

AN ASYMPTOTIC VALUATION FOR THE OPTION UNDER A GENERAL STOCHASTIC VOLATILITY

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Abstract This article examines the valuation problem for the European option under a general stochastic volatility in a certain approximate sense by adopting the small disturbance asymptotic theory developed by Kunitomo and Takahashi [25, 26]. The option value can be decomposed into the Black and Scholes value under deterministic volatility and adjustment terms driven by the randomness of the volatility, which also extends some portions of Kunitomo and Kim [24].

1. Introduction

Many option pricing models which incorporate the randomness of the volatility of underlying asset return have been proposed and tested as a natural extension of the celebrated Black and Scholes [7]. In continuous time setting, where trading takes place continuously over time, the volatility is basically assumed to evolve as a separate state process. The works in this regard include Hull and White [21, 22], Johnson and Shanno [23], Wiggins [38], Scott [32, 33, 34], Chesney and Scott [10], Melino and Turnbull [27], Stein and Stein [36], Heston [19], and Ball and Roma [5], to name only a few.¹ In general it is not easy to evaluate a fair price of the option under stochastic volatility.

Hitherto, several approaches to the valuation problem have been adopted. Firstly, the Monte Carlo simulation and/or numerical techniques are used. Johnson and Shanno [23], Wiggins [38], Melino and Turnbull [27], and Duan [12] utilized this approach, for example. Many variance reduction techniques and more streamlined numerical algorithms have been proposed to save the calculation time and increase the numerical accuracy.

Secondly, utilizing the Black-Scholes value as a stepping stone is helpful in some cases. For example, Hull and White [21] showed that the option value can be obtained as the expectation of the Black-Scholes value, where the constant volatility is replaced by the average volatility up to maturity. Romano and Touzi [29] extended Hull and White [21] in a more general setting. See also Amin and Ng [1] in a multiperiod equilibrium argument. However these valuation forms are not tractable ones and furthermore, need to be implemented by simulation for example.

¹On the other hand, the option pricing models under stochastic volatility have been developed in discrete time framework as an extension of Rubinstein [31] and Brennan [8]. The so called GARCH option model, and Amin and Ng [1] are the representative works. In particular the GARCH option pricing model has a merit, in that the parameters of option model can be estimated with ease since the innovations driving the asset return and volatility are same. Duan [12] and Heston and Nandi [20] are notable works. It is also well known that the stochastic volatility option model in continuous time is closely related to the GARCH option model. See Duan [13] on this point.

Thirdly, Fourier inversion methods (or transform analysis) proposed by Heston [19] are useful in a restricted class of volatility process. It can be thought that although the risk-neutralized probabilities are not immediately available in a closed form, their characteristic functions sometimes are. By inverting the characteristic function, we can obtain the desired probability and evaluate the option price. This approach has received much attention in the literature, which includes Bates [6], Bakshi and Chen [3], Scott [34], Bakshi and Madan [4], and Duffie, Pan and Singleton [14]. This method seems to be accurate when the closed form of characteristic function is available.

Finally, the approximation method can be considered. Some authors have tried to derive an approximate option value by imposing some restrictions on the structure of state processes. For example, see Hull and White [22], and Ball and Roma [5].

Our approach is associated with the approximation one with a general volatility process in continuous time. By a general volatility process it means that the volatility process can be described by an Ito process and furthermore the correlation between the return process and the volatility process is allowed. The basic framework is as follows. To begin with, the volatility process will be expanded with respect to the small disturbance parameter, that is, the volatility of the volatility. Next, the probability density of the underlying asset price at the maturity time of the option is obtained in an asymptotic sense. Finally, this allows one to evaluate some payoff functions in an asymptotic sense. This procedure is an application of the small disturbance asymptotic theory, which was originally developed by Kunitomo and Takahashi [25, 26]. In this context, Kunitomo and Kim [24] discussed the valuation of contingent claims, including options under stochastic volatility.

Although the resultant implication is similar to that of Kunitomo and Kim [24], this study has some extra features. This paper extends Kunitomo and Kim [24] in that the option value is obtained up to higher order term, by utilizing a kind of characteristic function inversion formula. The option pricing formula derived gives a neater representation. The hedging argument is also considered. As an application of the small disturbance asymptotic theory, Takahashi [37] also dealt with a kind of option valuation model under stochastic volatility. However, he did not derive an explicit form of option pricing formula and, his model is not same as that of this study.

Sircar and Papanicolaou [35], and Fouque, Papanicolaou and Sircar [15] developed an asymptotic formula of option value which is related to our approach. By focusing on the fast mean reverting property of volatility process, they asserted that the inverse of the rate of mean reversion in the volatility process is typically small. Expanding with respect to this small quantity parameter leads to Black-Scholes value, where the constant volatility is replaced by a long-run time average of volatility function, plus corrected terms. However their result heavily relies on the assumption of the ergodicity of volatility process.

Fournie, Lebuchoux, and Touzi [16] also derived an asymptotic expansion form of the solution of a parabolic problem around the small disturbance and applied this idea to the option pricing problem under stochastic volatility. The small disturbance parameter in them is the same as that of our approach, *i.e.*, the volatility of the volatility. They showed that the option value is composed of the Black-Scholes value and the derivatives of first term with stock prices. However, the approach of this study is different from them in that the asymptotic expansion is executed on the state process *per se* and consequently the conditional expectation is evaluated. It should also be added that the resultant valuation formulae of this study appeals to the intuitive interpretation.

The remainder of this paper is organized as follows. The asymptotic option valuation formulae under a general volatility process are provided in section 2. Section 3 gives some

examples and shows the numerical accuracies. Section 4 concludes. Finally, section 5 gives appendices.

2. Option Pricing under a General Stochastic Volatility

2.1. Setup

Let us consider the economy where two primitive assets, the risky stock and non-risky zero coupon bond are available. The stock does not pay cash dividends and its price process $S_t^{(\delta)}$ obeys

$$S_t^{(\delta)} = S_0 + \int_0^t \alpha(S_s^{(\delta)}, \sigma_s^{(\delta)}, s) S_s^{(\delta)} ds + \int_0^t \sigma_s^{(\delta)} S_s^{(\delta)} dW_{1s}, \quad (1)$$

$$\sigma_s^{(\delta)} = \sigma_0 + \int_0^s \mu^*(\sigma_u^{(\delta)}, u, \delta) du + \delta \int_0^s w(\sigma_u^{(\delta)}, u) dW_{2u}, \quad (2)$$

where $\sigma_t^{(\delta)}$ is the volatility process, and W_{1t} and W_{2t} are possibly correlated two Brownian motions. The usual notations such as S_t and σ_t without the superscript (δ) should be understood as deterministic counterparts. The volatility of the volatility δ plays a particularly important role in this study, which will be apparent later. The drift function $\mu^*(\cdot)$ of the volatility process may contain the parameter δ , which depends on the volatility process models. The covariation between stock price and volatility is assumed to be caught by ρ , a constant:

$$E[dW_{1t}dW_{2t}] = \rho dt.$$

The instantaneous interest rate r is assumed to be constant so that the time t price of a zero coupon bond maturing at time T is given by $e^{-r(T-t)}$. It can be verified that the above setting encompasses the existing option pricing models under stochastic volatility process by applying Ito's formula for the function of volatility process, if necessary.

The European stock call option value at time zero with exercise price K and maturity time T , denoted V , is determined by

$$V = e^{-rT} E[[S_T^{(\delta)} - K]^+], \quad (3)$$

where $[\cdot]^+$ is equal to $\max[\cdot, 0]$, E is the risk neutral expectations operator, and the stock price and volatility are now given by

$$S_t^{(\delta)} = S_0 + \int_0^t r S_s^{(\delta)} ds + \int_0^t \sigma_s^{(\delta)} S_s^{(\delta)} dW_{1s} \quad (4)$$

and

$$\sigma_s^{(\delta)} = \sigma_0 + \int_0^s \mu(\sigma_u^{(\delta)}, u, \delta) du + \delta \int_0^s w(\sigma_u^{(\delta)}, u) dW_{2u}. \quad (5)$$

Let $\lambda_t(\cdot)$ denote the risk premium of volatility process. Then the functional form of $\mu(\cdot)$ in (5) implies that it is implicitly assumed $\lambda_t = \lambda(\sigma_t^{(\delta)}, t)$. Of course, in general equilibrium context such as Cox, Ross, and Ingersoll [11] and Bakshi and Chen [3], the risk premium should be determined as the function of individuals preferences in a much more complicated way. Meanwhile, in the no arbitrage equilibrium context, since the stock is the only traded asset, the risk premiums respectively on the two sources of uncertainty W_{1t} and W_{2t} are not uniquely determined, which characterizes the non-uniqueness of the equivalent martingale measure in this incomplete market. In the setting of (1) and (2), Romano and Touzi [29] extended Ross [30] in a dynamic setup and discussed the conditions under which introducing

a contingent claim completes the market in the sense of Harrison and Pliska [18].² Their basic idea is very intuitive. If one could deduce the volatility process from the traded risky asset such as option, the volatility itself could be interpreted as if it is a traded asset. They showed that this ‘invertibility’ can be guaranteed if the contingent claim is strictly convex in stock price under some restricted payoff structure. If the conditions described in Romano and Touzi [29] hold, there exists a risk premium process λ_t which completes the market. The idea of spanning via options is also exploited by Bakshi and Madan [4] in a different framework.

