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A METHOD FOR SOLVING 0-1 MULTIPLE OBJECTIVE LINEAR PROGRAMMING PROBLEM USING DEA

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Abstract In this paper, by using Data Envelopment Analysis (DEA) technique a method is proposed to find efficient solutions of 0-1 Multiple Objective Linear Programming (MOLP) problem. In this method from a feasible solution of 0-1 MOLP problem, a Decision Making Unit (DMU) without input vector is constructed in which output vector for DMU is the values of objective functions. The method consists of a two-stage algorithm. In the first stage, some efficient solutions are generated. In the second stage, the DMUs corresponding to the generated efficient solutions in the first stage together with the generated DMUs in the previous iterations are evaluated by using the additive model without input.

Keywords: DEA; 0-1 multiple objective linear programming; efficient solution

1. Introduction

Data Envelopment Analysis (DEA) is a mathematical programming technique, which is used to evaluate relative efficiency of Decision Making Units (DMUs) and has been proposed by Charnes et al. [4]. This technique has extended by Banker et al. (BCC model) [2]. The additive model, which is used in this paper, has proposed by Charnes et al. [5]. In the literature of DEA some articles can be found in which application of DEA in Multiple Objective Linear Programming (MOLP) and application of MOLP in DEA have been discussed (see [7,8]).

To solve 0-1 MOLP problem some methods by Bitran [3] and Deckro et al. [6] have been proposed. Bitran used relaxation technique to generate efficient solutions. He defined a relaxation problem and proved the efficient solutions of the relaxation problem that are feasible to original problem would also be efficient in the original problem. Deckro et al. reported computational results in terms of implicit enumeration compared to Bitran's works. They claimed that their studies was compared favorably with Bitran's results. The proposed method by Deckro et al. solves 0-1 MOLP problem through implicit enumeration. Liu et al. [8] proposed another method and used DEA technique to generate efficient solutions. They defined a DMU corresponding to each feasible solution of the problem and developed a two-stage algorithm to generate and evaluate DMUs. They used BCC model to evaluate the generated DMUs. Since in each iteration of their method three problems are solved in stage 1, it is not computationally efficient. In their method for the following reasons some efficient solutions are lost.

1) Existence of the convexity constraint in BCC model,

2) adding the constraints $\sum_{q=1}^{Q}\Big[\sum_{r=1}^{s}u^*_{rj}c_{rq}-\sum_{i=1}^{m}v^*_{ij}a_{iq}\Big]w_{qd} > u^*_{oj}, \quad \forall j \in G'_k$ to problem in step 1.1 of algorithm,

3) omitting the DMUs with negative inputs or outputs.
As well, their method may find some efficient solutions, which are not indeed the desired solution according to the definition in literature.
The difficulties mentioned above are illustrated by numerical examples at the end of this paper. Since there is a close relation between DEA and MOLP, we use this relation for solving 0-1 MOLP problem. The proposed method in this paper, in comparison with the proposed method by Liu et al., is more computationally efficient and removes some difficulties of this method.

In the next section, 0-1 Multiple Objective Programming (MOP) and DEA are introduced. In section 3, a method for finding efficient solutions of 0-1 MOLP problem by using DEA and reference hyperplane is proposed. Section 4 illustrates the procedure with some numerical examples. In the last section, a conclusion and some remarks are put forward.

2. 0-1 MOP and DEA

DEA and MOP are introduced briefly in the following subsections.

2.1. 0-1 multiple objective programming

A multiple objective programming problem is defined in the following form:

$$\begin{array}{ll} Max & (f_1(W), f_2(W), \cdots, f_k(W)) \\ Min & (g_1(W), g_2(W), \cdots, g_t(W)) \\ s.t. & W \in \Omega \end{array} \tag{1}$$

where $f_1, f_2, \cdots, f_k$ and $g_1, g_2, \cdots, g_t$ are objective functions and Ω is a feasible region. If all objective functions are linear and Ω is a convex polyhedron, then problem (1) is called a multiple objective linear programming problem. In order to solve problem (1), there are some different methods in the literature.

Definition 1: $\overline{W} \in \Omega$ is said to be an efficient solution of problem (1) if and only if there does not exist a point $W^o \in \Omega$, such that:

$$(f_1(W^0), \cdots, f_k(W^0), -g_1(W^0), \cdots, -g_t(W^0)) \geq (f_1(\overline{W}), \cdots, f_k(\overline{W}), -g_1(\overline{W}), \cdots, -g_t(\overline{W}))$$

and inequality holds strictly for at least one index.
If in problem (1) all variables are restricted to be zero - one and all objective functions and constraints are linear then, problem (1) is called 0-1 MOLP problem and defined as follows:

$$\begin{array}{ll} Max & (f_1(W), f_2(W), \cdots, f_k(W)) \\ Min & (g_1(W), g_2(W), \cdots, g_t(W)) \\ s.t. & W \in \Omega \\ & w_j \in \{0, 1\}, \quad j = 1, 2, \cdots, n \end{array} \tag{2}$$

where $W = (w_1, w_2, \ldots, w_n)^T$.

