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A MARKOV CHAIN APPROACH TO OPTIMAL PINCH HITTING STRATEGIES IN A DESIGNATED HITTER RULE BASEBALL GAME

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Abstract A baseball game between teams consisting of non-identical players is modeled using a Markov chain, taking into account the number of runs by which the home team leads. Using the Markov model the probability of winning in any state in the course of a game is calculated directly by solving a set of over one million simultaneous equations. This approach makes it possible to obtain the optimal pinch hitting strategy under the 'Designated Hitter' rule by applying dynamic programming to this model. We demonstrate this method using a match based on the line-ups of the Anaheim Angels and the Oakland Athletics in the American League of Major League Baseball. We show how this approach may help to determine when to use a pinch hitter and how much the probability of winning increases if the optimal strategy is followed.

Keywords: Markov process, decision making, dynamic programming, baseball, sports

1. Introduction

Baseball has been quantitatively analyzed using various methods by a number of researchers in order to optimize strategic moves and batting orders. The strategic moves, 'sacrifice', 'stolen base' and 'intentional walk' have been well analyzed. Lindsey [6] computed the empirical probability distributions of the number of runs to be scored in the remainder of a half-inning for all combinations of number of outs and the occupation of the bases. Lindsey also evaluated the conditions under which these strategic moves result in a greater probability of scoring the runs needed to overcome the lead in that half-inning, as an aid to managerial decision-making. Truman [9] analyzed these strategic moves using the Monte Carlo method and later [10] developed his analysis by defining a half-inning as a 25-state Markov chain and thus obtained the breakeven success probabilities of these strategic moves.

Other researchers have focused on optimal batting orders. Freeze [4] simulated over 200,000 baseball games using different batting orders. Bukiet *et al.* [2] also analyzed the optimal batting orders by developing algorithms to find the optimal out of 9! possible orders. They devised a unique method to calculate the distribution of runs scored in a whole game using a 25-state Markov chain, and evaluated the batting orders of real teams by calculating the expected number of wins out of 162 games in a season.

Though there have been many analyses of baseball such as those mentioned above, there has been very little application to the substitution of players in baseball. This paper proposes the application of a Markov chain model to optimize the 'pinch hitting' strategy.

A 'pinch hitter' is a substitute introduced during the course of a game, sometimes as a bit of a gamble when a team is behind in the match. In a real baseball game, the substitution of a pinch hitter may occur for a variety of reasons. For example, a manager may substitute a pinch hitter for a pitcher about to come up to bat if there is a good chance of scoring.

Alternatively the manager may want to substitute a left-handed hitter for a right-handed hitter when the opposing pitcher is right-handed or vice versa. Another reason the manager may wish to bring on a pinch hitter could be to force the opposing team decision to substitute its pitcher. A manager may wish to give a new player a chance to bat in a real game. The reason why a pinch hitter is not used may be the defensive ability of a player.

For simplicity, in this paper, we do not consider the substitution of a pinch hitter for a pitcher. In a real game, this assumption corresponds to playing under the 'Designated Hitter (DH)' rule, which admits a skilled batter in the batting line-up instead of a pitcher. Under this condition, we solve the optimization problem of pinch hitting in terms of maximizing the probability of winning a game. In other words, we solve the pinch hitting problem as a pure combinatorial problem focusing on the offensive aspect of baseball, under the assumption that the probabilities of a batter to result in a single, double, triple, home run, walk or out are known. The answer will provide the decision in a particular situation such as whether the manager should, or should not, substitute a pinch hitter who has a high probability of hitting a home run, but also a high probability of being out, instead of a batter who has a low probability of a home run but also has a low probability of being out.

This is clearly a very simple model representing a complex reality. However, this provides the basic structure of a formulation to handle the pinch hitting problem as a first step on the way to develop a more complicated formulation, such as including the defensive ability of pitchers and taking into account the opposing team's substitution.

In this paper a baseball game is modeled as a Markov chain. We start off with 433 states for modeling a whole baseball game with identical players. Then, by means of introducing new states relating to the existence of non-identical players of both teams and the number of runs by which the home team leads, we finally formulate a whole game as a 1,434,673-state Markov chain. Using the Markov model the probability of winning in any state in the course of a game is calculated directly by solving a set of over one million simultaneous equations. This approach makes it possible to obtain the optimal pinch hitting strategy under the DH rule by applying dynamic programming to this model. We demonstrate this method using a match based on the line-ups of the Anaheim Angels and the Oakland Athletics in the American League of Major League Baseball. We show how this approach may help to determine when to use a pinch hitter and how much the probability of winning increases if the optimal strategy is followed.

2. The Markov Chain Approach

We first define the states of baseball from the view of a whole game with identical players. This makes it easier to formulate the calculation of the probability of winning in a whole nine innings. Here, we define 433 states which derive from the inning (9×2 possibilities), the number of outs (3 possibilities), the pattern of bases occupied (8 possibilities) and the end of the game ($9 \times 2 \times 3 \times 8 + 1$).

We define the transition matrix $\mathbf{P^{(H)}}$ for the batting of a home team player as follows.

$$
\mathbf{P^{(H)}} = \begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & & 2 & & \cdots & \cdots & \cdots & 9 & & End \\ T & B & T & B & \cdots & \cdots & \cdots & T & B & \end{array} \\
\begin{array}{cc} 1 & T \\ & B \\ 2 & T \\ & B \\ \vdots & \\ & \vdots \\ \vdots & \\ 9 & T \\ & B \\ End & \end{array} &
\left(\begin{array}{cccccccccc}
\mathbf{0} & \mathbf{0} & & & & & & & & \\
 & \mathbf{Q} & \mathbf{Q_0} & & & & & & & \\
 & & \mathbf{0} & \mathbf{0} & & & & & & \\
 & & & \mathbf{Q} & \mathbf{Q_0} & & & & & \\
 & & & & \ddots & \ddots & & & & \\
 & & & & & \ddots & \ddots & & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & \mathbf{0} & \mathbf{0} & \\
 & & & & & & & & \mathbf{Q} & \mathbf{F} \\
 & & & & & & & & \mathbf{0} & 1
\end{array}\right)
\end{array} \tag{1}
$$

In this expression, the numbers outside the matrix represent the innings. T represents the top of the inning and B represents the bottom of the inning. 'End' represents the end of the game as the absorbing state. The $\mathbf{Q}$ and $\mathbf{Q_0}$ blocks are 24×24 matrices, which represent any transitions inside a half-inning and any transitions to the next half-inning, respectively. The $\mathbf{F}$ block is a 24×1 vector, which represents any transitions from the bottom of the 9th inning to the end of the game. Because a home team starts batting from the bottom of the 1st inning, all blocks in the first 24 rows consist of zeroes. Non-zero entries appear from the second 24 rows represented as the $\mathbf{Q}$ block in the second 24 columns and as the $\mathbf{Q_0}$ block in the third 24 columns. The game ends following the transition from the bottom of the 9th inning to the end of the game through the entries of the $\mathbf{F}$ block. The entries of the matrix can be obtained from the individual player's batting statistics.

Similarly we define the transition matrix $\mathbf{P^{(V)}}$ for the batting of a visiting team player as follows:

$$
\mathbf{P^{(V)}} = \begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & & 2 & & \cdots & \cdots & \cdots & 9 & & End \\ T & B & T & B & \cdots & \cdots & \cdots & T & B & \end{array} \\
\begin{array}{cc} 1 & T \\ & B \\ 2 & T \\ & B \\ \vdots & \\ & \vdots \\ \vdots & \\ 9 & T \\ & B \\ End & \end{array} &
\left(\begin{array}{cccccccccc}
\mathbf{Q} & \mathbf{Q_0} & & & & & & & & \\
 & \mathbf{0} & \mathbf{0} & & & & & & & \\
 & & \mathbf{Q} & \mathbf{Q_0} & & & & & & \\
 & & & \mathbf{0} & \mathbf{0} & & & & & \\
 & & & & \ddots & \ddots & & & & \\
 & & & & & \ddots & \ddots & & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & \mathbf{Q} & \mathbf{Q_0} & \\
 & & & & & & & & \mathbf{0} & \mathbf{0} \\
 & & & & & & & & \mathbf{0} & 1
\end{array}\right)
\end{array} \tag{2}
$$

Because a visiting team starts batting at the top of the 1st inning, the $\mathbf{Q}$ block at the top-left corner of the matrix has non-zero entries. When the visiting team has 3 outs, the state transits from the top of the 1st inning to the beginning of the bottom of the 1st inning through the entries of the $\mathbf{Q_0}$ block in the first 24 rows and the second 24 columns, and the home team starts batting.

By setting up transition probabilities from any state to any other state inside a half-inning as entries in $\mathbf{Q}$ and from a half-inning to the next half-inning as entries in $\mathbf{Q_0}$, we can develop a number of complicated models including any possible transition in a game. However, for simplicity, we follow the D'Esopo and Lefkowitz model for runner advancement [3] as also used by Bukiet *et al.* [2]. The use of this model means that we only take into account six possibilities as shown in Table 1. Of course there are in practice other possibilities, but this surprisingly simple model predicts the number of runs scored for their test cases to within less than seven percent.

Table 1: The D'Esopo and Lefkowitz model

Play	Results
Single	Batter to first base. Runner on first reaches second. All other runners score.
Double	Batter to second base. Runner on first to third base. All other runners score.
Triple	Batter to third base. All base runners score.
Home Run	Batter scores. All base runners score.
Walk	Batter to first base. All base runners advance one base only if forced to do so.
Out	Base runners do not advance.

