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RISK MINIMIZATION IN OPTIMAL STOPPING PROBLEM AND APPLICATIONS

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Abstract We consider an optimal stopping problem with a discrete time stochastic process where a criterion is a threshold probability. We first obtain the fundamental characterization of an optimal value and an optimal stopping time as the result of the classical optimal stopping problem, but the optimal value and the optimal stopping time depend upon a threshold value. We also give the properties of the optimal value with respect to threshold value. These are applied to a secretary problem, a parking problem and job search problems and we explicitly find an optimal value and an optimal stopping time for each problem.

Keywords: Stochastic optimization, optimal stopping, threshold probability, secretary problem, parking problem, job search problem

1. Introduction

In the classical optimal stopping problem, a standard criterion function is the expected reward (e.g. Chow et al.[4], Neveu[10] and Shiryayev[15]). It is, however, known that the criterion is quite insufficient to characterize the decision problem from the point of view of the decision maker and it is necessary to select other criteria to reflect the variability of risk features for the problem. Indeed, in Markov decision processes many authors propose a variety of criteria (e.g. utility, probabilistic constraints and mean-variance) and investigate Markov decision processes for their criteria, instead of standard criteria, that is, the expected discounted total reward and the average expected reward per unit. White[17] reviews the decision problems with such criteria in detail. Especially, White[18], Wu and Lin[19] and Ohtsubo and Toyonaga[13] consider a problem in which we minimize a threshold probability. Wu and Lin[19] show that optimal values are distribution functions in the threshold value and Ohtsubo and Toyonaga[13] give two sufficient conditions for the existence of the optimal policy in an infinite horizon case. Ohtsubo[12] also applies such a problem to stochastic shortest path problems. On the other hand, many authors investigate optimal stopping problems with new criteria. Denardo and Rothblum[6] consider an optimal stopping problem with an exponential utility function as a criterion function in finite Markov decision chain and use a linear programming to compute an optimal policy. In Kadota et al.[8], they investigate an optimal stopping problem with a general utility function in a denumerable Markov chain. They give a sufficient condition for an one-step look ahead (OLA) stopping time to be optimal and characterize a property of an OLA stopping time for risk-averse and risk-seeking utilities. Bojdecki[1] formulates an optimal stopping problem which is concerned with maximizing the probability of a certain event and give necessary and sufficient conditions for existence of an optimal stopping time. He also applies the results to a version of the discrete-time disorder problem. Ohtsubo[11] considers optimal stopping problems with a threshold probability criterion in a Markov process, characterize optimal values and find optimal stopping times for finite and infinite horizon cases. Such a problem with a threshold probability criterion is available for applications to the percentile of the losses or Value-at-Risk (VaR) in finance (e.g. Filar[7] and Uryasev[16]). It is effective in a threshold value such that the threshold probability is small, e.g. 0.01 or 0.05. These references give us an important motivation for our risk minimizing problem.

In this paper we consider optimal stopping problems with a threshold probability in a random sequence. In Section 3 we characterize optimal values and optimal stopping times for finite and infinite horizon cases and show that optimal values are distribution functions in a threshold value. In Sections 4, 5 and 6 we investigate a secretary problem, a parking problem and job search problems, respectively, as applications of our problem, and we explicitly find an optimal value and an optimal stopping time for each problem.

2. Formulation of Problems

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n)_{n=1}^{\infty}$ be a filtration of $\mathcal{F}$, and let $X = (X_n)$ be a given stochastic process defined on $(\Omega, \mathcal{F}, P)$ and adapted to $(\mathcal{F}_n)$. We assume that $P(\sup_n X_n^+ < \infty) = 1$, where $x^+ = \max(0, x)$. This assumption holds if $E[\sup_n X_n^+] < \infty$, which is a condition given in the classical optimal stopping problem. For each $n \geq 1$, we also denote by Γ_n^N (resp. Γ_n) the class of $(\mathcal{F}_n)$–stopping times τ such that $n \leq \tau \leq N$ (resp. $n \leq \tau < \infty$) almost surely, where $n \leq N$.

We consider a minimizing problem for threshold probability $F_0(r;\tau) = P(X_\tau \leq r) = E[I_{(X_\tau \leq r)}]$ with respect to τ in Γ_0^N or Γ_0, where r is a real number, which is called a threshold (target) value, and I_A is the indicator function on a set A.