2.2. Data envelopment analysis

Consider n decision making units DMU$_j (j = 1, 2, \ldots, n)$ where each DMU consumes a m-vector input to produce a s-vector output. Suppose that $X_j = (x_{1j}, x_{2j} \ldots, x_{mj})^T$ and $Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^T$ are the vectors of inputs and outputs values respectively for DMU$_j$, in which it has been assumed that $X_j \geq \mathbf{0}$ & $X_j \neq \mathbf{0}$ and $Y_j \geq \mathbf{0}$ & $Y_j \neq \mathbf{0}$. Consider the set S and its convex hull as follows:

$$S = \left\{ \binom{Y_j}{X_j} \mid j = 1, 2, \ldots, n \right\}$$
$$C(S) = \left\{ \binom{Y}{X} \mid \binom{Y}{X} = \sum_{j=1}^{n} \lambda_j \binom{Y_j}{X_j} ,\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0,\ j = 1, \cdots, n \right\}.$$

Suppose that $\binom{Y_p}{X_p}$ corresponds to DMU$_p$. If a vector belonging to $C(S)$ can be found such that

$$\binom{Y}{-X} \geq \binom{Y_p}{-X_p} \ \&\ \binom{Y}{-X} \neq \binom{Y_p}{-X_p}$$

then DMU$_p$ is called inefficient, otherwise it is called efficient.
For evaluating relative efficiency of DMU$_p$, additive model is used which is as follows:

$$\begin{array}{lll} Min \quad h_p = & -\sum_{i=1}^{m} s_i^- - \sum_{r=1}^{s} s_r^+ & \\ s.t. & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp} , & r = 1, \cdots, s \\ & -\sum_{j=1}^{n} \lambda_j x_{ij} - s_i^- = -x_{ip} , & i = 1, \cdots, m \\ & \sum_{j=1}^{n} \lambda_j = 1 & \\ & \lambda_j \geq 0,\ s_i^- \geq 0,\ s_r^+ \geq 0, & j = 1, \cdots, n,\ i = 1, \cdots, m,\ r = 1, \cdots, s \end{array} \tag{3}$$

where s_i^- and s_r^+ are the slack variables of the corresponding constraints of i^{th} input and r^{th} output, respectively. The dual of model (3) is:

$$\begin{array}{lll} Max \quad g_p = & \sum_{r=1}^{s} u_{rp} y_{rp} - \sum_{i=1}^{m} v_{ip} x_{ip} + u_{op} & \\ s.t. & \sum_{r=1}^{s} u_{rp} y_{rj} - \sum_{i=1}^{m} v_{ip} x_{ij} + u_{op} \leq 0, & j = 1, \cdots, n \\ & -v_{ip} \leq -1, & i = 1, \cdots, m \\ & -u_{rp} \leq -1, & r = 1, \cdots, s. \end{array} \tag{4}$$

We know that DMU$_p$ is efficient in the additive model if and only if $h_p^* = g_p^* = 0$ (see [5]).

Lemma 1: *The additive model is translation invariant* (see [1]).

As additive model is translation invariant, it can be used for evaluating relative efficiency of DMUs with zero or negative component of input/output vector. The additive model also can be used for evaluating relative efficiency of DMUs without input. The models (5) and (6) which are the multiplier and envelopment sides of additive model without input respectively, formulated as:

$$\begin{array}{lll} Max \quad Q_p = & \sum_{r=1}^{s} u_{rp} y_{rp} + u_{op} & \\ s.t. & \sum_{r=1}^{s} u_{rp} y_{rj} + u_{op} \leq 0, & j = 1, \cdots, n \\ & u_{rp} \geq 1, & r = 1, \cdots, s. \end{array} \tag{5}$$

$$
\begin{array}{lll}
Min \quad Q'_p = & -\sum_{r=1}^{s} s_r^+ & \\
s.t. & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \cdots, s & \\
& \sum_{j=1}^{n} \lambda_j = 1 & \\
& \lambda_j, s_r^+ \geq 0, \qquad j = 1, \cdots, n, \ r = 1, \cdots, s. &
\end{array} \tag{6}
$$

Theorem 1: *Envelopment side of additive model without input is feasible and bounded.*

Proof : It can be easily verified that $\lambda = e_p = (0, 0, \ldots, 0, 1, 0, \ldots, 0, 0)$ where 1 is in p-position and $S^+ = (0, 0, \ldots, 0)$ is a feasible solution for this model.
Consider the model (5) which is the dual of envelopment side of additive model without input. Since $(u_{1p}, u_{2p}, \ldots, u_{sp}, u_{op}) = (1, 1, \ldots, 1, u_{op})$ where $u_{op} = \min_{1 \leq j \leq n}\{-\sum_{r=1}^{s} y_{rj}\}$ is feasible solution for (5), the dual is feasible. Therefore the envelopment side of the additive model without input is bounded. □

Lemma 2: *The additive model without input is translation invariant.*
The proof is straightforward.

Lemma 3: *In additive model without input, DMU_p is efficient if and only if $Q_p^{'*} = Q_p^* = 0$.*
The proof is straightforward.

3. Efficient Solutions for 0-1 MOLP Problem

Consider the following problem:

$$
\begin{array}{lll}
Max & \{C_1W, C_2W, \cdots, C_sW\} & \\
s.t. & A_iW \leq b_i, & i = 1, 2, \cdots, m \\
& w_j \in \{0, 1\}, & j = 1, 2, \cdots, n
\end{array} \tag{7}
$$

where $A_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ and $C_r = (c_{r1}, c_{r2}, \cdots, c_{rn})$ are the coefficients vector for the i^{th} constraint and the r^{th} objective function, respectively.

Corresponding to each feasible solution W_d in (7), the vector Y_d is defined as $Y_d = (y_{1d}, y_{2d}, \cdots, y_{sd})^T$ where,$Y_d \in R^s$ & $y_{rd} = C_rW_d = \sum_{j=1}^{n} C_{rj}w_{jd}, \quad r = 1, \cdots, s.$
In order to use DEA technique for finding efficient solutions of the problem (7), corresponding to each vector Y_d, we consider a DMU with output vector of Y_d and without input vector. By using additive model without input, the relative efficiency of constructed DMUs is evaluated.