Following the D'Esopo and Lefkowitz model, the 24×24 matrices $\mathbf{Q}$ and $\mathbf{Q_0}$ are given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{A} \end{pmatrix} \qquad \mathbf{Q_0} = \begin{pmatrix} \mathbf{F} & \mathbf{0} & \ldots & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{F_0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{3}$$

where $\mathbf{A}$ and $\mathbf{B}$ are blocks with

$$\mathbf{A} = \begin{pmatrix} P_H & P_S+P_W & P_D & P_T & 0 & 0 & 0 & 0 \\ P_H & 0 & 0 & P_T & P_S+P_W & 0 & P_D & 0 \\ P_H & P_S & P_D & P_T & P_W & 0 & 0 & 0 \\ P_H & P_S & P_D & P_T & 0 & P_W & 0 & 0 \\ P_H & 0 & 0 & P_T & P_S & 0 & P_D & P_W \\ P_H & 0 & 0 & P_T & P_S & 0 & P_D & P_W \\ P_H & P_S & P_D & P_T & 0 & 0 & 0 & P_W \\ P_H & 0 & 0 & P_T & P_S & 0 & P_D & P_W \end{pmatrix} \tag{4}$$

and

$$\mathbf{B} = P_{out}\mathbf{I}. \tag{5}$$

P_S, P_D, P_T, P_H, P_W and P_{out} are the probabilities of a player getting a single, double, triple, home run, walk or out, respectively. $\mathbf{I}$ is the 8×8 identity matrix. Off-diagonal elements of the $\mathbf{B}$ blocks are zeroes since this model does not allow runners to advance on an out. In expression (3),

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 & P_{out} & \dots & P_{out} \end{pmatrix}^T \tag{6}$$

is a 24×1 vector with the 8 series of P_{out} entries and

$$\mathbf{F_0} = \begin{pmatrix} P_{out} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ P_{out} & 0 & \dots & 0 \end{pmatrix} \tag{7}$$

is an 8×8 matrix. These P_{out} inside the $\mathbf{F}$ and $\mathbf{F_0}$ lead to the transition from a state of 2 outs in a half-inning to the beginning of the next half-inning, or lead to the end of the game.

3. Calculation of the Probability of Winning

We now extend the model by identifying the players and describe how to obtain the probability of winning from any state. First we simplify the model by deleting the 433rd row and the 433rd column of $\mathbf{P^{(H)}}$. Since all batters are now different, we define the 432×432 matrix $\mathbf{P_n^{(H)}}$, which represents transitions for the batting of player x_n of home team X. That is, if the current state is in the bottom of an inning and player x_n is batting, the current state will transit to the next state following the transition matrix $\mathbf{P_n^{(H)}}$.

$$\mathbf{P_n^{(H)}} = \begin{array}{cc} & \begin{array}{ccccccccc} 1\,T & 1\,B & 2\,T & 2\,B & \dots & \dots & \dots & 9\,T & 9\,B \end{array} \\ \begin{array}{c} 1\,T \\ 1\,B \\ 2\,T \\ 2\,B \\ \vdots \\ \\ \vdots \\ 9\,T \\ 9\,B \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & & & & & & & \\ & \mathbf{Q_n} & \mathbf{Q_{0n}} & & & & & & \\ & & \mathbf{0} & \mathbf{0} & & & & & \\ & & & \mathbf{Q_n} & \mathbf{Q_{0n}} & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \mathbf{Q_n} & \mathbf{Q_{0n}} & \\ & & & & & & & \mathbf{0} & \mathbf{0} \\ & & & & & & & & \mathbf{Q_n} \end{pmatrix} \end{array} \tag{8}$$

Each $\mathbf{P_n^{(H)}}$ is decomposed into five matrices, which correspond to the portions leading to zero runs scored, one run, two runs, three runs and four runs, following the results of the batting of player x_n. We define these portions $\mathbf{P0_n^{(H)}}$, $\mathbf{P1_n^{(H)}}$, $\mathbf{P2_n^{(H)}}$, $\mathbf{P3_n^{(H)}}$ and $\mathbf{P4_n^{(H)}}$, respectively. Thus the batting of x_n leads from any states where the team X leads by i runs to the next state where it leads by i runs, $i+1$ runs, $i+2$ runs, $i+3$ runs or $i+4$ runs following $\mathbf{P0_n^{(H)}}$, $\mathbf{P1_n^{(H)}}$, $\mathbf{P2_n^{(H)}}$, $\mathbf{P3_n^{(H)}}$ and $\mathbf{P4_n^{(H)}}$, respectively. Here, the relation

$$\mathbf{P_n^{(H)}} = \mathbf{P0_n^{(H)}} + \mathbf{P1_n^{(H)}} + \mathbf{P2_n^{(H)}} + \mathbf{P3_n^{(H)}} + \mathbf{P4_n^{(H)}} \tag{9}$$

holds.

Similarly, if the current state is in the top of an inning and player y_m of visiting team Y is batting, the current state will transit to the next state with the 432×432 transition matrix $\mathbf{P_m^{(V)}}$. This is also decomposed into five matrices, which correspond to zero runs scored, one run, two runs, three runs and four runs. We define these portions $\mathbf{P0_m^{(V)}}$, $\mathbf{P1_m^{(V)}}$, $\mathbf{P2_m^{(V)}}$, $\mathbf{P3_m^{(V)}}$ and $\mathbf{P4_m^{(V)}}$, respectively. Thus the batting of the player y_m leads from any states where team X leads by i runs to the next state where it leads by i runs, $i-1$ runs, $i-2$ runs, $i-3$ runs or $i-4$ runs following $\mathbf{P0_m^{(V)}}$, $\mathbf{P1_m^{(V)}}$, $\mathbf{P2_m^{(V)}}$, $\mathbf{P3_m^{(V)}}$ and $\mathbf{P4_m^{(V)}}$, respectively. Here, the relation

$$\mathbf{P_m^{(V)}} = \mathbf{P0_m^{(V)}} + \mathbf{P1_m^{(V)}} + \mathbf{P2_m^{(V)}} + \mathbf{P3_m^{(V)}} + \mathbf{P4_m^{(V)}} \tag{10}$$

also holds.

Let $\mathbf{\Omega_{nm}}(i) = \mathbf{\Omega_{nm}}(i)(x_1, x_2, \cdots, x_9, y_1, y_2, \cdots, y_9)$ be the 432×1 vector representing the probabilities of home team X winning in the remainder of the game from a current position where home team X leads by i runs, the nth batter x_n of team X is coming up in the bottom of the inning and mth batter y_m of visiting team Y is coming up in the top of the inning. Since the batting of home team X and that of visiting team Y are mutually exclusive, we get the equation for $\mathbf{\Omega_{nm}}(i)$ as follows:

$$\begin{aligned}
&\mathbf{\Omega_{nm}}(i) \\
&= \mathbf{P0_m^{(V)}}\mathbf{\Omega_{nm+1}}(i) + \mathbf{P1_m^{(V)}}\mathbf{\Omega_{nm+1}}(i-1) + \mathbf{P2_m^{(V)}}\mathbf{\Omega_{nm1+1}}(i-2) + \mathbf{P3_m^{(V)}}\mathbf{\Omega_{nm+1}}(i-3) + \mathbf{P4_m^{(V)}}\mathbf{\Omega_{nm+1}}(i-4) \\
&\quad + \mathbf{P0_n^{(H)}}\mathbf{\Omega_{n+1m}}(i) + \mathbf{P1_n^{(H)}}\mathbf{\Omega_{n+1m}}(i+1) + \mathbf{P2_n^{(H)}}\mathbf{\Omega_{n+1m}}(i+2) + \mathbf{P3_n^{(H)}}\mathbf{\Omega_{n+1m}}(i+3) + \mathbf{P4_n^{(H)}}\mathbf{\Omega_{n+1m}}(i+4) \\
&\quad + U(i)\mathbf{P_{out\,n}^{(H)}}
\end{aligned} \tag{11}$$

where $\mathbf{P_{out\,n}^{(H)}}$ is a 432×1 vector which leads to the end of the game from the batting of player x_n in the bottom of the 9th inning, represented in the following expression.

$$\mathbf{P_{out\,n}^{(H)}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{F_n} \end{pmatrix}^T. \tag{12}$$

Here, the $\mathbf{F_n}$ has the same structure as $\mathbf{F}$ in expression (6) such that

$$\mathbf{F_n} = \begin{pmatrix} 0 & 0 & \ldots & \ldots & 0 & P_{out\,n} & \ldots & P_{out\,n} \end{pmatrix}^T. \tag{13}$$

The $U(i)$ in expression (11) is a function defined as follows:

$$U(i) = \begin{cases} 1 & : i > 0 \\ Probability\,of\,home\,team\,X\,winning\,in\,extra\,innings & : i = 0. \\ 0 & : i < 0 \end{cases} \tag{14}$$

This function is the boundary condition at the end of the game. This probability is 1 if $i > 0$ (i.e. home team X wins), is the probability of team X winning in an extra inning if $i = 0$ and is 0 if $i < 0$ (i.e. home team X loses).

We simplify expression (11) by arranging it into a matrix representation. At first, we put the 432×1 vector $\mathbf{\Omega_{nm}}(i)$ into a $17,712(432 \times 41) \times 1$ vector $\mathbf{\Omega_{nm}}$ following the order of the number of runs by which home team X leads as follows:

$$\boldsymbol{\Omega}_{\mathbf{nm}} = \left(\boldsymbol{\Omega}_{\mathbf{nm}}(20) \;\; \boldsymbol{\Omega}_{\mathbf{nm}}(19) \;\; \ldots \;\; \boldsymbol{\Omega}_{\mathbf{nm}}(1) \;\; \boldsymbol{\Omega}_{\mathbf{nm}}(0) \;\; \boldsymbol{\Omega}_{\mathbf{nm}}(-1) \;\; \ldots \;\; \boldsymbol{\Omega}_{\mathbf{nm}}(-19) \;\; \boldsymbol{\Omega}_{\mathbf{nm}}(-20) \right)^T, \tag{15}$$

assuming that the number of runs by which either team may lead will never exceed 20.

