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BAYESIAN INTERPRETATION OF CONTINUOUS-TIME UNIVERSAL PORTFOLIOS

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Abstract Under the continuous-time framework with incomplete information on asset price processes, we show that the universal portfolio coincides with the optimal Bayes portfolio, which have been studied intensively in the financial economics literature. That is, we can interpret the universal portfolio as simultaneously estimating the drift and controlling the portfolio. This result holds in the finite terminal-time setting of the investment horizon. Moreover, we investigate the asymptotic behavior of the universal portfolio along its original definition and obtain a result that in the long run, the universal portfolio with incomplete information converges to the optimal portfolio with complete information.

1. Introduction

In this paper, we address the issue of how universal portfolios can be interpreted under the continuous-time framework that is familiar in the financial economics literature. Universal portfolios are initiated by Cover in 1991 [4], where the asset prices are observed as a sample path from unknown sources. If an investor were able to observe the entire sample path throughout the whole investment horizon a priori, he could obtain the *best constant rebalanced portfolio* which achieves the best growth-rate of the portfolio value. This is a portfolio with *hindsight* information, and no one in the market can actually obtain such an ideal portfolio. Cover's idea is to construct a portfolio based on the available sample path that assures to asymptotically achieve the growth rate of the best constant rebalanced portfolio. Universal portfolios possess such a *universal* property for a very general class of discrete-time asset price processes [4, 5]. Jamshidian extended the result to the continuous-time framework [19] and several other versions have been proposed [14, 2, 20].

On the other hand, the dynamic portfolio selection problem in the continuous-time framework has been studied since Merton [30, 31]. In his seminal work, asset price processes are assumed to follow certain stochastic differential equations and the objective for the investor is then set so as to maximize the expected utility from consumption and terminal wealth. This *Merton's problem* has been extended in numerous ways [21, 7] and particularly the settings with incomplete information have been studied intensively [10, 8, 13, 9, 23, 11, 25, 26, 27, 24, 15]. Under this setting, the drifts of the asset price processes are also described as stochastic differential equations that are unobservable. In [10, 8, 13, 9, 11, 27], the optimal consumption and portfolio selection problem under the continuous-time setting is considered owing to the same continuous Bayesian updating scheme of Liptser-Shiryaev [28] and *Separation Principle* [12].

The aim of this paper is to provide the relation between the universal portfolio and the dynamic portfolio selection problem in the continuous-time framework with incomplete information. Specifically, while the universal portfolio is oriented in the *non-parametric* approach and shown to possess the universal property without the model assumption on asset price processes, we try to reveal its property by using the familiar continuous *parametric* model. The main result implies that the universal portfolio simultaneously and continuously estimates the drift parameter and controls the portfolio choice as a optimal Bayes portfolio. Here we remark that the universal portfolio can be applied only to the portfolio selection problem of maximizing the expected log-utility from terminal wealth. Although the property of the universal portfolios may seem rather limited one, Luenberger has recently shown that the portfolio selection model of maximizing the expected log-utility has a valid reasoning in the sense of *tail preference* when the investment horizon is long enough [29].

This paper is organized as follows. In section 2, our model is described and the optimal portfolio under complete information is derived, which corresponds to the best constant rebalanced portfolio. In section 3, the information available to investors is restricted to be incomplete and the optimal Bayes portfolio and its asymptotic form are obtained. In section 4, the universal portfolio is defined and shown to be consistent with the optimal Bayes portfolio in its asymptotic form. Also, the asymptotic behavior of universal portfolios is investigated.

2. The Optimal Portfolio under Complete Information

We consider a market in which n risky assets and a risk-free asset are traded. We assume the following price processes for the risky assets and the risk-free asset, respectively:

$$\begin{aligned} (\mathrm{diag}(\boldsymbol{S}_t))^{-1}d\boldsymbol{S}_t &= \boldsymbol{\mu}dt + \Sigma d\mathbf{W}_t \ , \\ \frac{dr_t}{r_t} &= r_0 dt \ , \end{aligned} \tag{1}$$

where $\boldsymbol{S}_0$ is constant, $\mathrm{diag}(\boldsymbol{S}_t)$ is a diagonal matrix whose element is $\boldsymbol{S}_t$, $\boldsymbol{\mu}$ is a drift parameter, and $\Sigma = (\sigma_{ij})_{1\le i,j\le n}$ is a constant diffusion parameter. Here, $\mathbf{W}_t = (W_{1t}, \ldots, W_{nt})'$ denotes an n-dimensional standard Brownian motion on the filtered probability space $(\Omega_W, \{\mathcal{F}_{W,t}; t \ge 0\}, \eta)$ where $\mathcal{F}_{W,t}$ is generated by $\sigma(\{\mathbf{W}_u; 0 \le u \le t\})$. r_0 is a constant. Superscript $'$ shows the transpose of matrices and vectors. In this section, it is supposed that investors have the following class of information:

Information 1 (Complete Information)
Both the $\mathcal{F}_{W,0}$-measurable drift parameter $\boldsymbol{\mu}$ and diffusion parameter Σ being known, asset price processes follow the stochastic differential equation (1).

