	62	84
79	73	17
80	84	40
91	95	73

Table 2: Data table

$\dot{a}$	$\dot{u}_0$	$\dot{u}_1$	$\dot{u}_2$	$\dot{u}_3$	$\dot{u}_4$	$\dot{u}_5$
$\dot{a}_{01}$	1	-1	0	0	0	0
$\dot{a}_{12}$	0	1	-1	0	0	0
$\dot{a}_{23}$	0	0	1	-1	0	0
$\dot{a}_{34}$	0	0	0	1	-1	0
$\dot{a}_{45}$	0	0	0	0	1	-1
$\dot{a}_{05}$	1	0	0	0	0	-1
$\dot{a}_{02}$	1	0	-1	0	0	0
$\dot{a}_{13}$	0	1	0	-1	0	0
$\dot{a}_{24}$	0	0	1	0	-1	0
$\dot{a}_{35}$	0	0	0	1	0	-1
$\dot{a}_{04}$	1	0	0	0	-1	0
$\dot{a}_{15}$	0	1	0	0	0	-1
0	1	1	1	1	1	1

Table 3: Standard error

n	the best 2-cyclic design	the complete case
10	.4743	.3000
15	.5085	.2494
20	.5317	.2179
25	.5489	.1960
30	.5649	.1795
35	.5733	.1666
40	.5828	.1561
45	.5904	.1474
50	.5979	.1400
55	.6050	.1336
60	.6101	.1280

Table 4: Isomorphic relations of 2-cyclic designs for various values of n

(i) prime n

n	$i = 2 \sim \frac{n-1}{2}$ *isomorphic relations*
7	$i = 2 \sim 3$
	$2 \approx 3*$
11	$i = 2 \sim 5$
	$2 \approx 5$
	$3 \approx 4*$
13	$i = 2 \sim 6$
	$2 \approx 6$
	$3 \approx 4$
	$(5)*$
17	$i = 2 \sim 8$
	$2 \approx 8$
	$3 \approx 6$
	$5 \approx 7$
	$(4)*$
19	$i = 2 \sim 9$
	$2 \approx 9$
	$3 \approx 6$
	$4 \approx 5*$
	$7 \approx 8$
23	$i = 2 \sim 11$
	$2 \approx 11$
	$3 \approx 8$
	$4 \approx 6$
	$5 \approx 9*$
	$7 \approx 10$
29	$i = 2 \sim 14$
	$2 \approx 14$
	$3 \approx 10$
	$4 \approx 7$
	$5 \approx 6$
	$8 \approx 11$
	$9 \approx 13$
	$(12)*$
31	$i = 2 \sim 15$
	$2 \approx 15$
	$3 \approx 10$
	$4 \approx 8$
	$5 \approx 6$
	$7 \approx 9$
	$11 \approx 14$
	$12 \approx 13*$
37	$i = 2 \sim 18$
	$2 \approx 18$
	$3 \approx 12$
	$4 \approx 9$
	$5 \approx 15$
	$7 \approx 16$
	$8 \approx 14*$
	$10 \approx 11$
	$13 \approx 17$
	(6)
41	$i = 2 \sim 20$
	$2 \approx 20$
	$3 \approx 14$
	$4 \approx 10$
	$5 \approx 8$
	$6 \approx 7$
	$11 \approx 15$
	$12 \approx 17$
	$13 \approx 19$
	$16 \approx 18$
	$(9)*$
43	$i = 2 \sim 21$
	$2 \approx 21$
	$3 \approx 14$
	$4 \approx 11$
	$5 \approx 17$
	$6 \approx 7$
	$8 \approx 16$
	$9 \approx 19$
	$10 \approx 13$
	$12 \approx 18*$
	$15 \approx 20$
47	$i = 2 \sim 23$
	$2 \approx 23$
	$3 \approx 16$
	$4 \approx 12$
	$5 \approx 19$
	$6 \approx 8$
	$7 \approx 20*$
	$9 \approx 21$
	$10 \approx 14$
	$11 \approx 17$
	$13 \approx 18$
	$15 \approx 22$
53	$i = 2 \sim 26$
	$2 \approx 26$
	$3 \approx 18$
	$4 \approx 13$
	$5 \approx 21$
	$6 \approx 9$
	$7 \approx 15$
	$8 \approx 20*$
	$10 \approx 16$
	$11 \approx 24$
	$12 \approx 22$
	$14 \approx 19$
	$17 \approx 25$
	$21 \approx 23$
59	$i = 2 \sim 29$
	$2 \approx 29$
	$3 \approx 20$
	$4 \approx 15$
	$5 \approx 12$
	$6 \approx 10$
	$7 \approx 17$
	$8 \approx 22$
	$9 \approx 13*$
	$11 \approx 16$
	$14 \approx 21$
	$18 \approx 23$
	$19 \approx 28$
	$24 \approx 27$
	$25 \approx 26$