Using expression (15), expression (11) is represented as the form of a larger matrix as follows:

$$\begin{pmatrix} \boldsymbol{\Omega}_{\mathbf{nm}}(20) \\ \boldsymbol{\Omega}_{\mathbf{nm}}(19) \\ \boldsymbol{\Omega}_{\mathbf{nm}}(18) \\ \boldsymbol{\Omega}_{\mathbf{nm}}(17) \\ \boldsymbol{\Omega}_{\mathbf{nm}}(16) \\ \vdots \\ \vdots \\ \boldsymbol{\Omega}_{\mathbf{nm}}(0) \\ \vdots \\ \vdots \\ \boldsymbol{\Omega}_{\mathbf{nm}}(-16) \\ \boldsymbol{\Omega}_{\mathbf{nm}}(-17) \\ \boldsymbol{\Omega}_{\mathbf{nm}}(-18) \\ \boldsymbol{\Omega}_{\mathbf{nm}}(-19) \\ \boldsymbol{\Omega}_{\mathbf{nm}}(-20) \end{pmatrix} = \begin{pmatrix}
\mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})}\mathbf{P2}_\mathbf{m}^{(\mathbf{V})}\mathbf{P3}_\mathbf{m}^{(\mathbf{V})}\mathbf{P4}_\mathbf{m}^{(\mathbf{V})} & & & & & & & & \\
& \mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})}\mathbf{P2}_\mathbf{m}^{(\mathbf{V})}\mathbf{P3}_\mathbf{m}^{(\mathbf{V})}\mathbf{P4}_\mathbf{m}^{(\mathbf{V})} & & & & & & & \\
& & \mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})}\mathbf{P2}_\mathbf{m}^{(\mathbf{V})}\mathbf{P3}_\mathbf{m}^{(\mathbf{V})}\mathbf{P4}_\mathbf{m}^{(\mathbf{V})} & & & & & & \\
& & & \mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})}\mathbf{P2}_\mathbf{m}^{(\mathbf{V})}\mathbf{P3}_\mathbf{m}^{(\mathbf{V})}\mathbf{P4}_\mathbf{m}^{(\mathbf{V})} & & & & & \\
& & & & \mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})}\mathbf{P2}_\mathbf{m}^{(\mathbf{V})}\mathbf{P3}_\mathbf{m}^{(\mathbf{V})}\mathbf{P4}_\mathbf{m}^{(\mathbf{V})} & & & & \\
& & & & \ddots\;\ddots\;\ddots\;\ddots\;\ddots & & & & \\
& & & & & \ddots\;\ddots\;\ddots\;\ddots\;\ddots & & & \\
& & & & & \mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})}\mathbf{P2}_\mathbf{m}^{(\mathbf{V})}\mathbf{P3}_\mathbf{m}^{(\mathbf{V})}\mathbf{P4}_\mathbf{m}^{(\mathbf{V})} & & & \\
& & & & & & \ddots\;\ddots\;\ddots\;\ddots\;\ddots & & \\
& & & & & & & \ddots\;\ddots\;\ddots\;\ddots\;\ddots & \\
& & & & & & & \mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})}\mathbf{P2}_\mathbf{m}^{(\mathbf{V})}\mathbf{P3}_\mathbf{m}^{(\mathbf{V})} & \mathbf{P4}_\mathbf{m}^{(\mathbf{V})} \\
& & & & & & & \mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})}\mathbf{P2}_\mathbf{m}^{(\mathbf{V})} & \sum_{l=3}^4 \mathbf{P}l_\mathbf{m}^{(\mathbf{V})} \\
& & & & & & & \mathbf{P0}_\mathbf{m}^{(\mathbf{V})}\mathbf{P1}_\mathbf{m}^{(\mathbf{V})} & \sum_{l=2}^4 \mathbf{P}l_\mathbf{m}^{(\mathbf{V})} \\
& & & & & & & \mathbf{P0}_\mathbf{m}^{(\mathbf{V})} & \sum_{l=1}^4 \mathbf{P}l_\mathbf{m}^{(\mathbf{V})} \\
& & & & & & & & \sum_{l=0}^4 \mathbf{P}l_\mathbf{m}^{(\mathbf{V})}
\end{pmatrix} \begin{pmatrix} \boldsymbol{\Omega}_{\mathbf{nm+1}}(20) \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(19) \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(18) \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(17) \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(16) \\ \vdots \\ \vdots \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(0) \\ \vdots \\ \vdots \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(-16) \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(-17) \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(-18) \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(-19) \\ \boldsymbol{\Omega}_{\mathbf{nm+1}}(-20) \end{pmatrix}$$

$$+ \begin{pmatrix}
\sum_{l=0}^4 \mathbf{P}l_\mathbf{n}^{(\mathbf{H})} & & & & & & & & \\
\sum_{l=1}^4 \mathbf{P}l_\mathbf{n}^{(\mathbf{H})} \mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & & & & & & & & \\
\sum_{l=2}^4 \mathbf{P}l_\mathbf{n}^{(\mathbf{H})} \mathbf{P1}_\mathbf{n}^{(\mathbf{H})} \mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & & & & & & & & \\
\sum_{l=3}^4 \mathbf{P}l_\mathbf{n}^{(\mathbf{H})} \mathbf{P2}_\mathbf{n}^{(\mathbf{H})} \mathbf{P1}_\mathbf{n}^{(\mathbf{H})} \mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & & & & & & & & \\
\mathbf{P4}_\mathbf{n}^{(\mathbf{H})} & \mathbf{P3}_\mathbf{n}^{(\mathbf{H})}\mathbf{P2}_\mathbf{n}^{(\mathbf{H})}\mathbf{P1}_\mathbf{n}^{(\mathbf{H})}\mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & & & & & & & \\
& \ddots\;\ddots\;\ddots\;\ddots\;\ddots & & & & & & & \\
& & \ddots\;\ddots\;\ddots\;\ddots\;\ddots & & & & & & \\
& & \mathbf{P4}_\mathbf{n}^{(\mathbf{H})}\mathbf{P3}_\mathbf{n}^{(\mathbf{H})}\mathbf{P2}_\mathbf{n}^{(\mathbf{H})}\mathbf{P1}_\mathbf{n}^{(\mathbf{H})}\mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & & & & & & \\
& & & \ddots\;\ddots\;\ddots\;\ddots\;\ddots & & & & & \\
& & & & \ddots\;\ddots\;\ddots\;\ddots\;\ddots & & & & \\
& & & & \mathbf{P4}_\mathbf{n}^{(\mathbf{H})}\mathbf{P3}_\mathbf{n}^{(\mathbf{H})}\mathbf{P2}_\mathbf{n}^{(\mathbf{H})}\mathbf{P1}_\mathbf{n}^{(\mathbf{H})}\mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & & & & \\
& & & & & \mathbf{P4}_\mathbf{n}^{(\mathbf{H})}\mathbf{P3}_\mathbf{n}^{(\mathbf{H})}\mathbf{P2}_\mathbf{n}^{(\mathbf{H})}\mathbf{P1}_\mathbf{n}^{(\mathbf{H})}\mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & & & \\
& & & & & & \mathbf{P4}_\mathbf{n}^{(\mathbf{H})}\mathbf{P3}_\mathbf{n}^{(\mathbf{H})}\mathbf{P2}_\mathbf{n}^{(\mathbf{H})}\mathbf{P1}_\mathbf{n}^{(\mathbf{H})}\mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & & \\
& & & & & & & \mathbf{P4}_\mathbf{n}^{(\mathbf{H})}\mathbf{P3}_\mathbf{n}^{(\mathbf{H})}\mathbf{P2}_\mathbf{n}^{(\mathbf{H})}\mathbf{P1}_\mathbf{n}^{(\mathbf{H})}\mathbf{P0}_\mathbf{n}^{(\mathbf{H})} & \\
& & & & & & & & \mathbf{P4}_\mathbf{n}^{(\mathbf{H})}\mathbf{P3}_\mathbf{n}^{(\mathbf{H})}\mathbf{P2}_\mathbf{n}^{(\mathbf{H})}\mathbf{P1}_\mathbf{n}^{(\mathbf{H})}\mathbf{P0}_\mathbf{n}^{(\mathbf{H})}
\end{pmatrix} \begin{pmatrix} \boldsymbol{\Omega}_{\mathbf{n+1m}}(20) \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(19) \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(18) \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(17) \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(16) \\ \vdots \\ \vdots \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(0) \\ \vdots \\ \vdots \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(-16) \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(-17) \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(-18) \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(-19) \\ \boldsymbol{\Omega}_{\mathbf{n+1m}}(-20) \end{pmatrix} + \begin{pmatrix} U(20)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ U(19)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ U(18)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ U(17)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ U(16)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ \vdots \\ \vdots \\ U(0)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ \vdots \\ \vdots \\ U(-16)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ U(-17)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ U(-18)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ U(-19)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \\ U(-20)\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})} \end{pmatrix} \tag{16}$$

where the summations in the above matrices are needed because of the truncation at leads of 20 runs and −20 runs.