Theorem 2: *If DMU_d is efficient in model (5) then W_d is an efficient solution for the problem (7).*

Proof: If DMU_d is efficient in (5) then it will be efficient in (6). By contradiction, suppose that W_d is not an efficient solution of (7). So, there exists W_β such that

$$C_r(W_\beta) \geq C_r(W_d), \quad r = 1, \cdots, s$$

and inequality holds strictly for at least one index. That is, there will be at least one index, say l, such that $C_l(W_\beta) > C_l(W_d)$. Hence, $y_{l\beta} > y_{ld}$. From $\overline{\lambda} = e_l = (0, 0, \ldots, 0, 1, 0, \ldots, 0, 0)$ and $C_l(W_\beta) > C_l(W_d)$, we will have $\sum_{j=1}^{n} \overline{\lambda}_j y_{lj} > y_{ld}$. Hence, $s_l^+ > 0$. This means that there exists a feasible solution for model (6), say $(\overline{\lambda}, S^+)$, such that $Q_d = -s_l^+ < 0$ and this is a contradiction. □

3.1. The reference hyperplane

Let $g_r = C_r W_r^*,\ r = 1, 2, \ldots, s$ where W_r^* is the optimal solution of the following problem:

$$\begin{array}{lll} Max & C_r W & \\ s.t. & A_i W \leq b_i, & i = 1, 2, \cdots, m \\ & w_j \in \{0, 1\}, & j = 1, 2, \cdots, n. \end{array} \tag{8}$$

For $r = 1, 2, \cdots, s$, (8) denotes s problem. To state the following theorem, we consider the l^{th} problem from problems (8) ($r = l$).

Theorem 3: *If $O_l = \{W_{1l}^*, W_{2l}^*, \cdots, W_{fl}^*\}$ is the set of optimal solutions of l^{th} problem from the problems (8), then at least one of these is an efficient solution of the problem (7).*

Proof: First, suppose that O_l is singleton; namely, $O_l = \{W_l^*\}$. We prove that W_l^* is an efficient solution of the problem (7). By contradiction, suppose that W_l^* is not an efficient solution. So, there will be a feasible solution such as W^o such that

$$C_r W^o \geq C_r W_l^*,\quad r = 1, 2, \cdots, s\ \ \&\ \ \exists\, k\ (1 \leq k \leq s)\ ;\ \ C_k W^o > C_k W_l^*.$$

Hence, $C_l W^o \geq C_l W_l^*$ and this contradicts $C_l W^o < C_l W_l^*$.
Now, suppose $O_l = \{W_{1l}^*, W_{2l}^*, \cdots, W_{fl}^*\}$ where $f > 1$ and Ω is the feasible region of the problem (7). We prove that for each $W^o \in \Omega - O_l$ there is not $W_{ql}^* \in O_l$ such that:

$$C_r W^o \geq C_r W_{ql}^*\ \ , r = 1, 2, \cdots, s\quad \&\ \ \exists\, k\ \ ;\ \ C_k W^o > C_k W_{ql}^*.$$

By contradiction, suppose that there are $W^p \in \Omega - O_l$ and $W_{il}^* \in O_l$ such that:

$$C_r W^p \geq C_r W_{il}^*\ \ , r = 1, 2, \cdots, s\quad \&\ \ \exists\, k\ \ ;\ \ C_k W^p > C_k W_{il}^*$$

if $k = l$ then $C_l W^p > C_l W_{il}^*$ which is a contradiction. If $k \neq l$ then $C_l W^p = C_l W_{il}^*$ so, $W^p \in O_l$ and this contradicts $W^p \in \Omega - O_l$.
Let $K = \{Y_{1l}, Y_{2l}, \cdots, Y_{fl}\}$ where $Y_{ql} = (C_1 W_{ql}^*, C_2 W_{ql}^*, \cdots, C_s W_{ql}^*)^T (q = 1, 2, \cdots, f)$. It is evident that there is a member from K, say Y_{il}, such that $Y_{ql} \ngeq Y_{il}$ $(q = 1, 2, \cdots, f\ \ , q \neq i)$. Otherwise, for each member of K, say Y_{jl}, there should exist a member of K, say Y_{pl}, so that $Y_{jl} \leq Y_{pl}$ & $Y_{jl} \neq Y_{pl}$. Without loss of generality, suppose that

$$\begin{array}{lcll} Y_{1l} \leq Y_{2l} & \& & Y_{1l} \neq Y_{2l}, & \\ Y_{2l} \leq Y_{3l} & \& & Y_{2l} \neq Y_{3l}, & \\ & \ldots & & \\ Y_{(f-1)l} \leq Y_{fl} & \& & Y_{(f-1)l} \neq Y_{fl}, & \\ Y_{fl} \leq Y_{kl} & \& & Y_{fl} \neq Y_{kl}, & 1 \leq k < f \end{array}$$

which is a contradiction. Therefore, W_{il}^* is an efficient solution of the problem (7). □

Definition 2: The vector g, which is defined as

$$g = (g_1, g_2, \cdots, g_j, \cdots, g_s)^T = (C_1 W_1^*, C_2 W_2^*, \cdots, C_j W_j^*, \cdots, C_s W_s^*)^T,$$

is called the ideal vector.

Lemma 4: *The vector $g = (g_1, g_2, \cdots, g_s)^T$ dominates all DMUs ($\neq g$) which correspond to the feasible solutions of the problem (7).*

Proof: Suppose DMU$_o$ is a decision making unit corresponding to W^o where,

$$Y^o = (C_1 W^o, C_2 W^o, \cdots, C_s W^o)^T \quad \& \quad Y^o \neq g$$

and also suppose that $W_r^* (r = 1, 2, \cdots, s)$ is an optimal solution of r^{th} problem from the problems (8). So, $C_r W^o \leq C_r W_r^*$ $(r = 1, 2, \cdots, s)$. In other words, $g \geq Y^o$. If $\exists\, k (1 \leq k \leq s)\ ;\ g_k > y_k^o$ then g dominates the vector Y^o. Otherwise, $C_r W^o = C_r W_r^* (r = 1, 2, \cdots, s)$

and W^o is an optimal solution of all the problems (8). Therefore, $Y^o = g$ which contradicts $Y^o \neq g$. $\square$

Definition 3: The sets G and G' are defined as follows:

$$G = \{Y_d \mid Y_d = (C_1W_d, C_2W_d, \cdots, C_sW_d)^T,\ A_iW_d \leq b_i,\ i = 1, 2, \cdots, m,\ w_{jd} \in \{0, 1\}\}$$

and $G' = G \cup \{g\}$.

