Journal of the Operations Research
Society of Japan
2003, Vol. 46, No. 2, 220-241

SOJOURN TIME IN A QUEUE WITH CLUSTERED PERIODIC ARRIVALS

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(Received Received May 27, 2002; Revised December 24, 2002)

Abstract This paper considers a FIFO single-server queue with independent and homogeneous sources. Each source generates exactly one message every interval of fixed length. Each message is then divided into a constant number of fixed-size cells and these cells arrive to the queue back to back as if they form a train of cells. We call this arrival process clustered periodic arrivals. The queue with clustered periodic arrivals is an obvious generalization of $\sum D/D/1$ queue which corresponds to the case that each message consists of only one cell. This paper derives the stationary probability distributions of sojourn times of respective cells in a message. An interesting feature of these sojourn time distributions is that they are *not* continuous functions of time, but they have masses at multiples of the cell transmission time. This paper also derives the joint probability distribution of differences between sojourn times of successive cells in a message and the mean waiting times of respective cells in a message. At last, the overall mean waiting time in the queue with clustered periodic arrivals is compared with those in the corresponding queues with dispersed periodic arrivals and periodic batch arrivals, and the efficiency of dispersing cells is quantitatively shown by simple formulas.

Keywords: Queue, telecommunication, sojourn time, clustered periodic arrival

1. Introduction

ATM is considered as one of the most promising transfer technologies for implementing Broadband ISDN that provides service to such diverse traffic as video, still image, voice and data. In ATM, information flow is organized into fixed-size cells. CBR is one of service classes supported by ATM, where CBR sources generate bit streams at a constant rate. For example, video with fixed rate encoders and coding voice result in CBR traffic streams. Thus the performance of multiplexers with CBR traffic streams has been studied extensively.

So far, queues with the superposition of periodic arrival processes have been studied in a large number of research papers (see [9] and references therein). In particular, an algorithmic solution to the complementary distribution of queue length in the $\sum D/D/1$ queue was derived in [6], where all sources are independent and homogeneous and the number of sources is fixed. In [5] and [16], the steady-state delay distributions were derived in a discrete-time and continuous-time $\sum D/D/1$ queues, respectively. In [10], an analytical approach using the Ballot Theorems was shown to obtain steady-state delay distributions both in a discrete-time and continuous-time $\sum D/D/1$ queues. Note that in those papers, each source is assumed to generate cells one by one every time interval of fixed length.

According to a recent report on monitoring of CBR traffic generated by a CBR MPEG encoder, the cell stream generated by the CBR MPEG encoder turns not to be a CBR in cell-level, but it is a *clustered cell stream* at a constant rate [13]. More precisely, the CBR MPEG encoder generates eight cells (corresponding to the size of PDU of the upper layer)

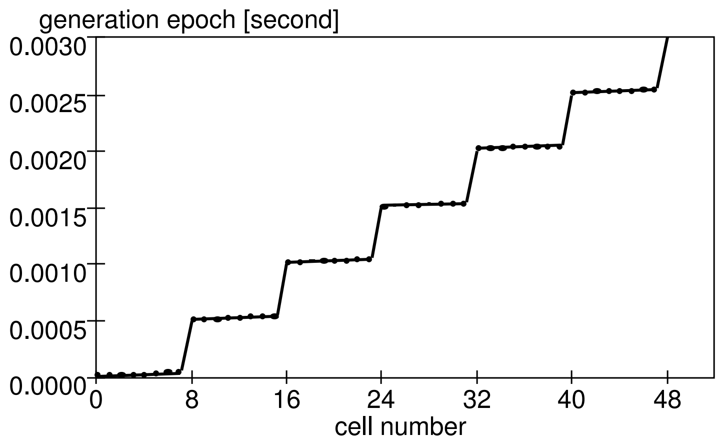


Figure 1: Cell generation epochs from a CBR MPEG encoder.

back to back at full rate every interval of fixed length (see Figure 1). Such a feature may be found in cell-level traffic of other streaming applications.

In this paper, we call clustered cells corresponding to a PDU of the upper layer a *message* and the arrival pattern in cell level *clustered periodic arrivals*. For this kind of arrivals, of particular interest is the length of time from a message generation epoch (i.e., a generation epoch of the first cell in a message) to the end of the last cell's transmission in the message. Cidon et al. [4] found the stationary probability distributions of unfinished work for queues with discrete-time clustered periodic arrivals and with fluid clustered periodic arrivals. Note that from those results, we can obtain only the stationary probability distribution of sojourn times of the first cells in messages (or the first part of fluid messages). Related work is also found in [1], [2], [3], [7], [8], [9], [12], [15] and [17], where each source is allowed to generate more than one cell every time interval of fixed length.

We consider a continuous-time single-server FIFO queue with independent and homogeneous clustered periodic arrivals. Note that our model is not a fluid model but a model with instantaneous arrivals, and therefore it is an obvious generalization of $\sum D/D/1$ queue which corresponds to the case that each message consists of only one cell. We obtain explicit formulas for the probability distributions of sojourn times of respective cells in a message. The message-level performance, which is important in dimensioning network resources, is then obtained by adding the constant lag (i.e., time interval between generation epochs of the first and last cells in a message) to the sojourn time of the last cell. As by-products, the stationary probability distributions of the amount of unfinished work and the queue length are also obtained. An interesting feature of the sojourn time distributions of respective cells in a message is that they are *not* continuous functions of time, but they have masses at multiples of the cell transmission time.

Moreover we obtain the joint probability distribution of differences between sojourn times of successive cells in a message, and using this result we derive the explicit expressions for the mean waiting times of respective cells. Finally, We will compare the overall mean waiting time of the queue with clustered periodic arrivals with those in the corresponding queues with dispersed periodic arrivals [9] and periodic batch arrivals, and the efficiency of dispersing cells is quantitatively shown by simple formulas.

The rest of this paper is organized as follows. In section 2, we describe the mathematical model. In section 3, we derives the probability distributions of sojourn times of respective cells. In section 4, we obtain the joint probability generating function of the distributions

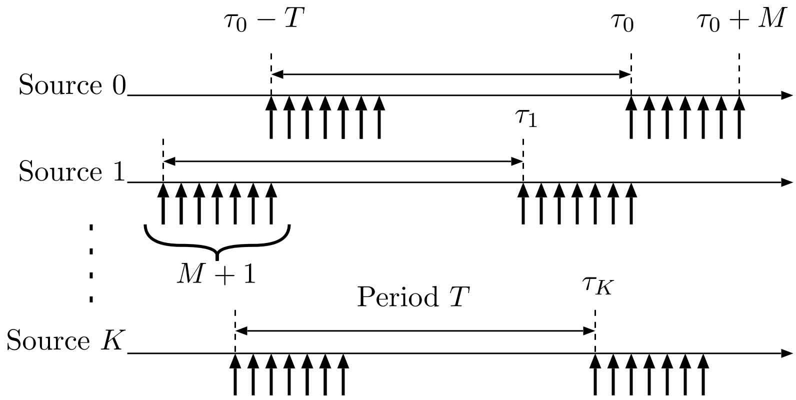


Figure 2: Original system.

of differences of sojourn times of successive cells. Finally, in section 5, we derive the overall mean waiting time and compare it with those in the corresponding systems with dispersed periodic arrivals and periodic batch arrivals.

2. Model and Equivalent Systems

2.1. Model description

We consider a work-conserving single-server queue fed by $K + 1$ ($K \geq 0$) independent and homogeneous sources labeled 0 to K. Each source generates messages of fixed length periodically with period of length T. We assume that source j ($j = 0, \ldots, K$) generates messages at time $\tau_j, \tau_j \pm T, \tau_j \pm 2T, \ldots$, where τ_j's are independent and identically distributed according to a uniform distribution in interval $[0, T)$. Each message is then divided into $M + 1$ ($M \geq 0$) cells of fixed length. When a message is generated at time t, the ith ($i = 1, \ldots, M + 1$) cell in the message arrives to the queue at time $t + i - 1$ (see Figure 2). The queue has a buffer of infinite capacity, so that no cell is lost. Cells are served by a single server on an FIFO basis. Service times of cells are constant and are chosen as a unit of time. Because the amount of work brought into the system during any interval of length T is equal to $(K + 1)(M + 1)$, we assume that

$$(K + 1)(M + 1) \leq T, \tag{1}$$

so that the system is stable.

Remark 1 *When the queue is stable, the maximum queue length is equal to $(M + 1)K + 1$ cells, so that no cells are lost if the capacity of the buffer is not less than $(M + 1)K + 1$ cells.*

We define U_t as the amount of unfinished work at time t. The amount U_t of unfinished work is decreasing at rate 1 while $U_t > 0$ and U_t has an upward jump by 1 when a cell arrives. We assume that U_t is right-continuous and has left-hand-side limits. Thus when a cell arrives at time t, U_t includes the service time of this cell, so that $U_t \geq 1$.