2.2. Asymptotic risk-neutral probability density

To begin with, the return process (4) can be written by

$$S_t^{(\delta)} = S_0 \exp \left\{ r t - \frac{1}{2} \int_0^t (\sigma_s^{(\delta)})^2 ds + \int_0^t \sigma_s^{(\delta)} dW_{1s} \right\}. \quad (6)$$

If one could obtain an asymptotic behavior of stock price at arbitrary time and its density function when a possibly small parameter δ goes down to zero, one could evaluate the option value (3) in an asymptotic sense. This approach is referred to as the small disturbance asymptotic theory, which is no ad hoc one and has rigorous mathematical foundations. In this study the mathematical validity of this approach is not delivered. The interested readers may consult Kunitomo and Takahashi [25, 26], and Yoshida [39] on this point.

Usually, the estimates of parameter δ in the volatility function of the unobservable volatility process has small values in financial markets. For example, in security markets the estimates of δ range from 0.15 to 0.30 in Stein and Stein [36] (where, the volatility is modelled as OU process) and 0.23 (res. 0.39, 0.06) in Scott [34] (res. Bakshi, Cao, and Chen [2], Chernov and Ghysels [9]) (, where the squared volatility is described as the square root process). In a foreign currency market, the estimates of δ is 0.073 in Chesney and Scott [10] and from 0.15 to 0.19 in Melino and Turnbull [27], when the log of volatility is proposed as the mean reverting process.

Motivated by these empirical observations, we shall derive the asymptotic stock price of (6), with respect to δ . Let us expand the volatility process (5) at any particular time t , with respect to δ as follows:

$$\sigma_t^{(\delta)} = \sigma_t + \delta A(t) + \delta^2 B(t) + o_p(\delta^2) \quad (7)$$

as $\delta \downarrow 0$, where σ_t is the solution of the ordinary differential equation described by

$$\sigma_t = \sigma_0 + \int_0^t \mu(\sigma_s, s, 0) ds, \quad (8)$$

$$A(t) = \frac{\partial \sigma_t^{(\delta)}}{\partial \delta} \Big|_{\delta=0}, \text{ and } B(t) = \frac{1}{2} \frac{\partial^2 \sigma_t^{(\delta)}}{\partial \delta^2} \Big|_{\delta=0}.$$

Furthermore, $A(t)$ can be represented in a formal way as follows.

$$A(t) = \int_0^t [\partial \mu(\sigma_s, s, 0) A(s) + \partial_\delta \mu(\sigma_s, s, 0)] ds + \int_0^t w(\sigma_s, s) dW_{2s},$$

where

$$\partial \mu(\sigma_s, s, 0) = \frac{\partial \mu(\sigma_s^{(\delta)}, s, \delta)}{\partial \sigma_s^{(\delta)}} \Big|_{\sigma_s^{(\delta)} = \sigma_s, \delta = 0},$$

²See also Pham and Touzi [28] for its economic justification.

and

$$\partial_\delta \mu(\sigma_s, s, 0) = \frac{\partial \mu(\sigma_s^{(\delta)}, s, \delta)}{\partial \delta} \Big|_{\delta=0}.$$

Let Y_t be the solution of $dY_t = \partial \mu(\sigma_t, t, 0)Y_t dt$ with $Y_0 = 1$. Then $A(t)$ can be expressed by

$$A(t) = \int_0^t Y_t Y_s^{-1} [w(\sigma_s, s) dW_{2s} + \partial_\delta \mu(\sigma_s, s, 0) ds].$$

In a similar fashion, we could write $B(t)$ as

$$B(t) = \frac{1}{2} \left\{ \int_0^t \left[\partial \mu(\sigma_s, s, 0) B(s) + \partial^2 \mu(\sigma_s, s, 0) A(s)^2 + \partial_\delta^2 \mu(\sigma_s, s, 0) \right] ds + \int_0^t 2 \partial w(\sigma_s, s) A(s) dW_{2s} \right\},$$

where

$$\partial^2 \mu(\sigma_s, s, 0) = \frac{\partial^2 \mu(\sigma_s^{(\delta)}, s, \delta)}{\partial (\sigma_s^{(\delta)})^2} \Big|_{\sigma_s^{(\delta)} = \sigma_s, \delta=0},$$

$$\partial_\delta^2 \mu(\sigma_s, s, 0) = \frac{\partial^2 \mu(\sigma_s^{(\delta)}, s, \delta)}{\partial \delta^2} \Big|_{\delta=0},$$

and

$$\partial w(\sigma_s, s) = \frac{\partial w(\sigma_s^{(\delta)}, s)}{\partial \sigma_s^{(\delta)}} \Big|_{\sigma_s^{(\delta)} = \sigma_s}.$$

Thus from the definition of Y_t , $B(t)$ is represented as

$$B(t) = \frac{1}{2} \int_0^t Y_t Y_s^{-1} \left[\partial^2 \mu(\sigma_s, s, 0) A(s)^2 ds + \partial_\delta^2 \mu(\sigma_s, s, 0) ds + 2 \partial w(\sigma_s, s) A(s) dW_{2s} \right].$$

Therefore plugging (7) into (6), gives an asymptotic price behavior of the underlying asset at arbitrary time t as follows:

$$S_t^{(\delta)} = S_0 \exp \left\{ rt + \int_0^t \sigma_s dW_{1s} - \frac{1}{2} \int_0^t \sigma_s^2 ds + \delta \left[\int_0^t A(s) dW_{1s} - \int_0^t \sigma_s A(s) ds \right] + \delta^2 \left[\int_0^t B(s) dW_{1s} - \frac{1}{2} \int_0^t A(s)^2 ds - \int_0^t \sigma_s B(s) ds \right] + o_p(\delta^2) \right\}. \tag{9}$$

In this study, since we are concerned with the price evolution of the underlying asset at maturity time T we rewrite (9) at time T :

$$S_T^{(\delta)} = S_T^* \exp \left\{ X_{1T} + \delta [X_{2T} - X_{3T}] + \delta^2 [X_{4T} - \frac{1}{2} X_{5T} - X_{6T}] + o_p(\delta^2) \right\},$$

where

$$S_T^* \equiv S_0 \exp \left\{ rT - \frac{1}{2} \int_0^T \sigma_t^2 dt \right\},$$

$X_{1T} = \int_0^T \sigma_t dW_{1t}$, $X_{2T} = \int_0^T A(t) dW_{1t}$, $X_{3T} = \int_0^T \sigma_t A(t) dt$, $X_{4T} = \int_0^T B(t) dW_{1t}$, $X_{5T} = \int_0^T A(t)^2 dt$, and $X_{6T} = \int_0^T \sigma_t B(t) dt$. If we introduce the notation $Z_T^{(\delta)}$ which collects all stochastic integrals in (9), we have

$$\begin{aligned} Z_T^{(\delta)} &\equiv \log \frac{S_T^{(\delta)}}{S_T^*} \\ &= X_{1T} + \delta [X_{2T} - X_{3T}] + \delta^2 [X_{4T} - \frac{1}{2} X_{5T} - X_{6T}] + o_p(\delta^2). \end{aligned} \tag{10}$$

As mentioned before, investigating the density asymptotics of $Z_t^{(\delta)}$ plays a crucial role in evaluating option value. If we denote the characteristic function of $Z_T^{(\delta)}$ by $q(\psi) = E[e^{i\psi Z_T^{(\delta)}}]$, we have the following expressions:

$$\begin{aligned}
 q(\psi) &= E[e^{i\psi(X_{1T} + \delta[X_{2T} - X_{3T}] + \delta^2[X_{4T} - \frac{1}{2}X_{5T} - X_{6T}] + \dots)}] \\
 &= E\left[e^{i\psi X_{1T}} \left\{ 1 + \delta(i\psi)[X_{2T} - X_{3T}] + \delta^2(i\psi)[X_{4T} - \frac{1}{2}X_{5T} - X_{6T}] \right. \right. \\
 &\quad \left. \left. + \frac{1}{2}\delta^2(i\psi)^2[X_{2T} - X_{3T}]^2 + \dots \right\}\right] \\
 &= E[e^{i\psi X_{1T}}] + \delta(i\psi)E[e^{i\psi X_{1T}} h_1(x)] - \delta(i\psi)E[e^{i\psi X_{1T}} h_2(x)] \\
 &\quad + \delta^2(i\psi)E[e^{i\psi X_{1T}} h_3(x)] - \frac{1}{2}\delta^2(i\psi)E[e^{i\psi X_{1T}} h_4(x)] - \delta^2(i\psi)E[e^{i\psi X_{1T}} h_5(x)] \\
 &\quad + \frac{1}{2}\delta^2(i\psi)^2 E[e^{i\psi X_{1T}} h_6(x)] + \frac{1}{2}\delta^2(i\psi)^2 E[e^{i\psi X_{1T}} h_7(x)] \\
 &\quad - \delta^2(i\psi)^2 E[e^{i\psi X_{1T}} h_8(x)] + \dots, \tag{11}
 \end{aligned}$$

where $h_1(x) \equiv E[X_{2T}|X_{1T} = x]$, $h_2(x) \equiv E[X_{3T}|X_{1T} = x]$, $h_3(x) \equiv E[X_{4T}|X_{1T} = x]$, $h_4(x) \equiv E[X_{5T}|X_{1T} = x]$, $h_5(x) \equiv E[X_{6T}|X_{1T} = x]$, $h_6(x) \equiv E[X_{2T}^2|X_{1T} = x]$, $h_7(x) \equiv E[X_{3T}^2|X_{1T} = x]$, and $h_8(x) \equiv E[X_{2T}X_{3T}|X_{1T} = x]$. At this stage, we need to evaluate each conditional expectation, h_i s for $i = 1, \dots, 8$ in (11). These conditional expectations have polynomial forms and their derivations are given in Appendix A.