Definition 4: The hyperplane $H = \{Y \mid \mathbf{1}^T(Y - g) = 0\}$ where $\mathbf{1}^T = (1, 1, \ldots, 1)$ is called reference hyperplane.
H is a supporting hyperplane on the convex hull of G' at g because 1) $\mathbf{1}^T(Y - g) = \mathbf{1}^T(g - g) = 0$ and 2) if $Y = (y_1, y_2, \cdots, y_s) \in G'$, then according to Lemma (4), we will have $y_r \leq g_r \ \ , r = 1, 2, \cdots, s$. So, $y_1 + y_2 + \cdots + y_s \leq g_1 + g_2 + \cdots + g_s$; that is, $\mathbf{1}^TY \leq \mathbf{1}^Tg$. Instead of using gradient $\mathbf{1}$, a vector, say $a = (|g_1|, |g_2|, \cdots, |g_s|)^T$, may be used to find the supporting hyperplane for G' at g, but from the computational point of view, this is a difficult task.

3.2. Distance function

For each point $Y^o \in G$, the distance function is defined as follows:

$$D(Y^o) = \frac{|a^TY^o - a^Tg|}{\sqrt{a^Ta}}.$$

Note that for each $Y^o \in G,\ a^TY^o \leq a^Tg$ so, $D(Y^o) = \frac{1}{\sqrt{a^Ta}}(a^Tg - a^TY^o)$.

For illustration, consider the following problem:

$$\begin{array}{lll} Max & 4w_1 - 3w_2 + 5w_3 & \\ Max & 2w_1 + 7w_2 - w_3 & \\ s.t. & w_1 + 2w_2 + w_3 & \leq \ 7 \\ & 3w_1 + w_2 + 2w_3 & \leq \ 6 \\ & w_1,\ w_2,\ w_3 \in \{0, 1\}. & \end{array}$$

The feasible solutions of the above problem and its corresponding output vectors are shown in Table 1.

Table 1: The feasible solutions and corresponding outputs

DMU_j	1	2	3	4	5	6	7	8
W_i	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	(1,1,1)
Output	(0,0)	(4,2)	(-3,7)	(5,-1)	(1,9)	(9,1)	(2,6)	(6,8)

It is evident that $W_5 = (1, 1, 0)$ and $W_6 = (1, 0, 1)$ are the optimal solutions of (8) for $r = 1, 2$, respectively. For this problem, the reference hyperplane, the ideal vector and the distance function are $y_1 + y_2 = 18$, $g = (C_1W_1^*, C_2W_2^*)^T = (9, 9)^T$ and $D(Y^o) = \frac{1}{\sqrt{2}}(18 - y_1^o - y_2^o)$, respectively which have been depicted in Figure 1.

3.3. Efficient solutions generation

Let G'' be the set of DMUs corresponding to all optimal solutions of the problems (8). To find efficient solutions of the problem (7), two steps should be taken. First, the members of G'' are evaluated by using the model (5) and efficient DMUs are specified. Suppose that G_o

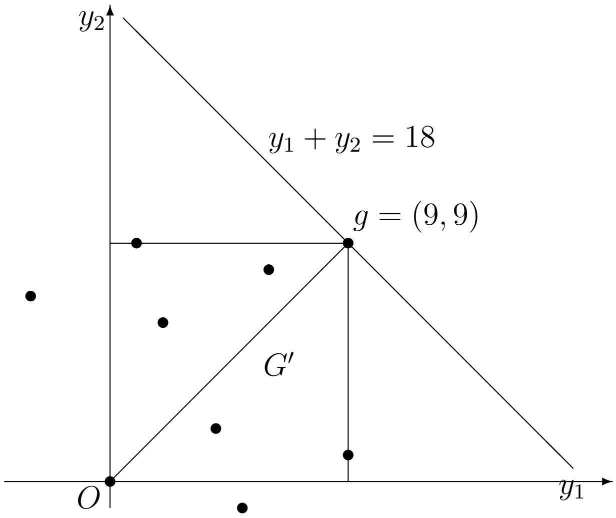

Figure 1: Reference hyperplane and ideal vector

is the set of efficient DMUs in G''. Based on the theorem (2), each W corresponding to a member of G_o is an efficient solution of the problem (7) and these are not the only efficient solutions of it. Let (U_p^*, u_{op}^*) denote an optimal solution of the problem which corresponds to DMU$_p$ (DMU$_p \in G''$). If DMU$_p$ is efficient in the model (5), $U_p^* Y + u_{op}^* = 0$ is the supporting hyperplane on the production possibility set constructed by efficient DMUs (the members of G_o) which is defined as follows:

$$T(G_o) = \{Y \mid Y \leq \sum_{j=1}^{l} \lambda_j Y_j, \quad \sum_{j=1}^{l} \lambda_j = 1, \lambda_j \geq 0, j = 1, 2, \ldots, l\}.$$

The supporting hyperplanes on $T(G_o)$ separate the set G into two sets which are as follows:

$$G^H = \{Y_d \mid U_p^* Y_d + u_{op}^* \leq 0, \quad p = 1, 2, \cdots, l\}, \quad G^E = G - G^H.$$