Let $A(x, y]$, $A[x, y]$ and $A(x, y)$ be the number of cells arriving in interval $(x, y]$, $[x, y]$ and (x, y), respectively. We now assume that the queueing process U_t begins at time $-\tau^*$ ($\tau^* > 0$) with $U_{-\tau^*} = 0$. We then have [4]

$$U_t = \max_{0 \leq u \leq t+\tau^*} \{A(t - u, t] - u\}, \qquad t \geq -\tau^*. \tag{2}$$

Proposition 1 *Suppose the stability condition (1) holds and $U_{-\tau^*} = 0$. Then $U_{t+T} = U_t$ for all $t \geq -\tau^* + T$.*

Proof. Substituting t in (2) by $t+T$, we obtain for all $t \geq -\tau^* + T$,

$$\begin{aligned} U_{t+T} &= \max_{0\leq u\leq t+\tau^*+T} (A(t+T-u, t+T] - u) \\ &= \max\left\{ \max_{0\leq u\leq t+\tau^*} (A(t+T-u, t+T] - u), \max_{t+\tau^*\leq u\leq t+\tau^*+T} (A(t+T-u, t+T] - u) \right\} \\ &= \max\left\{ \max_{0\leq u\leq t+\tau^*} (A(t-u, t] - u), \max_{t+\tau^*-T\leq u'\leq t+\tau^*} (A(t-u', t+T] - u' - T) \right\}, \end{aligned} \quad (3)$$

where we use the periodicity $A(t-u,t] = A(t+T-u,t+T]$ in the last equality. Note here that

$$\begin{aligned} &\max_{t+\tau^*-T\leq u'\leq t+\tau^*} (A(t-u', t+T] - u' - T) \\ &\qquad = \max_{t+\tau^*-T\leq u\leq t+\tau^*} (A(t-u, t] - u + A(t, t+T] - T) \\ &\qquad \leq \max_{t+\tau^*-T\leq u\leq t+\tau^*} (A(t-u, t] - u) \\ &\qquad \leq \max_{0\leq u\leq t+\tau^*} (A(t-u, t] - u), \end{aligned}$$

where the first inequality follows from the stability condition $A(t, t+T] = (K+1)(M+1) \leq T$ (see (1)). It then follows from (3) that

$$U_{t+T} = \max_{0\leq u\leq t+\tau^*} (A(t-u, t] - u) = U_t,$$

for all $t \geq -\tau^* + T$, which completes the proof. □

Because $K+1$ sources are independent and homogeneous, we focus on the sojourn time distribution of the $(i+1)$st ($i = 0, \ldots, M$) cells in messages generated by source 0. To do so, we shift the time axis by τ_0 in such a way that source 0 generates messages at time $0, \pm T, \pm 2T, \ldots$. Further, we assume that $U_{-\tau^*} = 0$ for some $\tau^* \geq 2T$ in the shifted time axis. It is easy to see from Proposition 1 that the system is periodic after epoch $-\tau^* + T$ ($\leq -T$) when the stability condition (1) holds. Thus the sojourn times for the $(i+1)$st cells is equivalent to the amount U_i of unfinished work at time i because cells are served on a FIFO basis and U_t is assumed to be right-continuous.

2.2. Equivalent systems

To obtain the probability distribution of the amount U_i ($i = 0, \ldots, M$) of unfinished work at time i in the original system, we follow an idea in [4]. Namely, we introduce an *equivalent system for the* $(i+1)$*st cells* ($i = 0, \ldots, M$) which is also called system E_i in short. In system E_i ($i = 0, \ldots, M$), nothing is changed except that all cells in messages generated in interval $(-\tau^*, i-M]$ in the original system arrive at the same time when the first cells arrive, i.e., batch arrivals (see Figure 3). Note that arrival epochs of cells in messages generated after time $i-M$ in system E_i are identical to those in the original system.

We define $U_t^{(i)}$ ($i = 0, \ldots, M$) as the amount of unfinished work at time t in the equivalent system E_i for the $(i+1)$st cells. Let $A^{(i)}(x,y]$, $A^{(i)}[x,y]$ and $A^{(i)}(x,y)$ denote the number of cells arriving in interval $(x,y]$, $[x,y]$ and (x,y), respectively, in system E_i. We then have

$$U_t^{(i)} = \max_{0\leq u\leq t+\tau^*} \left(A^{(i)}(t-u, t] - u \right), \qquad t \geq -\tau^*. \quad (4)$$

Note here that

$$A^{(i)}(-\tau^*, t] = A(-\tau^*, t], \qquad t \geq i. \quad (5)$$

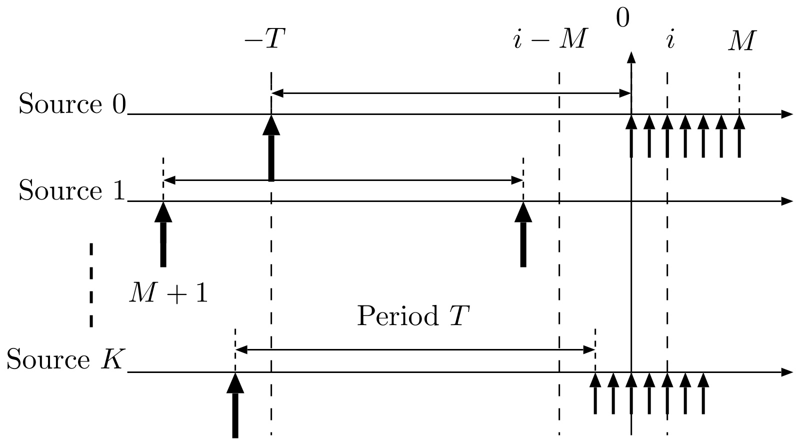


Figure 3: System E_i: Equivalent system for the $(i+1)$st cells.

Proposition 2 *The amount $U_i^{(i)}$ $(i=0,\dots,M)$ of unfinished work at time i in the equivalent system for the $(i+1)$st cells is identical to the amount U_i of unfinished work at time i in the original system.*

Proof. To prove the theorem, we first consider the corresponding system with batch arrivals, where it is empty at time $-\tau^*$, all messages are generated at the same time as in the original system and $M+1$ cells in each message simultaneously arrive to the system when the message is generated. Let $C(x,y]$ denote the number of messages generated in interval $(x,y]$. We define $U_t^{(B)}$ as the amount of unfinished work at time t in the corresponding system with batch arrivals. We then have

$$U_t^{(B)} = \max_{0\le u\le t+\tau^*} \left((M+1)C(t-u,t]-u\right), \qquad t\ge -\tau^*. \tag{6}$$

Note that $U_t^{(i)} = U_t^{(B)}$ $(i=0,\dots,M)$ for $t\in[-\tau^*, i-M]$ because $A^{(i)}(t-u,t]=(M+1)C(t-u,t]$ for all $t\in(-\tau^*,i-M]$ and $u\in(0,t+\tau^*]$, and in general,

$$U_t \le U_t^{(i)} \le U_t^{(B)}, \qquad t\ge -\tau^*, i=0,\dots,M, \tag{7}$$

because $A(t-u,t]\le A^{(i)}(t-u,t]\le (M+1)C(t-u,t]$ for all $t>-\tau^*$ and $u\in(0,t+\tau^*]$.

It is easy to see from (7) that if $U_t^{(B)}=0$, $U_t=U_t^{(i)}=0$. Conversely, if $U_t=0$, this implies that all *messages* generated in $(-\tau^*,t]$ have been already served before time t, so that $U_t^{(i)}=U_t^{(B)}=0$. Thus we have the following lemma.

Lemma 1 *Both the original system and the equivalent system E_i for the $(i+1)$st cells are busy if and only if $U_t^{(B)}>0$.*

When the stability condition (1) holds, it is easy to see that there exists a time instant t, at which $U_{t-}^{(B)}=0$ and $U_t^{(B)}=M+1>0$, in any interval of length T beginning after time $-T$ $(\ge -\tau^*+T)$. Because a message from source 0 is generated at time 0, it follows that $U_t^{(B)}>0$ for all $t\in[0,i]$. Thus there exists a time instant $t^*\in[i-T,0)$ such that $U_{t^*-}^{(B)}=0$ and $U_\tau^{(B)}>0$ for all $\tau\in[t^*,i]$. Thus it follows from Lemma 1 that both the original system and system E_i are busy in $[t^*,i]$. Further, Lemma 1 (or (7)) implies that $U_{t^*-}=U_{t^*-}^{(i)}=0$. Therefore we obtain

$$U_i = A[t^*,i]-(i-t^*), \tag{8}$$
$$U_i^{(i)} = A^{(i)}[t^*,i]-(i-t^*). \tag{9}$$

Note here that $U_{t^*-} = U^{(i)}_{t^*-} = 0$ implies that all cells arriving in interval $(-\tau^*, t^*)$ have already been served before time t^* both in the original system and the equivalent system E_i for the $(i+1)$st cells. Thus

$$A(-\tau^*, t^*) = A^{(i)}(-\tau^*, t^*),$$

from which and (5), it follows that

$$A[t^*, i] = A^{(i)}[t^*, i]. \tag{10}$$

Finally, with (8), (9) and (10), we have

$$\begin{aligned} U_i &= A[t^*, i] - (i - t^*) \\ &= A^{(i)}[t^*, i] - (i - t^*) = U^{(i)}_i, \end{aligned}$$

which complete the proof. □

3. Sojourn Time Distributions

In the preceding section, we showed that sojourn times of the $(i+1)$st cells have the same probability distribution as $U^{(i)}_i$ does. Thus, in this section, we consider the amount $U^{(i)}_i$ of unfinished work at time i in the equivalent system for the $(i+1)$st cells to obtain the sojourn time distribution of the $(i+1)$st cells in the original system.

3.1. Equation for $U^{(i)}_i$

Let $B(x, y]$ denote the number of messages arriving in interval $(x, y]$ from sources other than source 0. We define $G(x, y]$ as the length of an interval from time x to the first generation epoch of messages in interval $(x, y]$.

$$G(x, y] = \arg\min\{q \in (x, y] \mid B(x, q] > 0\} - x.$$

Let $G(x, y] = y - x$ if $B(x, y] = 0$.