Table 4: (continued)

(ii) odd n (not prime)

n	$i = 2 \sim \frac{n-1}{2}$ *isomorphic relations*
9	$i = 2 \sim 4$ $2 \approx 4$ $3*$
15	$i = 2 \sim 7$ $2 \approx 7$ $(4)*$
21	$i = 2 \sim 10$ $2 \approx 10$ $4 \approx 5$ $(8)*$
25	$i = 2 \sim 12$ $2 \approx 12$ $3 \approx 8$ $4 \approx 6$ $9 \approx 11$ $(7)*$
27	$i = 2 \sim 13$ $2 \approx 13$ $4 \approx 7$ $5 \approx 11$ $8 \approx 10$ $6*$
33	$i = 2 \sim 16$ $2 \approx 16$ $4 \approx 8$ $5 \approx 13$ $7 \approx 14$ (10) $6*$
35	$i = 2 \sim 17$ $2 \approx 17$ $3 \approx 12$ $4 \approx 9$ $8 \approx 13$ $11 \approx 16$ (6) $10*$
39	$i = 2 \sim 19$ $2 \approx 19$ $4 \approx 10$ $5 \approx 8$ $7 \approx 11*$ (14) $16 \approx 17$
45	$i = 2 \sim 22$ $2 \approx 22$ $4 \approx 11$ $7 \approx 13$ $8 \approx 17$ $14 \approx 16$ $(19)*$
49	$i = 2 \sim 24$ $2 \approx 24$ $3 \approx 16$ $4 \approx 12$ $5 \approx 10$ $6 \approx 8$ $9 \approx 11$ $13 \approx 15$ $17 \approx 23$ $18 \approx 19$ $20 \approx 22$ $14*$
51	$i = 2 \sim 25$ $2 \approx 25$ $4 \approx 13$ $5 \approx 10$ $7 \approx 22$ $8 \approx 19$ $11 \approx 14$ $20 \approx 23$ (16) $9*$
55	$i = 2 \sim 27$ $2 \approx 27$ $3 \approx 18$ $4 \approx 14$ $6 \approx 9$ $7 \approx 8$ $11 \approx 22$ $12 \approx 23$ $13 \approx 17$ $16 \approx 24$ $19 \approx 26$ $(21)*$
57	$i = 2 \sim 28$ $2 \approx 28$ $4 \approx 14$ $5 \approx 23$ $7 \approx 8$ $10 \approx 17$ $11 \approx 26$ $13 \approx 22$ $16 \approx 25$ $24*$

Table 4: (continued)

(iii) even n

n	$i = 2 \sim \frac{n-2}{2}$ *isomorphic relations*
8	$i = 2 \sim 3$ (3)∗
10	$i = 2 \sim 4$ (3) 4∗
12	$i = 2 \sim 5$ (5) 3∗
14	$i = 2 \sim 6$ $3 \approx 5$ 4∗
16	$i = 2 \sim 7$ $3 \approx 5$ (7) 6∗
18	$i = 2 \sim 8$ $5 \approx 7$∗
20	$i = 2 \sim 9$ $3 \approx 7$ (9) 8∗
22	$i = 2 \sim 10$ $3 \approx 7$ $5 \approx 9$ 6∗
24	$i = 2 \sim 11$ $7 \approx 9$ (5)∗ (11)
26	$i = 2 \sim 12$ $3 \approx 9$ $7 \approx 11$ (5) 10∗
28	$i = 2 \sim 13$ $3 \approx 9$ $5 \approx 11$ (13) 6∗
30	$i = 2 \sim 14$ $7 \approx 13$ (11) 5∗
32	$i = 2 \sim 15$ $3 \approx 11$ $5 \approx 13$ $7 \approx 9$∗ (15)
34	$i = 2 \sim 16$ $3 \approx 11$ $5 \approx 7$ $9 \approx 15$ (13) 6∗
36	$i = 2 \sim 17$ $5 \approx 7$ $11 \approx 13$ (17) 15∗
38	$i = 2 \sim 17$ $5 \approx 7$ $11 \approx 13$ (17) 15∗
40	$i = 2 \sim 19$ $3 \approx 13$ $7 \approx 17$ (9) (11)
42	$i = 2 \sim 20$ $5 \approx 17$ $11 \approx 19$ (13) 12∗
44	$i = 2 \sim 21$ $3 \approx 15$ $5 \approx 9$ $7 \approx 19$ $13 \approx 17$ 8∗
46	$i = 2 \sim 22$ $3 \approx 15$ $5 \approx 9$ $7 \approx 13$∗ $11 \approx 21$ $17 \approx 19$
48	$i = 2 \sim 23$ $5 \approx 19$ $11 \approx 13$ (17) 18∗
50	$i = 2 \sim 24$ $3 \approx 17$ $9 \approx 11$ $19 \approx 21$∗ (7)
52	$i = 2 \sim 25$ $3 \approx 17$ $5 \approx 21$ $7 \approx 15$ $9 \approx 23$ $11 \approx 19$ 8∗
54	$i = 2 \sim 26$ $5 \approx 11$ $7 \approx 23$ $17 \approx 19$ 12∗
56	$i = 2 \sim 27$ $3 \approx 19$ $5 \approx 11$ $9 \approx 25$ $17 \approx 23$ (13) (15) 21∗
58	$i = 2 \sim 28$ $3 \approx 19$ $5 \approx 23$ $7 \approx 25$ $9 \approx 13$ $11 \approx 21$ $15 \approx 27$ (17) 22∗
60	$i = 2 \sim 29$ $7 \approx 17$ $13 \approx 23$ (11) (19) 8∗
	—