To obtain the density function of $Z_T^{(\delta)}$, we need to invert $q(\psi)$. The inversion formula in Fujikoshi et. al. [17], which is also reproduced as Lemma 6.2 in Kunitomo and Takahashi [25], is very useful in this regard. This formula is provided in Appendix B for convenience. By applying this formula to (11), we have the following density asymptotics of $Z_T^{(\delta)}$, which is denoted by $f_{Z_T^{(\delta)}}$:

$$\begin{aligned}
 f_{Z_T^{(\delta)}} &\sim \phi_{\Sigma_{11}}(x) + \delta \left\{ -\frac{\partial}{\partial x}(h_1(x)\phi_{\Sigma_{11}}(x)) + \frac{\partial}{\partial x}(h_2(x)\phi_{\Sigma_{11}}(x)) \right\} \\
 &+ \delta^2 \left\{ -\frac{\partial}{\partial x}(h_3(x)\phi_{\Sigma_{11}}(x)) + \frac{1}{2}\frac{\partial}{\partial x}(h_4(x)\phi_{\Sigma_{11}}(x)) + \frac{\partial}{\partial x}(h_5(x)\phi_{\Sigma_{11}}(x)) \right. \\
 &\quad \left. + \frac{1}{2}\frac{\partial}{\partial x^2}(h_6(x)\phi_{\Sigma_{11}}(x)) + \frac{1}{2}\frac{\partial}{\partial x^2}(h_7(x)\phi_{\Sigma_{11}}(x)) - \frac{\partial}{\partial x^2}(h_8(x)\phi_{\Sigma_{11}}(x)) \right\} + \dots,
 \end{aligned}$$

where Σ_{11} stands for the variance of X_{1T} , i.e., $\int_0^T \sigma_t^2 dt$.

To be more precise, with the expressions of $h_i(x)$ for $i = 1, \dots, 8$ in hand, we could also represent $f_{Z_T^{(\delta)}}$ as follows.

$$f_{Z_T^{(\delta)}} = \phi_{\Sigma_{11}}(x) + \delta \phi_{\Sigma_{11}}(x) \sum_{i=0}^3 c_{1i} x^i + \delta^2 \phi_{\Sigma_{11}}(x) \sum_{i=0}^6 c_{2i} x^i + o(\delta^2), \tag{12}$$

where

$$c_{10} = \frac{a_{11} - a_{12}}{\Sigma_{11}}, \quad c_{11} = -\frac{3a_{11}}{\Sigma_{11}^2} - \frac{a_{12}}{\Sigma_{11}}, \quad c_{12} = \frac{a_{12} - a_{11}}{\Sigma_{11}^2}, \quad c_{13} = \frac{a_{11}}{\Sigma_{11}^3},$$

$$c_{20} = \frac{1}{2\Sigma_{11}^3} \left[-15a_{61} + \Sigma_{11} [6a_{31} + 6a_{36} + 3a_{62} + 3a_{63} + 3a_{64} + 3a_{67} + 3a_{72} \right.$$

$$\left. \begin{aligned} & -6a_{84} - 6a_{85} + \Sigma_{11}(-2a_{32} - 2a_{33} - 2a_{35} + a_{44} + 2a_{54} + 2a_{57} - a_{65} - a_{66} \\ & - a_{71} - a_{73} + 2a_{86}) \end{aligned} \right] ,$$

$$c_{21} = -\frac{1}{2\Sigma_{11}^3} \left[-15a_{68} + 30a_{81} + \Sigma_{11} [6a_{34} + 6a_{37} - 3a_{42} - 6a_{52} - 6a_{56} \right. \\ \left. + 3a_{69} + 3a_{610} + 3a_{74} - 6a_{82} - 6a_{83} - 6a_{87} + \Sigma_{11}(a_{41} + a_{43} + 2a_{51} + 2a_{53} + 2a_{55}) \right] ,$$

$$c_{22} = \frac{1}{2\Sigma_{11}^4} \left[45a_{61} + \Sigma_{11} [-12a_{31} - 12a_{36} - 6a_{62} - 6a_{63} - 6a_{64} - 6a_{67} - 6a_{72} \right. \\ \left. + 12a_{84} + 12a_{85} + \Sigma_{11}(2a_{32} + 2a_{33} + 2a_{35} - a_{44} - 2a_{54} - 2a_{57} + a_{65} + a_{66} \right. \\ \left. + a_{71} + a_{73} - 2a_{86}) \right] ,$$

$$c_{23} = \frac{1}{2\Sigma_{11}^4} \left[-10a_{68} + 20a_{81} + \Sigma_{11}(2a_{34} + 2a_{37} - a_{42} - 2a_{52} - 2a_{56} \right. \\ \left. + a_{69} + a_{610} + a_{74} - 2a_{82} - 2a_{83} - 2a_{87}) \right] ,$$

$$c_{24} = \frac{1}{2\Sigma_{11}^5} \left[-15a_{61} + \Sigma_{11}(2a_{31} + 2a_{36} + a_{62} + a_{63} + a_{64} + a_{67} + a_{72} - 2a_{84} - 2a_{85}) \right] ,$$

$$c_{25} = \frac{a_{68} - 2a_{81}}{2\Sigma_{11}^5} ,$$

and

$$c_{26} = \frac{a_{61}}{2\Sigma_{11}^6} .$$

Furthermore, from the relation between $Z_T^{(\delta)}$ and $S_T^{(\delta)}$ the asymptotic probability density of underlying asset at time T , denoted by $f_{S_T^{(\delta)}}$ can be described as

$$f_{S_T^{(\delta)}} = \frac{1}{S_T^{(\delta)}} f_{Z_T^{(\delta)}} . \quad (13)$$

The first term in (13) is equivalent to the risk-neutral probability density (or state price density) of the Black-Scholes model under deterministic volatility, and the remaining terms are adjustment ones due to the randomness of volatility.

2.3. Option value

It is stressed that the call option value V of (3) can be rewritten as

$$V = S_0 \exp \left(-\frac{1}{2} \int_0^T \sigma_t^2 dt \right) E \left[[e^{Z_T^{(\delta)}} + y]^+ \right] , \quad (14)$$

where $y \equiv -\frac{K}{S_T}$. Hence exploiting the asymptotic density of (12) makes the asymptotic valuation of (14) feasible.

Theorem 2.1 : *The European call option value at time zero, V , with maturity time T and exercise price K , when the volatility is given by (5), can be represented by*

$$V = [S_0\Phi(d_1) - e^{-rT}K\Phi(d_2)] + \delta \sum_{i=0}^3 \beta_{1i} c_{1i} + \delta^2 \sum_{i=0}^6 \beta_{2i} c_{2i} + o(\delta^2), \quad (15)$$

where $\Phi(\cdot)$ is the distribution function of the standard normal variable and $\phi(\cdot)$ is its density function. In addition,

$$d_1 = \frac{1}{\sqrt{\Sigma_{11}}} \left[\log \frac{S_0}{K} + rT + \frac{1}{2}\Sigma_{11} \right],$$

where $\Sigma_{11} = \int_0^T \sigma_t^2 dt$ and σ_t is the solution of (8), and $d_2 = d_1 - \sqrt{\Sigma_{11}}$. Finally, c_i for $i = 1, 2$ are given in (12) and β_i for $i = 1, 2$ are given below. In the following expressions, B_D denotes the square brackets term of RHS in (15).