In the second step, in order to find other efficient solutions, we specify a point from G^E that has the minimal distance from the reference hyperplane. To do so, the following problem may be solved.

$$\begin{array}{lll} Min & D(Y_d) & \\ s.t. & U_p^* Y_d + u_{op}^* > -Mt_p, & p = 1, 2, \cdots, l \\ & t_1 + t_2 + \cdots + t_l \leq l - 1 & \\ & Y_d \in G & \\ & t_p \in \{0, 1\}, & p = 1, 2, \cdots, l. \end{array} \tag{9}$$

where $D(Y_d)$ is the distance of the vector Y_d from the reference hyperplane and M is a positive large number. Note that if $t_p = 1$, then constraint $U_p^* Y_d + u_{op}^* > -M$ is redundant. Otherwise, this constraint is not redundant. The constraint $t_1 + t_2 + \cdots + t_l \leq l - 1$ implies that at least one of the constraints $U_p^* Y_d + u_{op}^* > 0, p = 1, 2, \cdots, l$ is not redundant.

Based on the definition of the vector $Y_d = (y_{1d}, y_{2d}, \cdots, y_{sd})^T$, we will have:

$$y_{rd} = \sum_{j=1}^{n} C_{rj} w_{jd}, \quad r = 1, 2, \cdots, s,$$

$$D(Y_d) = \frac{1}{\sqrt{a^T a}}(a^T g - a^T Y_d) = \frac{1}{\sqrt{\mathbf{1}^T\mathbf{1}}}(\mathbf{1}^T g - \mathbf{1}^T Y_d)$$
$$= \frac{1}{\sqrt{s}}[(g_1 + g_2 + \cdots + g_S) - (y_{1d} + y_{2d} + \cdots + y_{sd})],$$

$$Min \ \ D(Y_d) = Max \ \ (y_{1d} + y_{2d} + \cdots + y_{sd}) = Max \ \ (\sum_{j=1}^{n}\sum_{r=1}^{s} C_{rj} w_{jd}),$$

$$U_p^* Y_d + u_{op}^* = \sum_{r=1}^{s} u_{rp}^* y_{rd} + u_{op}^* = \sum_{j=1}^{n}\sum_{r=1}^{s} u_{rp}^* C_{rj} w_{jd} + u_{op}^*.$$

Hence, the problem (9) is transformed to a single objective 0-1 linear programming problem which is as follows:

$$\begin{array}{lll} Max \ \ Z_W = & \sum_{j=1}^{n}\sum_{r=1}^{s} C_{rj} w_{jd} & \\ s.t. & \sum_{j=1}^{n}\sum_{r=1}^{s} u_{rp}^* C_{rj} w_{jd} > -u_{op}^* - t_p M, & p = 1, 2, \cdots, l \\ & \sum_{j=1}^{n} a_{ij} w_{jd} \le b_i, & i = 1, 2, \cdots, m \\ & t_1 + t_2 + \cdots + t_l \le l - 1 & \\ & t_p \in \{0, 1\}, & p = 1, 2, \cdots, l \\ & w_{jd} \in \{0, 1\}, & j = 1, 2, \cdots, n. \end{array} \tag{10}$$

Theorem 4: *Each optimal solution of the problem (10) is an efficient solution for the problem (7).*

Proof: Let W_d^* be an optimal solution of the problem (10). By contradiction, suppose that W_d^* is not efficient for the problem (7). Hence, the problem (7) has a feasible solution W^o so that:

$$C_r W^o \ge C_r W_d^*, \quad r = 1, 2, \cdots, s \ \text{ and } \exists\ k\ ;\ C_k W^o > C_k W_d^* \tag{11}$$

By multiplying u_{rp}^* in $C_r W^o \ge C_r W_d^*$ $(r = 1, 2, \cdots, s)$ and summing them, we will have:

$$\begin{array}{ll} \sum_{r=1}^{s} u_{rp}^* C_r W^o \ge \sum_{r=1}^{s} u_{rp}^* C_r W_d^*, & p = 1, 2, \cdots, l \\ \sum_{j=1}^{n}\sum_{r=1}^{s} u_{rp}^* C_{rj} w_j^o \ge \sum_{j=1}^{n}\sum_{r=1}^{s} u_{rp}^* C_{rj} w_{jd}^*, & p = 1, 2, \cdots, l \\ \sum_{j=1}^{n}\sum_{r=1}^{s} u_{rp}^* C_{rj} w_j^o \ > \ -u_{op}^* - t_p M, & p = 1, 2, \cdots, l. \end{array}$$

Also, W^o holds in the inequalities $\sum_{j=1}^{n} a_{ij} w_j \le b_i, \quad i = 1, 2, \cdots, m$. Therefore, W^o is a feasible solution of the problem (10). From (11), we have $\sum_{r=1}^{s} C_r W^o > \sum_{r=1}^{s} C_r W_d^* (Z_{W_o} > Z_{W_d^*})$ which is a contradiction. □

After obtaining W_d^*, the corresponding DMU is constructed and the set of G_1 is defined as $G_1 = G_o \cup \{Y_d\}$. By evaluating the members of G_1 and continuing the above process, the efficient solutions of (7) are obtained. Since the number of the efficient solutions is finite and at least one efficient solution is found at each iteration, the algorithm is terminated in the finite number of the iteration. The aforementioned concepts have been illustrated in Figure 2. This Figure corresponds to a problem which has two objective functions.

In the Figure 2: 1) L_1 is the reference hyperplane, 2)$D(Y_d)$ is the distance Y_d from L_1, 3) The points (1) and (3) are the members of G_o and the point (2) is corresponding to an efficient solution which is obtained in iteration 1(that is $G_1 = \{1, 2, 3\}$), 4) L_2 and L_3 are the supporting hyperplanes on $T(G_1)$ and 5) G^H and G^E are the set of the points $\bullet$ and $\circ$, respectively.