Table 5: Standard error for 2-cyclic design

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18
complete	0.4000	0.3726	0.3499	0.3307	0.3143	0.3000	0.2875	0.2764	0.2665	0.2575	0.2494	0.2421	0.2353	0.2291
Q1+Q1	0.4472	0.4930	0.5345	0.5728	0.6086	0.6423	0.6742	0.7047	0.7338	0.7618	0.7888	0.8149	0.8402	0.8647
Q1+Q2	*0.4	*0.4249	0.4448	0.4645	0.4828	0.5005	0.5173	0.5336	0.5493	0.5646	0.5794	0.5938	0.6078	0.6215
Q1+Q3			*0.4447	*0.4506	*0.4671	0.4752	*0.4832	*0.4934	0.5022	0.5104	0.5191	0.5273	0.5353	0.5433
Q1+Q4	0.8656	1.0353			0.4828	*0.4743	0.4833	0.4994	0.5022	*0.5029	*0.5085	0.5163	*0.5210	0.5247
Q1+Q5	0.6720	0.7375	0.8535	1.0190			0.5173	0.4965	*0.4961	0.5104	0.5280	0.5273	0.5222	*0.5229
Q1+Q6	0.6185	0.6335	0.6658	0.7287	0.8412	1.0025			0.5493	0.5175	0.5093	*0.5158	0.5353	0.5545
Q1+Q7	0.6120	0.6102	0.6151	0.6289	0.6594	0.7199	0.8287	0.9858			0.5793	0.5376	0.5222	0.5229
Q1+Q8	0.6121	0.6102	0.6102	0.6077	0.6117	0.6243	0.6531	0.7109	0.8160	0.9687			0.6078	0.5569
Q1+Q9	0.6492	0.6335	0.6136	0.6069	0.6076	0.6054	0.6082	0.6196	0.6466	0.7018	0.8031	0.9513		
Q1+Q10	0.6120	0.6233	0.6402	0.6377	0.6173	0.6051	0.6042	0.6032	0.6046	0.6149	0.6401	0.6926	0.7900	0.9336
Q1+Q11	0.6146	0.6096	0.6076	0.6119	0.6266	0.6361	0.6220	0.6052	0.6006	0.6006	0.6012	0.6102	0.6335	0.6054
Q1+Q12	0.6852	0.6445	0.6213	0.6100	*0.6056	*0.6050	0.6133	0.6275	0.6249	0.6072	0.5979	*0.5974	0.5978	0.6054
Q1+Q13	0.8365	0.7997	0.7331	0.6750	0.6367	0.6153	0.6054	0.6016	0.6035	0.6149	0.6228	0.6102	0.5966	*0.5938
Q1+Q14	0.6852	0.7375	0.7944	0.8196	0.7838	0.7199	0.6646	0.6288	0.6093	0.6006	*0.5979	0.6032	0.6144	0.6118
Q1+Q15	0.6145	0.6233	0.6419	0.6750	0.7243	0.7785	0.8025	0.7677	0.7064	0.6540	0.6207	0.6032	*0.5957	0.5950
Q1+Q16	0.6127	0.6114	0.6089	0.6099	0.6173	0.6341	0.6646	0.7109	0.7622	0.7849	0.7512	0.6926	0.6433	0.6126
Q1+Q17	0.6284	0.6153	0.6091	*0.6077	0.6074	0.6054	0.6054	0.6113	0.6261	0.6540	0.6972	0.7456	0.7669	0.7343
Q1+Q18	0.7069	0.6926	0.6658	0.6393	0.6200	0.6086	0.6038	0.6027	0.6015	0.6009	0.6052	0.6180	0.6433	0.6832
Q1+Q19	0.6284	0.6445	0.6658	0.6854	0.6919	0.6782	0.6531	0.6288	0.6115	0.6019	0.5982	0.5970	0.5962	0.5991
Q1+Q20	0.6120	0.6089	0.6088	0.6119	0.6200	0.6341	0.6531	0.6709	0.6766	0.6635	0.6401	0.6180	0.6028	0.5940
Q1+Q21	0.6492	0.6427	0.6312	0.6193	0.6103	0.6051	*0.6034	0.6052	0.6115	0.6234	0.6401	0.6560	0.6609	0.6485
Q1+Q22	0.6121	0.6153	0.6213	0.6289	0.6352	0.6361	0.6300	0.6196	0.6093	0.6019	0.5983	0.5984	0.6028	0.6126
Q1+Q23	0.6235	0.6202	0.6151	0.6099	0.6064	0.6053	0.6069	0.6113	0.6173	0.6223	0.6228	0.6171	0.6078	0.5991
Q1+Q24	0.6104	*0.6089	*0.6088	0.6099	0.6117	0.6129	0.6122	0.6091	0.6046	0.6006	0.5983	0.5984	0.6011	0.6054
Q1+Q25	*0.6101	0.6096	0.6094	0.6090	0.6078	0.6058	0.6035	*0.6016	*0.6005	0.6006	0.6013	0.6017	0.6006	0.5977
Q1+Q26	0.6127	0.6114	0.6102	0.6090	0.6075	0.6058	0.6042	0.6027	0.6015	*0.6006	0.5995	0.5980	0.5960	0.5938
Q1+Q27	0.6221	0.6202	0.6184	0.6165	0.6147	0.6129	0.6110	0.6091	0.6072	0.6053	0.6035	0.6017	0.5997	0.5977
Q1+Q28	0.6452	0.6427	0.6402	0.6377	0.6352	0.6326	0.6300	0.6275	0.6249	0.6223	0.6197	0.6171	0.6144	0.6118
Q1+Q29	0.6961	0.6926	0.6890	0.6854	0.6818	0.6782	0.6745	0.6709	0.6672	0.6635	0.6598	0.6560	0.6522	0.6485
	0.8049	0.7997	0.7944	0.7891	0.7838	0.7785	0.7731	0.7677	0.7622	0.7567	0.7512	0.7456	0.7399	0.7343
	1.0433	1.0353	1.0272	1.0190	1.0108	1.0025	0.9942	0.9858	0.9773	0.9687	0.9600	0.9513	0.9425	0.9336
	1.5809	1.5677	1.5543	1.5409	1.5273	1.5136	1.4997	1.4858	1.4717	1.4575	1.4431	1.4286	1.4139	1.3991
	0.1280	0.1290	0.1302	0.1313	0.1324	0.1336	0.1348	0.1361	0.1373	0.1386	0.1400	0.1414	0.1428	0.1443
	60	59	58	57	56	55	54	53	52	51	50	49	48	47