Letting $Y_n(r) = I_{(X_n \leq r)}$, we generally define

$$F_n(r;\tau) = E[Y_\tau(r)|\mathcal{F}_n],$$

and we define optimal value processes $(V_n^N(r))$ and $(V_n(r))$ for finite and infinite horizon cases by

$$V_n^N(r) = \operatorname*{ess\,inf}_{\tau \in \Gamma_n^N} F_n(r;\tau), \quad V_n(r) = \operatorname*{ess\,inf}_{\tau \in \Gamma_n} F_n(r;\tau),$$

respectively. We also define optimal value sequences $(v_n^N(r))$ and $(v_n(r))$ for finite and infinite horizon cases by

$$v_n^N(r) = \inf_{\tau \in \Gamma_n^N} E[Y_\tau(r)], \quad v_n(r) = \inf_{\tau \in \Gamma_n} E[Y_\tau(r)],$$

respectively. For $n \geq 1$ and $\varepsilon \geq 0$, we say that a stopping time τ_ε in Γ_n^N (resp. Γ_n) is ε–optimal at (n, r) if $v_n^N(r) \geq E[Y_{\tau_\varepsilon}(r)] - \varepsilon$ (resp. $v_n(r) \geq E[Y_{\tau_\varepsilon}(r)] - \varepsilon$).

Then our problem is to characterize optimal values and to find ε–optimal stopping time. We note that our model is a special one of the classical stopping problems for a *fixed* threshold value r, but it is exactly different from those as optimal values and stopping times depend upon r.

3. General Results

In this section we give fundamental properties of optimal values and optimal stopping times for finite and infinite horizon cases and show that optimal values are distribution functions in r.

We first have three propositions below for a fixed threshold value r from the classical theory of optimal stopping (e.g. see Chow et al.[4]).

Proposition 3.1 *Let r be any real number. Then $\lim_{N\to\infty} V_n^N(r) = V_n(r)$ a.s.. Moreover, $v_n^N(r) = E[V_n^N(r)]$ and $v_n(r) = E[V_n(r)]$ for each n and N with $1 \le n \le N$.* $\square$

Proposition 3.2 *Let r be any real number. The optimal value process $(V_n^N(r))$ in a finite horizon case satisfies a recursive relation*

$$V_N^N(r) = Y_N(r), \quad V_n^N(r) = \min(Y_n(r), E[V_{n+1}^N(r)|\mathcal{F}_n]), \ 1 \le n \le N-1.$$

Also, the stopping time $\sigma_N^*(r) = \inf\{1 \le k \le N | V_k^N(r) = Y_k(r)\}$ is 0–optimal in Γ_1^N at $(1, r)$, where $\inf \phi = N$. $\square$

Proposition 3.3 *Let r be any real number. The optimal value process $(V_n(r))$ in an infinite horizon case satisfies a recursive relation*

$$V_n(r) = \min(Y_n(r), E[V_{n+1}(r)|\mathcal{F}_n]).$$

Also, for each $\varepsilon > 0$ the stopping time $\tau_n^\varepsilon(r) = \inf\{k \ge n | V_k(r) \ge Y_k(r) - \varepsilon\}$ is ε–optimal in Γ_n at (n, r), where $\inf \phi = \infty$. Furthermore if $\tau_n^0(r) = \inf\{k \ge n | V_k(r) = Y_k(r)\}$ is a.s. finite, then $\tau_n^0(r)$ is 0–optimal stopping time in Γ_n at (n, r). $\square$

We next give a sufficient condition for the stopping time $\tau_n^0(r)$ to be 0-optimal.

Theorem 3.1 *Let r be any real number and set $A_n(r) = \{X_n > r\}$. If $P(\limsup_{n\to\infty} A_n(r)) = 1$, then the stopping time $\tau_n^0(r)$ is 0–optimal in Γ_n at (n, r) for every n.*

Proof: From Proposition 3.3, it suffices to show that $\tau_n^0(r)$ is a.s. finite. Let $\omega \in \limsup_n A_n(r)$. Then for each $n \ge 1$ there is a $k \ge n$ such that $\omega \in A_k(r)$, that is, $X_k(\omega) > r$. Thus we have $V_k(r)(\omega) = Y_k(r)(\omega) = 0$, which implies that $\tau_n^0(r) \le k$ on the set $A_k(r)$, and hence it follows that $\tau_n^0(r) < \infty$ a.s.. $\square$

In Theorem 3.2 below, we show that the optimal value $v_1(r)$ is a distribution function in r in the sense that $v_1(\cdot)$ is nondecreasing and right continuous, $\lim_{r\to-\infty} v_1(r) = 0$ and $\lim_{r\to\infty} v_1(r) = 1$. However, we notice that a criterion function $P(X_\tau \le r)$ is not necessarily a distribution function in r for any given $\tau \in \Gamma_n$, as in the following example.