Theorem 1 $U^{(i)}_i$ *($i = 0, \ldots, M$) is given by*

$$\begin{aligned} U^{(i)}_i = \max\Big\{ &G(i-M, 0], \max_{0 \le u \le T-M} ((M+1)B(i-M-u, i-M] - u) \Big\} \\ &+ \sum_{k=1}^{M} kB(i-k, i-k+1] + i - M + 1. \end{aligned} \tag{11}$$

Proof. Setting $t = i$ in (4) yields

$$\begin{aligned} U^{(i)}_i &= \max_{0 \le u \le i+\tau^*} \left(A^{(i)}(i-u, i] - u \right) \\ &= \max\left\{ \max_{0 \le u \le T} \left(A^{(i)}(i-u, i] - u \right), \max_{T \le u \le i+\tau^*} \left(A^{(i)}(i-u, i] - u \right) \right\}. \end{aligned} \tag{12}$$

Note here that $\tau^* \ge 2T$ and according to the same argument as in the proof of Proposition 1,

$$\begin{aligned} &\max_{T \le u \le i+\tau^*} \left(A^{(i)}(i-u, i] - u \right) \\ &\quad = \max_{0 \le u \le i+\tau^*-T} \left(A^{(i)}(i-T-u, i] - u - T \right) \\ &\quad = \max_{0 \le u \le i+\tau^*-T} \left(A^{(i)}(i-T-u, i-u] - T + A^{(i)}(i-u, i] - u \right) \\ &\quad \le \max_{0 \le u \le i+\tau^*-T} \left(A^{(i)}(i-u, i] - u \right) \\ &\quad = \max\left\{ \max_{0 \le u \le T} \left(A^{(i)}(i-u, i] - u \right), \max_{T \le u \le i+\tau^*-T} \left(A^{(i)}(i-u, i] - u \right) \right\}. \end{aligned}$$

Thus when $i+\tau^* = kT + x$ for some positive integer k and $0 \le x < T$, we repeat the above procedure and obtain

$$\begin{aligned}\max_{T\le u\le i+\tau^*}\left(A^{(i)}(i-u,i]-u\right) &\le \max\left\{\max_{0\le u\le T}\left(A^{(i)}(i-u,i]-u\right), \max_{0\le u\le x}\left(A^{(i)}(i-u,i]-u\right)\right\}\\ &= \max_{0\le u\le T}\left(A^{(i)}(i-u,i]-u\right). \qquad (13)\end{aligned}$$

Therefore we obtain from (12) and (13)

$$\begin{aligned}U_i^{(i)} &= \max_{0\le u\le T}\left\{A^{(i)}(i-u,i]-u\right\}\\ &= \max\left\{\max_{0\le u\le M}\left(A^{(i)}(i-u,i]-u\right), \max_{M\le u\le T}\left(A^{(i)}(i-u,i]-u\right)\right\}\\ &= \max\left\{\max_{0\le u\le M}\left(A^{(i)}(i-u,i]-u\right),\right.\\ &\qquad\left.\max_{0\le u\le T-M}\left(A^{(i)}(i-M-u,i-M]-u+A^{(i)}(i-M,i]-M\right)\right\}. \qquad (14)\end{aligned}$$

We now consider the first component $\max_{0\le u\le M}\left(A^{(i)}(i-u,i]-u\right)$ on the right hand side of (14). Let $g^{(i)}(u) = A^{(i)}(i-u,i]-u$ for $0 \le u \le M$. Suppose cells arrive at $i-t_j$ $(j = 0,\dots,m)$ during interval $(i-M,i]$, where $t_{j-1} < t_j$ $(j = 1,\dots,m)$. Note that $t_0 = 0$ and $t_m = M-G(i-M,0]$. Because $M+1$ cells arrive one by one with intervals of length one once a message is generated, we have $t_j-t_{j-1} \le 1$ for all $j = 1,\dots,m$. Further each cell brings one unit of unfinished work into the system. Thus the sequence $g^{(i)}(t_0+), g^{(i)}(t_1+),\dots,g^{(i)}(t_m+)$ is nondecreasing, while $g^{(i)}(u)$ is a locally decreasing function of u in interval $(t_{j-1},t_j]$ $(j = 1,\dots,m)$. Further $g^{(i)}(u)$ is a decreasing function of u in interval $(t_m,M]$. As a result, the first component in (14) attains its maximum at $u = (M-G(i-M,0])+$. Thus we have

$$\begin{aligned}\max_{0\le u\le M}\left(A^{(i)}(i-u,i]-u\right) &= A^{(i)}[i-M+G(i-M,0],i]-(M-G(i-M,0])\\ &= A^{(i)}(i-M,i]-(M-G(i-M,0]), \qquad (15)\end{aligned}$$

where the second equality follows from the fact that there are no arrivals in $(i-M, i-M+G(i-M,0])$. Note also that

$$A^{(i)}(i-M-u,i-M] = (M+1)B(i-M-u,i-M], \qquad u\in[0,i-M+\tau^*], \qquad (16)$$

$$A^{(i)}(i-M,i] = \sum_{k=1}^{M} kB(i-k,i-k+1]+(i+1), \qquad (17)$$

where cells arriving from source 0 contribute to the last term, $i+1$, in the second equation. Thus it follows from (14), (15), (16) and (17) that

$$\begin{aligned}U_i^{(i)} = \max\Bigg\{&\left(\sum_{k=1}^{M} kB(i-k,i-k+1]+(i+1)-(M-G(i-M,0])\right),\\ &\max_{0\le u\le T-M}\left(\sum_{k=1}^{M} kB(i-k,i-k+1]+(i+1)-M\right.\\ &\qquad\left.+(M+1)B(i-M-u,i-M]-u\right)\Bigg\},\end{aligned}$$

which is equivalent to (11). □

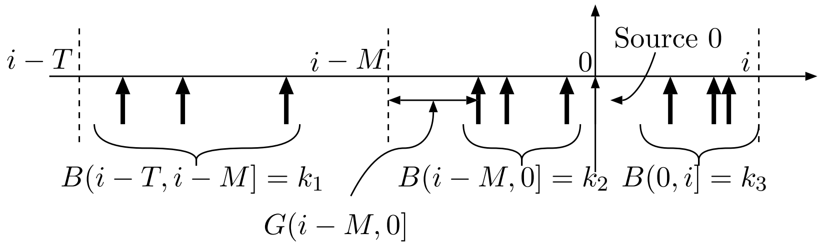


Figure 4: Events $I_1(k_1)$, $I_2(k_2)$ and $I_3(k_3)$.

3.2. Distribution function of $U_i^{(i)}$

In this subsection, we derive the distribution function of the amount $U_i^{(i)}$ of unfinished work at time i in the equivalent system for the $(i+1)$st cells. Let $I_1(k_1)$, $I_2(k_2)$ and $I_3(k_3)$ denote the events $B(i-T, i-M] = k_1$, $B(i-M, 0] = k_2$ and $B(0, i] = k_3$, respectively. See Figure 4. By definition, $k_1 + k_2 + k_3 = B(i-T, i] = K$, $B(i-M, 0] = k_2 = 0$ for $i = M$, and $B(0, i] = k_3 = 0$ for $i = 0$.

We first consider the distribution function of $U_i^{(i)}$ by conditioning those three events. That is,

$$\Pr\left(U_i^{(i)} \le v\right) = \sum_{\substack{k_1+k_2+k_3=K \\ k_1,k_2,k_3 \ge 0}} \Pr(I_1(k_1), I_2(k_2), I_3(k_3)) \Pr\left(U_i^{(i)} \le v \mid I_1(k_1), I_2(k_2), I_3(k_3)\right), \tag{18}$$

where $\Pr(I_1(k_1), I_2(k_2), I_3(k_3))$ follows a multinomial distribution:

$$\Pr(I_1(k_1), I_2(k_2), I_3(k_3)) = \frac{K!}{k_1!k_2!k_3!}\left(\frac{T-M}{T}\right)^{k_1}\left(\frac{M-i}{T}\right)^{k_2}\left(\frac{i}{T}\right)^{k_3},$$

because message generation epochs of K sources are independent and identically distributed according to a uniform distribution over any interval of length T.

It follows from (11) that the event $U_i^{(i)} \le v$ happens if and only if

$$\begin{cases} G(i-M,0] + \displaystyle\sum_{k=1}^{M} kB(i-k, i-k+1] + i - M + 1 \le v, \\ \displaystyle\max_{0 \le u \le T-M} ((M+1)B(i-M-u, i-M] - u) + \sum_{k=1}^{M} kB(i-k, i-k+1] + i - M + 1 \le v, \end{cases}$$

or equivalently,

$$\begin{cases} \displaystyle\sum_{k=1}^{M} kB(i-k, i-k+1] \le \bar{j}^{(i)}\left(v - G(i-M,0]\right), \\ \displaystyle\max_{0 \le u \le T-M} ((M+1)B(i-M-u, i-M] - u) \le v + M - i - 1 - \sum_{k=1}^{M} kB(i-k, i-k+1], \end{cases} \tag{19}$$

where

$$\bar{j}^{(i)}(x) = \lfloor x \rfloor + M - i - 1, \tag{20}$$

and $\lfloor x \rfloor$ represents the maximum integer which is not greater than x.

We define $q^{(i)}(j \mid k_2, k_3, w)$ and $R^{(i)}(y \mid k_1)$ as

$$q^{(i)}(j \mid k_2, k_3, w) = \Pr\left(\sum_{k=1}^{M} kB(i-k, i-k+1] = j \;\middle|\; I_2(k_2), I_3(k_3), G(i-M, 0] = w\right), \quad (21)$$

$$R^{(i)}(y \mid k_1) = \Pr\left(\max_{0\le u\le T-M}((M+1)B(i-M-u, i-M]-u) \le y \;\middle|\; I_1(k_1)\right). \quad (22)$$

Note here that

$$\begin{aligned}
&\Pr\left(\sum_{k=1}^{M} kB(i-k, i-k+1] = j \;\middle|\; I_1(k_1), I_2(k_2), I_3(k_3), G(i-M, 0] = w\right)\\
&\qquad = \Pr\left(\sum_{k=1}^{M} kB(i-k, i-k+1] = j \;\middle|\; I_2(k_2), I_3(k_3), G(i-M, 0] = w\right)\\
&\qquad = q^{(i)}(j \mid k_2, k_3, w),\\
&\Pr\left(\max_{0\le u\le T-M}((M+1)B(i-M-u, i-M]-u) \le y \;\middle|\; I_1(k_1), I_2(k_2), I_3(k_3)\right)\\
&\qquad = \Pr\left(\max_{0\le u\le T-M}((M+1)B(i-M-u, i-M]-u) \le y \;\middle|\; I_1(k_1)\right)\\
&\qquad = R^{(i)}(y \mid k_1).
\end{aligned}$$

Thus the two probabilities $q^{(i)}(j \mid k_2, k_3, w)$ and $R^{(i)}(y \mid k_1)$ are conditionally independent given $I_j(k_j)$ $(j = 1, 2, 3)$ and $G(i-M, 0] = w$. Note here that (22) is rewritten to be

$$R^{(i)}(y \mid k_1) = \Pr((M+1)B(i-M-u, i-M] - u \le y,\ \forall u \in [0, T-M] \mid I_1(k_1)). \quad (23)$$

In what follows, an empty sum is defined as zero.