19	20	21	22	23	24	25	26	27	28	29	30	31	32	
0.2233	0.2179	0.2130	0.2083	0.2039	0.1998	0.1960	0.1923	0.1889	0.1856	0.1825	0.1795	0.1767	0.1740	
0.8885	0.9117	0.9344	0.9564	0.9780	0.9991	1.0198	1.0401	1.0599	1.0794	1.0986	1.1174	1.1359	1.1541	
0.6349	0.6480	0.6609	0.6735	0.6858	0.6979	0.7098	0.7215	0.7330	0.7444	0.7555	0.7665	0.7774	0.7880	
0.5511	0.5587	0.5663	0.5737	0.5811	0.5883	0.5954	0.6025	0.6094	0.6163	0.6231	0.6298	0.6364	0.6430	
*0.5295	0.5350	0.5398	0.5441	0.5487	0.5534	0.5580	0.5624	0.5668	0.5712	0.5756	0.5799	0.5842	0.5884	Q1+Q29
0.5295	0.5371	0.5398	0.5402	*0.5421	*0.5461	0.5506	*0.5537	0.5558	0.5583	0.5615	*0.5649	0.5679	0.5705	Q1+Q28
0.5511	0.5407	0.5364	*0.5398	0.5487	0.5568	0.5580	0.5556	*0.5547	*0.5569	0.5615	0.5659	0.5679	0.5684	Q1+Q27
0.5346	0.5587	0.5796	0.5737	0.5585	0.5494	*0.5489	0.5556	0.5668	0.5757	0.5756	0.5707	0.5669	*0.5668	Q1+Q26
0.5346	*0.5317	*0.5363	0.5526	0.5811	0.6035	0.5954	0.5756	0.5621	0.5579	*0.561	0.5707	0.5842	0.5938	Q1+Q25
0.6349	0.5755	0.5467	0.5402	0.5421	0.5494	0.5700	0.6025	0.6264	0.6163	0.5923	0.5744	0.5669	0.5668	Q1+Q24
		0.6608	0.5935	0.5585	0.5480	0.5495	0.5519	0.5621	0.5868	0.6231	0.6484	0.6364	0.6084	Q1+Q23
0.7767	0.9155			0.6857	0.6110	0.5700	0.5556	0.5558	0.5583	0.5610	0.5744	0.6031	0.6430	Q1+Q22
0.6269	0.6737	0.7631	0.8971			0.7098	0.6278	0.5812	0.5631	0.5608	0.5655	*0.5655	0.5699	Q1+Q21
0.5944	0.6006	0.6201	0.6641	0.7493	0.8782			0.7330	0.6443	0.5923	0.5707	0.5655	0.5705	Q1+Q20
0.5970	*0.5904	0.5908	0.5957	0.6133	0.6543	0.7352	0.8590			0.7555	0.6603	0.6031	0.5780	Q1+Q19
0.6030	0.6091	0.5982	*0.5878	0.5869	0.5909	0.6065	0.6444	0.7209	0.8393			0.7773	0.6760	Q1+Q18
0.5970	0.5909	0.5929	0.6011	0.5984	0.5865	0.5828	0.5860	0.5995	0.6344	0.7062	0.8192			Q1+Q17
0.6785	0.6323	0.6043	0.5906	*0.5864	0.5909	0.5951	0.5860	*0.5790	0.5809	0.5924	0.6241	0.6913	0.7985	Q1+Q16
0.7286	0.7485	0.7170	0.6641	0.6212	0.5959	0.5843	0.5824	0.5874	0.5847	0.5760	0.5757	0.5853	0.6137	Q1+Q15
0.6098	0.6323	0.6689	0.7111	0.7297	0.6992	0.6494	0.6099	0.5874	0.5780	0.5784	0.5806	0.5735	0.5705	Q1+Q14
0.5921	0.5912	0.5929	0.6015	0.6212	0.6543	0.6932	0.7103	0.6810	0.6344	0.5983	0.5787	0.5718	0.5732	Q1+Q13
0.6269	0.6071	0.5941	0.5878	0.5858	0.5865	0.5931	0.6099	0.6394	0.6749	0.6903	0.6623	0.6189	0.5865	Q1+Q12
0.6269	0.6408	0.6448	0.6330	0.6133	0.5959	0.5851	0.5805	0.5800	0.5845	0.5983	0.6241	0.6559	0.6697	Q1+Q11
0.5934	0.5916	0.5941	0.6015	0.6133	0.6252	0.6283	0.6172	0.5995	0.5845	*0.576	*0.5733	0.5758	0.5865	Q1+Q10
0.6092	0.6091	0.6038	0.5957	0.5886	*0.5848	0.5852	0.5902	0.5995	0.6091	0.6113	0.6009	0.5853	0.5729	Q1+Q9
0.5939	0.5909	*0.5896	0.5906	0.5933	0.5957	0.5951	0.5902	0.5833	*0.5780	0.5762	0.5787	0.5853	0.5926	Q1+Q8
*0.5921	0.5912	0.5908	0.5903	0.5888	0.5860	*0.5828	*0.5805	0.5800	0.5809	0.5819	0.5807	0.5761	0.5705	Q1+Q7
0.5957	0.5938	0.5921	0.5903	0.5883	0.5860	0.5837	0.5818	0.5804	0.5789	0.5768	0.5740	*0.5710	*0.5691	Q1+Q6
0.6092	0.6065	0.6038	0.6011	0.5984	0.5957	0.5930	0.5902	0.5874	0.5847	0.5819	0.5791	0.5761	0.5732	Q1+Q5
0.6446	0.6408	0.6369	0.6330	0.6291	0.6252	0.6212	0.6172	0.6132	0.6091	0.6051	0.6009	0.5968	0.5926	Q1+Q4
0.7286	0.7228	0.7170	0.7111	0.7052	0.6992	0.6932	0.6872	0.6810	0.6749	0.6686	0.6623	0.6559	0.6495	Q1+Q3
0.9246	0.9155	0.9064	0.8971	0.8878	0.8783	0.8687	0.8591	0.8493	0.8394	0.8294	0.8192	0.8090	0.7986	Q1+Q2
1.3841	1.3690	1.3537	1.3382	1.3225	1.3066	1.2906	1.2743	1.2579	1.2412	1.2243	1.2071	1.1897	1.1721	Q1+Q1
0.1458	0.1474	0.1490	0.1507	0.1525	0.1543	0.1561	0.1581	0.1601	0.1622	0.1643	0.1666	0.1690	0.1714	complete
46	45	44	43	42	41	40	39	38	37	36	35	34	33	n