Example 3.1. Let (X_n) be a stochastic process such that $X_1 = 0$ and $X_n = 2$ for each $n \ge 2$, and let $\tau = \tau(r)$ be a stopping time such that $\tau(r) = 1$ if $r \le 1$ and $\tau(r) = 2$ otherwise. Then we easily have

$$P(X_\tau \le r) = I_{[0,1]\cup[2,\infty)}(r).$$

Hence, $P(X_\tau \le r)$ is not a distribution function in r. However, from Propositions 3.1–3.3 the optimal value $v_1(r)$ is represented by

$$v_1(r) = I_{[2,\infty)}(r),$$

which is a distribution function in r, and an optimal stopping time is $\tau_1^0(r) = 2$ for every $r \in R$.

Lemma 3.1 *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. If bounded and $\mathcal{F}$-measurable random variable $Z(r)$ is a.s. a distribution function in r, then $E[Z(r)|\mathcal{G}]$ is a.s. a distribution function in r and so is $E[Z(r)]$.*

Proof: From the definition of a conditional expectation and the dominated convergence theorem it easily follows that this lemma is true. □

Lemma 3.2 *For each n and N with $1 \leq n \leq N$, $\lim_{r\to\infty} V_n^N(r) = \lim_{r\to\infty} V_n(r) = 1$ a.s., and $\lim_{r\to\infty} v_n^N(r) = \lim_{r\to\infty} v_n(r) = 1$.*

Proof: We have

$$1 \geq V_n^N(r) \geq V_n(r) = \operatorname*{ess\,inf}_{\tau\in\Gamma_n} P(X_\tau \leq r|\mathcal{F}_n) \geq P(\sup_{k\geq n} X_k^+ \leq r|\mathcal{F}_n).$$

However, since $P(\sup_k X_k^+ < \infty) = 1$, we have $\lim_{r\to\infty} P(\sup_{k\geq n} X_k^+ \leq r|\mathcal{F}_n) = 1$ a.s. for each n, which implies that $\lim_{r\to\infty} V_n(r) = \lim_{r\to\infty} V_n^N(r) = 1$ a.s.. Similarly, the relations for the convergences of $v_n(r)$ and $v_n^N(r)$ hold. □

Theorem 3.2 *For each n and N with $1 \leq n \leq N$, $V_n(\cdot)$ and $V_n^N(\cdot)$ are distribution functions on R, and so are $v_n(\cdot)$ and $v_n^N(\cdot)$.*

Proof: We shall prove by induction regarding to n that $V_n^N(\cdot)$ is a distribution function a.s.. We first notice that $Y_n(r) = I_{(X_n\leq r)}$ is a distribution function in r for each $n \in \mathbf{N}$. Let $N \geq 1$ be arbitrarily fixed. When $n = N$, we see that $V_N^N(r) = Y_N(r)$ is a distribution function in r. Assume that $V_{n+1}^N(r)$ is a.s. a distribution function in r for $n \leq N-1$. From Lemma 3.1, $E[V_{n+1}^N(r)|\mathcal{F}_n]$ is also a distribution function a.s.. Hence $V_n^N(r) = \min(Y_n(r), E[V_{n+1}^N(r)|\mathcal{F}_n])$ is a distribution function a.s., which implies that $v_n^N(r) = E[V_n^N(r)]$ is a distribution function in r. Thus it follows by induction that $V_n^N(r)$ and $v_n^N(r)$ is a distribution function in r for each n and N with $1 \leq n \leq N$. Next, since a sequence $\{V_n^N(r)\}_{N=1}^\infty$ of functions is nonincreasing and $V_n(r) = \lim_{N\to\infty} V_n^N(r)$, it follows that $V_n(r)$ is nondecreasing, right continuous at r and $\lim_{r\to-\infty} V_n(r) = 0$. Combining those facts with Lemma 3.2, we obtain that $V_n(r)$ is a.s. a distribution function in r. Similarly, $v_n(\cdot) = \lim_{N\to\infty} v_n^N(\cdot)$ is a distribution function. □

4. A Secretary Problem

In this section we apply our minimizing risk model to a secretary problem (a best choice problem with no information) and explicitly give an optimal value and an optimal stopping time.

Let $A_1, A_2, \ldots, A_N$ denote a permutation of the integers $1, 2, \ldots, N$, where all permutations are equally likely and integer 1 corresponds to the best candidate, 2 to the second best one, and N to the worst one, that is, A_n denotes absolute rank of the nth candidate to appear. Also, let Z_n denote relative rank of the nth candidate to appear, and let $\mathcal{F}_n = \sigma(Z_1, Z_2, \ldots, Z_n)$. We define a process (X_n) by $X_n = P(A_n = 1|\mathcal{F}_n)$, $n = 1, 2, \ldots, N$. A criterion in a classical secretary problem is $E(X_\tau) = P(A_\tau = 1)$, which we maximize with respect to stopping time τ. We define an optimal value on the classical problem by $w_n^N = \inf_{\tau\in\Gamma_n^N} E(X_\tau)$. Then it follows from Chow et al.[4] that $w_1^N \geq w_2^N \geq \cdots \geq w_N^N = E(X_N) = N^{-1}$ and an optimal stopping time is represented by $\tau^* = \inf\{n \geq t^*|Z_n = 1\}$ where $t^* = t^*(N) = \inf\{t|\sum_{k=t}^N k^{-1} \leq 1\}$. Furthermore it is known that $\lim_{N\to\infty} w_1^N = \lim_{N\to\infty} t^*/N = e^{-1}$.