Lemma 2 *$R^{(i)}(y \mid k_1)$ is given by*

$$R^{(i)}(y \mid k_1) = 1 - \sum_{j=\lfloor \frac{y}{M+1} \rfloor+1}^{k_1} \frac{(T-M)-(M+1)k_1+y}{(T-M)-(M+1)j+y} \cdot \binom{k_1}{j}\left(\frac{(M+1)j-y}{T-M}\right)^{j}\left(1-\frac{(M+1)j-y}{T-M}\right)^{k_1-j}. \quad (24)$$

The proof is given in Appendix A.

Remark 2 *Consider the special case of $M = 0$ and $k_1 = K$ for $i = 0$ in (24).*

$$R^{(0)}(y \mid K) = \sum_{j=\lfloor y \rfloor+1}^{K} \frac{T+y-K}{T+y-j}\binom{K}{j}\left(\frac{j-y}{T}\right)^{j}\left(1-\frac{j-y}{T}\right)^{K-j}. \quad (25)$$

Note that $R^{(0)}(y \mid K)$ for $M = 0$ represents the probability distribution function of the amount of unfinished work in the conventional $\sum D/D/1$ queue with K sources of period T, which is considered in section 3-C of [10]. Conversely, (24) is obtained by substituting $y/(M+1)$, k_1 and $(T-M)/(M+1)$ for y, K and T, respectively, in (25). Thus (24) itself is considered as the the distribution function of the amount of unfinished work in the conventional $\sum D/D/1$ queue with a particular setting of parameters.

In what follows, we derive the conditional distribution $\Pr(U_i^{(i)} \le v \mid I_1(k_1), I_2(k_2), I_3(k_3))$ by considering two cases, $k_2 = 0$ and $k_2 \ge 1$, separately.

Lemma 3 $\Pr(U_i^{(i)} \le x \mid I_1(k_1), I_2(0), I_3(k_3))$ *is given by*

$$\Pr(U_i^{(i)} \le v \mid I_1(k_1), I_2(0), I_3(k_3)) = \sum_{j=0}^{\lfloor v \rfloor -1} q^{(i)}(j \mid 0, k_3, M-i) R^{(i)}(v+M-i-1-j \mid k_1). \quad (26)$$

Proof. Note first that when the event $I_2(0)$ happens, $B(i-M, 0] = 0$, so that the first arrival in $(i-M, 0]$ occurs from source 0 at time 0. Thus $G(i-M, 0] = M-i$. It then follows from (19) that

$$\begin{aligned}
&\Pr\left(U_i^{(i)} \le v \;\middle|\; I_1(k_1), I_2(0), I_3(k_3)\right) \\
&\quad = \Pr\Bigg(\sum_{k=1}^{M} kB(i-k, i-k+1] \le \bar{j}^{(i)}\left(v-(M-i)\right), \\
&\qquad\qquad \max_{0\le u\le T-M} ((M+1)B(i-M-u, i-M] - u) \\
&\qquad\qquad\qquad \le v + M - i - 1 - \sum_{k=1}^{M} kB(i-k, i-k+1] \;\Bigg|\; I_1(k_1), I_2(0), I_3(k_3)\Bigg) \\
&\quad = \sum_{j=0}^{\bar{j}^{(i)}(v-M+i)} q^{(i)}(j \mid 0, k_3, M-i) \Pr\Bigg(\max_{0\le u\le T-M} ((M+1)B(i-M-u, i-M] - u) \\
&\qquad\qquad\qquad \le v + M - i - 1 - j \;\Bigg|\; I_1(k_1)\Bigg).
\end{aligned}$$

Thus (26) follows from (20), (22) and the above equation. □

Lemma 4 *For* $k_2 \ge 1$, $\Pr(U_i^{(i)} \le v \mid I_1(k_1), I_2(k_2), I_3(k_3))$ *is given by*

$$\begin{aligned}
&\Pr\left(U_i^{(i)} \le v \mid I_1(k_1), I_2(k_2), I_3(k_3)\right) \\
&\quad = \sum_{l=0}^{M-i-1} \sum_{k_4=1}^{k_2} \frac{k_2!}{k_4!(k_2-k_4)!} \frac{(M-i-l-1)^{k_2-k_4}}{(M-i)^{k_2}} \\
&\qquad \cdot \Bigg\{ \sum_{j=k_4(M-l)}^{\lfloor v \rfloor - l + M - i - 2} q^{(i)}\left(j - k_4(M-l) \mid k_2 - k_4, k_3, l+1,\right) R^{(i)}(v+M-i-1-j \mid k_1) \\
&\qquad\quad + \left[1 - (1 - v + \lfloor v \rfloor)^{k_4}\right] q^{(i)}\left(\lfloor v \rfloor - (k_4-1)(M-l) - i - 1 \mid k_2 - k_4, k_3, l+1\right) \\
&\qquad\qquad \cdot R^{(i)}(v - \lfloor v \rfloor + l \mid k_1) \Bigg\}. \qquad (27)
\end{aligned}$$

Proof. Let $f(y \mid k_2)$ denote the conditional probability density function of $G(i-M, 0]$ given $I_2(k_2)$ for $k_2 \ge 1$. Because k_2 arrivals are independent and uniformly distributed over the interval $(i-M, 0]$, we have

$$f(y \mid k_2) = \frac{k_2}{M-i}\left(\frac{M-i-y}{M-i}\right)^{k_2-1}, \qquad 0 \le y \le M-i.$$

To obtain $\Pr(U_i^{(i)} \le v \mid I_1(k_1), I_2(k_2), I_3(k_3))$, let $I_4(k_4)$ denote the following event:

$$B\Big(i - M + G(i-M, 0], i - M + \lceil G(i-M, 0] \rceil\Big] = k_4 - 1,$$

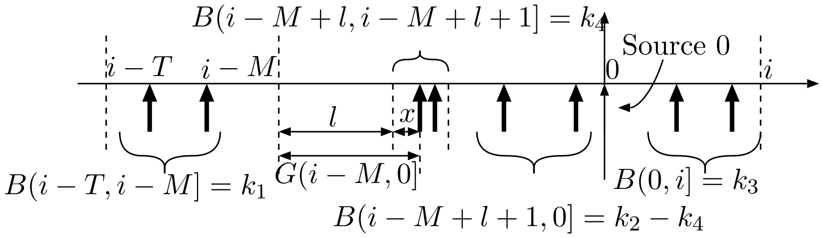


Figure 5: Events $I_i(k_i)$ $(i = 1, \ldots, 4)$ in the case $k_2 \geq 1$.

where $\lceil y \rceil$ represents the minimum integer which is not smaller than y. Suppose the interval $(i-M, 0]$ is divided into $M-i$ slots of length one. Then the event $I_4(k_4)$ implies that $k4$ messages arrive in the first slot among those having message arrivals. Thus, given the event $I_2(k_2)$, we have $1 \leq k_4 \leq k_2$, and the number of messages arriving in the interval from the end of the first slot with message arrivals to time 0 is given by $k_2 - k_4$. See Figure 5.

We now divide $G(i-M, 0]$ into the integer part l and the decimal part x, i.e., $G(i-M,0] = l + x$ with $0 \leq x < 1$. Then, using (19) and conditioning events $G(i-M, 0] = l + x$ and $I_4(k_4)$, we obtain

$$\begin{aligned}
&\Pr\left(U_i^{(i)} \leq v \,\middle|\, I_1(k_1), I_2(k_2), I_3(k_3)\right) \\
&\quad = \sum_{l=0}^{M-i-1} \int_0^1 dx f(l + x \mid k_2) \sum_{k_4=1}^{k_2} \Pr\left(I_4(k_4) \mid I_2(k_2), G(i-M, 0] = l + x\right) \\
&\qquad \cdot \Pr\Biggl(k_4(M-l) + \sum_{k=1}^{M-l-1} kB(i-k, i-k+1] \leq \bar{j}^{(i)}\left(v - (l+x)\right), \\
&\qquad\qquad \max_{0 \leq u \leq T-M} ((M+1)B(i-M-u, i-M] - u) \\
&\qquad\qquad\qquad \leq v + M - i - 1 - k_4(M-l) - \sum_{k=1}^{M-l-1} kB(i-k, i-k+1] \\
&\qquad\qquad\qquad \Bigg| \; I_1(k_1), I_2(k_2), I_3(k_3), I_4(k_4), G(i-M, 0] = l + x\Biggr).
\end{aligned}$$

Further, by conditioning the value of $\sum_{k=1}^{M-l-1} kB(i-k, i-k+1]$ and using (21) and (22), we rewrite the above equation to be

$$\begin{aligned}
&\Pr\left(U_i^{(i)} \leq v \,\middle|\, I_1(k_1), I_2(k_2), I_3(k_3)\right) \\
&\quad = \sum_{l=0}^{M-i-1} \int_0^1 dx f(l + x \mid k_2) \sum_{k_4=1}^{k_2} \Pr\left(I_4(k_4) \mid I_2(k_2), G(i-M, 0] = l + x\right) \\
&\qquad \cdot \sum_{j=k_4(M-l)}^{\bar{j}^{(i)}(v-(l+x))} q^{(i)}\left(j - k_4(M-l) \mid k_2 - k_4, k_3, l+1\right) R^{(i)}(v + M - i - 1 - j \mid k_1). \qquad (28)
\end{aligned}$$