We assume the investors' utility is expressed as the log-utility function

$$u(x) = \log x \ (x > 0) \ .$$

Then the investors having the log-utility continuously select the optimal portfolios within all the $\mathcal{F}_{W,t}$-predictable portfolios. The portfolio selection is made within the following feasible region:

$$\left\{ \boldsymbol{b} = (b_0, b_1, \ldots, b_n)' \in \mathbf{R}^{n+1} \ \middle| \ \sum_{i=0}^{n} b_i = 1 \right\} \ . \tag{2}$$

The instantaneous return of the portfolio value process is given by

$$\begin{aligned}\frac{dV_t}{V_t} &= \boldsymbol{b}_t'(\mathrm{diag}(\boldsymbol{S}_t))^{-1}d\boldsymbol{S}_t + (1-\boldsymbol{b}_t'\mathbf{1})\frac{dr_t}{r_t} \\ &= \{\boldsymbol{b}_t'(\boldsymbol{\mu}-r_0\mathbf{1})+r_0\}\,dt + \boldsymbol{b}_t'\Sigma d\mathbf{W}_t\ ,\end{aligned}$$

where $\mathbf{1}$ is a vector of ones. Along the above expression, the portfolio selection is made without constraints in the following discussion.

In this paper, the objective of the investor is to select the optimal portfolio continuously so as to maximize the expected utility from the terminal wealth. To achieve this, the investor's portfolio selection problem is stated as follows:

$$\mathbf{P}_1 \left| \begin{array}{ll} \underset{\mathbf{b}_\bullet}{\text{maximize}} & E\left[u\left(V_T(\boldsymbol{b}_\bullet)\right)\right] \\ \text{subject to} & \boldsymbol{b}_t \text{ is } \mathcal{F}_{W,t}\text{–predictable process}\ . \end{array}\right.$$

To solve Problem $\mathbf{P}_1$, define the value function at t as :

$$J(V_t,t) \triangleq \max_{\{\boldsymbol{b}_u;\ t\le u\le T\}} E_t[u(V_T(\boldsymbol{b}_\bullet))]\ . \tag{3}$$

Then the Hamilton-Jacobi-Bellman (HJB) equation is:

$$0 = \underset{\mathbf{b}_t}{\text{maximize}}\ J_V V_t\left\{\boldsymbol{b}_t'\left(\boldsymbol{\mu}-r_0\mathbf{1}\right)+r_0\right\} + J_t + \frac{1}{2}J_{VV}V_t^2\boldsymbol{b}_t'\Sigma\Sigma'\boldsymbol{b}_t\ . \tag{4}$$

By the first order condition, we can characterize the optimal solution:

$$\boldsymbol{b}_t^* = -\frac{J_V}{J_{VV}V_t}\left(\Sigma\Sigma'\right)^{-1}\left(\boldsymbol{\mu}-r_0\mathbf{1}\right)\ . \tag{5}$$

Finally we obtain the following theorem:

Theorem 1 (The optimal portfolio under complete information)
Under Complete Information 1, the optimal portfolio for Problem $\mathbf{P}_1$ is given by the constant portfolio

$$\boldsymbol{b}^* = \left(\Sigma\Sigma'\right)^{-1}\left(\boldsymbol{\mu}-r_0\mathbf{1}\right)\ . \tag{6}$$

Proof. As in [30, 31, 6], the value function of Eq. (3) is conjectured as

$$J(V_t,t) = f(t)\log V_t\ . \tag{7}$$

Here the function of time, $f(t)$, is given as a solution to the p.d.e. of Eq. (4), in which $\boldsymbol{b}_t^*$ of Eq. (5) is substituted for $\boldsymbol{b}_t$ and Eq. (7) for the value function. The boundary condition is $f(T)=1$. Then we have $\frac{J_V}{J_{VV}V_t} = -1$ in Eq. (5) to complete the proof. □

This result, i.e. the optimal portfolio choice of the log-utility investor is complete myopia, is well-known in the literature [1, 32, 30]. We repeat it here, however, to use this optimal portfolio $\boldsymbol{b}^*$ in defining universal portfolios in the later section.

3. The Optimal Bayes Portfolio under Incomplete Information

In this section, we assume investors only have incomplete information on the drift parameter $\boldsymbol{\mu}$ in Eq. (1). This situation is more close to the practical market than the one with Complete Information 1. Under this setting, we investigate how log-utility investors' optimal portfolio selection would change.