Now we consider our minimizing risk problem with a threshold probability criterion $E[Y_\tau(r)|\mathcal{F}_n] = P[X_\tau \le r|\mathcal{F}_n]$. We then have the following theorem for an optimal value and an optimal stopping time. By the way, it is easy to see that

$$E[Y_n(r)] = (n-1)/n \cdot I_{[0,n/N)}(r) + I_{[n/N,\infty)}(r).$$

Theorem 4.1 *For each N, the optimal value $v_1^N(r) = E(V_1^N)$ is represented by*

$$v_1^N(r) = \sum_{k=1}^{N}(k-1)/N \cdot I_{[(k-1)/N,k/N)}(r) + I_{[1,\infty)}(r),$$

and an optimal stopping time is $\sigma_N^*(r) = \inf\{k \le n \le N|Z_n = 1\}$ if $(k-1)/N \le r < k/N, k = 1,\ldots,N$ and $\sigma_N^*(r) = 1$ otherwise, where $\inf\phi = N$.

Proof: From Proposition 3.2, we first have the recursive equation

$$V_N^N(r) = Y_N(r), \quad V_n^N(r) = \min(Y_n(r), E[V_{n+1}^N(r)|\mathcal{F}_n]), \quad 1 \le n \le N-1.$$

Since $V_{n+1}^N(r)$ is depend upon only $Z_{n+1},\ldots,Z_N$, it is independent of $\mathcal{F}_n$. Hence we obtain

$$V_n^N(r) = \min(Y_n(r), E[V_{n+1}^N(r)]) = \min(Y_n(r), v_{n+1}^N(r)),$$

where $v_{N+1}^N(r) = 1$ for every r. It also follows from the definition of v_n^N that

$$v_1^N \le v_2^N \le \cdots \le v_N^N(r) = E[Y_N(r)] = (N-1)/N \cdot I_{[0,1)}(r) + I_{[1,\infty)}(r),$$

since

$$V_N^N(r) = Y_N(r) = I_{(X_N \le r)} = \begin{cases} 0 & \text{if } (r<0) \text{ or } (0 \le r < 1, Z_N = 1) \\ 1 & \text{if } (0 \le r < 1, Z_N > 1) \text{ or } (r \ge 1). \end{cases}$$

By backward induction, we shall show that for each $n = 1,\ldots,N$

$$v_n^N(r) = (n-1)/N \cdot I_{[0,n/N)}(r) + \sum_{k=n+1}^{N}(k-1)/N \cdot I_{[(k-1)/N,k/N)}(r) + I_{[1,\infty)}(r),$$

where $\sum_{k=N+1}^{N} \cdot = 0$. When $n = N$, it is true. Assume that the relation holds for $n = N,\ldots,2$. We notice that

$$V_{n-1}^N(r) = \min(Y_{n-1}(r), v_n^N(r))$$

and

$$Y_{n-1}(r) = \begin{cases} 0 & \text{if } (r<0) \text{ or } (0 \le r < (n-1)/N, Z_{n-1} = 1) \\ 1 & \text{if } (0 \le r < (n-1)/N, Z_{n-1} > 1) \text{ or } (r \ge (n-1)/N). \end{cases}$$

When $r < 0$, then we have $Y_{n-1}(r) = 0$ and hence $V_{n-1}^N(r) = 0$. Thus $v_{n-1}^N(r) = 0$. When $0 \le r < (n-1)/N$, we obtain $V_{n-1}^N(r) = v_n^N(r)I_{(Z_{n-1}>1)}$, and hence $v_{n-1}^N(r) = v_n^N(r)P(Z_{n-1} > 1) = (n-2)/N$, since $v_n^N(r) = (n-1)/N$ and $P(Z_{n-1} > 1) = (n-2)/(n-1)$. If $(n-1)/N \le r < n/N$, we have $Y_{n-1}(r) = 1$ and hence $V_{n-1}^N(r) = v_n^N(r) = (n-1)/N$. Thus $v_{n-1}^N(r) = (n-1)/N$. If $r \ge n/N$, we similarly have $v_{n-1}^N(r) = v_n^N(r)$. Therefore we obtain the desired relation for $n-1$, that is, the above relation holds for each $1 \le n \le N$. In particular, we have

$$v_1^N(r) = \sum_{k=1}^{N}(k-1)/N \cdot I_{[(k-1)/N,k/N)}(r) + I_{[1,\infty)}(r).$$