Note here that

$$\begin{aligned}
&\Pr\left(I_4(k_4) \mid I_2(k_2), G(i-M, 0] = l + x\right) \\
&\quad = \Pr(B(i-M+l+x, i-M+l+\lceil x \rceil] = k_4 - 1 \mid B(i-M+l+x, 0] = k_2 - 1) \\
&\quad = \frac{(k_2-1)!}{(k_4-1)!(k_2-k_4)!} \left(\frac{\lceil x \rceil - x}{M-i-l-x}\right)^{k_4-1} \left(\frac{M-i-l-\lceil x \rceil}{M-i-l-x}\right)^{k_2-k_4},
\end{aligned}$$

and

$$\bar{j}^{(i)}(v-l-x) = \begin{cases} \bar{j}^{(i)}(v-l), & x \le v - \lfloor v \rfloor, \\ \bar{j}^{(i)}(v-l-1), & x > v - \lfloor v \rfloor. \end{cases}$$

We define $\overline{x}^{(i)}(v)$ as

$$\overline{x}^{(i)}(v) = v - \lfloor v \rfloor. \tag{29}$$

Then (28) is rewritten to be

$$\begin{aligned}
&\Pr\left(U_i^{(i)} \le v \,\middle|\, I_1(k_1), I_2(k_2), I_3(k_3)\right) \\
&= \sum_{l=0}^{M-i-1} \left(\int_0^{\overline{x}^{(i)}(v)} + \int_{\overline{x}^{(i)}(v)}^1 \right) dx \frac{k_2}{M-i} \\
&\quad \cdot \sum_{k_4=1}^{k_2} \frac{(k_2-1)!}{(k_4-1)!(k_2-k_4)!} \left(\frac{\lceil x \rceil - x}{M-i} \right)^{k_4-1} \left(\frac{M-i-l-\lceil x \rceil}{M-i} \right)^{k_2-k_4} \\
&\quad \cdot \sum_{j=k_4(M-l)}^{\bar{j}^{(i)}(v-l-x)} q^{(i)}\left(j - k_4(M-l) \mid k_2-k_4, k_3, l+1\right) R^{(i)}(v+M-i-1-j \mid k_1) \\
&= \sum_{l=0}^{M-i-1} \sum_{k_4=1}^{k_2} \frac{k_2!}{k_4!(k_2-k_4)!} \frac{(M-i-l-1)^{k_2-k_4}}{(M-i)^{k_2}} \\
&\quad \cdot \Bigg\{ \left[1 - (1-\overline{x}^{(i)}(v))^{k_4}\right] \sum_{j=k_4(M-l)}^{\bar{j}^{(i)}(v-l)} q^{(i)}\left(j - k_4(M-l) \mid k_2-k_4, k_3, l+1\right) \\
&\quad \cdot R^{(i)}(v+M-i-1-j \mid k_1) + (1-\overline{x}^{(i)}(v))^{k_4} \\
&\quad \cdot \sum_{j=k_4(M-l)}^{\bar{j}^{(i)}(v-l-1)} q^{(i)}\left(j - k_4(M-l) \mid k_2-k_4, k_3, l+1\right) R^{(i)}(v+M-i-1-j \mid k_1) \Bigg\}.
\end{aligned}$$

Thus (27) follows from (20), (29) and the above equation. □

Finally, using (18), Lemmas 3 and 4, we obtain the following theorem.

Theorem 2 *Let S_i ($i = 0, \dots, M$) denote a generic random variable representing sojourn times of the $(i+1)$st cells. We then have*

$$\begin{aligned}
\Pr(S_i \le v) = &\sum_{\substack{k_1+k_2+k_3+k_4=K \\ k_1,k_2,k_3 \ge 0, k_4 \ge 1}} \frac{K!}{k_1!k_2!k_3!k_4!} \frac{(T-M)^{k_1}}{T^K} \\
&\cdot \sum_{l=0}^{M-i-1} \Bigg\{ \sum_{j=k_4(M-l)}^{\lfloor v \rfloor - l + M - i - 2} n^{(i)}\left(j - k_4(M-l) \mid k_2, k_3, l+1\right) \\
&\cdot R^{(i)}(v+M-i-1-j \mid k_1) - \left[1 - (1-v+\lfloor v \rfloor)^{k_4}\right] \\
&\cdot n^{(i)}\left(\lfloor v \rfloor - (k_4-1)(M-l) - i - 1 \mid k_2, k_3, l+1\right) \\
&\cdot R^{(i)}(v - \lfloor v \rfloor + l \mid k_1) \Bigg\} + \sum_{\substack{k_1+k_3=K \\ k_1,k_3 \ge 0}} \frac{K!}{k_1!k_3!} \frac{(T-M)^{k_1}}{T^K} \\
&\cdot \sum_{j=0}^{\lfloor v \rfloor - 1} n^{(i)}(j \mid 0, k_3, M-i) R^{(i)}(v+M-i-1-j \mid k_1),
\end{aligned} \tag{30}$$

where $R^{(i)}(y \mid k_1)$ *is given in (24),* $n^{(i)}(j \mid k_2, k_3, l)$ *is defined as*

$$n^{(i)}(j \mid k_2, k_3, l) = (M - i - l)^{k_2} i^{k_3} q^{(i)}(j \mid k_2, k_3, l), \tag{31}$$

and they are recursively computed by

$$n^{(i)}(j \mid 0, 0, l) = \begin{cases} 1 & (j = 0) \\ 0 & (j \neq 0) \end{cases}, \tag{32}$$

$$n^{(i)}(j \mid k_2 + 1, k_3, l), = \sum_{m=i+1}^{M-l} n^{(i)}(j - m \mid k_2, k_3, l), \tag{33}$$

$$n^{(i)}(j \mid k_2, k_3 + 1, l) = \sum_{m=1}^{i} n^{(i)}(j - m \mid k_2, k_3, l). \tag{34}$$

Proof. Recall that $\Pr(S_i \le v) = \Pr(U_i^{(i)} \le v)$. Thus we obtain (30) by substituting (26) and (27) into (18), using (31) and noting with $k_2' = k_2 - k_4$

$$\sum_{\substack{k_1+k_2+k_3=K \\ k_1,k_3 \ge 0, k_2 \ge 1}} \frac{K!}{k_1! k_2! k_3!} \sum_{k_4=1}^{k_2} \frac{k_2!}{k_4!(k_2 - k_4)!} = \sum_{\substack{k_1+k_2'+k_3+k_4=K \\ k_1,k_2',k_3 \ge 0, k_4 \ge 1}} \frac{K!}{k_1! k_2'! k_3! k_4!}.$$

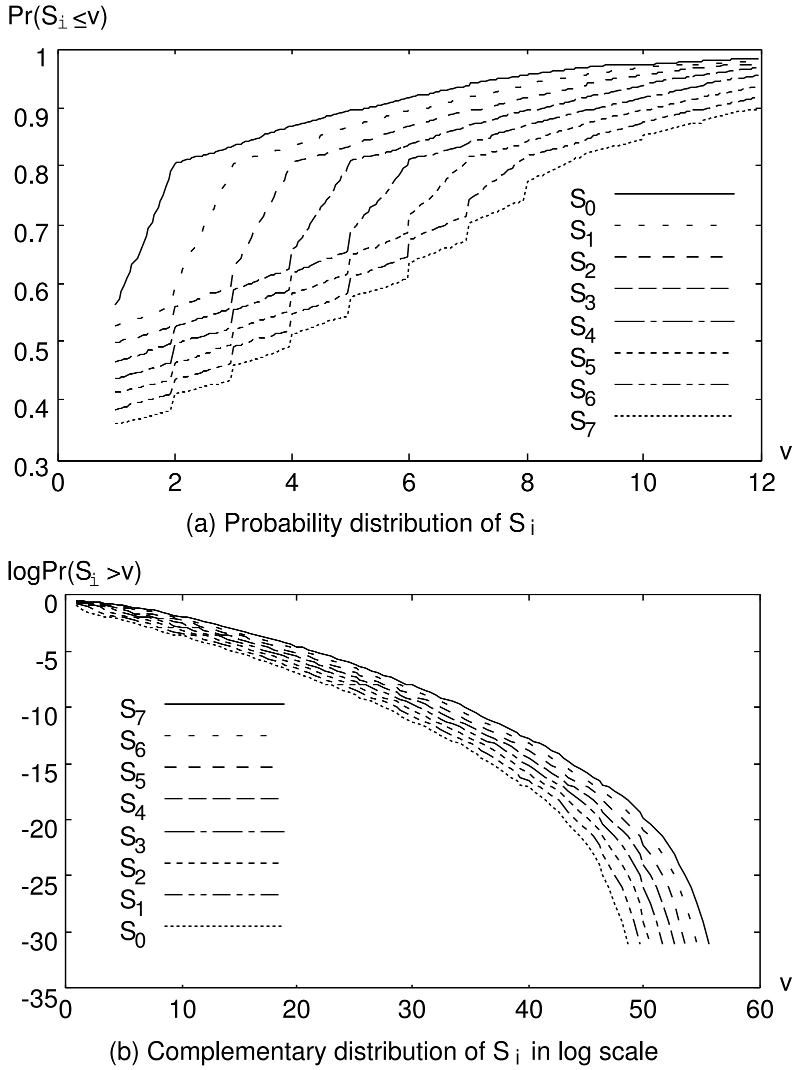
On the other hand, from (21) and (31), $n^{(i)}(j \mid k_2, k_3, l)$ can be regarded as the number of ways such that $\sum_{k=1}^{M-l} kB(i-k, i-k+1]$ is equal to j given that three events $I_2(k_2)$, $I_3(k_3)$ and $G(i - M, 0] = l$ occur. With this observation, we have (32)-(34). □

From Theorem 2, we can derive the probability distributions of the amount of unfinished work and the queue length. Note that the first cells in respective messages from source 0 see the time average of unfinished work in the system with source 0 removed [18]. On the other hand, the queue length is uniquely determined by the amount of unfinished work because service times of all cells are equal to one. Thus we have the following corollary.