$$\beta_{10} = \beta_{20} = B_D,$$

$$\beta_{11} = \beta_{21} = S_0\Sigma_{11}\Phi(d_1),$$

$$\beta_{12} = \beta_{22} = \Sigma_{11}B_D + S_0\Sigma_{11}^2\Phi(d_1) + S_0\Sigma_{11}^{\frac{3}{2}}\phi(d_1),$$

$$\beta_{13} = \beta_{23} = S_0\Phi(d_1)\Sigma_{11}^2(3 + \Sigma_{11}) + S_0\phi(d_1)\Sigma_{11}^{\frac{3}{2}}[4\Sigma_{11} + \sqrt{\Sigma_{11}}(2d_2 - 3d_1)],$$

$$\beta_{24} = 3\Sigma_{11}^2B_D + S_0\Sigma_{11}^3\Phi(d_1)(6 + \Sigma_{11}) + S_0\Sigma_{11}^{\frac{5}{2}}\phi(d_1)(5 + d_2^2 - d_2\sqrt{\Sigma_{11}} + 3\Sigma_{11}),$$

$$\begin{aligned} \beta_{25} &= S_0\Sigma_{11}^3\Phi(d_1)(15 + 10\Sigma_{11} + \Sigma_{11}^2) + S_0\Sigma_{11}^3\phi(d_1) \\ &\times [8d_2 - 15d_1 - d_1^3 + \sqrt{\Sigma_{11}}(24 + 10d_1^2 + 6d_2^2) + \Sigma_{11}(4d_2 - 10d_1) + 6\Sigma_{11}^{\frac{3}{2}}], \end{aligned}$$

$$\begin{aligned} \beta_{26} &= 15\Sigma_{11}^3B_D + S_0\Sigma_{11}^4(45 + 15\Sigma_{11} + \Sigma_{11}^2) + S_0\Sigma_{11}^{\frac{7}{2}}\phi(d_1) \\ &\times [33 + 24d_1^2 - 15d_2^2 + 6d_1^4 - 5d_2^4 - \sqrt{\Sigma_{11}}(15d_2 + 10d_2^3 + 45d_1 + 15d_1^3) \\ &+ \Sigma_{11}(35 - 10d_2^2 + 20d_1^2) - \Sigma_{11}^{\frac{3}{2}}(5d_2 + 15d_1) + 5\Sigma_{11}^2]. \end{aligned}$$

Proof: See Appendix C.

The option value up to order $o(\delta)$ in (15) can be represented more succinctly by rearranging the coefficients of δ .

Corollary 2.1 *The option value up to order of δ in Theorem 2.1 is equivalent to*

$$V = [S_0\Phi(d_1) - e^{-rT}K\Phi(d_2)] + \delta S_0\phi(d_1) \left[\frac{a_{12}}{\sqrt{\Sigma_{11}}} - \frac{a_{11}}{\Sigma_{11}}d_2 \right] + o(\delta), \quad (16)$$

where a_{11} and a_{12} are given by (24) and (25), respectively.

The option value of (16) corresponds to Theorem 3.3 in Kunitomo and Kim [24] for the case of constant interest rate, where it was derived in a slightly different manner.

Let the original Black-Scholes option value be B_O . If we define the coefficient of δ (respectively δ^2) in (15) by R_1 (respectively R_2), we could represent the option value as

$$V = B_O + [B_D - B_O] + \delta R_1 + \delta^2 R_2 + o(\delta^2).$$

This decomposition structure allows one to catch the effects of randomness of volatility on option value explicitly.

By a similar procedure or put-call parity, we can derive the value of European put option, whose payoff function is $[K - S_T^{(\delta)}]^+ = [-e^{Z_T^{(\delta)}} - y]^+$ at maturity time T .

Theorem 2.2 : *The European put option value at time zero with maturity time T and exercise price K when the volatility is given by (5), V_* , can be represented by*

$$V_* = [e^{-rT} K \Phi(-d_2) - S_0 \Phi(-d_1)] + \delta \sum_{i=0}^3 \beta_{1i} c_{1i} + \delta^2 \sum_{i=0}^6 \beta_{2i} c_{2i} + o(\delta^2), \quad (17)$$

where all parameters are the same as those of Theorem 2.1.

2.4. Option hedging

To hedge a given option, whose price is denoted by V_t^1 in the stochastic volatility context, the so called delta-sigma hedging strategy can be considered. If we introduce any other option whose price is denoted by V_t^2 , the hedging ratios (Δ_t^*, Σ_t^*) are defined as the solution to

$$\begin{aligned} \frac{\partial V_t^1}{\partial S_t^{(\delta)}} - \Delta_t - \Sigma_t \frac{\partial V_t^2}{\partial S_t^{(\delta)}} &= 0, \\ \frac{\partial V_t^1}{\partial \sigma_t^{(\delta)}} - \Sigma_t \frac{\partial V_t^2}{\partial \sigma_t^{(\delta)}} &= 0. \end{aligned} \quad (18)$$

Under the condition $\frac{\partial V_t^2}{\partial \sigma_t^{(\delta)}} \neq 0$, the hedging ratios are uniquely determined and hence we can hedge a given option of price V_t^1 by Δ_t^* units of the underlying asset and Σ_t^* units of any other option of price V_t^2 . From Theorem 2.1, the two Greeks in (18) can be obtained in an asymptotic sense, since we can obtain the partial derivatives in (18) with ease. Furthermore, we can verify that these partial derivatives are the sum of Greeks in the Black-Scholes model, under deterministic volatility and adjustment ones. For example, $\frac{\partial V_t^i}{\partial S_t^{(\delta)}}$ (respectively, $\frac{\partial V_t^i}{\partial \sigma_t^{(\delta)}}$) is the sum of $\Phi(d_1)$ (respectively, $\frac{1}{2} \frac{\partial \Sigma_{11}}{\partial \sigma_t} / \sqrt{\Sigma_{11}}$) and adjustment terms caused by the randomness of volatility for $i = 1, 2$.

3. Examples

Basically, the economic reasoning gives little guidance for the choice of volatility process. Therefore, the researchers have selected the volatility function judiciously, based on empirical observations and/or computational tractability. Two models among them will be examined. For the simplicity of exposition, we calculate the option value up to order $o(\delta)$.

3.1. Lognormal volatility of $\sigma_t^{(\delta)}$

As a volatility dynamics, Johnson and Shanno [23] proposed the lognormal process

$$d\sigma_t^{(\delta)} = \kappa \sigma_t^{(\delta)} dt + \delta \sigma_t^{(\delta)} dW_{2t}. \quad (19)$$

They also assumed that the volatility risk can be diversified away.

The inputs for the asymptotic valuation can be obtained as follows: $\sigma_t = \sigma_0 e^{\kappa t}$, $Y_t = e^{\kappa t}$, $\partial \mu(\sigma_s, s, 0) = \kappa$, $\partial_s \mu(\sigma_s, s, 0) = 0$, and $\partial w(\sigma_s, s) = 1$. The deterministic version of the volatility up to maturity, Σ_{11} is given by

$$\frac{\sigma_0^2}{2\kappa} [e^{2\kappa T} - 1].$$

Finally , we have

$$a_{11} = \frac{\rho\sigma_0^3}{6\kappa^2}[e^{2\kappa T}(2e^{\kappa T} - 3) + 1]$$

and $a_{12} = 0$.

Table 1 describes the option values under the lognormal volatility process up to order $o(\delta)$. As the benchmark values, the simulation results of Johnson and Shanno [27] are reproduced. This numerical experiment shows that even the first order approximate option values are very close to the benchmark values.

Table 1: Option value under lognormal volatility

The values of parameters are adopted from Table 2 of Johnson and Shanno [27]: $S_0 = 40$, $\kappa = -0.1$, $r = 0.0488$, $\sigma_0 = 0.4$, $\delta = 0.3$, and $T = 1/3$ year. The call prices and their standard errors are the simulation results of Johnson and Shanno [27]. 1st order approx. is the approximation result up to order $o(\delta)$.

	$\rho = -0.5$			$\rho = 0.5$		
K	35	40	45	35	40	45
Call Price	6.93	3.93	2.00	6.80	3.92	2.15
(Standard Error)	(0.01)	(0.009)	(0.008)	(0.01)	(0.01)	(0.01)
1st order approx.	6.9126	3.9132	1.9627	6.7862	3.9260	2.1283

3.2. Square-root process of $(\sigma_t^{(\delta)})^2$

Heston [19], Ball and Roma [5], Bakshi and Chen [4], Scott [34], Heston and Nandi [20] proposed the volatility process as

$$d(\sigma_t^{(\delta)})^2 = \kappa(\theta - (\sigma_t^{(\delta)})^2)dt + \delta \sigma_t^{(\delta)} dW_{2t}. \tag{20}$$

This model seems to be widely used, mainly because the closed form expression of option value can be obtained by utilizing the transform analysis.

Heston [19] and Scott [34] specified the risk premium of volatility as $\lambda \cdot (\sigma_t^{(\delta)})^2$ for constant λ . Hence, the volatility process (20) is described as

$$d(\sigma_t^{(\delta)})^2 = \kappa^*(\theta^* - (\sigma_t^{(\delta)})^2)dt + \delta \sigma_t^{(\delta)} dW_{2t}, \tag{21}$$

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa \theta / (\kappa + \lambda)$ under the risk-adjusted probability.³

By Ito's lemma, we could rewrite the volatility process (21) as

$$d\sigma_t^{(\delta)} = \left[\frac{1}{2\sigma_t^{(\delta)}} [\kappa^*(\theta^* - (\sigma_t^{(\delta)})^2) - \frac{1}{4}\delta^2] \right] dt + \frac{1}{2}\delta dW_{2t}. \tag{22}$$

Since

$$\sigma_t = (e^{-\kappa^*t}(\sigma_0^2 - \theta^*) + \theta^*)^{\frac{1}{2}}$$

and

$$Y_t = \frac{e^{-\kappa^*t} \sigma_0}{(e^{-\kappa^*t}(\sigma_0^2 - \theta^*) + \theta^*)^{\frac{1}{2}}},$$

³In our δ -expansion approach, we notice that the risk premium $\lambda \cdot (\sigma_t^{(\delta)})^2$ should go down to zero as $\delta \downarrow 0$. This implies that λ subsumes parameter δ proportionally. For example, in Heston [19], λ can be represented as $\gamma \rho' \sigma_C \delta$, where γ is the relative-risk aversion of an investor, σ_C is the volatility parameter of consumption process, and ρ' is constant correlation between spot asset return and consumption growth process. However, in this study, we proceed to analyze the option value, assuming λ is constant.

we have $\partial\mu(\sigma_s, s, 0) = -\frac{\kappa\theta}{2\sigma_s^2} - \frac{\kappa}{2}$, $\partial_\delta\mu(\sigma_s(\delta), s, 0) = 0$, and $\partial w(\sigma_s, s) = 0$.