Next we shall find the optimal stopping time $\sigma_N^*(r)$. From Proposition 3.2, we have $\sigma_N^*(r) = \inf\{1 \le n \le N | V_n^N(r) = Y_n(r)\}$, where $\inf \phi = N$. Since $V_n^N(r) = \min(Y_n(r), v_{n+1}^N(r))$ where $v_{N+1}^N(r) = 1$ for every r, we have $\sigma_N^*(r) = \inf\{1 \le n \le N | Y_n(r) \le v_{n+1}^N(r)\}$. When $r < 0$, we have $Y_1(r) = 0$ and hence $\sigma_N^*(r) = 1$. If $r \ge 1$, then $Y_1(r) = v_2^N(r) = 1$ and hence $\sigma_N^*(r) = 1$. Let $0 \le r < 1$. Since $0 \le v_n^N(r) < 1 (1 \le n \le N)$ from the above result, it follows that for $n = 1, \cdots, N-1$ $Y_n(r) \le v_{n+1}^N(r)$ if and only if $Y_n(r) = 0$, that is, $Z_n = 1$ and $r < n/N$. Thus we have $\sigma_N^*(r) = \inf\{1 \le n \le N | Z_n = 1, r < n/N\}$, where $\inf \phi = N$. Hence if $(k-1)/N \le r < k/N (k = 1, \cdots, N)$, then $\sigma_N^*(r) = \inf\{k \le n \le N | Z_n = 1\}$, where $\inf \phi = N$. □

Remark 4.1. From the above theorem we easily see that $v_1^N(r)$ converges to a uniform distribution function on $[0,1]$ as $N \to \infty$. Let W be a random variable which is distributed uniformly on $[0,1]$. Then we have $P(W \le r) = r$ if $0 \le r \le 1$, in particular, $P(W \le e^{-1}) = e^{-1}$ and $E[W] = 1/2$, which is larger than $\lim_{n\to\infty} w_1^N = e^{-1}$.

5. A Parking Problem

In this section we consider a parking problem as a minimizing risk model.

A motorist is driving along a straight highway from a starting place 1 toward his destination N, and he is looking for a parking place. As he drives along, he can observed only one parking place at a time, and he notes whether or not it is occupied. We assume that unoccupied places occur independently and that the probability that any given place will be occupied is $p (0 < p < 1)$. In other words, let Z_n be a random variable such that $Z_n = 1$ if nth parking place is occupied and $Z_n = 0$ otherwise. Then $Z_1, Z_2, \ldots, Z_N, \ldots$ is independently and identically distributed and $P(Z_n = 1) = p = 1 - P(Z_n = 0)$. Let $q = 1 - p$. If a space is unoccupied, he may stop and park there; if it is occupied, he is forced to continue. His loss when he parks in nth place is the distance $|N - n|$ he must walk to his destination N. When $Z_N = 1$, that is, Nth parking space is occupied, he justly stop at the first unoccupied place $n \ge N$ such that $Z_n = 0$. Thus his loss at N when $Z_N = 1$ is

$$P(Z_{N+1} = 0) + \sum_{k=2}^{\infty} kP(Z_{N+k} = 0) \prod_{j=1}^{k-1} P(Z_{N+j} = 1) = \sum_{k=1}^{\infty} kqp^{k-1} = 1/q.$$

Hence, to keep up the original problem, we define the reward process (X_n) by

$$\begin{cases} X_n = (-M) \cdot I_{(Z_n=1)} + (n-N) \cdot I_{(Z_n=0)}, \ 1 \le n < N, \\ X_N = -1/q \cdot I_{(Z_N=1)}, \end{cases}$$

where M is a sufficiently large number, and let $\mathcal{F}_n$ be a σ–field generated by $Z_1, \ldots, Z_n$. Then we can formulate our parking problem as an optimal stopping problem in a finite horizon case.

A criterion in a classical parking problem is the expectation $E[X_\tau]$ which we maximize with respect to stopping time τ. For this problem, for example, see Chow et al.[4] and DeGroot[5].

Now we apply our minimizing risk model to the above parking problem with a criterion $E[Y_\tau(r)|\mathcal{F}_n] = P[X_\tau \le r|\mathcal{F}_n]$. We first set $m = 1$ if $1 - N > -1/q$ and define an integer $m \in \{2, \ldots, N\}$ satisfying $m - N - 1 \le -1/q < m - N$ otherwise. Then we can give an optimal value and an optimal stopping time in the following theorem.