Table 6: Standard Error of $Q_i + Q_j$ type

$n = 10$	
$Q_2 + Q_5$	0.48713
$n = 12$	
$Q_2 + Q_3$	*0.48897
$Q_3 + Q_4$	0.50020
$n = 15$	
$Q_3 + Q_5$	0.52814
$n = 18$	
$Q_2 + Q_3$	0.52841
$Q_3 + Q_4$	0.52841
$Q_3 + Q_8$	0.52841
$n = 20$	
$Q_2 + Q_5$	0.53679
$Q_4 + Q_5$	0.53754
$Q_5 + Q_6$	0.53679
$Q_5 + Q_8$	0.53754
$n = 24$	
$Q_2 + Q_3$	0.56424
$Q_2 + Q_9$	0.56424
$Q_3 + Q_4$	*0.54531
$Q_3 + Q_8$	0.60351
$Q_3 + Q_{10}$	0.56424
$Q_4 + Q_9$	0.54531
$Q_9 + Q_{10}$	0.56424
$n = 28$	
$Q_2 + Q_7$	0.57564
$Q_4 + Q_7$	0.57569
$Q_6 + Q_7$	0.57564
$Q_7 + Q_8$	0.57569
$Q_7 + Q_{10}$	0.57564
$Q_4 + Q_{12}$	0.57569
$n = 30$	
$Q_2 + Q_3$	0.59768
$Q_2 + Q_5$	0.56276
$Q_2 + Q_9$	0.56391
$Q_3 + Q_4$	0.56391
$Q_3 + Q_5$	*0.56177
$Q_3 + Q_8$	0.59768
$Q_3 + Q_{10}$	0.64844
$Q_3 + Q_{14}$	0.56391
$Q_4 + Q_5$	0.56276
$Q_4 + Q_9$	0.59768
$Q_5 + Q_6$	0.56608
$Q_5 + Q_8$	0.56276
$Q_5 + Q_9$	0.56177
$Q_5 + Q_{12}$	0.56608
$Q_5 + Q_{14}$	0.56276
$Q_8 + Q_9$	0.56391
$Q_9 + Q_{10}$	0.64844
$Q_9 + Q_{14}$	0.59768
$n = 35$	
$Q_5 + Q_7$	0.58075
$Q_5 + Q_{14}$	0.58075
$Q_7 + Q_{10}$	0.58075
$Q_7 + Q_{15}$	0.58075
$n = 36$	
$Q_2 + Q_3$	0.62925
$Q_2 + Q_9$	0.61132
$Q_2 + Q_{15}$	0.62925
$Q_3 + Q_4$	0.58161
$Q_3 + Q_8$	0.58161
$Q_3 + Q_{10}$	0.62925
$Q_3 + Q_{14}$	0.62925
$Q_3 + Q_{16}$	0.58161
$Q_4 + Q_9$	0.61133
$Q_4 + Q_{15}$	0.58161
$Q_8 + Q_9$	0.61133
$Q_8 + Q_{15}$	0.58161
$Q_9 + Q_{10}$	0.61132
$Q_9 + Q_{14}$	0.61132
$Q_9 + Q_{16}$	0.61133
$n = 40$	
$Q_2 + Q_5$	0.58832
$Q_2 + Q_{15}$	0.58832
$Q_4 + Q_5$	*0.58274
$Q_4 + Q_{15}$	0.58274
$Q_5 + Q_6$	0.58832
$Q_5 + Q_8$	0.59510
$Q_5 + Q_{12}$	0.58274
$Q_5 + Q_{14}$	0.58832
$Q_5 + Q_{16}$	0.59510