Finally, the following relations hold:

$$\Sigma_{11} = \frac{1}{\kappa^*} \left[\sigma_0^2(1 - e^{-\kappa^*T}) + \theta^*(e^{-\kappa^*T} + \kappa^*T - 1) \right],$$

$$a_{11} = \frac{\rho e^{-\kappa^*T}}{2(\kappa^*)^2} \left[\sigma_0^2(e^{\kappa^*T} - \kappa^*T - 1) + \theta^*[2 - 2e^{\kappa^*T} + \kappa^*T(1 + e^{\kappa^*T})] \right],$$

and $a_{12} = 0$.

Table 2 shows the option values up to order $o(\delta)$ when the squared volatility follows a mean reverting square-root process. We also provide the benchmark values by implementing Heston’s approach. Table 2 also indicates that even the first order approximations are close to the benchmark values.

Table 2: Option value under square-root volatility $((\sigma_t^{(\delta)})^2)$

The values of parameters are adopted from Heston [19]: $S_0 = 100$, $r = 0.0$, $T = 0.5$ year, $\kappa^* = 2$, $\theta^* = 0.01$, $\sigma_0 = 0.1$, and $\delta = 0.1$. The call option prices are calculated by using Heston’s approach. 1st order approx. is the approximation result up to order $o(\delta)$.

K	$\rho = -0.5$			$\rho = 0.5$		
	90	100	110	90	100	110
Call Price	10.2879	2.7841	0.2088	10.1307	2.7968	0.4128
1st order approx.	10.2844	2.8139	0.1973	10.1176	2.8268	0.4118

4. Concluding Remarks

When the volatility of asset return follows a fairly general stochastic process, an asymptotic valuation for the European option is discussed. The decomposition structure of the option value allows one to seize the effects of randomness of asset return volatility on the option value, explicitly, in an asymptotic sense.

It is observed that the approximate option values up to order $o(\delta)$ have shown good numerical accuracies. However, to achieve more stability and precision of numerical accuracy, one should make tedious calculations since the option valuation up to second or higher order in δ involves a considerable number of (but elementary!) integrals. Considering this fact, in case the precision in numerical accuracy is highly demanded, one can utilize the first order approximation as an accelerator in Monte Carlo simulation, for example.

5. Appendices

5.1. Appendix A

In this appendix, we calculate several conditional expectations and their densities which have been used in section 2. To begin with, we provide some useful properties for conditional stochastic integrals, which will be used repeatedly below. For simplicity, one-dimensional cases are rendered.

Lemma 5.1 *Let $q_i(t)$ for $i = 1, \dots, 5$ be deterministic functions of time t . We denote x by $\int_0^T q_1(t)dW_t$ and Σ_{11} by its variance. Then the following relations hold.*

(1) *Property 1*

$$E \left[\int_0^T \int_0^t q_2(s)dW_s q_3(t)dW_t \mid x \right]$$

$$= \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T \int_0^t q_1(s)q_2(s)q_1(t)q_3(t) ds dt$$

(2) *Property 2*

$$\begin{aligned} & E \left[\int_0^T q_2(t)dW_t \int_0^T q_3(t)dW_t \mid x \right] \\ &= \int_0^T q_2(t)q_3(t)dt + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T q_1(t)q_2(t) dt \int_0^T q_1(t)q_3(t) dt \end{aligned}$$

(3) *Property 3*

$$\begin{aligned} & E \left[\int_0^T \int_0^t q_2(s)dW_s \int_0^t q_3(s)dW_s q_4(t)dW_t \mid x \right] \\ &= \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] \int_0^T \int_0^t q_1(s)q_2(s) ds \int_0^t q_1(s)q_3(s) ds q_1(t)q_4(t) dt \\ &+ \frac{x}{\Sigma_{11}} \int_0^T \int_0^t q_2(s)q_3(s)q_1(t)q_4(t) ds dt \end{aligned}$$

(4) *Property 4*

$$\begin{aligned} & E \left[\int_0^T \int_0^t \int_0^s q_2(u)dW_u q_3(s)dW_s q_4(t)dW_t \mid x \right] \\ &= \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] \int_0^T \int_0^t \int_0^s q_1(u)q_2(u)q_1(s)q_3(s)q_1(t)q_4(t)du ds dt \end{aligned}$$

(5) *Property 5*

$$\begin{aligned} & E \left[\int_0^T \int_0^t q_2(s)dW_s q_3(t)dW_t \int_0^T \int_0^t q_4(s)ds q_5(t)dW_t \mid x \right] \\ &= \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] \int_0^T \int_0^t q_1(s)q_2(s)ds q_1(t)q_3(t)dt \int_0^T \int_0^t q_4(s)ds q_1(t)q_5(t) dt \\ &+ \frac{x}{\Sigma_{11}} \int_0^T \left(\int_0^t q_1(s)q_2(s)ds \right) \left(\int_0^t q_4(s)ds \right) q_3(t)q_5(t) dt \\ &+ \frac{x}{\Sigma_{11}} \int_0^T q_1(t)q_3(t) \int_0^t q_2(s)q_5(s) \int_0^s q_4(u)du ds dt \end{aligned}$$

(6) *Property 6*

$$\begin{aligned} & E \left[\int_0^T \int_0^t q_2(s)dW_s q_3(t)dW_t \int_0^T \int_0^t q_4(s)dW_s q_5(t)dt \mid x \right] \\ &= \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] \int_0^T \int_0^t q_1(s)q_2(s)ds q_1(t)q_3(t)dt \int_0^T \int_0^t q_1(s)q_4(s)q_5(t)dt \\ &+ \frac{x}{\Sigma_{11}} \int_0^T q_1(t)q_3(t) \int_0^t q_5(s) \int_0^s q_2(u)q_4(u)du ds dt \\ &+ \frac{x}{\Sigma_{11}} \int_0^T q_5(t) \int_0^t q_1(s)q_3(s) \int_0^s q_2(u)q_4(u)du ds dt \\ &+ \frac{x}{\Sigma_{11}} \int_0^T q_5(t) \int_0^t q_3(s)q_4(s) \int_0^s q_1(u)q_2(u)du ds dt \end{aligned}$$

(7) *Property 7*

$$\begin{aligned}
& E \left[\int_0^T \int_0^t \int_0^s q_2(u) dW_u \int_0^s q_3(u) dW_u q_4(s) ds q_5(t) dW_t \mid x \right] \\
&= \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] \int_0^T \int_0^t \int_0^s q_1(u) q_2(u) du \int_0^s q_1(u) q_3(u) du q_4(s) ds q_1(t) q_5(t) dt \\
&+ \frac{x}{\Sigma_{11}} \int_0^T \int_0^t \int_0^s q_2(u) q_3(u) du q_4(s) ds q_1(t) q_5(t) dt
\end{aligned}$$

(8) *Property 8*

$$\begin{aligned}
& E \left[\int_0^T \int_0^t \int_0^s q_2(u) dW_u \int_0^s q_3(u) du q_4(s) ds q_5(t) dW_t \mid x \right] \\
&= \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T \int_0^t \int_0^s q_1(u) q_2(u) du \int_0^s q_3(u) du q_4(s) ds q_1(t) q_5(t) dt
\end{aligned}$$

(9) *Property 9*

$$\begin{aligned}
& E \left[\int_0^T \int_0^t q_2(s) dW_s q_3(t) dt \int_0^T \int_0^t q_4(s) dW_s q_5(t) dt \mid x \right] \\
&= \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T \int_0^t q_1(s) q_2(s) ds q_3(t) dt \int_0^T \int_0^t q_1(s) q_4(s) ds q_5(t) dt \\
&+ \int_0^T q_3(t) \int_0^t q_5(s) \int_0^s q_2(s) q_4(s) du ds dt \\
&+ \int_0^T q_5(t) \int_0^t q_3(s) \int_0^s q_2(s) q_4(s) du ds dt
\end{aligned}$$

(10) *Property 10*

$$\begin{aligned}
& E \left[\int_0^T \int_0^t q_2(s) ds q_3(t) dW_t \int_0^T \int_0^t q_4(s) dW_s q_5(t) dt \mid x \right] \\
&= \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T \int_0^t q_2(s) ds q_1(t) q_3(t) dt \int_0^T \int_0^t q_1(s) q_4(s) ds q_5(t) dt \\
&+ \int_0^T q_5(t) \int_0^t q_3(s) q_4(s) \int_0^s q_2(u) du ds dt
\end{aligned}$$