Theorem 5.1 *For each N, the optimal value $v_1^N(r)$ is represented by*

$$v_1^N(r) = p^{N-m+1} \cdot I_{[-1/q,m-N)}(r) + \sum_{k=1}^{N-m} p^k \cdot I_{[-k,-k+1)}(r) + I_{[1,\infty)}(r),$$

where $\sum_{k=1}^{0} \cdot = 0$, and an optimal stopping time is $\sigma_N^*(r) = \inf\{k \le n < N | Z_n = 0\}$ if $k-N-1 \le r < k-N$ $(k = 1, \ldots, N)$, $\sigma_N^*(r) = \inf\{1 \le n < N | Z_n = 0\}$ if $-M \le r < -N$ and $\sigma_N^*(r) = 1$ otherwise, where $\inf \phi = N$.

Proof: Since $V_{n+1}^N(r)$ is independent of $\mathcal{F}_n$, we have

$$V_N^N(r) = Y_N(r), \ V_n^N(r) = \min(Y_n(r), v_{n+1}^N(r)).$$

Also, since

$$Y_N(r) = I_{(X_N \le r)} = \begin{cases} 0 & \text{if } \ (r < -1/q) \text{ or } (-1/q \le r < 0, Z_N = 0) \\ 1 & \text{if } \ (-1/q \le r < 0, Z_N = 1) \text{ or } (r \ge 0). \end{cases}$$

we obtain

$$v_1^N \le v_2^N \le \ldots \le v_N^N(r) = E[Y_N(r)] = p \cdot I_{[-1/q,0)}(r) + I_{[0,\infty)}(r).$$

We shall show by induction that for each $n = m, \ldots, N$

$$v_n^N(r) = p^{N-n+1} \cdot I_{[-1/q,n-N)}(r) + \sum_{k=1}^{N-n} p^k \cdot I_{[-k,-k+1)}(r) + I_{[1,\infty)}(r).$$

When $n = N$, it holds from the above fact. Assume that the relation holds for $n = N, \ldots, m+1$. We first notice that

$$Y_{n-1}(r) = \begin{cases} 0 & \text{if } \ (r < -M) \text{ or } (-M \le r < n-N-1, Z_{n-1} = 0) \\ 1 & \text{if } \ (-M \le r < n-N-1, Z_{n-1} = 1) \text{ or } (r \ge n-N-1). \end{cases}$$

When $r < -M$, then we have $V_{n-1}^N(r) = Y_{n-1}(r) = 0$ and hence $v_{n-1}^N(r) = 0$. When $-M \le r < n-N-1$, we obtain $V_{n-1}^N(r) = v_n^N(r) I_{(Z_{n-1}=1)}$, and hence

$$v_{n-1}^N(r) = v_n^N(r) P(Z_{n-1} = 1) = p v_n^N(r) = \begin{cases} 0 & \text{if } r < -1/q \\ p^{N-(n-1)+1} & \text{if } -1/q \le n-N-1. \end{cases}$$

If $n-N-1 \le r < 0$, we have $Y_{n-1}(r) = 1$ and hence $V_{n-1}^N(r) = v_n^N(r)$. Thus $v_{n-1}^N(r) = p^k$ when $-k \le r < -k+1$ $(k = N-(n-1), \ldots, 1)$. If $r \ge 0$, we similarly have $v_{n-1}^N(r) = v_n^N(r) = 1$. Therefore, by induction, the desired relation holds for each $m \le n \le N$. By the same method, we have $v_n^N(r) = v_m^N(r)$ for any $n = m-1, \ldots, 2, 1$. In particular, we have

$$v_1^N(r) = v_m^N(r) = p^{N-m+1} \cdot I_{[-1/q,m-N)}(r) + \sum_{k=1}^{N-m} p^k \cdot I_{[-k,-k+1)}(r) + I_{[1,\infty)}(r).$$

Next we find the optimal stopping time $\sigma_N^*(r)$. By the same argument as in Theorem 4.1 we have $\sigma_N^*(r) = \inf\{1 \le n \le N | Y_n(r) \le v_{n+1}^N(r)\}$, where $v_{N+1}^N(r) = 1$ for every r. When $r < -M$, we have $Y_1(r) = 0$ and hence $\sigma_N^*(r) = 1$. If $r \ge 0$, then $Y_1(r) = v_2^N(r) = 1$ and hence $\sigma_N^*(r) = 1$. Let $-M \le r < 0$. Since $0 \le v_n^N(r) < 1, 1 \le n \le N$, we easily see that for $n = 1, \ldots, N-1$ $Y_n(r) \le v_{n+1}^N(r)$ if and only if $Y_n(r) = 0$, that is, $Z_n = 0$ and $r < n-N$.