$Q_5 + Q_{18}$	0.58832
$Q_8 + Q_{15}$	0.59510
$Q_{14} + Q_{15}$	0.58832
$Q_{15} + Q_{16}$	0.59510
$n = 42$	
$Q_2 + Q_3$	0.65925
$Q_2 + Q_7$	0.58850
$Q_2 + Q_9$	0.59870
$Q_2 + Q_{15}$	0.58707
$Q_3 + Q_4$	0.59870
$Q_3 + Q_7$	0.58835
$Q_3 + Q_8$	0.58707
$Q_3 + Q_{10}$	0.59870
$Q_3 + Q_{14}$	0.72965
$Q_3 + Q_{16}$	0.65925
$Q_3 + Q_{20}$	0.58707
$Q_4 + Q_7$	0.58850
$Q_4 + Q_9$	0.58707
$Q_4 + Q_{15}$	0.65925
$Q_6 + Q_7$	0.58896
$Q_7 + Q_8$	0.58850
$Q_7 + Q_9$	0.58835
$Q_7 + Q_{10}$	0.58850
$Q_7 + Q_{12}$	0.58896
$Q_7 + Q_{15}$	0.58835
$Q_7 + Q_{16}$	0.58850
$Q_7 + Q_{18}$	0.58896
$Q_7 + Q_{20}$	0.58850
$Q_8 + Q_9$	0.65925
$Q_8 + Q_{15}$	0.59870
$Q_9 + Q_{10}$	0.58707
$Q_9 + Q_{14}$	0.72965
$Q_9 + Q_{16}$	0.59870
$Q_9 + Q_{20}$	0.65925
$Q_{14} + Q_{15}$	0.72965
$Q_{15} + Q_{16}$	0.58707
$n = 44$	
$Q_2 + Q_{11}$	0.64483
$Q_4 + Q_{11}$	0.64483
$Q_6 + Q_{11}$	0.64483
$Q_8 + Q_{11}$	0.64483
$Q_{10} + Q_{11}$	0.64483
$Q_{11} + Q_{12}$	0.64483
$Q_{11} + Q_{14}$	0.64483
$Q_{11} + Q_{16}$	0.64483
$Q_{11} + Q_{18}$	0.64483
$Q_{11} + Q_{20}$	0.64483
$n = 45$	
$Q_5 + Q_9$	0.60912
$Q_5 + Q_{18}$	0.60912
$Q_9 + Q_{10}$	0.60912
$Q_9 + Q_{20}$	0.60912
$n = 48$	
$Q_2 + Q_3$	0.68790
$Q_3 + Q_4$	0.61528
$Q_3 + Q_8$	0.60043
$Q_3 + Q_{10}$	0.59606
$Q_3 + Q_{14}$	0.68790
$Q_3 + Q_{16}$	0.76693
$Q_3 + Q_{20}$	0.61528
$Q_3 + Q_{22}$	0.59606
$n = 50$	
$Q_2 + Q_5$	0.61256
$Q_2 + Q_{15}$	0.60019
$n = 51$	
$Q_3 + Q_{17}$	0.78490
$Q_6 + Q_{17}$	0.78490
$Q_9 + Q_{17}$	0.78490
$Q_{12} + Q_{17}$	0.78490
$Q_{15} + Q_{17}$	0.78490
$Q_{17} + Q_{18}$	0.78490
$Q_{17} + Q_{21}$	0.78490
$Q_{17} + Q_{24}$	0.78490
$n = 52$	
$Q_2 + Q_{13}$	0.67658