We now introduce the case of extra innings into this formulation. Since an extra inning is considered as a repetition of the 9th inning, the probability of winning the game at the beginning of the extra inning is therefore equal to that at the beginning of the 9th inning with a tied score ($i = 0$), and so we add the extra 432×432 matrix $\mathbf{P0}_{\mathbf{n_{extra}}}^{(\mathbf{H})}$ in the row representing 0 runs by which team X leads in order to return the state from the bottom of the 9th inning to the top of the 9th inning when entering extra innings. Moreover, by putting the value 1 or 0 of the function $U(i)$ following expression (14), we obtain the following expression.

$$
\begin{pmatrix}
\boldsymbol{\Omega}_{\mathbf{nm}}(20)\\
\boldsymbol{\Omega}_{\mathbf{nm}}(19)\\
\boldsymbol{\Omega}_{\mathbf{nm}}(18)\\
\boldsymbol{\Omega}_{\mathbf{nm}}(17)\\
\boldsymbol{\Omega}_{\mathbf{nm}}(16)\\
\vdots\\
\vdots\\
\boldsymbol{\Omega}_{\mathbf{nm}}(0)\\
\vdots\\
\vdots\\
\boldsymbol{\Omega}_{\mathbf{nm}}(-16)\\
\boldsymbol{\Omega}_{\mathbf{nm}}(-17)\\
\boldsymbol{\Omega}_{\mathbf{nm}}(-18)\\
\boldsymbol{\Omega}_{\mathbf{nm}}(-19)\\
\boldsymbol{\Omega}_{\mathbf{nm}}(-20)
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P2}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P3}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P4}_{\mathbf{m}}^{(\mathbf{V})} & & & & \\
\quad\mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P2}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P3}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P4}_{\mathbf{m}}^{(\mathbf{V})} & & & & \\
\qquad\mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P2}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P3}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P4}_{\mathbf{m}}^{(\mathbf{V})} & & & & \\
\qquad\quad\mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P2}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P3}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P4}_{\mathbf{m}}^{(\mathbf{V})} & & & & \\
\qquad\qquad\mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P2}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P3}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P4}_{\mathbf{m}}^{(\mathbf{V})} & & & & \\
 & \ddots\ \ddots\ \ddots\ \ddots\ \ddots & & & \\
 & \ddots\ \ddots\ \ddots\ \ddots\ \ddots & & & \\
 & & \mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P2}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P3}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P4}_{\mathbf{m}}^{(\mathbf{V})} & & \\
 & & & \ddots\ \ddots\ \ddots\ \ddots\ \ddots & \\
 & & & \ddots\ \ddots\ \ddots\ \ddots\ \ddots & \\
 & & & & \mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P2}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P3}_{\mathbf{m}}^{(\mathbf{V})}\quad\mathbf{P4}_{\mathbf{m}}^{(\mathbf{V})}\\
 & & & & \mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P2}_{\mathbf{m}}^{(\mathbf{V})}\sum_{l=3}^{4}\mathbf{P}l_{\mathbf{m}}^{(\mathbf{V})}\\
 & & & & \mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\mathbf{P1}_{\mathbf{m}}^{(\mathbf{V})}\sum_{l=2}^{4}\mathbf{P}l_{\mathbf{m}}^{(\mathbf{V})}\\
 & & & & \mathbf{P0}_{\mathbf{m}}^{(\mathbf{V})}\sum_{l=1}^{4}\mathbf{P}l_{\mathbf{m}}^{(\mathbf{V})}\\
 & & & & \sum_{l=0}^{4}\mathbf{P}l_{\mathbf{m}}^{(\mathbf{V})}
\end{pmatrix}
\begin{pmatrix}
\boldsymbol{\Omega}_{\mathbf{nm+1}}(20)\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(19)\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(18)\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(17)\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(16)\\
\vdots\\
\vdots\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(0)\\
\vdots\\
\vdots\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(-16)\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(-17)\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(-18)\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(-19)\\
\boldsymbol{\Omega}_{\mathbf{nm+1}}(-20)
\end{pmatrix}
$$

$$
+\begin{pmatrix}
\sum_{l=0}^{4}\mathbf{P}l_{\mathbf{n}}^{(\mathbf{H})} & & & & \\
\sum_{l=1}^{4}\mathbf{P}l_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & & & & \\
\sum_{l=2}^{4}\mathbf{P}l_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & & & & \\
\sum_{l=3}^{4}\mathbf{P}l_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P2}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & & & & \\
\mathbf{P4}_{\mathbf{n}}^{(\mathbf{H})}\quad\mathbf{P3}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P2}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & & & & \\
\ddots\ \ddots\ \ddots\ \ddots\ \ddots & & & & \\
 & \ddots\ \ddots\ \ddots\ \ddots\ \mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & & & \\
 & \mathbf{P4}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P3}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P2}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})}+\mathbf{P0}_{\mathbf{n}_{\mathbf{extra}}}^{(\mathbf{H})} & & & \\
 & & \ddots\ \ddots\ \ddots\ \ddots\ \mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & & \\
 & & \ddots\ \ddots\ \ddots\ \ddots\ \ddots & & \\
 & & \mathbf{P4}_{\mathbf{n}}^{(\mathbf{H})}\quad\mathbf{P3}_{\mathbf{n}}^{(\mathbf{H})}\quad\mathbf{P2}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & & \\
 & & \mathbf{P4}_{\mathbf{n}}^{(\mathbf{H})}\quad\mathbf{P3}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P2}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & & \\
 & & & \mathbf{P4}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P3}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P2}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})} & \\
 & & & & \mathbf{P4}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P3}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P2}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})}\\
 & & & & \mathbf{P4}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P3}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P2}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P1}_{\mathbf{n}}^{(\mathbf{H})}\mathbf{P0}_{\mathbf{n}}^{(\mathbf{H})}
\end{pmatrix}
\begin{pmatrix}
\boldsymbol{\Omega}_{\mathbf{n+1m}}(20)\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(19)\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(18)\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(17)\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(16)\\
\vdots\\
\vdots\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(0)\\
\vdots\\
\vdots\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(-16)\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(-17)\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(-18)\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(-19)\\
\boldsymbol{\Omega}_{\mathbf{n+1m}}(-20)
\end{pmatrix}
+
\begin{pmatrix}
\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})}\\
\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})}\\
\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})}\\
\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})}\\
\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})}\\
\vdots\\
\mathbf{P}_{\mathbf{out\,n}}^{(\mathbf{H})}\\
\mathbf{0}\\
\vdots\\
\vdots\\
\mathbf{0}\\
\mathbf{0}\\
\mathbf{0}\\
\mathbf{0}\\
\mathbf{0}
\end{pmatrix}
\tag{17}
$$

Here, the 432×432 matrix $\mathbf{P0}_{\mathbf{n}_{\mathbf{extra}}}^{(\mathbf{H})}$ at the center of the second matrix is represented in the following expression.

$$
\mathbf{P0}_{\mathbf{n}_{\mathbf{extra}}}^{(\mathbf{H})} =
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & & 2 & & \dots & \dots & 9 & \\ T & B & T & B & & \dots & T & B \end{array}\\
\begin{array}{c} 1\ \begin{array}{c} T \\ B \end{array} \\ 2\ \begin{array}{c} T \\ B \end{array} \\ \vdots\ \vdots \\ \vdots\ \vdots \\ 9\ \begin{array}{c} T \\ B \end{array} \end{array}
&
\begin{pmatrix}
\mathbf{0} & \dots & \dots & & & \mathbf{0}\\
\vdots & & & & & \vdots\\
\vdots & & & & & \vdots\\
\vdots & & & & & \vdots\\
\mathbf{0} & & & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{Q_{0n}} & \mathbf{0}
\end{pmatrix}
\end{array}
\tag{18}
$$

We simply represent expression (17) by the following expression.

$$\mathbf{\Omega_{nm}} = \mathbf{P_{Vm}}\mathbf{\Omega_{nm+1}} + \mathbf{P_{Hn}}\mathbf{\Omega_{n+1m}} + \mathbf{P_{out\,n}} \tag{19}$$

Further, by arranging the above expression (19) following the order of n and m, we obtain the whole expression of transition for the batting of x_n and y_m as follows:

$$
\begin{pmatrix} \mathbf{\Omega_{11}}\\ \mathbf{\Omega_{12}}\\ \vdots\\ \mathbf{\Omega_{18}}\\ \mathbf{\Omega_{19}}\\ \hline \mathbf{\Omega_{21}}\\ \vdots\\ \vdots\\ \mathbf{\Omega_{29}}\\ \hline \mathbf{\Omega_{31}}\\ \vdots\\ \vdots\\ \vdots\\ \mathbf{\Omega_{79}}\\ \hline \mathbf{\Omega_{81}}\\ \vdots\\ \vdots\\ \mathbf{\Omega_{89}}\\ \hline \mathbf{\Omega_{91}}\\ \vdots\\ \vdots\\ \mathbf{\Omega_{99}} \end{pmatrix}
=
\left(\begin{array}{c|c|c|c|c}
\begin{matrix} \mathbf{0} & \mathbf{P_{V1}} & & & \\ & \mathbf{0} & \mathbf{P_{V2}} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{0} & \mathbf{P_{V8}} \\ \mathbf{P_{V9}} & & & & \mathbf{0} \end{matrix} &
\begin{matrix} \mathbf{P_{H1}} & & & \\ & \mathbf{P_{H1}} & & \\ & & \ddots & \\ & & & \mathbf{P_{H1}} \\ & & & & \mathbf{P_{H1}} \end{matrix} & & & \\
\hline
 & \begin{matrix} \mathbf{0} & \mathbf{P_{V1}} & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots\ \mathbf{P_{V8}} \\ \mathbf{P_{V9}} & & & \mathbf{0} \end{matrix} &
\begin{matrix} \mathbf{P_{H2}} & & \\ & \ddots & \\ & & \ddots \\ & & & \mathbf{P_{H2}} \end{matrix} & & \\
\hline
 & & \begin{matrix} \mathbf{0} & \mathbf{P_{V1}} & \\ \ddots & \ddots & \\ \ddots & \ddots & \ddots \\ & \ddots & \ddots \end{matrix} &
\begin{matrix} \mathbf{P_{H3}} & & \\ \ddots & & \\ \ddots & \ddots & \\ \ddots & & \ddots \\ \ddots & \ddots & & \mathbf{P_{H7}} \end{matrix} & \\
\hline
 & & & \begin{matrix} \ddots\ \ddots & \\ & \ddots\ \ddots \end{matrix} &
\begin{matrix} \mathbf{P_{H8}} & & \\ & \ddots & \\ & & \\ & & \mathbf{P_{H8}} \end{matrix} \\
\hline
\begin{matrix} \mathbf{P_{H9}} & & \\ & \ddots & \\ & & \ddots \\ & & & \mathbf{P_{H9}} \end{matrix} & & & &
\begin{matrix} & \mathbf{0} & \mathbf{P_{V1}} \\ & & \ddots \\ & & & \mathbf{P_{V8}} \\ \mathbf{P_{V9}} & & & \mathbf{0} \end{matrix}
\end{array}\right)
\begin{pmatrix} \mathbf{\Omega_{11}}\\ \mathbf{\Omega_{12}}\\ \vdots\\ \mathbf{\Omega_{18}}\\ \mathbf{\Omega_{19}}\\ \hline \mathbf{\Omega_{21}}\\ \vdots\\ \vdots\\ \mathbf{\Omega_{29}}\\ \hline \mathbf{\Omega_{31}}\\ \vdots\\ \vdots\\ \vdots\\ \mathbf{\Omega_{79}}\\ \hline \mathbf{\Omega_{81}}\\ \vdots\\ \vdots\\ \mathbf{\Omega_{89}}\\ \hline \mathbf{\Omega_{91}}\\ \vdots\\ \vdots\\ \mathbf{\Omega_{99}} \end{pmatrix}
+
\begin{pmatrix} \mathbf{P_{out1}}\\ \mathbf{P_{out1}}\\ \vdots\\ \mathbf{P_{out1}}\\ \mathbf{P_{out1}}\\ \hline \mathbf{P_{out2}}\\ \vdots\\ \vdots\\ \mathbf{P_{out2}}\\ \hline \mathbf{P_{out3}}\\ \vdots\\ \vdots\\ \vdots\\ \mathbf{P_{out7}}\\ \hline \mathbf{P_{out8}}\\ \vdots\\ \vdots\\ \mathbf{P_{out8}}\\ \hline \mathbf{P_{out9}}\\ \vdots\\ \vdots\\ \mathbf{P_{out9}} \end{pmatrix}
\tag{20}$$

By defining a $1,434,672(432 \times 41 \times 9 \times 9) \times 1$ vector $\mathbf{\Omega}$ as

$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{\Omega_{11}} & \mathbf{\Omega_{12}} & \ldots & \mathbf{\Omega_{19}} & \mathbf{\Omega_{21}} & \mathbf{\Omega_{22}} & \ldots & \mathbf{\Omega_{29}} & \mathbf{\Omega_{31}} & \ldots & \ldots & \mathbf{\Omega_{89}} & \mathbf{\Omega_{91}} & \ldots & \mathbf{\Omega_{99}} \end{pmatrix}^T, \tag{21}$$

and by defining the $1,434,672 \times 1,434,672$ matrix as $\mathbf{P_{ns}}$ and the $1,434,672 \times 1$ vector as $\mathbf{P_{out}}$ in expression (20), we finally obtain the simple equation as follows:

$$\mathbf{\Omega} = \mathbf{P_{ns}}\mathbf{\Omega} + \mathbf{P_{out}} \tag{22}$$

By adding the state representing the end of the game, we complete the model of a baseball game as the 1,434,673-state Markov chain.

We note that expression (22) is equivalent to a set of 1,434,672 simultaneous equations with 1,434,672 unknown variables as the probabilities of team X winning in any state in the course of a game. We can solve these equations by entering the transition probabilities based on the performance of all nine players in both teams into the matrix $\mathbf{P_{ns}}$ and the vector

$\mathbf{P_{out}}$. That is, we have converted the problem of obtaining the probability of winning into a problem of solving a set of the simultaneous equations. Once we formulate the problem in terms of simultaneous equations, it is relatively easy to solve it using a numerical method. Moreover, as this method has an advantage of simultaneously obtaining the probability of winning in every state in a game, we can extend this method to develop a dynamic programming formulation for identifying the optimal pinch hitting strategy, to be shown in the next section.

In order to test our formulation, we calculated the probability of the 1989 Braves winning using the same data as Bukiet *et al.* [2] and obtained the same result with a difference of about 0.0017. This small difference seems to be caused by the different methods used to evaluate the probability of winning in the extra innings.

4. Modeling for Pinch Hitting Strategy

We now describe how to extend the method described above to identify the optimal pinch hitting strategy. We explain the case when there is just one substitute available but this formulation is easily extended to handle the case of more than one substitute. Let $\mathbf{\Omega}[x_p] = \mathbf{\Omega}(x_1, x_2, \cdots, x_9, y_1, y_2, \cdots, y_9|x_p)$ be the $1,434,672(432 \times 41 \times 9 \times 9) \times 1$ vector representing the probabilities of home team X winning with a substitute x_p available.

If the manager does not make a substitution in the current state, the state transits to the next state indicated by the transition matrix $\mathbf{P_{ns}}$ as shown in expression (22). After this transition, the probability of team X winning in the next state is expressed by one of the elements of the same vector $\mathbf{\Omega}[x_p]$ since team X still has a substitute x_p available. On the other hand, if the manager substitutes x_p for x_1 in the current state, the state transits to the next state indicated by the transition matrix $\mathbf{P^{(1\to p)}}$, which represents the matrix modified by substituting the $\mathbf{P_{Hp}}$ blocks for the $\mathbf{P_{H1}}$ blocks in the matrix $\mathbf{P_{ns}}$ in expression (20). After this transition, the probability of winning in this next state is expressed by one of the elements of a $1,434,672 \times 1$ vector $\mathbf{\Omega^{(1\to p)}} = \mathbf{\Omega^{(1\to p)}}(x_p, x_2, \cdots, x_9, y_1, y_2, \cdots, y_9)$. This represents the probability of team X winning without any substitutes in the remainder of the game after the substitution of x_p for x_1.

Similarly, if the manager substitutes x_p for x_2 in the current state, then the state transits to the next state by the transition matrix $\mathbf{P^{(2\to p)}}$, which represents the matrix modified by substituting the $\mathbf{P_{Hp}}$ blocks for the $\mathbf{P_{H2}}$ blocks in the matrix $\mathbf{P_{ns}}$. After this transition, the probability of winning in the next state is expressed by one of the elements of a $1,434,672 \times 1$ vector $\mathbf{\Omega^{(2\to p)}} = \mathbf{\Omega^{(2\to p)}}(x_1, x_p, \cdots, x_9, y_1, y_2, \cdots, y_9)$. This represents the probability of team X winning without any substitutes in the remainder of the game after the substitution of x_p for x_2. In the same way, we define the transition matrices $\mathbf{P^{(3\to p)}}, \cdots, \mathbf{P^{(9\to p)}}$ and the vectors $\mathbf{\Omega^{(3\to p)}}, \cdots, \mathbf{\Omega^{(9\to p)}}$.

By comparing the probabilities of team X winning between the cases of non-substitution and the 9 possible case of substitutions and by taking the maximum of them in each state, we obtain the formulation to calculate the probability of team X winning with a substitute x_p in the remainder of the game using dynamic programming as follows:

$$\mathbf{\Omega}[x_p] = \max \begin{cases} \mathbf{P_{ns}}\mathbf{\Omega}[x_p] + \mathbf{P_{out}} & : Non - substitution \\ \mathbf{P^{(1\to p)}}\mathbf{\Omega^{(1\to p)}} + \mathbf{P^{(1\to p)}_{out}} & : Substitution\, of\, x_p\, for\, x_1 \\ \mathbf{P^{(2\to p)}}\mathbf{\Omega^{(2\to p)}} + \mathbf{P^{(2\to p)}_{out}} & : Substitution\, of\, x_p\, for\, x_2 \\ \qquad \vdots & \qquad \vdots \\ \mathbf{P^{(9\to p)}}\mathbf{\Omega^{(9\to p)}} + \mathbf{P^{(9\to p)}_{out}} & : Substitution\, of\, x_p\, for\, x_9 \end{cases} \tag{23}$$

where $\mathbf{P_{out}^{(1\rightarrow p)}}$,$\cdots$, $\mathbf{P_{out}^{(9\rightarrow p)}}$ represent the $1,434,672 \times 1$ vector modified by substituting the $\mathbf{P_{outp}}$ blocks for the $\mathbf{P_{out1}}$ blocks,$\cdots$, $\mathbf{P_{out9}}$ blocks in the vector $\mathbf{P_{out}}$ in expression (20), respectively. By solving the recursive simultaneous equation (23) above using the method described in Appendix, we obtain the probability of team X winning with one substitute.

In order to modify this formulation to calculate the probability of the visiting team winning, we just exchange the notations between team X and team Y in the formulation. Further, we can extend this formulation to the case of more than one substitute along the same line by adding the possible substitutions of other substitutes, for example x_q for x_n, into expression (23), and taking the maximum of them. Here, we note that even in the case of one substitute available there are 9 possible places in the batting order. So, there are 9 possible combinations for a substitute. In the case of 2 substitutes available, there are $9^2 \times 2$ possible combinations for two substitutes. That is, a substitute x_p has 9 possible places in the batting order for a substitution and another substitute x_q has also 9 possible places for a substitution. By considering also which substitute is used first, we get $9^2 \times 2$ possible combinations. In general, if there are k substitutes available, there are $9^k \times k!$ combinations of possible substitutions. Therefore, if we add one substitute, the computing time to complete a whole calculation increases by a factor of 9 times the number of substitutes. For example, when the number of substitutes increases as 0,1,2,3,4 and 5, the computing time increases by a factor of 1, 9, 162, 4,374, 157,464 and 7,085,880 respectively.