(11) *Property 11*

$$\begin{aligned}
& E \left[\int_0^T \int_0^t q_2(s) dW_s q_3(t) dW_t \int_0^T \int_0^t q_4(s) dW_s q_5(t) dW_t \mid x \right] \\
&= \left[\frac{x^4}{\Sigma_{11}^4} - \frac{6x^2}{\Sigma_{11}^3} + \frac{3}{\Sigma_{11}^2} \right] \int_0^T \int_0^t q_1(s) q_2(s) q_1(t) q_3(t) ds dt \int_0^T \int_0^t q_1(s) q_4(s) q_1(t) q_5(t) ds dt \\
&+ \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T q_1(t) q_3(t) \int_0^t q_1(s) q_5(s) \int_0^s q_2(u) q_4(u) du ds dt
\end{aligned}$$

$$\begin{aligned}
 & + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T q_1(t)q_5(t) \int_0^t q_1(s)q_3(s) \int_0^s q_2(u)q_4(u)du ds dt \\
 & + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T q_1(t)q_3(t) \int_0^t q_2(s)q_5(s) \int_0^s q_1(u)q_4(u)du ds dt \\
 & + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T q_1(t)q_5(t) \int_0^t q_3(s)q_4(s) \int_0^s q_1(u)q_2(u)du ds dt \\
 & + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] \int_0^T q_3(t)q_5(t) \left[\int_0^t q_1(s)q_2(s)ds \right] \left[\int_0^t q_1(s)q_4(s)ds \right] dt \\
 & + \int_0^T \int_0^t q_2(s)q_4(s)q_3(t)q_5(s)ds dt
 \end{aligned}$$

The properties 1 to 4 and property 11 have also appeared as multi-dimensional versions in Takahashi [37], and Kunitomo and Takahashi [25]. The remaining ones are the slight modifications thereof.

We set up the following basic relation:

$$dW_{2t} = \rho dW_{1t} + \sqrt{1 - \rho^2} dW_{1t}^*, \tag{23}$$

where the Brownian motion W_{1t}^* is independent of W_{1t} . Also recall that $X_{1T} = \int_0^T \sigma_t dW_{1t}$.

(1) $X_{2T} = \int_0^T A(t) dW_{1t}$

Using the relation (23), we have

$$\begin{aligned}
 \int_0^T A(t) dW_{1t} & = \int_0^T \int_0^t Y_t Y_s^{-1} [\rho w(\sigma_s, s) dW_{1s} + \partial_\delta \mu(\sigma_s, s, 0) ds] dW_{1t} \\
 & \quad + \int_0^T \int_0^t Y_t Y_s^{-1} w(\sigma_s, s) \sqrt{1 - \rho^2} dW_{1s}^* dW_{1t}.
 \end{aligned}$$

The conditional expectation of X_{2T} given X_{1T} can be expressed by

$$\begin{aligned}
 E[X_{2T} | X_{1T} = x] & = \frac{a_{11}}{\Sigma_{11}^2} x^2 - \frac{a_{11}}{\Sigma_{11}} + \frac{a_{12}}{\Sigma_{11}} x \\
 & \equiv h_1(x),
 \end{aligned}$$

where Σ_{11} is given by $\int_0^T \sigma_t^2 dt$,

$$a_{11} = \rho \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} w(\sigma_s, s) \sigma_s ds dt, \tag{24}$$

and

$$a_{12} = \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt. \tag{25}$$

Hence we have

$$-\delta \frac{\partial}{\partial x} (h_1(x) \phi_{\Sigma_{11}}(x)) = \delta \phi_{\Sigma_{11}}(x) \left\{ \frac{a_{11}}{\Sigma_{11}^3} x^3 + \frac{a_{12}}{\Sigma_{11}^2} x^2 - \frac{3a_{11}}{\Sigma_{11}^2} x - \frac{a_{12}}{\Sigma_{11}} \right\}.$$

(2) $X_{3T} = \int_0^T \sigma_t A(t) dt$

Using the relation (23), we have

$$\begin{aligned} \int_0^T \sigma_t A(t) dt &= \int_0^T \sigma_t \int_0^t Y_t Y_s^{-1} [w(\sigma_s, s) dW_{2s} + \partial_\delta \mu(\sigma_s, s, 0) ds] dt \\ &= \int_0^T \sigma_t \int_0^t Y_t Y_s^{-1} [\rho w(\sigma_s, s) dW_{1s} + \partial_\delta \mu(\sigma_s, s, 0) ds] dt \\ &\quad + \int_0^T \sigma_t \int_0^t Y_t Y_s^{-1} w(\sigma_s, s) \sqrt{1 - \rho^2} dW_{1s}^* dt. \end{aligned}$$

The conditional expectation of X_{3T} given X_{1t} can be expressed by

$$\begin{aligned} E[X_{3T} | X_{1T} = x] &= \frac{a_{11}}{\Sigma_{11}} x + a_{12} \\ &\equiv h_2(x). \end{aligned}$$

Hence we have

$$\delta \frac{\partial}{\partial x} (h_2(x) \phi_{\Sigma_{11}}(x)) = \delta \phi_{\Sigma_{11}}(x) \left\{ -\frac{a_{11}}{\Sigma_{11}^2} x^2 - \frac{a_{12}}{\Sigma_{11}} x + \frac{a_{11}}{\Sigma_{11}} \right\}.$$

For the remaining $X_{iT}, i = 4, \dots, 6, X_{iT}^2, i = 2, 3$ and $X_{2T} X_{3T}$, the conditional expectations could be calculated in an entirely similar fashion.

(3) $X_{4T} = \int_0^T B(t) dW_{1t}$

$$\begin{aligned} h_3(x) &\equiv \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] a_{31} + \frac{x}{\Sigma_{11}} a_{32} + \frac{x}{\Sigma_{11}} a_{33} + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{34} \\ &\quad + \frac{x}{\Sigma_{11}} a_{35} + \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] a_{36} + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{37}, \end{aligned}$$

where

$$\begin{aligned} a_{31} &= \frac{1}{2} \rho^2 \int_0^T \sigma_t Y_t \int_0^t Y_s \partial^2 \mu(\sigma_s, s, 0) \left(\int_0^s Y_u^{-1} w(\sigma_u, u) \sigma_u du \right)^2 ds dt, \\ a_{32} &= \frac{1}{2} \int_0^T \sigma_t Y_t \int_0^t Y_s \partial^2 \mu(\sigma_s, s, 0) \left(\int_0^s Y_u^{-2} w(\sigma_u, u)^2 du \right) ds dt, \\ a_{33} &= \frac{1}{2} \int_0^T \sigma_t Y_t \int_0^t Y_s \partial^2 \mu(\sigma_s, s, 0) \left(\int_0^s Y_u^{-1} \partial_\delta \mu(\sigma_u, u, 0) du \right)^2 ds dt, \\ a_{34} &= \rho \int_0^T \sigma_t Y_t \int_0^t Y_s \partial^2 \mu(\sigma_s, s, 0) \int_0^s Y_u^{-1} w(\sigma_u, u) \sigma_u du \int_0^s Y_u^{-1} \partial_\delta \mu(\sigma_u, u, 0) du ds dt, \\ a_{35} &= \frac{1}{2} \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta^2 \mu(\sigma_s, s, 0) ds dt, \\ a_{36} &= \rho^2 \int_0^T \sigma_t Y_t \int_0^t \sigma_s \partial w(\sigma_s, s) \int_0^s Y_u^{-1} w(\sigma_u, u) \sigma_u du ds dt, \end{aligned}$$

and

$$a_{37} = \rho \int_0^T \sigma_t Y_t \int_0^t \sigma_s \partial w(\sigma_s, s) \int_0^s Y_u^{-1} \partial_\delta \mu(\sigma_u, u, 0) du ds dt.$$