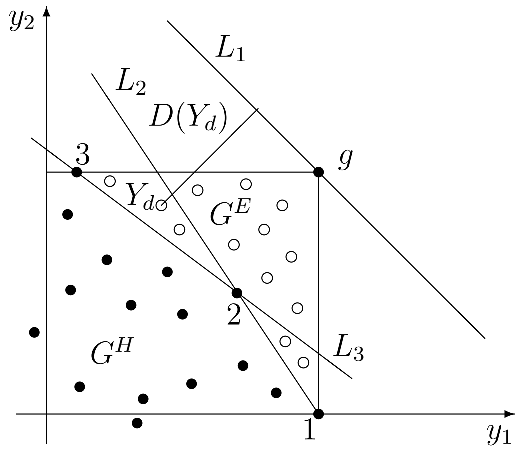

Figure 2: Distance $D(Y_d)$ and sets G^H and G^E

In order to develop the algorithm of finding efficient solutions, the sets G_k, G_k^H, G_k^E and $T(G_k)$ are defined as follows:
G_k: G_k is a subset of G which denotes the set of the evaluated DMUs by the model (5) at the beginning of the k^{th} iteration.
G_k^H: $G_k^H = \{Y_d \mid U_p^* Y_d + u_{op}^* \leq 0,\ p = 1, 2, \cdots, l, l+1, \cdots, \eta\}$ where η is the number of the elements G_k and Y_d is corresponding to a feasible solution of (7), say W_d.
$G^E : G^E = G - G^H$.
$T(G_k) : T(G_k) = \{Y \mid Y \leq \sum_{j=1}^{\eta} \lambda_j Y_j,\ \ \sum_{j=1}^{\eta} \lambda_j = 1, \lambda_j \geq 0, j = 1, 2, \ldots, \eta\}$.

3.4. An algorithm for generating efficient solutions

Stage 0: Initialization
Step 0-1 : Find the output vector of DMUs corresponding to all optimal solutions of (8) and put, $G'' = \{Y_1, Y_2, \cdots, Y_h\}$,
Step 0-2 : By evaluating the members of G'' by means of the model (5), determine the members of G_o and the supporting hyperplanes on $T(G_o)$.
Stage 1 : The generation of an Efficient Solution
Step 1-1 : Solve the problem (10). If it has an optimal solution, go to stage 2; otherwise, stop,
Stage 2 : The evaluation of DMUs and Generation of the Supporting Hyperplane
Step 2-1 : Determine the vector $Y_d = (C_1 W_d^*, C_2 W_d^*, \ldots, C_s W_d^*)$ where, W_d^* is an efficient solution of the problem (10), and put $G_{k+1} = G_k \cup \{Y_d\}$,
Step 2-2 : Evaluate the members of G_{k+1} by using model (5) and identify the supporting hyperplanes on $T(G_{k+1})$ and go to stage 1.

4. Numerical Examples

Example 1: Consider the following 0-1 MOLP problem:

$$\begin{array}{lll} Max & 2w_1 - 5w_2 + w_3 & \\ Max & -7w_1 + 4w_2 + w_3 & \\ s.t. & 2w_1 + 3w_2 - w_3 & \leq 7 \\ & -3w_1 + w_2 + 5w_3 & \leq 6 \\ & w_1,\ w_2,\ w_3 \in \{0,1\}. & \end{array}$$

Table 2 denotes the feasible solutions of the problem, the outputs vector and the inputs vector of DMU corresponding to these feasible solutions.

Table 2: The feasible solutions, outputs and inputs

DMU_j	1	2	3	4	5	6	7	8
W_i	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	(1,1,1)
Input	(0,0)	(2,-3)	(3,1)	(-1,5)	(5,-2)	(1,2)	(2,6)	(4,3)
Output	(0,0)	(2,-7)	(-5,4)	(1,1)	(-3,-3)	(3,-6)	(-4,5)	(-2,-2)

As it can be seen, all DMUs(except DMU_1) have at least one negative input or output. Thus, the proposed method by Liu et al. cannot determine the efficient solutions of this example. But the suggested method in this paper obtains efficient solutions, which are $W_3 = (0,1,0)$, $W_4 = (0,0,1)$, $W_6 = (1,0,1)$ and $W_7 = (0,1,1)$.

Example 2: Consider the following 0-1 MOLP problem in which $W = (1,1)$ is its efficient solution.

$$\begin{array}{lll} Max & 4w_1 + w_2 & \\ Max & 6w_1 + w_2 & \\ s.t. & 3w_1 + 5w_2 & \leq 9 \\ & 4w_1 + 7w_2 & \leq 12 \\ & w_1,\ w_2 \in \{0,1\}. & \end{array}$$

This example has been solved by the proposed method in [8] and the presented method in this paper. To solve the above example by Liu's method, we choose feasible solutions $W_1 = (1,0)$, $W_2 = (0,1)$, $W_3 = (1,1)$ and $W_4 = (0,0)$ in step 0-1 of the presented algorithm in [8]. The results of the evaluation of DMUs corresponding to these feasible solutions by BCC model are represented in Table 3.

Table 3: The data set and feasible solutions

No	W_i	Input	Output	Eff-BCC
1	(1,0)	(3,4)	(4,6)	1
2	(0,1)	(5,7)	(1,1)	0.15
3	(1,1)	(8,11)	(5,7)	1
4	(0,0)	(0,0)	(0,0)	-

Based on the proved theorem in [8], $W_1 = (1,0)$ and $W_3 = (1,1)$ are the efficient solutions of the above problem. But according to the definition of the efficient solution in literature (in [8]), W_1 is not efficient; that is, the Liu's method obtains a solution which is not efficient. The proposed method in this paper introduces only $W_3 = (1,1)$ as the efficient solution.