We developed the C code for calculating the probability of winning under the D'Esopo and Lefkowitz model for solving recursive equation (23). We avoid any unnecessary multiplication for the sake of efficiency, since the transition matrix is fairly sparse. Our C code takes about half a CPU day on an HP VISUALIZE C3600 after compilation with an HP C compiler with maximum optimization (+O2) in case of 3 substitutes available, and it will take more than half a CPU month to complete the calculation in case of 4 substitutes available. The computing time in the case of 4 substitutes is too long not only to occupy our computer, but also to reflect the information from the latest game when considering the next game in a real baseball league or tournament. Thus, we demonstrate the following example with up to 3 substitutes because of the computing time and the usefulness for a real game. We note that usually less than 4 substitutions for pinch hitting or fielding are used in a DH rule game. In fact, the Anaheim Angels (to be focused on later) made 4 or more substitutions for pinch hitting or fielding (not pinch runners or pitching) in just 6 games in the 2000 season. Thus, 3 substitutes being available in this paper would give a useful result to baseball teams.

5. An Example of the Procedure for Identifying the Optimal Pinch Hitting Strategy

5.1. Sample data

We now present an example of our procedure to find the optimal pinch hitting strategy. We chose the Angels because they are an average team as shown by their 82-80 won-lost record in the American League in the 2000 season. We demonstrate the game of the Angels against the Oakland Athletics, the winner of the West Division. Both teams of players in this example are listed in Table 2. We selected three Angels players Palmeiro, Stocker and Walbeck as substitutes because they played in a high number of games during the season.

Using the statistics taken from Neft *et al.* [7] and STATS, Inc.[8], we calculated the probability of each player in both teams achieving the following results - a single, double, triple, home run, walk and out (see Table 2). To set up the players' transition matrices, we used the D'Esopo and Lefkowitz model. We allowed the Angels players to be allocated to their fielding positions if they played these positions in more than 10 games in the 2000 season.

Table 2: The line-ups of the Anaheim Angels and the Oakland Athletics

(a) Anaheim Angels

Order	Players	Single	Double	Triple	Home Run	Walk	Out	Positions
1	Erstad	0.2297	0.0527	0.0081	0.0338	0.0865	0.5892	LF (CF)
2	Kennedy	0.1693	0.0527	0.0176	0.0144	0.0447	0.7013	2B
3	Vaughn	0.1443	0.0447	0.0000	0.0519	0.1140	0.6450	1B
4	Salmon	0.1384	0.0536	0.0030	0.0506	0.1548	0.5997	RF
5	Anderson	0.1595	0.0596	0.0045	0.0522	0.0358	0.6885	CF (RF)
6	Glaus	0.1111	0.0548	0.0015	0.0696	0.1659	0.5970	3B
7	Spiezio	0.1246	0.0326	0.0059	0.0504	0.1187	0.6677	DH (1B,3B)
8	Molina	0.1956	0.0403	0.0040	0.0282	0.0464	0.6855	C
9	Gil	0.1541	0.0423	0.0030	0.0181	0.0906	0.6918	SS
Sub	Palmeiro	0.1815	0.0712	0.0071	0.0000	0.1352	0.6050	LF (RF)
Sub	Stocker	0.1111	0.0498	0.0115	0.0000	0.1226	0.7050	SS
Sub	Walbeck	0.1176	0.0327	0.0000	0.0392	0.0458	0.7647	C

Remarks : The positions shown inside ( ) represent the alternative positions.
: Any pinch hitter for a DH himself becomes a DH.
: No multiple substitutions may be made that will alter the batting rotation of the DH.

(b) Oakland Athletics

Order	Players	Single	Double	Triple	Home Run	Walk	Out
1	Long	0.1786	0.0542	0.0064	0.0287	0.0686	0.6635
2	Velarde	0.1855	0.0427	0.0000	0.0223	0.1002	0.6494
3	Giambi	0.1499	0.0448	0.0015	0.0665	0.2117	0.5255
4	Grieve	0.1469	0.0600	0.0015	0.0405	0.1094	0.6417
5	Saenz	0.1841	0.0502	0.0084	0.0377	0.1046	0.6151
6	Stairs	0.1101	0.0469	0.0000	0.0379	0.1408	0.6643
7	Tejada	0.1545	0.0475	0.0015	0.0446	0.0981	0.6538
8	Chavez	0.1528	0.0409	0.0071	0.0462	0.1101	0.6430
9	Hernandez	0.1488	0.0416	0.0000	0.0306	0.0832	0.6958

If the Angels fill all the fielding positions (DH for pitcher, catcher (C), first baseman (1B), second baseman (2B), third baseman (3B), shortstop (SS), left fielder (LF), center fielder (CF) and right fielder (RF)), then we used the value listed in Table 2 (b) to calculate the probability of the Angels winning. However, if the Angels failed to fill all these fielding positions during the game, the probability of all the Athletics players getting a single, double, triple, home run and walk was more than doubled as a penalty against the Angels. We did this because we must avoid maximizing the probability of the Angels winning when they break the fielding conditions (however, in the bottom of the 9th inning, making a substitution can maximize the probability of winning in spite of incurring a penalty).

5.2. The optimal pinch hitting strategy

An example of the calculated results for finding the optimal pinch hitting strategy based on the line-up in Table 2 is shown in Table 3. In this table, B in the second column represents the bottom of the innings. The digits at the head of the columns represent the base runner condition. Three digits $d_3d_2d_1$ are 1 or 0 corresponding to whether there is or is not a runner on third base, second base and first base, respectively. For example, 010 represents the situation where a runner is only on second base. The numbers such as 5 or 7 indicate the batting orders of starting players who come up but should be substituted by a pinch hitter. In this case, only Palmeiro is chosen as a pinch hitter, and he should be substituted instead of the 5th batter Anderson, the 7th batter Spiezio, and so on.

Table 3: An example of the situations for pinch hitting in the case where the Angels and the Athletics are running equal (5:Anderson 7: Spiezio, 9: Gil)

Inning	T/B	Out	000	001	010	100	011	101	110	111
1	B	0			7	7	5,7	5,7	7	5,7
1	B	1							7	5,7
1	B	2							7	5,7,9
2	B	0			7	7	5,7	5,7	7	5,7,9
2	B	1							7	5,7,9
2	B	2							7	5,7,9
3	B	0			7	7	7	7	7	5,7,9
3	B	1							7	5,7,9
3	B	2							7	5,7,9
...	...	...								
9	B	0		9	9	9	9	9	9	5,9
9	B	1		9	7,9	7,9	5,7,9	5,7,9	5,7,9	5,9
9	B	2		9	7,9	7,9	5,7,9	5,7,9	5,7,9	5,7,8,9

Remark: Innings 4 - 8 are omitted.

According to this result, the 5th batter (Anderson) and the 7th batter (Spiezio) should be substituted by Palmeiro even in the 1st inning with no outs. In terms of the fielding position, Spiezio, a DH, can be substituted by Palmeiro because any pinch hitter can be substituted for a DH. On the other hand, the substitution of Anderson may appear a little surprising because he is a regular as a CF, and Palmeiro is not a CF but a LF. However, this is possible because Palmeiro can also play as an RF, and Erstad can also play as a CF. So, Palmeiro can be substituted for Anderson in the bottom of the 1st inning without disturbing the fielding positions, because Palmeiro plays as a RF and Erstad moves to be a CF from the beginning of the top of the 2nd inning. Although we do not show the next optimal substitution after the first substitution in Table 3, Palmeiro is substituted, for example, by Stocker after the substitution of Palmeiro for Gil in order to fill the position of SS.

Since the substitution in the early innings improves the probability of the Angels winning, it is reasonable to infer that it is better to start off with Palmeiro instead of Anderson, and use Anderson as a substitute. Alternately, they can start off with Palmeiro instead of Spiezio, and use Spiezio as a substitute. In fact Table 4 shows that if they start off with Palmeiro as the 5th batter instead of Anderson, the probability of the Angels winning is 0.4900. This is greater than 0.4878, which is the case when starting off with Anderson. It is also greater than 0.4880, when starting off with Palmeiro as the 7th batter instead of Spiezio.

Table 4: The probability of Angels winning at the beginning of the game

Starting player	Substitutes	Probability of winning
Anderson(5th), Spiezio(7th)	Palmeiro, Stocker, Walbeck	0.4878
Anderson(5th), Palmeiro(7th)	Spiezio, Stocker, Walbeck	0.4880
Palmeiro(5th), Spiezio(7th)	Anderson, Stocker, Walbeck	0.4900

We note that using Palmeiro in the starting line-up and Anderson as a substitute improves their probability of winning by 0.0022, which corresponds to 0.36 wins out of 162 games in a season.