Thus we have $\sigma_N^*(r) = \inf\{1 \le n \le N | Z_n = 0, r < n - N\}$, where $\inf \phi = N$. Hence if $k - N - 1 \le r < k - N (k = 1, \ldots, N)$, then $\sigma_N^*(r) = \inf\{k \le n \le N | Z_n = 0\}$, where $\inf \phi = N$. If $-M \le r < -N$, then $\sigma_N^*(r) = \inf\{1 \le n \le N | Z_n = 0\}$, where $\inf \phi = N$. □

Remark 5.1. Let W_N is a random variable which corresponds to $v_1^N(r)$, that is, $v_1^N(r) = P(W_N \le r)$. Since $N - m$ is constant, say L, for every sufficiently large N, it follows from the above theorem that $P(W_N = -n) = qp^n, n = 0, 1, \ldots, L$ and $P(W_N = -1/q) = p^{L+1}$, which implies that $v_1^N(r)$ is a truncated geometric distribution.

6. Job Search Problems

Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables with a distribution function $F(z)$ and $E[|Z_n|] < \infty$. We observe Z_n sequentially and stop at any time. If we stop at time n we receive the payoff $X_n = f_n(Z_1, \cdots, Z_n) - c_n$, where c_n is a cost for observations till time n and we assume that $c_n \le c_{n+1}$ for each $n \ge 1$. A well known problem is to find a stopping time τ which maximizes the expected payoff $E[X_\tau]$ and it is called a job search problem or a house buying problem when $f_n(Z_1, \cdots, Z_n) = Z_n$ or $f_n(Z_1, \cdots, Z_n) = \max(Z_1, \ldots, Z_n)$, which are investigated by MacQueen and Miller[9], Sakaguchi[14] and Chow and Robbins[2, **?**].

In this section we consider our minimizing risk problem with a threshold probability criterion $P[X_\tau \le r]$ for two cases : $f_n(Z_1, \cdots, Z_n) = Z_n$ and $f_n(Z_1, \cdots, Z_n) = \max(Z_1, \ldots, Z_n)$.

For each r, we define an integer $L(r)$ by

$$L(r) = \inf\{n \ge 1 | F(r + c_{n+1}) = 1\},$$

where $\inf \phi = \infty$.

Theorem 6.1 *Let $X_n = Z_n - c_n$ for every $n \ge 1$ or $X_n = \max(Z_1, \ldots, Z_n) - c_n$ for every $n \ge 1$. For each N, the optimal value $v_1^N(r)$ is represented by*

$$v_1^N(r) = \prod_{k=1}^{N} F(r + c_k),$$

and an optimal stopping time is $\sigma_N^*(r) = \min(L(r), \inf\{1 \le n \le N | Z_n > r + c_n\})$ where $\inf \phi = N$.

Proof: First, let $X_n = Z_n - c_n$ for every $n \ge 1$. It follows from Proposition 3.2 that

$$V_N^N(r) = Y_N(r), \quad V_n^N(r) = \min(Y_n(r), v_{n+1}^N(r)), \quad 1 \le n \le N - 1,$$

since $V_{n+1}^N(r)$ is independent of $\mathcal{F}_n$. We shall show by induction that $v_n^N(r) = \prod_{k=n}^N F(r + c_k), 1 \le n \le N$. When $n = N$, we have

$$v_N^N(r) = E[Y_N(r)] = P(Z_N \le r + c_N) = F(r + c_N)$$

and hence it is true. Assume that it holds for $n+1$. Then we have $v_{n+1}^N(r) = \prod_{k=n+1}^N F(r+c_k)$. Since $Y_n(r) = I_{(Z_n \le r + c_n)}$, it follows that

$$V_n^N(r) = v_{n+1}^N(r) I_{(Z_n \le r + c_n)},$$

which implies that

$$v_n^N(r) = E[V_n^N(r)] = v_{n+1}^N(r) P(Z_n \le r + c_n) = \prod_{k=n}^{N} F(r + c_k).$$

Thus by induction we have the desired solution $v_n^N(r)$ for every $n : 1 \le n \le N$, and hence $v_1^N(r) = \prod_{k=1}^N F(r+c_k)$.

Next, since $V_n^N(r) = \min(Y_n(r), v_{n+1}^N(r))$ where $v_{N+1}^N(r) = 1$, we have $\sigma_N^*(r) = \inf\{1 \le n \le N | Y_n(r) \le v_{n+1}^N(r)\}$. When $F(r+c_N) < 1$, it follows that $L(r) \ge N$ and $v_{n+1}^N(r) = \prod_{k=n+1}^N F(r+c_k) < 1$ and hence

$$\begin{aligned}\sigma_N^*(r) &= \inf\{1 \le n \le N | Y_n(r) = 0\}\\ &= \inf\{1 \le n \le N | Z_n > r + c_n\}\\ &= \min(L(r), \inf\{1 \le n \le N | Z_n > r + c_n\}).\end{aligned}$$