Example 3: Consider a 0-1 MOLP problem which the corresponding DMUs of its feasible solutions have been depicted in Figure 3.

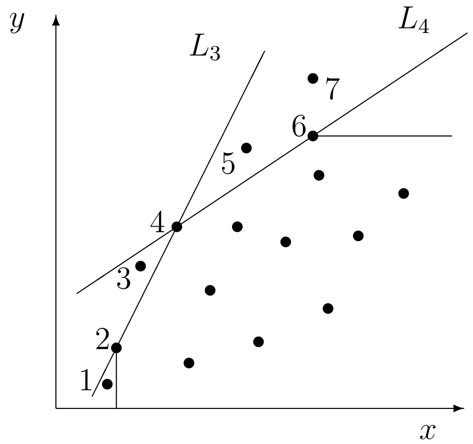

Figure 3: Omitting of efficient solutions in the Liu's method

To solve the corresponding problem of Figure 3 by Liu's method, if the points 2, 4 and 6 are chosen in step 0-1, the constraints $L_3 : U_1^*Y - V_1^*X > u_{01}^*$ and $L_4 : U_2^*Y - V_2^*X > u_{02}^*$ will be added to the problem in step 1-1. Figure 3 denotes that if these constraints are imposed to the problem, the points 3, 5 and 7 will be omitted. Hence, some efficient solutions may be lost.

Example 4: Consider the following 0-1 MOLP problem:

$$\begin{array}{lll} Max & 4w_1 - 3w_2 + 5w_3 + w_4 & \\ Max & 2w_1 + 7w_2 - w_3 - w_4 & \\ s.t. & 2w_1 + 5w_2 - w_3 + 2w_4 & \leq \; 10 \\ & 4w_1 + 3w_2 + 5w_3 - 4w_4 & \leq \; 12 \\ & -2w_1 + 4w_2 + 7w_3 + w_4 & \leq \; 15 \\ & w_1,\ w_2,\ w_3,\ w_4 \in \{0,1\}. & \end{array}$$

Stage 0, step 0-1 : By solving P_1 and P_2, the set G'' is identified.

$$\begin{array}{lll} P_1) & Max & 4w_1 - 3w_2 + 5w_3 + w_4 \\ & s.t. & 2w_1 + 5w_2 - w_3 + 2w_4 \leq 10 \\ & & 4w_1 + 3w_2 + 5w_3 - 4w_4 \leq 12 \\ & & -2w_1 + 4w_2 + 7w_3 + w_4 \leq 15 \\ & & w_1,\ w_2,\ w_3,\ w_4 \in \{0,1\}. \end{array}$$

$$\begin{array}{lll} P_2) & Max & 2w_1 + 7w_2 - w_3 - w_4 \\ & \text{s.t.} & 2w_1 + 5w_2 - w_3 + 2w_4 \leq 10 \\ & & 4w_1 + 3w_2 + 5w_3 - 4w_4 \leq 12 \\ & & -2w_1 + 4w_2 + 7w_3 + w_4 \leq 15 \\ & & w_1,\ w_2,\ w_3,\ w_4 \in \{0,1\}. \end{array}$$

The optimal solutions of P_1 and P_2 are $W_1 = (1, 0, 1, 1)$ and $W_2 = (1, 1, 0, 0)$ respectively. Therefore, $G'' = \left\{ \binom{10}{0}, \binom{1}{9} \right\}$.

Step 0-2: By evaluating the members of G'' by model (5), the members of G_o and the supporting hyperplanes on the $T(G_o)$ are specified.

$$\begin{array}{llll} Max & Q_1 = 10u_{11} + u_{01} & & \\ s.t. & 10u_{11} + u_{01} & \leq & 0 \\ & u_{11} + 9u_{21} + u_{01} & \leq & 0 \\ & u_{11}, u_{21} & \geq & 1 \end{array}$$

$$\begin{array}{llll} Max & Q_2 = u_{12} + 9u_{22} + u_{02} & & \\ \text{s.t.} & 10u_{12} + u_{02} & \leq & 0 \\ & u_{12} + 9u_{22} + u_{02} & \leq & 0 \\ & u_{12}, u_{22} & \geq & 1 \end{array}$$

The results of evaluation have been presented in Table 4.

So, we have $G_o = \left\{ \binom{10}{0}, \binom{1}{9} \right\}$ and $y_1 + y_2 - 10 = 0,\ y_1 + y_2 - 10 = 0$. The equation of the above hyperplane in terms of $w_j (j = 1, 2, 3, 4)$ is as follows:

Table 4: The results of evaluation

variable	DMU_1	DMU_2
u^*_{1p}	1	1
u^*_{2p}	1	1
u^*_{0p}	-10	-10
Q^*_p	0	0
result	efficient	efficient

$y_1 + y_2 - 10 = (4w_1 - 3w_2 + 5w_3 + w_4) + (2w_1 + 7w_2 - w_3 - w_4) - 10 = 0,$
$6w_1 + 4w_2 + 4w_3 - 10 = 0.$

Iteration (1)
Stage 1, step 1-1: In this step, the following problem must be solved:

$$\begin{array}{lll} Max & 6w_1 + 4w_2 + 4w_3 & \\ s.t. & 6w_1 + 4w_2 + 4w_3 & > 10 \\ & 2w_1 + 5w_2 - w_3 + 2w_4 & \leq 10 \\ & 4w_1 + 3w_2 + 5w_3 - 4w_4 & \leq 12 \\ & -2w_1 + 4w_2 + 7w_3 + w_4 & \leq 15 \\ & w_1,\ w_2,\ w_3,\ w_4 \in \{0, 1\}. & \end{array}$$

The alternative optimal solutions of the above problem are $W_3 = (1,1,1,1)$ and $W_4 = (1,1,1,0)$ which both are the efficient solutions of the problem.
Stage (2), step 2-1: $Y_3 = \binom{7}{7}$ and $Y_4 = \binom{6}{8}$ are the vectors corresponding to W_3 and W_4 respectively. Hence,

$$G_1 = G_o \cup \left\{Y_3, Y_4\right\} = \left\{ \binom{10}{0}, \binom{1}{9}, \binom{7}{7}, \binom{6}{8} \right\}.$$

Step 2-2: The results of the evaluation of G_1's members have been presented in Table 5.