Corollary 1 *For the system with K sources, the probability distribution function of the amount of the stationary unfinished work is given by* $\Pr(S_0 \le v)$*, and the stationary queue length distribution* $\Pr(L \le k)$ *is given by* $\Pr(S_0 \le k+1)$ $(k = 0, 1, \ldots, (M+1)(K-1)+1)$*, where L denotes a generic random variable representing the stationary queue length.*

Remark 3 *Consider the special case of $M = 0$ (i.e., the conventional $\sum$ D/D/1) and* $\Pr(S_0 \le v)$*. In this special case, the first term on the right hand side of (30) vanishes. Further in the summation in the second term, only the term for $k_1 = K$ and $j = 0$ remains because* $\Pr(I_3(0)) = 1$ *and* $n^{(j)}(0, 0, 0) = 0$ *for all $j \ge 1$. Thus* $\Pr(S_0 \le v)$ *in (30) is reduced to* $R^{(0)}(v - 1 \mid K)$*. Thus, the sojourn time is given by the sum of the amount of unfinished work in the system with K sources and the service time in this special case.*

For $K + 1 = 8$, $T = 128$ and $M + 1 = 8$, we plot the distribution function $\Pr(S_i \le v)$ and its complement $\Pr(S_i > v)$ in Figure 6 (a) and (b), respectively. From Figure 6 (a), we observe that $\Pr(S_i \le v)$ $(i = 1, \ldots, M)$ is a discontinuous function of v. This is due to the fact that sojourn times of all cells in a message which starts a busy period always take integer values. Thus the sojourn time distributions of the second to the last cells have masses at integers. Further, from Figure 6 (b), we observe that the formula in Theorem 2 is numerically stable.



(a) Probability distribution of S i

(b) Complementary distribution of S i in log scale

Figure 6: Sojourn time distributions ($K + 1 = 8$, $T = 128$ and $M + 1 = 8$).

4. Joint Distribution of Differences of Sojourn Times

In this section, we consider the joint distribution of the differences of sojourn times S_i ($i = 0, \ldots, M$) of the $(i + 1)$st cells in a message. Recall that $S_i = U_i$. Thus it follows from (2) that

$$S_i = S_{i-1} + A(i-1, i] - 1, \qquad i = 1, \ldots, M,$$

because $U_t > 0$ for all $t \in [0, M]$. Let Δ_i ($i = 1, \ldots, M$) denote the difference between sojourn times of the ith and $(i - 1)$st cells in the message generated at time 0:

$$\Delta_i = S_i - S_{i-1}, \qquad i = 1, \ldots, M.$$

We then have

$$\Delta_i = A(i-1, i] - 1 = (B(i - M - 1, i] + 1) - 1 = B(i - M - 1, i]. \tag{35}$$

Thus Δ_i ($i = 1, \ldots, M$) takes integer values and $0 \le \Delta_i \le K$. We define $P(z_1, \ldots, z_M)$ as the probability generating function of the joint distribution of Δ_i ($i = 0, \ldots, M$).

$$P(z_1, \ldots, z_M) = E\left[z_1^{\Delta_1} \cdots z_M^{\Delta_M}\right].$$

Theorem 3 *$P(z_1, \ldots, z_M)$ is given by*

$$P(z_1, \ldots, z_M) = \left(\frac{T - 2M}{T} + \frac{1}{T} \sum_{j=1}^{M} \prod_{l=1}^{j} z_l + \frac{1}{T} \sum_{j=1}^{M} \prod_{l=j}^{M} z_l \right)^K. \tag{36}$$

Proof. It follows from (35) that

$$\Delta_i = \sum_{k=i-M-1}^{i-1} B(k, k+1], \qquad i = 1, \ldots, M.$$

Thus we have

$$z_i^{\Delta_i} = \prod_{k=i-M-1}^{i-1} z_i^{B(k,k+1]},$$

from which it follows that

$$\begin{aligned} P(z_1, \ldots, z_M) &= E\left[\prod_{k_1=-M}^{0} z_1^{B(k_1,k_1+1]} \cdots \prod_{k_i=i-M-1}^{i-1} z_i^{B(k_i,k_i+1]} \cdots \prod_{k_M=-1}^{M-1} z_M^{B(k_M,k_M+1]} \right] \\ &= E\left[\prod_{k=-M}^{-1} (z_1 \cdots z_{M+1+k})^{B(k,k+1]} \cdot \prod_{k=0}^{M-1} (z_{k+1} \cdots z_M)^{B(k,k+1]} \right]. \end{aligned} \tag{37}$$

Note here that the probability generating function of the joint distribution of $B(k, k+1]$ $(k = -M, \ldots, M-1)$ is given by

$$E\left[\prod_{k=-M}^{M-1} \omega_k^{B(k,k+1]} \right] = \left(\frac{T - 2M}{T} + \sum_{k=-M}^{M-1} \frac{\omega_k}{T} \right)^K, \tag{38}$$

because message generation epochs of sources 1 to K are independent and identically distributed according to a uniform distribution over any interval of length T. Theorem 3 then follows from (37) and (38). □

Corollary 2 *The probability mass function of Δ_i $(i = 1, \ldots, M)$ and its mean are given by*

$$\Pr(\Delta_i = j) = \binom{K}{j} \left(\frac{M+1}{T} \right)^j \left(1 - \frac{M+1}{T} \right)^{K-j}, \qquad j = 1, \ldots, K, \tag{39}$$

$$E[\Delta_i] = \frac{K(M+1)}{T}, \tag{40}$$

respectively.

Proof. Setting $z_i = z$ for a specific i and $z_j = 1$ for all $j \neq i$ in (36), we have

$$E\left[z^{\Delta_i} \right] = \left(\frac{(M+1)z + T - M - 1}{T} \right)^K,$$

from which Corollary 2 follows immediately. □

5. Mean Waiting Time and Its Comparison to Related Systems

In this section, we first derive an explicit formula for the mean waiting time of the $(i+1)$st cells. Next, we compare the overall mean waiting time with those in the corresponding systems with dispersed periodic arrivals and with periodic batch arrivals.

5.1. Mean waiting time

Let W_i $(i = 0, \ldots, M)$ denote a generic random variable representing waiting times of the $(i + 1)$st cells. Note here that $W_i = S_i - 1$, and $W_i = 0$ when $K = 0$. Thus we assume $K \geq 1$ in the rest of this subsection. We define $E[W]$ as the overall mean waiting time, i.e.,

$$E[W] = \frac{1}{M+1} \sum_{i=0}^{M} E[W_i]. \tag{41}$$

Theorem 4 *When $K \geq 1$, the mean waiting time $E[W_i]$ $(i = 0, \ldots, M)$ of the $(i + 1)$st cells is given by*

$$E[W_i] = E[W] + \frac{K(M+1)}{T} \left(i - \frac{M}{2} \right), \tag{42}$$

where the overall mean waiting time $E[W]$ is given by

$$E[W] = \frac{M+1}{2} \sum_{j=0}^{K-1} \frac{K!}{j!} \left(\frac{M+1}{T} \right)^{K-j}. \tag{43}$$

Proof. From (40), we have

$$\begin{aligned} E[W_i] - E[W_0] &= E[S_i] - E[S_0] \\ &= \sum_{j=1}^{i} E[\Delta_j] \\ &= \frac{K(M+1)}{T} i. \end{aligned} \tag{44}$$

It then follows from (41) and (44) that

$$\begin{aligned} E[W] &= \frac{1}{M+1} \sum_{i=0}^{M} \left(E[W_0] + \frac{K(M+1)}{T} i \right) \\ &= E[W_0] + \frac{KM(M+1)}{2T}. \end{aligned}$$

Thus substituting $E[W] - KM(M+1)/(2T)$ for $E[W_0]$ in (44) and rearranging terms yield (42).

To show (43), we employ an induction with respect to the number of sources. For $k = 1, \ldots, K + 1$, let $E[W_i \mid k]$ $(i = 0, \ldots, M)$ and $E[W \mid k]$ denote the mean waiting time of the $(i + 1)$st cells and the overall mean waiting time, respectively, in the system with k sources. Note that, by definition, $E[W_i] = E[W_i \mid K + 1]$ and $E[W] = E[W \mid K + 1]$. We also define $E[U \mid k]$ $(k = 1, \ldots, K + 1)$ as the time-average of the amount of unfinished work in the system with k sources. Because of periodicity of U_t (see Proposition 1), we have

$$E[U \mid K+1] = \frac{1}{T} \int_{t}^{t+T} U_u du,$$

for any $t \geq -\tau^* + T$, where $U_{-\tau^*} = 0$.