$Q_2 + Q_{13}$	0.67658
$Q_4 + Q_{13}$	0.67658
$Q_6 + Q_{13}$	0.67658
$Q_8 + Q_{13}$	0.67658
$Q_{10} + Q_{13}$	0.67658
$Q_{12} + Q_{13}$	0.67658
$Q_{13} + Q_{14}$	0.67658
$Q_{13} + Q_{16}$	0.67658
$Q_{13} + Q_{18}$	0.67658
$Q_{13} + Q_{20}$	0.67658
$Q_{13} + Q_{22}$	0.67658
$Q_{13} + Q_{24}$	0.67658
$n = 54$	
$Q_2 + Q_3$	0.71539
$Q_2 + Q_9$	0.61213
$Q_2 + Q_{15}$	0.63141
$Q_2 + Q_{21}$	0.60825
$Q_3 + Q_4$	0.63141
$Q_3 + Q_8$	0.60825
$Q_3 + Q_{10}$	0.60825
$Q_3 + Q_{14}$	0.63141
$Q_3 + Q_{16}$	0.71539
$Q_3 + Q_{20}$	0.71539
$Q_3 + Q_{22}$	0.63141
$Q_3 + Q_{26}$	0.60825
$Q_4 + Q_9$	0.61213
$Q_4 + Q_{15}$	0.60825
$Q_4 + Q_{21}$	0.71539
$Q_8 + Q_9$	0.61213
$Q_8 + Q_{15}$	0.71539
$Q_8 + Q_{21}$	0.63141
$Q_9 + Q_{10}$	0.61213
$Q_9 + Q_{14}$	0.61213
$Q_9 + Q_{16}$	0.61213
$Q_9 + Q_{20}$	0.61213
$Q_9 + Q_{22}$	0.61213
$Q_9 + Q_{26}$	0.61213
$Q_{10} + Q_{21}$	0.63141
$Q_{14} + Q_{15}$	0.60825
$Q_{14} + Q_{21}$	0.71539
$Q_{15} + Q_{16}$	0.63141
$Q_{15} + Q_{22}$	0.60825
$Q_{15} + Q_{26}$	0.71539
$Q_{16} + Q_{21}$	0.60825
$Q_{20} + Q_{21}$	0.60825
$n = 57$	
$Q_3 + Q_{19}$	0.81963
$Q_6 + Q_{19}$	0.81963
$Q_9 + Q_{19}$	0.81963
$Q_{12} + Q_{19}$	0.81963
$Q_{15} + Q_{19}$	0.81963
$Q_{18} + Q_{19}$	0.81963
$Q_{19} + Q_{21}$	0.81963
$Q_{19} + Q_{24}$	0.81963
$Q_{19} + Q_{27}$	0.81963
$n = 60$	
$Q_2 + Q_3$	0.74184
$Q_2 + Q_5$	0.63575
$Q_2 + Q_9$	0.61452
$Q_2 + Q_{15}$	0.70685
$Q_2 + Q_{21}$	0.61452
$Q_2 + Q_{25}$	0.63575
$Q_2 + Q_{27}$	0.74184
$Q_3 + Q_4$	0.64712
$Q_3 + Q_5$	0.62525
$Q_3 + Q_{10}$	0.62350
$Q_3 + Q_{14}$	0.61452
$Q_3 + Q_{20}$	0.83645
$Q_3 + Q_{22}$	0.74184
$Q_3 + Q_{25}$	0.62525
$Q_3 + Q_{26}$	0.61452
$Q_3 + Q_{28}$	0.61160
$Q_4 + Q_5$	0.61695
$Q_4 + Q_9$	0.61160
$Q_4 + Q_{15}$	0.70685
$Q_4 + Q_{21}$	0.61160
$Q_4 + Q_{25}$	0.61695
$Q_4 + Q_{27}$	0.64712
$Q_5 + Q_6$	0.61050
$Q_5 + Q_8$	0.61695
$Q_5 + Q_9$	0.62525
$Q_5 + Q_{12}$	0.64921
$Q_5 + Q_{14}$	0.63575
$Q_5 + Q_{16}$	0.61695
$Q_5 + Q_{18}$	0.61050
$Q_5 + Q_{21}$	0.62525
$Q_5 + Q_{22}$	0.63575
$Q_5 + Q_{24}$	0.64921
$Q_5 + Q_{26}$	0.63575
$Q_5 + Q_{27}$	0.62525
$Q_5 + Q_{28}$	0.61695
$Q_6 + Q_{25}$	0.61050
$Q_8 + Q_9$	0.64712
$Q_8 + Q_{15}$	0.70685
$Q_8 + Q_{21}$	0.64712
$Q_8 + Q_{25}$	0.61695
$Q_8 + Q_{27}$	0.61160
$Q_9 + Q_{10}$	0.62350
$Q_9 + Q_{14}$	0.74184
$Q_9 + Q_{16}$	0.61160
$Q_9 + Q_{20}$	0.62525
$Q_9 + Q_{22}$	0.74184
$Q_9 + Q_{26}$	0.64712
$Q_{10} + Q_{21}$	0.62350
$Q_{10} + Q_{27}$	0.62350
$Q_{12} + Q_{25}$	0.64921
$Q_{14} + Q_{15}$	0.70685
$Q_{14} + Q_{25}$	0.63575
$Q_{14} + Q_{27}$	0.61452
$Q_{15} + Q_{16}$	0.70685
$Q_{15} + Q_{22}$	0.70685
$Q_{15} + Q_{26}$	0.70685
$Q_{15} + Q_{28}$	0.70685
$Q_{16} + Q_{21}$	0.61160
$Q_{16} + Q_{25}$	0.61695
$Q_{16} + Q_{27}$	0.64712
$Q_{18} + Q_{25}$	0.61050
$Q_{20} + Q_{21}$	0.83645
$Q_{20} + Q_{27}$	0.83645
$Q_{21} + Q_{22}$	0.61452
$Q_{21} + Q_{26}$	0.74184
$Q_{21} + Q_{28}$	0.64712
$Q_{22} + Q_{25}$	0.63575
$Q_{22} + Q_{27}$	0.74180
$Q_{24} + Q_{25}$	0.64921
$Q_{25} + Q_{26}$	0.63575
$Q_{25} + Q_{27}$	0.62525
$Q_{25} + Q_{28}$	0.61695
$Q_{26} + Q_{27}$	0.61452
$Q_{26} + Q_{28}$	0.61160