Table 5: The results for example 4

variable	DMU_1	DMU_2	DMU_3	DMU_4
u^*_{1p}	2.333	1	1	1
u^*_{2p}	1	5	1	1
u^*_{0p}	-23.33	-46	-14	-14
Q^*_p	0	0	0	0

The equations of the supporting hyperplanes on $T(G_1)$ are as follows:

$$\begin{aligned} 2.333y_1 + y_2 - 23.33 &= 0 \\ y_1 + 5y_2 - 46 &= 0 \\ y_1 + y_2 - 14 &= 0 \end{aligned}$$

and the equations of them in terms of $w_j (j = 1, 2, 3, 4)$ are as follows:

$$\begin{aligned} 11.332w_1 + 0.001w_2 + 10.665w_3 + 1.333w_4 - 23.33 &= 0 \\ 14w_1 + 32w_2 - 4w_4 - 46 &= 0 \\ 6w_1 + 4w_2 + 4w_3 - 14 &= 0 \end{aligned}$$

Iteration (2)
Stage 1 , step 1-1: In this step, the following problem must be solved:

$$\begin{array}{lllll} Max & 6w_1+4w_2+4w_3 & & \\ s.t. & 6w_1+4w_2+4w_3 & > & 14-t_1M \\ & 14w_1+32w_2-w_4 & > & 46-t_2M \\ & 11.332w_1+0.001w_2+10.665w_3+1.333w_4 & > & 23.33-t_3M \\ & 2w_1+5w_2-w_3+2w_4 & \leq & 10 \\ & 4w_1+3w_2+5w_3-4w_4 & \leq & 12 \\ & -2w_1+4w_2+7w_3+w_4 & \leq & 15 \\ & t_1+t_2+t_3 & \leq & 2 \\ & w_1,w_2,w_3,w_4,t_1,t_2,t_3 \in \{0,1\}. & & \end{array}$$

The above problem is infeasible, so the algorithm is terminated and the efficient solutions are as: $W_1 = (1,0,1,1), W_2 = (1,1,0,0), W_3 = (1,1,1,1), W_4 = (1,1,1,0)$.

Example 5: The following 0-1 MOLP problem is an adaptation of an example from [8]:

$$\begin{array}{llll} Max & 3w_1+6w_2+5w_3-2w_4+3w_5 & & \\ Max & 6w_1+7w_2+4w_3+3w_4-8w_5 & & \\ Max & 5w_1-3w_2+8w_3-4w_4+3w_5 & & \\ s.t. & -2w_1+3w_2+8w_3-w_4+5w_5 & \leq & 13 \\ & 6w_1+2w_2+4w_3+4w_4-3w_5 & \leq & 15 \\ & 4w_1-2w_2+6w_3-2w_4+w_5 & \leq & 11 \\ & w_1,\ w_2,\ w_3,\ w_4,\ w_5 \in \{0,1\}. & & \end{array}$$

Stage 0, step 0-1: By solving the problems corresponding to objective functions of the example 5 (problems P_1, P_2 and P_3), we will have $G'' = \{(14,17,10)^T, (11,2,16)^T\}$.

Step 0-2: By evaluating the members of G'' by model (5), the set G_o and the supporting hyperplane on $T(G_o)$ are determined as $G_o = \{(14,17,10)^T, (11,2,16)^T\}$ and $y_1+y_2+3y_3-61=0$.
The equation $y_1+y_2+3y_3-61=0$ in terms of $w_j (j=1,2,3,4,5)$ is as
$24w_1+4w_2+33w_3-11w_4+4w_5-61=0$.

Iteration (1)
Stage 1, step 1-1: In this step, we solve the following problem:

$$\begin{array}{llll} Max & 14w_1+10w_2+17w_3-3w_4-2w_5 & & \\ s.t. & -2w_1+3w_2+8w_3-w_4+5w_5 & \leq & 13 \\ & 6w_1+2w_2+4w_3+4w_4-3w_5 & \leq & 15 \\ & 4w_1-2w_2+6w_3-2w_4+w_5 & \leq & 11 \\ & 24w_1+4w_2+33w_3-11w_4+4w_5 & > & 61 \\ & w_1,\ w_2,\ w_3,\ w_4,\ w_5 \in \{0,1\}. & & \end{array}$$

The above problem is infeasible. So, the algorithm is terminated. $W_1 = (1,1,1,0,0)$ and $W_2 = (1,0,1,0,1)$ are the efficient solutions of the example 5.
As it can be seen, the suggested method obtains the efficient solutions of the example 5 in one iteration while the Liu's method obtains the same solutions in three iterations.

5. Conclusion

This paper presents a method for solving 0-1 MOLP problem. In the proposed method for finding the efficient solutions of 0-1 MOLP problem, full enumeration is not used. In each iteration of the suggested algorithm, at least one efficient solution is found. Since the number of feasible solutions is finite, the algorithm is convergent. The examples 1, 2, 3 and 5 illustrate the advantage of our method in comparison with Liu's method. The existence of the convexity constraint in the additive model without input, which is used in this paper for evaluating constructed DMUs, may eliminate some efficient solutions of the problem. This deficiency should be studied in the future. A modified version of this algorithm can be used for solving integer MOLP problem.

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