We first observe that the sojourn time of the first cell in a message generated at time t from a particular source is not affected by cells generated before time t from the same source. Further the arrival epochs of the first cells are independent and identically distributed according to a uniform distribution over any interval of length T, while the amount of

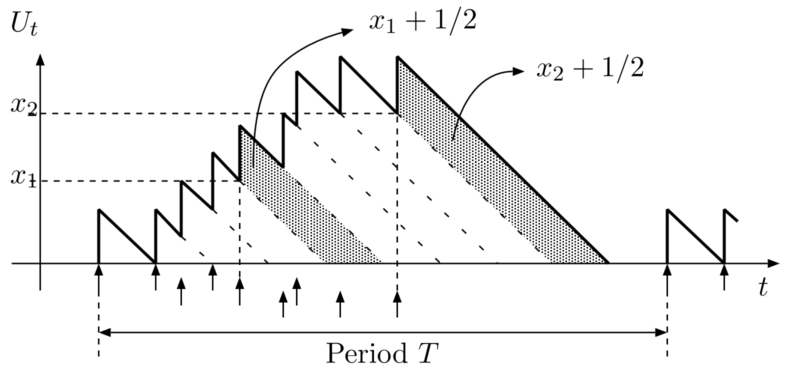


Figure 7: Contribution of respective waiting times to unfinished work.

unfinished work has a period of length T. Thus the first cell in a message from a particular source sees the time average unfinished work in the system with this source removed [18]. We then have

$$E[W_0 \mid k+1] = E[U \mid k], \qquad k = 1, \ldots, K. \tag{45}$$

Next we consider the relationship between the amount of unfinished work and waiting times of respective cells. It is easy to see from Figure 7 that a customer whose waiting time is equal to x contributes the amount $(x+1)^2/2 - x^2/2 = x + 1/2$ to unfinished work. Thus

$$\begin{aligned} E[U \mid k] &= \frac{1}{T} \sum_{i=0}^{M} k(E[W_i \mid k] + \frac{1}{2}) \\ &= \frac{k(M+1)}{T} E[W \mid k] + \frac{k(M+1)}{2T}, \end{aligned} \tag{46}$$

where we use (41), i.e., $\sum_{i=0}^{M} E[W_i \mid k] = (M+1)E[W \mid k]$. It then follows from (45) and (46) that

$$E[W_0 \mid k+1] = \frac{k(M+1)}{T} E[W \mid k] + \frac{k(M+1)}{2T},$$

and therefore from (42) with $i = 0$ and $K = k$, we obtain

$$\begin{aligned} E[W \mid k+1] &= E[W_0 \mid k+1] + \frac{kM(M+1)}{2T} \\ &= \frac{k(M+1)}{T} E[W \mid k] + \frac{k(M+1)^2}{2T}, \qquad k = 1, \ldots, K. \end{aligned} \tag{47}$$

Thus (43) is obtained by a straightforward induction with $E[W \mid 1] = 0$. □

5.2. Mean waiting time comparison to related systems

In this subsection, we compare the overall mean waiting time $E[W]$ to those in the related queueing systems with $K+1$ periodic sources, each of which generates exactly $M+1$ cells in any interval of length T. In particular, we consider (i) the corresponding system with dispersed periodic arrivals, where successive cells from each source are spread with an equal distance $T/(M+1)$, and (ii) the corresponding system with periodic batch arrivals introduced in the proof of Proposition 2. See Figure 8. Let $E\left[W^D\right]$ and $E\left[W^B\right]$ denote the overall mean waiting times in the corresponding systems with dispersed periodic arrivals and periodic batch arrivals, respectively.

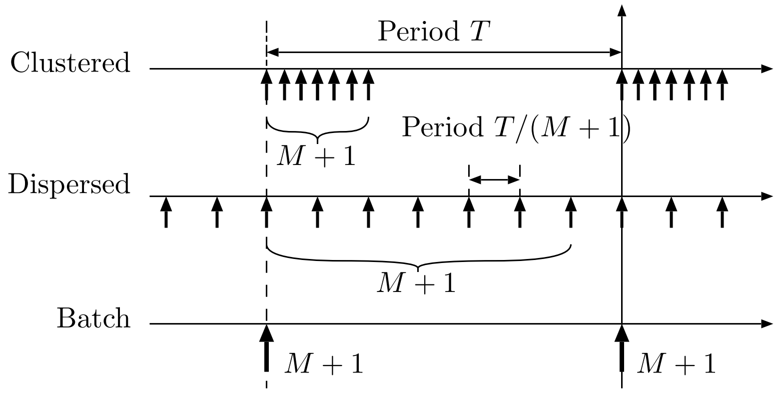


Figure 8: Clustered periodic, dispersed periodic and periodic batch arrivals.

Theorem 5 *$E\left[W^D\right]$ and $E\left[W^B\right]$ are given in terms of $E[W]$:*

$$E\left[W^D\right] = \frac{E[W]}{M+1}, \tag{48}$$

$$E\left[W^B\right] = E[W] + \frac{M}{2}. \tag{49}$$

Proof. Note that the system with dispersed periodic arrivals is a special case of $M=0$ and period $T/(M+1)$ in the system with clustered periodic arrivals. Thus substituting 0 and $T/(M+1)$ for M and T, respectively, in (43) yields

$$E\left[W^D\right] = \frac{1}{2}\sum_{j=0}^{K-1}\frac{K!}{j!}\left(\frac{M+1}{T}\right)^{K-j}. \tag{50}$$

Thus (48) immediately follows from (43) and (50).

Next we consider $E\left[W^B\right]$. We define $E\left[W_i^B\right]$ $(i=0,\ldots,M)$ as the mean waiting time of the $(i+1)$st cells in the corresponding system with periodic batch arrivals. Note here that

$$E\left[W_i^B\right] = E\left[W_{i-1}^B\right] + 1, \qquad i = 1,\ldots,M,$$

from which it follows that

$$\begin{aligned} E\left[W^B\right] &= \frac{1}{M+1}\sum_{i=0}^{M} E\left[W_i^B\right] \\ &= \frac{1}{M+1}\sum_{i=0}^{M}\left(E\left[W_0^B\right] + i\right) \\ &= E\left[W_0^B\right] + \frac{M}{2}. \end{aligned} \tag{51}$$

As in the proof of Theorem 4, we consider $E\left[W_0^B\right]$ by an induction with respect to the number of sources. For $k=1,\ldots,K+1$, let $E\left[W_i^B \mid k\right]$ and $E\left[U^B \mid k\right]$ $(i=0,\ldots,M)$ denote the mean waiting time of the $(i+1)$st cells and the time-average of the amount of unfinished work, respectively, in the corresponding system with periodic batch arrivals when the system has k sources. Note, by definition, that $E\left[W_i^B\right] = E\left[W_i^B \mid K+1\right]$. According to

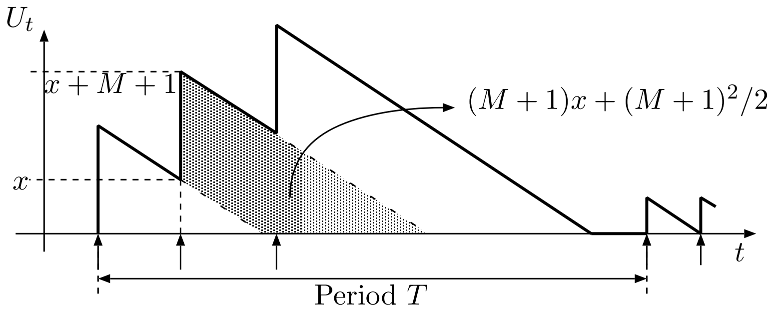


Figure 9: Contribution of respective waiting times to unfinished work.

an argument very similar to that in the proof of Theorem 4, we can show that $E\left[W_0^B \mid k+1\right]$ is identical to $E\left[U^B \mid k\right]$. Further, since each batch arrival whose first cell's waiting time is equal to x contributes the amount $(x+M+1)^2/2 - x^2/2 = (M+1)x + (M+1)^2/2$ to unfinished work (see Figure 9), we have

$$\begin{aligned} E\left[W_0^B \mid k+1\right] &= E\left[U^B \mid k\right] \\ &= k\left((M+1)E\left[W_0^B \mid k\right] + \frac{(M+1)^2}{2}\right)\Big/ T \\ &= \frac{k(M+1)}{T} E\left[W_0^B \mid k\right] + \frac{k(M+1)^2}{2T}. \end{aligned} \tag{52}$$

Because the recursion (52) for $E\left[W_0^B \mid k\right]$ is the same as (47) for $E[W \mid k]$ and $E[W \mid 1] = E\left[W_0^B \mid 1\right] = 0$, we obtain $E\left[W_0^B\right] = E[W]$, and from (51), (49) immediately follows. □

Finally, we consider the mean waiting times $E[W_i]$ $(i = 0, \ldots, M)$ and the overall mean waiting time $E[W]$ in the limit $K \to \infty$ while fixing $\rho = (M+1)(K+1)/T < 1$. In this limit, T also goes to infinity, and the queue with clustered periodic arrivals converges to the M/D/1 queue with clustered arrivals, where the size of a cluster is equal to $M+1$. Let $E[W^*]$ (resp. $E[W_i^*]$ $(i = 0, \ldots, M)$) denote the overall mean waiting time (resp. the mean waiting time of the $(i+1)$st cells) in the queue with clustered periodic arrivals in this limit. Then $E[W^*]$ is obtained by either (48) or (49), because the limits of the mean waiting times in the corresponding systems are known. For example, the queue with dispersed periodic arrivals converges to the M/D/1 queue with arrival rate ρ whose mean waiting time is given by $\rho/\{2(1-\rho)\}$ [11]. Further from Theorem 4, we can obtain $E[W_i^*]$. The results are summarized in the following corollary.

Corollary 3 *Suppose $\rho = (M+1)(K+1)/T < 1$. Then $E[W^*]$ and $E[W_i^*]$ $(i = 0, \ldots, M)$ are given by*

$$\begin{aligned} E[W^*] &= \frac{(M+1)\rho}{2(1-\rho)}, \\ E[W_i^*] &= E[W^*] + \rho\left(i - \frac{M}{2}\right), \qquad i = 0, \ldots, M. \end{aligned}$$

A. Proof of Lemma 2

For a fixed t_1, we define $R(y, t \mid k)$ as

$$R(y, t \mid k) = \Pr\left(B(t_1 - u, t_1] - u \le y,\ \forall u \in [0, t] \mid B(t_1 - t, t_1] = k\right).$$

Note that $R(y, k \mid t)$ is equivalent to the probability distribution function of the amount of unfinished work in the conventional $\sum D/D/1$ queue with k sources of period t.