(4) $X_{5T} = \int_0^T A(t)^2 dt$

$$h_4(x) \equiv a_{41} + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{42} + a_{43} + \frac{x}{\Sigma_{11}} a_{44},$$

where

$$\begin{aligned} a_{41} &= \int_0^T Y_t^2 \int_0^t Y_s^{-2} w(\sigma_s, s)^2 ds dt, \\ a_{42} &= \rho^2 \int_0^T Y_t^2 \left(\int_0^t Y_s^{-1} w(\sigma_s, s) \sigma_s ds \right)^2 dt, \\ a_{43}(t) &= \int_0^T Y_t^2 \left(\int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds \right)^2 dt, \end{aligned}$$

and

$$a_{44} = 2\rho \int_0^T Y_t^2 \int_0^t Y_s^{-1} w(\sigma_s, s) \sigma_s ds \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt.$$

$$(5) X_{6T} = \int_0^T \sigma_t B(t) dt$$

$$h_5(x) \equiv a_{51} + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{52} + a_{53} + \frac{x}{\Sigma_{11}} a_{54} + a_{55} + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{56} + \frac{x}{\Sigma_{11}} a_{57},$$

where

$$\begin{aligned} a_{51} &= \frac{1}{2} \int_0^T \sigma_t Y_t \int_0^t Y_s \partial^2 \mu(\sigma_s, s, 0) \int_0^s Y_u^{-2} w(\sigma_u, u)^2 du ds dt, \\ a_{52} &= \frac{1}{2} \rho^2 \int_0^T \sigma_t Y_t \int_0^t Y_s \partial^2 \mu(\sigma_s, s, 0) \left(\int_0^s Y_u^{-1} w(\sigma_u, u) \sigma_u du \right)^2 ds dt, \\ a_{53} &= \frac{1}{2} \int_0^T \sigma_t Y_t \int_0^t Y_s \partial^2 \mu(\sigma_s, s, 0) \left(\int_0^s Y_u^{-1} \partial_\delta \mu(\sigma_u, u, 0) du \right)^2 ds dt, \\ a_{54} &= \rho \int_0^T \sigma_t Y_t \int_0^t Y_s \partial^2 \mu(\sigma_s, s, 0) \int_0^s Y_u^{-1} w(\sigma_u, u) \sigma_u du \int_0^s Y_u^{-1} \partial_\delta \mu(\sigma_u, u, 0) du ds dt, \\ a_{55} &= \frac{1}{2} \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta^2 \mu(\sigma_s, s, 0) ds dt, \\ a_{56} &= \rho^2 \int_0^T \sigma_t Y_t \int_0^t \sigma_s \partial w(\sigma_s, s) \int_0^s Y_u^{-1} w(\sigma_u, u) \sigma_u du ds dt, \end{aligned}$$

and

$$a_{57} = \rho \int_0^T \sigma_t Y_t \int_0^t \partial w(\sigma_s, s) \sigma_s \int_0^s Y_u^{-1} \partial_\delta \mu(\sigma_u, u, 0) du ds dt.$$

$$(6) X_{2T}^2 = \left(\int_0^T A(t) dW_{1t} \right)^2$$

$$\begin{aligned} h_6(x) &\equiv \left[\frac{x^4}{\Sigma_{11}^4} - \frac{6x^2}{\Sigma_{11}^3} + \frac{3}{\Sigma_{11}^2} \right] a_{61} + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] (a_{62} + a_{63} + a_{64}) \\ &+ a_{65} + a_{66} + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{67} + \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] a_{68} + \frac{x}{\Sigma_{11}} (a_{69} + a_{610}), \end{aligned}$$

where

$$a_{61} = \rho^2 \left(\int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} w(\sigma_s, s) \sigma_s ds dt \right)^2,$$

$$\begin{aligned}
a_{62} &= 2 \int_0^T \sigma_t Y_t \int_0^t \sigma_s Y_s \int_0^s Y_u^{-2} w(\sigma_u, u)^2 du ds dt, \\
a_{63} &= 2 \rho^2 \int_0^T \sigma_t Y_t \int_0^t w(\sigma_s, s) \int_0^s \sigma_u Y_u^{-1} w(\sigma_u, u) du ds dt, \\
a_{64} &= \rho^2 \int_0^T Y_t^2 \left(\int_0^t \sigma_s Y_s^{-1} w(\sigma_s, s) ds \right)^2 dt, \\
a_{65} &= \int_0^T Y_t^2 \int_0^t Y_s^{-2} w(\sigma_s, s)^2 ds dt, \\
a_{66} &= \int_0^T Y_t^2 \left(\int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds \right)^2 dt, \\
a_{67} &= \left(\int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt \right)^2, \\
a_{68} &= 2 \rho \int_0^T \sigma_t Y_t \int_0^t \sigma_s Y_s^{-1} w(\sigma_s, s) ds dt \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt, \\
a_{69} &= 2 \rho \int_0^T Y_t^2 \int_0^t \sigma_s Y_s^{-1} w(\sigma_s, s) ds \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt,
\end{aligned}$$

and

$$a_{610} = 2 \rho \int_0^T \sigma_t Y_t \int_0^t w(\sigma_s, s) \int_0^s Y_u^{-1} \partial_\delta \mu(\sigma_u, u, 0) du ds dt.$$

$$(7) X_{3T}^2 = \left(\int_0^T \sigma_t A(t) dt \right)^2$$

$$h_7(x) \equiv a_{71} + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{72} + a_{73} + \frac{x}{\Sigma_{11}} a_{74},$$

where

$$\begin{aligned}
a_{71} &= 2 \int_0^T \sigma_t Y_t \int_0^t \sigma_s Y_s \int_0^s Y_s^{-2} w(\sigma_s, s)^2 du ds dt, \\
a_{72} &= \rho^2 \left(\int_0^T \sigma_t Y_t \int_0^t \sigma_s Y_s^{-1} w(\sigma_s, s) ds dt \right)^2, \\
a_{73} &= \left(\int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt \right)^2,
\end{aligned}$$

and

$$a_{74} = 2 \rho \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} w(\sigma_s, s) \sigma_s ds.$$

$$(8) X_{2T} X_{3T} = \int_0^T A(t) dW_{1t} \int_0^T \sigma_t A(t) dt$$

$$\begin{aligned}
h_8(x) &\equiv \left[\frac{x^3}{\Sigma_{11}^3} - \frac{3x}{\Sigma_{11}^2} \right] a_{81} + \frac{x}{\Sigma_{11}} (a_{82} + a_{83}) + \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{84} \\
&+ \left[\frac{x^2}{\Sigma_{11}^2} - \frac{1}{\Sigma_{11}} \right] a_{85} + a_{86} + \frac{x}{\Sigma_{11}} a_{87},
\end{aligned}$$

where

$$a_{81} = \rho^2 \left(\int_0^T \sigma_t Y_t \int_0^t \sigma_s Y_s^{-1} w(\sigma_s, s) ds dt \right)^2,$$

$$\begin{aligned}
 a_{82} &= 2 \int_0^T \sigma_t Y_t \int_0^t \sigma_s Y_s \int_0^s \sigma_u^{-2} w(\sigma_u, u)^2 du ds dt, \\
 a_{82} &= \int_0^T \sigma_t^2 Y_t^2 \int_0^t Y_s^{-2} w(\sigma_s, s)^2 ds dt, \\
 a_{83} &= \rho^2 \int_0^T \sigma_t Y_t \int_0^t w(\sigma_s, s) \int_0^s \sigma_u Y_u^{-1} w(\sigma_u, u) du ds dt, \\
 a_{84} &= \rho \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} w(\sigma_s, s) \sigma_s ds dt \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt \\
 a_{85} &= \rho \int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt \int_0^T \sigma_t Y_t \int_0^t \sigma_s Y_s^{-1} w(\sigma_s, s) ds dt, \\
 a_{86} &= \rho \int_0^T \sigma_t Y_t \int_0^t w(\sigma_s, s) \int_0^s Y_u^{-1} \partial_\delta \mu(\sigma_u, u, 0) du ds dt,
 \end{aligned}$$

and

$$a_{87} = \left(\int_0^T \sigma_t Y_t \int_0^t Y_s^{-1} \partial_\delta \mu(\sigma_s, s, 0) ds dt \right)^2.$$

5.2. Appendix B

Lemma 5.2 (Fujikoshi et. al. [17]): Suppose that \mathbf{x} follows an n -dimensional normal distribution with mean \mathbf{o} and variance-covariance matrix Σ . The density function of \mathbf{x} is denoted by $\phi_\Sigma(\cdot)$. Then for any polynomial functions $g(\cdot)$ and $h(\cdot)$,

$$\mathcal{F}^{-1} \left[g(-i\psi) \mathbf{E} \left[h(\mathbf{x}) e^{i\psi' \mathbf{x}} \right] \right]_{\langle \xi \rangle} = g \left[\frac{\partial}{\partial \xi} \right] h(\xi) \phi_\Sigma(\xi),$$

where

$$\mathcal{F}^{-1} \left[g(-i\psi) \mathbf{E} \left[h(\mathbf{x}) e^{i\psi' \mathbf{x}} \right] \right]_{\langle \xi \rangle} = \left(\frac{1}{2\pi} \right)^n \int_{\mathbf{R}^n} e^{-i\psi' \xi} g(-i\psi) \mathbf{E} \left[h(\mathbf{x}) e^{i\psi' \mathbf{x}} \right] d\psi,$$

and the expectation operation $\mathbf{E}[\cdot]$ is taken over $\mathbf{x} = (x_i) \in \mathbf{R}^n$, $\mathcal{F}^{-1}[\cdot]_{\langle \xi \rangle}$ denotes $\mathcal{F}^{-1}[\cdot]$ being evaluated at ξ , and $\psi' \mathbf{x} = \sum_{i=1}^n \psi_i x_i$ for $\psi = (\psi_i) \in \mathbf{R}^n$.