We recalculated to obtain the optimal pinch hitting strategy in the case where Palmeiro is placed as the 5th batter instead of Anderson, and Anderson is a substitute. Table 5 shows the result of recalculation in the case where the Angels lead by 1 run, tie or are losing by

Table 5: An example of the situations for pinch hitting (2:Kennedy, 3:Vaughn, 5:Palmeiro, 7:Spiezio, 8:Molina, or 9:Gil)

(a) Case where the Angels lead by 1 run

Inning	T/B	Out	000	001	010	100	011	101	110	111
1	B	0							7	
1	B	1							7	
1	B	2			7	7	7	7	7	
2	B	0							7	
2	B	1							7	
2	B	2			7	7	7	7	7	
3	B	0							7	
3	B	1			7	7			7	
3	B	2			7	7	7	7	7	
4	B	0							7	
4	B	1							7	
4	B	2			7	7	7	7	7	
5	B	0							7	
5	B	1			7	7			7	
5	B	2			7	7	7	7	7,9	
6	B	0								
6	B	1			7	7			7,9	
6	B	2			7	7	7,9	7,9	7,9	
7	B	0							9	
7	B	1			7	7	9	9	7,9	
7	B	2			7,9	7,9	7,9	7,9	7,9	
8	B	0			9	9	9	9	9	9
8	B	1		9	7,9	7,9	9	9	7,9	9
8	B	2	9	9	7,9	7,9	7,9	7,9	7,9	9
9	B	0	-	-	-	-	-	-	-	-
9	B	1	-	-	-	-	-	-	-	-
9	B	2	-	-	-	-	-	-	-	-

(b) Case where the scores are level

Inning	T/B	Out	000	001	010	100	011	101	110	111
1	B	0								
1	B	1							7	
1	B	2			7	7	7	7	7	
2	B	0								
2	B	1							7	
2	B	2			7	7			7	
3	B	0							7	
3	B	1							7	
3	B	2			7	7	7	7	7	
4	B	0							7	
4	B	1							7	
4	B	2			7	7	7	7	7	
5	B	0								
5	B	1			7	7			7	
5	B	2			7	7	7	7	7,9	
6	B	0								
6	B	1			7	7			7,9	
6	B	2			7	7	7,9	7,9	7,9	
7	B	0							9	
7	B	1			7	7	9	9	7,9	
7	B	2			7,9	7,9	7,9	7,9	7,9	
8	B	0			9	9	9	9	9	
8	B	1			7,9	7,9	7,9	7,9	7,9	
8	B	2	9	9	7,9	7,9	7,9	7,9	7,9	9
9	B	0	9		7,9	7,9	9	9	9	
9	B	1	5,9	9	7,9	7,9	7,9	7,9	7,9	
9	B	2	5,9	9	7,9	7,9	7,9	7,9	7,9	

(c) Case where the Angels are losing by 1 run

Inning	T/B	Out	000	001	010	100	011	101	110	111
1	B	0								
1	B	1							7	
1	B	2			7	7			7	
2	B	0								
2	B	1							7	
2	B	2			7	7			7	
3	B	0								
3	B	1							7	
3	B	2			7	7			7	
4	B	0								
4	B	1							7	
4	B	2			7	7			7	
5	B	0								
5	B	1							7	
5	B	2			7	7			7,9	
6	B	0								
6	B	1							7,9	
6	B	2			7	7	9	9	7,9	9
7	B	0							9	
7	B	1					9	9	7,9	9
7	B	2			7,9	7,9	7,9	7,9	7,9	9
8	B	0	9	9	9	9	9	9	9	9
8	B	1	9	9	9	9	9	9	7,9	9
8	B	2	9	5,9	7,9	7,9	9	9	7,9	9
9	B	0	9	9	9	9	9	9	2,9	9
9	B	1	9	5,9	7,9	7,9	9	9	2,7,9	7,9
9	B	2	5,8,9	2,5,8,9	5,7,8,9	5,7,8,9	7,8,9	7,8,9	2,3,7,8,9	2,7,8,9

1 run. At any situation in Table 5, only Anderson is chosen as a pinch hitter, and the first pinch hitting chance may occur in the bottom of the 1st inning when the 7th batter (Spiezio, a DH), is coming up to bat. In this case, after Anderson bats as a pinch hitter instead of Spiezio, Anderson continues to play as a DH.

Table 6: The situations leading to more than a 0.01 improvement by pinch hitting in the case where the Angels lead by from -4 to 4

				Number of runs the Angels leading by								
Inning	T/B	Out	Runner	-4	-3	-2	-1	0	1	2	3	4
7	B	2	110				7 (0.396→0.407)					
8	B	1	011			9 (0.297→0.307)						
8	B	1	101			9 (0.297→0.307)						
8	B	1	110			9 (0.366→0.378)	9 (0.522→0.534)					
8	B	1	111		9 (0.277→0.289)							
8	B	2	001				9 (0.256→0.269)					
8	B	2	010				9 (0.294→0.311)					
8	B	2	100				9 (0.294→0.311)					
8	B	2	011			9 (0.190→0.204)	9 (0.317→0.333)					
8	B	2	101			9 (0.190→0.204)	9 (0.317→0.333)					
8	B	2	110		9 (0.120→0.131)	7 (0.218→0.229) 9 (0.228→0.246)	7 (0.337→0.351) 9 (0.355→0.377)	9 (0.601→0.611)				
8	B	2	111	9 (0.090→0.101)	9 (0.154→0.170)	9 (0.255→0.270)	9 (0.381→0.393)					
9	B	0	000				9 (0.225→0.236)					
9	B	0	001			9 (0.225→0.236)	9 (0.352→0.364)					
9	B	0	010			9 (0.225→0.236)	9 (0.462→0.479)					
9	B	0	100			9 (0.225→0.236)	9 (0.462→0.479)					
9	B	0	011		9 (0.225→0.236)	9 (0.352→0.364)	9 (0.531→0.542)					
9	B	0	101		9 (0.225→0.236)	9 (0.352→0.364)	9 (0.531→0.542)					
9	B	0	110		9 (0.225→0.236)	9 (0.462→0.479)	9 (0.653→0.664)					
9	B	0	111	9 (0.225→0.236)	9 (0.352→0.364)	9 (0.531→0.542)						
9	B	1	001				9 (0.222→0.243)					
9	B	1	010				9 (0.338→0.363)					
9	B	1	100				9 (0.338→0.363)					
9	B	1	011			9 (0.222→0.243)	9 (0.393→0.411)					
9	B	1	101			9 (0.222→0.243)	9 (0.393→0.411)					
9	B	1	110			9 (0.338→0.363)	7 (0.465→0.487) 9 (0.510→0.536)					
9	B	1	111		9 (0.222→0.243)	9 (0.393→0.411)	9 (0.553→0.565)					
9	B	2	000				9 (0.047→0.063)					
9	B	2	001			9 (0.047→0.063)	2 (0.087→0.099) 5 (0.096→0.115) 8 (0.095→0.106) 9 (0.095→0.123)					
9	B	2	010			9 (0.047→0.063)	7 (0.167→0.185) 9 (0.158→0.192)	7 (0.607→0.621) 9 (0.623→0.637)				
9	B	2	100			9 (0.047→0.063)	7 (0.167→0.185) 9 (0.158→0.192)	7 (0.607→0.621) 9 (0.623→0.637)				
9	B	2	011		9 (0.047→0.063)	2 (0.087→0.099) 5 (0.096→0.115) 8 (0.095→0.106) 9 (0.095→0.123)	7 (0.191→0.204) 9 (0.187→0.215)	7 (0.608→0.621) 9 (0.625→0.637)				
9	B	2	101		9 (0.047→0.063)	2 (0.087→0.099) 5 (0.096→0.115) 8 (0.095→0.106) 9 (0.095→0.123)	7 (0.191→0.204) 9 (0.187→0.215)	7 (0.608→0.621) 9 (0.625→0.637)				
9	B	2	110		9 (0.047→0.063)	7 (0.167→0.185) 9 (0.158→0.192)	2 (0.268→0.286) 7 (0.249→0.286) 9 (0.252→0.289)	7 (0.608→0.621) 9 (0.625→0.638)				
9	B	2	111	9 (0.047→0.063)	2 (0.087→0.099) 5 (0.096→0.115) 8 (0.095→0.106) 9 (0.095→0.123)	7 (0.191→0.204) 9 (0.187→0.215)	9 (0.281→0.301)					

Remark: The improvement of probability of the Angels winning is represented in ( ).

As there is not enough space to list all the numerical results here, we have selected the situations which lead to more than a 0.01 improvement of probability of the Angels winning by pinch hitting and show the numerical results in Table 6. In any situation in this table only Anderson is chosen as a pinch hitter. Note that the value of the probability of the Angels winning changes slightly depending on the situation of the Athletics' next batter, that is, who the Athletics' next batter is in the next top of the inning. Here, the change is not enough to affect the decision to substitute a pinch hitter. In Table 6 we represent the probability of the Angels winning in the situation where the Athletics' next batter is the 1st batter (Long).

The selection of the situations in Table 6 helps us to understand the calculated results more intuitively. For example, when the 9th batter (Gil) is about to come up to bat in the situation where the Angles are behind by two runs in the bottom of the 8th inning with two outs and runners on first and second base, the manager should substitute Anderson for Gil, and this substitution will provide the improvement of the probability of winning in the reminder of the game from 0.190 to 0.204. This is reasonable because in this situation the manager generally wants the batter to hit a home run to turn the tables. Here, as shown in Table 2 (a), Anderson has the probability to hit a home run (0.0522), which is about three times larger than Gil's (0.0181).

As Table 5 and 6 show, there is a tendency for more pinch hitting chances to occur in the case where the Angels are behind than in the case where the Angels lead. In fact, there are no states which lead to more than a 0.01 improvement of the probability of the Angels winning when the Angels lead. Moreover, the pinch hitting chances which lead to more than a 0.01 improvement tend to occur in the late innings, even though this does not imply that the manager should wait until the late innings to improve the probability of winning. We note that under our formulation when the manager encounters any situations which improve the probability of winning, he should make a substitution. Otherwise, the manager will fail to maximize the probability of winning in the remainder of the game.

Another interesting point in this example is that the Angels substitute for 2nd batter Kennedy, a 2B, in several situations in the bottom of the 9th inning, even though no other players can fill Kennedy's fielding position 2B. In this case it is worth paying the cost of a penalty by substituting Anderson for Kennedy in the bottom of the 9th inning, although the Angels will almost certainly lose if they fail to win in the bottom of the 9th inning and they enter extra innings.