Lemma 5 ([10]) *$R(y,t \mid k)$ is given by*

$$R(y,t \mid k) = 1 - \sum_{j=\lfloor y \rfloor+1}^{k} \frac{t-k+y}{t-j+y} \binom{k}{j} \left(\frac{j-y}{t}\right)^j \left(1-\frac{j-y}{t}\right)^{k-j}. \tag{53}$$

Recall that

$$R^{(i)}(y \mid k_1) = \Pr((M+1)B(i-M-u, i-M] - u \le y,\ \forall u \in [0, T-M] \mid I_1(k_1)).$$

Thus it is easy to see that

$$R^{(i)}(y \mid k_1) = R\left(\frac{y}{M+1}, \frac{T-M}{M+1} \,\middle|\, k_1\right).$$

Therefore Lemma 2 immediately follows from Lemma 5.

Because Lemma 5 is provided in [10] without proof, we prove it below for completeness. We first note that

$$\begin{aligned} R(y,t \mid k) &= 1 - \Pr(B(t_1-u, t_1] > u+y,\ \exists u \in [0,t] \mid B(t_1-t, t_1] = k) \\ &= 1 - \int_0^t \Pr(\arg\max\{q \in [0,t] \mid B(t_1-q, t_1] > q+y\} = u \mid B(t_1-t,t_1]=k)du. \end{aligned}$$

Suppose $\arg\max\{q \in [0,t] \mid B(t_1-q,t_1] > q+y\} = u$ for some u ($u \in (0,t]$). This event is equivalent to (i) $B(t_1-u, t_1] = y+u+\delta > y+u$, (ii) $B(t_1-(u+\delta), t_1] = y+u+\delta$, and (iii) $B(t_1-q, t_1] \le q+y$ for all $q \in [u+\delta, t]$, where δ denotes an infinitesimal positive value. Note here that the conditional joint probability of the first two events is given by

$$\begin{aligned} &\Pr(B(t_1-u,t_1] = y+u+\delta, B(t_1-(u+\delta),t_1] = y+u+\delta) \mid B(t_1-t,t_1]=k) \\ &\ = \Pr(B(t_1-(u+\delta),t_1] = y+u+\delta \mid B(t_1-t,t_1]=k) \\ &\qquad \cdot \Pr(B(t_1-(u+\delta), t_1-u] = 0 \mid B(t_1-(u+\delta),t_1] = y+u+\delta, \mid B(t_1-t,t_1]=k)), \end{aligned}$$

and the second term on the right hand side of the above equation converges to one when δ goes to zero. Further $B(t_1-q,t_1]$ takes a nonnegative integer value and $\lfloor y \rfloor + 1 \le B(t_1-(u+\delta),t_1] = y+u+\delta \le k$. Thus letting $B(t_1-(u+\delta),t_1] = y+u+\delta = j$, we have

$$\begin{aligned} R(y,t \mid k) = 1 - \sum_{j=\lfloor y \rfloor+1}^{k} &\Pr\left(B(t_1-(j-y),t_1] = j \mid B(t_1-t,t_1]=k\right) \\ &\cdot \Pr(B(t_1-q,t_1] \le q+y,\ \forall q \in [j-y, t] \\ &\qquad \mid B(t_1-(j-y),t_1] = j, B(t_1-t,t_1]=k). \end{aligned} \tag{54}$$

Note here that

$$\Pr\left(B(t_1-(j-y),t_1] = j \mid B(t_1-t,t_1]=k\right) = \sum_{j=\lfloor y \rfloor+1}^{k} \binom{k}{j} \left(\frac{j-y}{t}\right)^j \left(1-\frac{j-y}{t}\right)^{k-j}.$$

Therefore (54) is rewritten to be

$$\begin{aligned} R(y,t \mid k) = 1 - \sum_{j=\lfloor y \rfloor+1}^{k} &\binom{k}{j} \left(\frac{j-y}{t}\right)^j \left(1-\frac{j-y}{t}\right)^{k-j} \\ &\cdot \Pr\Big(B(t_1-q,t_1] \le q+y,\ \forall q \in [j-y, t] \\ &\qquad \mid B(t_1-(j-y),t_1] = j, B(t_1-t,t_1]=k\Big). \end{aligned} \tag{55}$$

To obtain the expression of the probability in the last line in (55), we define $a(u)$ as a function which satisfies $\int_0^r a(u)du = B(t_1 - (j-y) - r, t_1 - (j-y)]$. With $a(u)$, we have

$$\begin{aligned}
&\Pr(B(t_1 - q, t_1] \le q + y,\ \forall q \in [j-y, t] \mid B(t_1 - (j-y), t_1] = j, B(t_1 - t, t_1] = k) \\
&\quad = \Pr(B(t_1 - q, t_1 - (j-y)] \le q + y - j\ \forall q \in [j-y, t] \mid B(t_1 - t, t_1 - (j-y)] = k - j) \\
&\quad = \Pr\left(\int_0^r a(u)du \le r\ \forall r \in [0, t-(j-y)] \;\middle|\; \int_0^{t-(j-y)} a(u)du = k - j\right).
\end{aligned}$$

Lemma 6 (Ballot Theorem [10]) *Let $a(u)$ be an integrable periodic real function on the real numbers, with period T and values that are either 0 or greater than or equal to some $L > 0$. Then, for arbitrary selected $s \in [0, T)$*

$$\Pr\left(\int_s^{s+r} a(u)du \le Lr,\ \forall r \in [0, T] \;\middle|\; \int_0^T a(u)du = W \le LT\right) = 1 - \frac{W}{LT}.$$

Substituting $t - (j-y)$, 1, $k - j$ and 0 for T, L, W and s, respectively, in Lemma 6, we obtain

$$\begin{aligned}
\Pr(B(t_1 - q, t_1] \le q + y,\ \forall q \in [j-y, t] \mid B(t_1 - (j-y), t_1] = j, B(t_1 - t, t_1] = k)& \\
= 1 - \frac{k - j}{t - (j-y)}& \\
= \frac{t - k + y}{t - j + y}.& \qquad (56)
\end{aligned}$$

(53) now follows from (55) and (56).

References

[1] J. M. Barcelo and J. Garcia: Multiplexing periodic sources in a tree network of ATM multiplexers. In *Proc. IFIP Broadband Communications '98* (1998) 577–588.

[2] J. M. Barcelo, J. Garcia, and O. Casals: Comparison of models for the multiplexing of worst case traffic sources. In *Proc. IFIP 1st Workshop on ATM Traffic Management* (1995) 41–49.

[3] B. Bensaou, J. Guibert, J. W. Roberts and A. Simonian: Performance of an ATM multiplexer queue in the fluid approximation using the Beneš approach. *Annals of Operations Research*, **49** (1994) 137–160.

[4] I. Cidon, R. Guerin, I. Kessler and A. Khamisy: Analysis of a statistical multiplexer with generalized periodic sources. *Queueing Systems*, **20** (1995) 139–169.

[5] L. G. Dron, G. Ramamurthy and B. Sengupta: Delay analysis of continuous bit rate traffic over an ATM network. *IEEE Journal on Selected Areas in Communications*, **9** (1991) 402–407.

[6] A. E. Eckberg: The single server queue with periodic arrival process and deterministic service times. *IEEE Transactions on Communications*, **27** (1997) 556–562.

[7] J. Garcia, J. M. Barcelo and O. Casals: An exact model for the multiplexing of worst case traffic sources. In *Proc. IFIP Conference on Performance of Computer Networks* (1995) 1–18.

[8] J. He and K. Sohraby: End-to-end analysis of CBR tandem networks using ballot theorems. In *Proc. ITC'16* (1999) 707–716.

[9] J. He and K. Sohraby: On the queueing analysis of dispersed periodic messages. *Proc. IEEE INFOCOM 2000* (2000) 1046–1054.

[10] P. Humblet, A. Bhargava and M. G. Hluchyj: Ballot theorems applied to the transient analysis of nD/D1 queues. *IEEE/ACM Transactions on Networking*, **1** (1993) 81–95.

[11] L. Kleinrock, *Queueing Systems, Vol. I: Theory* (John Wiley & Sons, New York 1975).

[12] K. Kvols and S. Blaabjerg: Bounds and approximations for the periodic on/off queue with application to ATM traffic control. In *Proc. IEEE INFOCOM '92* (1992) 487–494.

[13] T. Morita, H. Tatezumi, Y. Kawanishi, S. Kasahara, T. Takine and Y. Takahashi: Simulation and experimental study on ATM multi-node integrated connection. In *Proc. the 7th International Conference on Telecommunication Systems* (1999) 380–386.

[14] A. Privalov and K. Sohraby: Per-stream jitter analysis in CBR ATM multiplexors. *IEEE/ACM Transactions on Networking*, **6** (1998) 141–149.

[15] G. Ramamurthy and R. S. Dighe: Distributed source control: A network access control for integrated broadband packet networks. *IEEE Journal on Selected Areas in Communications*, **9** (1991) 990–1002.

[16] G. Ramamurthy and B. Sengupta: Delay analysis of a packet voice multiplexer by the ΣD_i/D/1 queue. *IEEE Transactions on Communications*, **39** (1991) 1107–1114.

[17] J. W. Roberts, B. Bensaou and Y. Canetti: A traffic control framework for high speed data transmission. In *Proc. IFIP Workshop on Modelling and Performance Evaluation of ATM Technology* (1993) 243–262.

[18] B. Sengupta: A queue with superposition of arrival streams with an application to packet voice technology. In *Proc. Performance '90* (1990) 53–59.

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