Let $F(r+c_N) = 1$. Since $v_{n+1}^N(r) = 1$ for every $n \ge L(r)$, we have

$$\begin{aligned}\sigma_N^*(r) &= \min(L(r), \inf\{1 \le n \le N | Y_n(r) = 0\})\\ &= \min(L(r), \inf\{1 \le n \le N | Z_n > r + c_n\}).\end{aligned}$$

We shall next consider another case : $X_n = M_n - c_n, n \ge 1$, where $M_n = \max(Z_1, \ldots, Z_n)$. Proposition 3.2 implies that

$$V_N^N(r) = Y_N(r), \quad V_n^N(r) = \min(Y_n(r), E[V_{n+1}^N(r)|\mathcal{F}_n]), \quad 1 \le n \le N-1.$$

We shall prove by induction that $V_n^N(r) = I_{(M_n \le r+c_n)} \prod_{k=n+1}^N F(r+c_k)$ where $\prod_{k=N+1}^N \cdot = 1$. Since $V_N^N(r) = I_{(X_N \le r)} = I_{(M_N \le r+c_N)}$ it is true for $n = N$. Assume that it holds for $n+1$. Then we have $V_{n+1}^N(r) = I_{(M_{n+1} \le r+c_{n+1})} \prod_{k=n+2}^N F(r+c_k)$. Hence it follows that

$$\begin{aligned}E[V_{n+1}^N(r)|\mathcal{F}_n] &= E[I_{(\max(M_n, Z_{n+1}) \le r+c_{n+1})}|\mathcal{F}_n] \prod_{k=n+2}^N F(r+c_k)\\ &= I_{(M_n \le r+c_{n+1})} \prod_{k=n+1}^N F(r+c_k).\end{aligned}$$

Thus we have

$$\begin{aligned}V_n^N(r) &= \min(I_{(M_n \le r+c_n)}, I_{(M_n \le r+c_{n+1})} \prod_{k=n+1}^N F(r+c_k))\\ &= I_{(M_n \le r+c_n)} \prod_{k=n+1}^N F(r+c_k),\end{aligned}$$

which, by induction, implies that the relation is true for every $1 \le n \le N$. Therefore we easily see that

$$v_n^N(r) = E[V_n^N] = P(M_n \le r+c_n) \prod_{k=n+1}^N F(r+c_k) = (F(r+c_n))^n \prod_{k=n+1}^N F(r+c_k)$$

and hence

$$v_1^N(r) = \prod_{k=1}^N F(r+c_k).$$

We shall finally find an optimal stopping time for this case. When $F(r+c_N) < 1$ we have

$$\begin{aligned}\sigma_N^*(r) &= \inf\{1 \le n \le N | Y_n(r) \le E[V_{n+1}^N(r)|\mathcal{F}_n]\}\\ &= \inf\{1 \le n \le N | Y_n(r) = 0\}\\ &= \min(L(r), \inf\{1 \le n \le N | M_n > r + c_n\}),\end{aligned}$$

since $L(r) \geq N$ and $E[V_{n+1}^N(r)|\mathcal{F}_n] < 1$ for each $n : 1 \leq n \leq N$. When $F(r + c_N) = 1$, we have $\prod_{k=n+1}^N F(r + c_k) = 1$ for each $n \geq L(r)$ and hence

$$\sigma_N^*(r) = \min(L(r), \inf\{1 \leq n \leq N | M_n > r + c_n\}).$$

However it is easily checked that $\inf\{1 \leq n \leq N|M_n > r+c_n\} = \inf\{1 \leq n \leq N|Z_n > r+c_n\}$. Thus we derive the desired optimal stopping time. □

Remark 6.1. From the proof of the above theorem we see that optimal values $v_1^N(r)$ and optimal stopping times $\sigma_N^*(r)$ for two cases coincide with, though $v_n^N(r), n \geq 2$, are different from. By the way, Chow and Robbins[2] show that optimal values as well as optimal stopping times for two cases agree with in the problem with the expectation criterion $E[X_\tau]$.

In infinite horizon case, it is easily obtained that the optimal value is $v_1(r) = \lim_{N\to\infty} v_1^N(r) = \prod_{k=1}^{L(r)} F(r+c_k)$ and the optimal stopping time is $\tau_1^0(r) = \min(L(r), \inf\{n \geq 1|Z_n > r + c_n\})$.

Finally we give a simple example.

Example 6.1. Let F be a uniform distribution function on $[0, 1]$ and let $c_n = nc, n \geq 1$, where c is a positive constant. Then, for any sufficiently large N, the optimal value is

$$v_1(r) = v_1^N(r) = \prod_{k=1}^{L(r)} (r + kc) I_{(-c,1-c))}(r) + I_{[1-c,\infty)}(r)$$

and the optimal stopping time is

$$\tau_1^0(r) = \sigma_N^*(r) = \min(L(r), \inf\{n \geq 1|Z_n > r + nc\}).$$

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