5.3. Effect of the number of substitutes

Table 7 shows the improvement in the probability of the Angels winning depending on the number of substitutes. As this table shows, the more substitutes, the higher the probability of winning. In this example, having three substitutes increases the probability of the Angels winning by 0.0041. However, adding Walbeck as a substitute does not much improve the probability of winning, since Walbeck is not better at batting than any other players in the Angels line-up.

Table 7: Probability of Angels winning depending on a number of substitutes

Substitutes	Probability of winning
None	0.4859
Anderson	0.4887
Anderson, Stocker	0.4899
Anderson, Stocker, Walbeck	0.4900

5.4. Effect of home and visiting

Since a home team can choose their strategy after a visiting team finishes batting in every inning, the home team should have an advantage. We calculated the effect of being a home or visiting team on optimal pinch hitting by making the Angels the visiting team. Table 8 shows the probability of the Angels winning both as a home and as a visiting team at the beginning of the top of each inning. This is under the conditions where the Angels are still withholding these 3 substitutes at the beginning of the top of the inning, the next batters of both teams are 1st batter (i.e. Erstad for the Angels and Long for the Athletics) and the Angels are running equal.

As shown in Table 8, the home team has a slight advantage of winning at the beginning of each inning as we expected. We can also see that as the inning progresses, the probability of winning gradually increases. This is not surprising for the following reason. The Angels have less than a 50% chance of winning at the beginning of the game. Thus, as the game progresses they are likely to be losing. However, as we have assumed that they are running equal at the beginning of the inning, this means that the Angels have improved their chance of winning. Note that this reasoning does not apply to the progression from the 8th inning to the 9th inning. This may be because there are fewer chances of making a substitution in the remainder of the game. In other words, according to Table 5 (b) the 5th, the 7th and the 9th batter should be substituted in the 8th and the 9th innings in some situations. Here, as the probabilities in Table 8 are in the situation where the next batter is 1st, so there are fewer chances for the 5th, the 7th and the 9th batters to come up in the 9th inning rather than in the 8th inning later in the game.

Table 8: The probability of the Angels winning as a home and as a visiting team at the beginning of the top of each inning withholding the 3 substitutes in the case where they are running equal

Inning	Probability of winning	
	As home team	As visiting team
1	0.4900	0.4899
2	0.4912	0.4911
3	0.4923	0.4921
4	0.4939	0.4937
5	0.4950	0.4948
6	0.4976	0.4973
7	0.4985	0.4983
8	0.5029	0.5025
9	0.4979	0.4970

6. Conclusions

We have formulated a method to calculate the probability of winning a baseball game using a Markov chain. This method has been extended to obtain the optimal pinch hitting strategy using dynamic programming and has been applied to the real line-up of the Anaheim Angels in the American League. We have shown that the availability of substitutes improves the probability of the Angels winning against the Athletics if the optimal strategy is followed.

In this paper, we have used the D'Esopo and Lefkowitz model for runner advancement, but we could easily apply a more complicated runner advancement model by modifying the entries of the block corresponding to the batter in the transition matrix. Further, though we do not show the optimal pinch hitting strategy when considering the handedness of batters and pitchers, it is possible to include the handedness simply by setting up batting probabilities with handedness (i.e. based on batting statistics against left- and right-handed pitchers). For example, by setting up the batting probabilities of the Angels' players based on the batting statistics against a left-handed pitcher, we can obtain their optimal pinch hitting strategy against a left-handed pitcher using the same formulation.

As part of our future work on this topic, we will be proposing a method to cater for the substitution of pitchers of differing abilities. Furthermore, by integrating these extended methods we could study quantitatively the real pinch hitting or substitution decided upon by managers. For example, we could evaluate real pinch hitting decisions by comparing the decisions by the managers with the recommendations produced by our calculations. This work will need a lot of analysis of real games but would provide a very interesting insight into baseball games in terms of the managerial decision-making.

Another study could also incorporate the opposing team's substitution using game theory.

Appendix: Iteration Method to Solve the Recursive Equation

To solve the recursive simultaneous equation (23), we can use the value iteration method of dynamic programming [5]. Here, we explain the case of one substitute. Let $\mathbf{\Omega}[N] = \mathbf{\Omega}[N](x_1, x_2, \cdots, x_9, y_1, y_2, \cdots, y_9|x_p)$ be a $1,434,672 \times 1$ vector representing the probabilities of winning in every state in the remainder of the game after N iterations starting off with the initial value $\mathbf{\Omega}[0]$ following expression (24).

$$\mathbf{\Omega}[N] = \max \begin{cases} \mathbf{P_{ns}}\mathbf{\Omega}[N{-}1] + \mathbf{P_{out}} & : Non-substitution \\ \mathbf{P^{(1\to p)}}\mathbf{\Omega^{(1\to p)}}[N{-}1] + \mathbf{P_{out}^{(1\to p)}} & : Substitution\, of\, x_p\, for\, x_1 \\ \mathbf{P^{(2\to p)}}\mathbf{\Omega^{(2\to p)}}[N{-}1] + \mathbf{P_{out}^{(2\to p)}} & : Substitution\, of\, x_p\, for\, x_2 \\ \vdots & \vdots \\ \mathbf{P^{(9\to p)}}\mathbf{\Omega^{(9\to p)}}[N{-}1] + \mathbf{P_{out}^{(9\to p)}} & : Substitution\, of\, x_p\, for\, x_9 \end{cases} \tag{24}$$

where $\mathbf{\Omega^{(1\to p)}}[N-1]$ represents the probability of winning without any substitutes in the remainder of the game after the substitution of x_p for x_1 and so on. We obtain $\mathbf{\Omega^{(1\to p)}}[N{-}1]$ by iteratively calculating from 0 to $N-1$ as follows:

$$\begin{aligned} \mathbf{\Omega^{(1\to p)}}[N{-}1] &= \mathbf{P^{(1\to p)}}\mathbf{\Omega^{(1\to p)}}[N{-}2] + \mathbf{P_{out}^{(1\to p)}} \\ &= \mathbf{P^{(1\to p)}}(\mathbf{P^{(1\to p)}}\mathbf{\Omega^{(1\to p)}}[N{-}3] + \mathbf{P_{out}^{(1\to p)}}) + \mathbf{P_{out}^{(1\to p)}} \\ &= \mathbf{P^{(1\to p)}}\mathbf{P^{(1\to p)}}\mathbf{\Omega^{(1\to p)}}[N{-}3] + \mathbf{P^{(1\to p)}}\mathbf{P_{out}^{(1\to p)}} + \mathbf{P_{out}^{(1\to p)}} \\ &= \qquad \vdots \\ &= \mathbf{P^{(1\to p)}}^{N-1}\mathbf{\Omega^{(1\to p)}}[0] + (\mathbf{P^{(1\to p)}}^{N-2} + \mathbf{P^{(1\to p)}}^{N-3} + \mathbf{P^{(1\to p)}}^{N-4} + \ldots + \mathbf{I})\mathbf{P_{out}^{(1\to p)}} \end{aligned} \tag{25}$$

The same expressions hold for $\mathbf{\Omega^{(2\to p)}}[N-1]$, $\cdots$, $\mathbf{\Omega^{(9\to p)}}[N-1]$. When the norm of these transition matrices satisfy $||\mathbf{P_{ns}}|| < 1$,$||\mathbf{P^{(1\to p)}}|| < 1$,$\cdots$,$||\mathbf{P^{(9\to p)}}|| < 1$, the $\mathbf{\Omega}[N{-}1]$ converges to the solution $\mathbf{\Omega}$ as $N \to \infty$, for example, by starting off with $\mathbf{\Omega}[0] = \mathbf{\Omega^{(1\to p)}}[0] = \cdots = \mathbf{\Omega^{(9\to p)}}[0] = \mathbf{0}$. Practically it is better firstly to solve the following 9 simultaneous equations individually by an iterative method for solving simultaneous equations.

$$\begin{aligned} \mathbf{\Omega^{(1\to p)}} &= \mathbf{P^{(1\to p)}}\mathbf{\Omega^{(1\to p)}} + \mathbf{P_{out}^{(1\to p)}} \\ \mathbf{\Omega^{(2\to p)}} &= \mathbf{P^{(2\to p)}}\mathbf{\Omega^{(2\to p)}} + \mathbf{P_{out}^{(2\to p)}} \\ \vdots & \qquad \vdots \\ \mathbf{\Omega^{(9\to p)}} &= \mathbf{P^{(9\to p)}}\mathbf{\Omega^{(9\to p)}} + \mathbf{P_{out}^{(9\to p)}} \end{aligned} \tag{26}$$

Then, we can solve the following recursive equation by starting off with the $\mathbf{\Omega}[0] = \mathbf{0}$ using the solutions $\mathbf{\Omega^{(1\to p)}}$, $\mathbf{\Omega^{(2\to p)}}$, $\cdots$,$\mathbf{\Omega^{(9\to p)}}$ obtained above.

$$\mathbf{\Omega}[N] = \max \begin{cases} \mathbf{P_{ns}}\mathbf{\Omega}[N\text{–}1] + \mathbf{P_{out}} & : Non - substitution \\ \mathbf{\Omega^{(1\to p)}} & : Substitution\, of\, x_p\, for\, x_1 \\ \mathbf{\Omega^{(2\to p)}} & : Substitution\, of\, x_p\, for\, x_2 \\ \quad\vdots & \qquad\qquad\vdots \\ \mathbf{\Omega^{(9\to p)}} & : Substitution\, of\, x_p\, for\, x_9 \end{cases} \tag{27}$$

We used the Gauss-Seidel method for firstly solving individual simultaneous equations, and reduced the CPU time to about 1/10 compared to the value iteration method shown in (24). This method is easily extended to the case of more than one substitute being available.

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