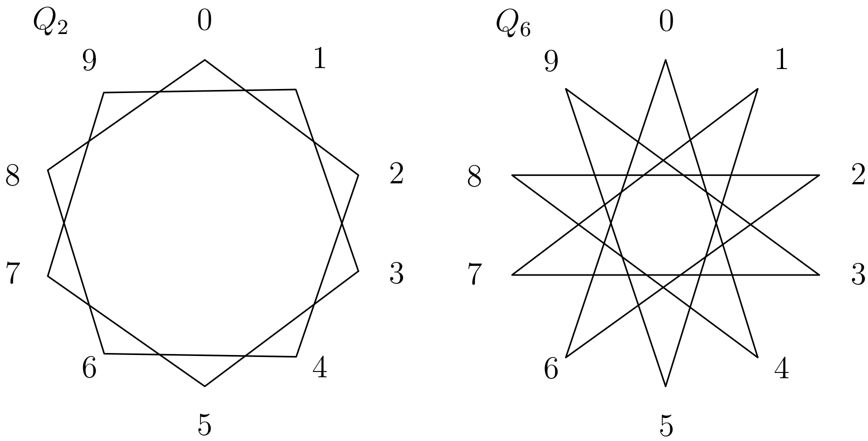

Figure 1: Isomorphic design

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