5.3. Appendix C

Proof of Theorem 2.1: From (14), we could rewrite the option value V as

$$V = S_0 \exp \left(-\frac{1}{2} \int_0^T \sigma_t^2 dt \right) \left[\int_{x \geq -\log y} y f_{Z_T^{(s)}}(x) dx + \int_{x \geq -\log y} e^x f_{Z_T^{(s)}}(x) dx \right]. \tag{26}$$

Thanks to (12), the first term in square brackets of (26) can be described by

$$\int_{x \geq -\log y} y \left(1 + \delta \sum_{i=0}^3 c_{1i} x^i + \delta^2 \sum_{i=0}^6 c_{2i} x^i \right) \phi_{\Sigma_{11}}(x) dx \tag{27}$$

and the second one by

$$\int_{x \geq -\log y} e^x \left(1 + \delta \sum_{i=0}^3 c_{1i} x^i + \delta^2 \sum_{i=0}^6 c_{2i} x^i \right) \phi_{\Sigma_{11}}(x) dx. \tag{28}$$

To evaluate (27), firstly we transform x by $z \equiv \frac{x}{\sqrt{\Sigma_{11}}}$. If we exploit the relations such that

$$\int_{z \geq -z_*} z^n \phi(z) dz = \sum_{j=0}^{\frac{1}{2}(n-1)} 2^{\frac{1}{2}(n-2j-1)} \left(\frac{n-1}{2}\right)! \frac{z_*^{2j}}{j!} \phi(z_*)$$

for odd n and

$$\int_{z \geq -z_*} z^n \phi(z) dz = (n-1)(n-3) \cdots 3 \cdot 1 \left[\Phi(z_*) - \sum_{j=0}^{\frac{1}{2}n-1} 2^j z_*^{2j+1} \frac{j!}{(2j+1)!} \phi(z_*) \right]$$

for even n , we could express (27) as

$$y \left(\Phi(z_*) + \delta \sum_{i=0}^3 m_{1i} c_{1i} + \delta^2 \sum_{i=0}^6 m_{2i} c_{2i} \right),$$

where $z_* \equiv \frac{\log y}{\sqrt{\Sigma_{11}}}$ and $m_{10} = m_{20} = \Phi(z_*)$, $m_{11} = m_{21} = \Sigma_{11}[\Phi(z_*) - z_*\phi(z_*)]$, $m_{12} = m_{22} = \Sigma_{11}^{\frac{3}{2}}[2 + z_*^2]\phi(z_*)$, $m_{13} = m_{23} = \Sigma_{11}^2[2 + z_*^2]\phi(z_*)$, $m_{24} = \Sigma_{11}^2[3\Phi(z_*) - (3z_* + z_*^3)\phi(z_*)]$, $m_{25} = \Sigma_{11}^{\frac{5}{2}}(8 + 4z_*^2 + z_*^4)\phi(z_*)$, and $m_{26} = \Sigma_{11}^3[15\Phi(z_*) - z_*(15 + 5z_*^2 + z_*^4)\phi(z_*)]$. We rename z_* as d_2 .

Similarly, if we transform x by $z = \frac{x}{\sqrt{\Sigma_{11}}} - \sqrt{\Sigma_{11}}$ and use binomial formula, we could express (28) as

$$e^{\frac{\Sigma_{11}}{2}} \left(\Phi(z_*) + \delta \sum_{i=0}^3 n_{1i} c_{1i} + \delta^2 \sum_{i=0}^6 n_{2i} c_{2i} \right),$$

where $z_* \equiv \frac{\log y}{\sqrt{\Sigma_{11}}} + \sqrt{\Sigma_{11}}$ and $n_{10} = n_{20} = \Phi(z_*)$, $n_{11} = n_{21} = \sqrt{\Sigma_{11}}[\phi(z_*) + \sqrt{\Sigma_{11}}\Phi(z_*)]$, $n_{12} = n_{22} = \Sigma_{11}[(1 + \Sigma_{11})\Phi(z_*) + \phi(z_*)(2\sqrt{\Sigma_{11}} - z_*)]$, $n_{13} = n_{23} = \Sigma_{11}^{\frac{3}{2}}[\Phi(z_*)\sqrt{\Sigma_{11}}(3 + \Sigma_{11}) + \phi(z_*)(3\Sigma_{11} - 3\sqrt{\Sigma_{11}}z_* + 2 + z_*^2)]$, $n_{24} = \Sigma_{11}^2[\Phi(z_*)(3 + 6\Sigma_{11} + \Sigma_{11}^2) + \sigma(z_*)(4\sqrt{\Sigma_{11}}(2 + z_*^2) - (3z_* + z_*^3) - 6\Sigma_{11}z_* + 4\Sigma_{11}^{\frac{3}{2}})]$, $n_{25} = \Sigma_{11}^{\frac{5}{2}}[\Phi(z_*)(15\sqrt{\Sigma_{11}} + 10\Sigma_{11}^{\frac{3}{2}} + \Sigma_{11}^{\frac{5}{2}}) + \phi(z_*)(8 + 4z_*^2 + z_*^4 - 5\sqrt{\Sigma_{11}}(3z_* + z_*^3) + 10\Sigma_{11}(2 + z_*^2) - 10\Sigma_{11}^{\frac{3}{2}}z_* + 5\Sigma_{11}^2)]$, and $n_{26} = \Sigma_{11}^3[\Phi(z_*)(5 + 45\Sigma_{11} + 15\Sigma_{11}^2 + \Sigma_{11}^3) + \phi(z_*)(6\sqrt{\Sigma_{11}}(8 + 4z_*^2 + z_*^4) - (15z_* + 5z_*^3 + z_*^5) - 15\Sigma_{11}(3z_* + z_*^3) + 20\Sigma_{11}^{\frac{3}{2}}(2 + z_*^2) - 15\Sigma_{11}^2d_1 + 6\Sigma_{11}^{\frac{5}{2}})]$. We rename z_* as d_1 .

If we remind the definitions of y , S_T^* , and Σ_{11} , and note the resultant relation $d_2 = d_1 - \sqrt{\Sigma_{11}}$, we can sum up the coefficients of c_{ij} which are denoted by β_{ij} , as follows.

$$\beta_{10} = \beta_{20} = S_0\Phi(d_1) - e^{-rT} K\Phi(d_2),$$

$$\beta_{11} = \beta_{21} = S_0\sqrt{\Sigma_{11}}[\phi(d_1) + \sqrt{\Sigma_{11}}\Phi(d_1)] - e^{-rT} K\sqrt{\Sigma_{11}}\phi(d_2),$$

$$\beta_{12} = \beta_{22} = S_0\Sigma_{11}[(1 + \Sigma_{11})\Phi(d_1) + \phi(d_1)(2\sqrt{\Sigma_{11}} - d_1)] - e^{-rT} K\Sigma_{11}[\Phi(d_2) - d_2\phi(d_2)],$$

$$\begin{aligned} \beta_{13} = \beta_{23} &= S_0\Sigma_{11}^{\frac{3}{2}}[\Phi(d_1)\sqrt{\Sigma_{11}}(3 + \Sigma_{11}) + \phi(d_1)(3\Sigma_{11} - 3\sqrt{\Sigma_{11}}d_1 + 2 + d_1^2)] \\ &\quad - e^{-rT} K\Sigma_{11}^{\frac{3}{2}}(2 + d_2^2)\phi(d_2), \end{aligned}$$

$$\begin{aligned} \beta_{24} &= S_0\Sigma_{11}^2[\Phi(d_1)(3 + 6\Sigma_{11} + \Sigma_{11}^2) + \phi(d_1)(4\sqrt{\Sigma_{11}}(2 + d_1^2) - (3d_1 + d_1^3) \\ &\quad - 6\Sigma_{11}d_1 + 4\Sigma_{11}^{\frac{3}{2}})] - e^{-rT} K\Sigma_{11}^2[3\Phi(d_2) - (3d_2 + d_2^3)\phi(d_2)], \end{aligned}$$

$$\beta_{25} = S_0 \Sigma_{11}^{\frac{5}{2}} [\Phi(d_1)(15\sqrt{\Sigma_{11}} + 10\Sigma_{11}^{\frac{3}{2}} + \Sigma_{11}^{\frac{5}{2}}) + \phi(d_1)(8 + 4d_1^2 + d_1^4 - 5\sqrt{\Sigma_{11}}(3d_1 + d_1^3) + 10\Sigma_{11}(2 + d_1^2) - 10\Sigma_{11}^{\frac{3}{2}}d_1 + 5\Sigma_{11}^2)] - e^{-rT} K \Sigma_{11}^{\frac{5}{2}} (8 + 4d_2^2 + d_2^4)\phi(d_2),$$

$$\beta_{26} = S_0 \Sigma_{11}^3 [\Phi(d_1)(15 + 45\Sigma_{11} + 15\Sigma_{11}^2 + \Sigma_{11}^3) + \phi(d_1)(6\sqrt{\Sigma_{11}}(8 + 4d_1^2 + d_1^4) - (15d_1 + 5d_1^3 + d_1^5) - 15\Sigma_{11}(3d_1 + d_1^3) + 20\Sigma_{11}^{\frac{3}{2}}(2 + d_1^2) - 15\Sigma_{11}^2d_1 + 6\Sigma_{11}^{\frac{5}{2}})] - e^{-rT} K \Sigma_{11}^3 [15\Phi(d_2) - d_2(15 + 5d_2^2 + d_2^4)\phi(d_2)].$$

If we utilize the relation $S_0\phi(d_1) = e^{-rT}K\phi(d_2)$, we could further simplify the expressions of β_{ij} . Now arranging terms gives the desired result. Q.E.D.

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