The drift $\boldsymbol{\mu}$ in Eq. (1) is now a random vector on $\left(\Omega_\mu, \mathcal{F}_\mu, \nu^{(k)}\right)$ such that induced measure $\nu^{(k)} \circ \boldsymbol{\mu}^{-1}$ by $\boldsymbol{\mu}$ is multivariate normal distribution $N\left(\boldsymbol{m}_0^{(k)}, \boldsymbol{\Gamma}_0^{(k)}\right)$ for each $k \in \mathbf{Z}$. Then we can construct a sequence of the filtered probability space set $\left\{\left(\Omega, \{\mathcal{F}_t; t \geq 0\}, \mathcal{P}^{(k)}\right); k \geq 1\right\}$ such that $\Omega = \Omega_\mu \times \Omega_W, \mathcal{F}_t = \mathcal{F}_\mu \otimes \mathcal{F}_{W,t}$ and $\mathcal{P}^{(k)} = \nu^{(k)} \otimes \eta$. We formally state the following class of incomplete information.

Information 2 (Incomplete Information)
Investors asymptotically have diffuse prior distribution information for the drift parameter $\boldsymbol{\mu}$. That is, we consider the limit of probability measure sequence $\mathcal{P}^{(k)}$, such that each prior distribution induced by $\boldsymbol{\mu}$ follows $N\left(\boldsymbol{m}_0^{(k)}, \boldsymbol{\Gamma}_0^{(k)}\right) \triangleq N(\boldsymbol{m}_0, k\boldsymbol{\Gamma}_0)$ for each $k \geq 1$, where $\boldsymbol{m}_0$ and $\boldsymbol{\Gamma}_0$ are given constant vector and covariance matrix.

They are only provided with the information $\mathcal{G}_t \subset \mathcal{F}_t$ generated by a realized asset price process of Eq. (1):

$$\mathcal{G}_t \triangleq \sigma(\boldsymbol{S}_u; 0 \leq u \leq t) \ . \tag{8}$$

Remark 1
The information $\mathcal{G}_t$ is enough to derive $\boldsymbol{\Sigma\Sigma}'$ exactly, since the Doob-Meyer decomposition of the quadratic process $(d\boldsymbol{S}_t/\boldsymbol{S}_t)(d\boldsymbol{S}_t/\boldsymbol{S}_t)'$ yields the finite process $\boldsymbol{\Sigma\Sigma}'t$.

Remark 2
We postulate the reason why we introduce a sequence of filtered probability spaces. To estimate the drift $\boldsymbol{\mu}$ by the Bayesian updating, one should set a suitable prior distribution. In our model, the prior is set as $N\left(\boldsymbol{m}_0^{(k)}, \Gamma_0^{(k)}\right) = N(\boldsymbol{m}_0, k\Gamma_0)$ $(k \geq 1)$. As k becomes large enough, i.e. asymptotically, we can make investors to have diffuse prior information for $\boldsymbol{\mu}$, in the sense that the differential entropy of the prior is infinite. Since it is well known that if $\boldsymbol{\mu}$ follows the multivariate normal distribution $N(\boldsymbol{m}, \boldsymbol{\Gamma})$ with the density

$$\varphi(\boldsymbol{x}) \triangleq \frac{1}{\left(\sqrt{2\pi}\right)^n |\boldsymbol{\Gamma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m})' \boldsymbol{\Gamma}^{-1} (\boldsymbol{x} - \boldsymbol{m})\right) ,$$

its differential entropy $h(\boldsymbol{\mu})$ [Th.9.4.1, 3] is given by:

$$h(\boldsymbol{\mu}) \triangleq - \int \varphi(\boldsymbol{x}) \log \varphi(\boldsymbol{x}) d\boldsymbol{x} = \frac{1}{2} \log(2\pi e)^n |\boldsymbol{\Gamma}| \ .$$

Then we can guarantee that the differential entropy of the prior distribution for $\boldsymbol{\mu}$ is asymptotically infinite, since $1/2 \log(2\pi e)^n |\boldsymbol{\Gamma}_0^{(k)}| = k/2 \log(2\pi e)^n |\boldsymbol{\Gamma}_0| \to \infty$ as $k \to \infty$.

Now the log-utility investor is to maximize the expected utility from the terminal wealth under Incomplete Information 2:

$$\mathbf{P}_2 \left| \begin{array}{ll} \underset{\boldsymbol{b}_\bullet}{\text{maximize}} & \lim_{k\to\infty} E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet)\ )] \\ \text{subject to} & \boldsymbol{b}_t \text{ is } \mathcal{G}_t\text{-predictable process} \ , \end{array} \right.$$

where $E^{(k)}[\cdot]$ is the expectation under the probability measure $\mathcal{P}^{(k)}$.

We remark that Separation Principle [12, 13] holds when we are to solve Problem $\mathbf{P}_2$. It asserts that the inference for the drift and the dynamic control of the portfolio can be treated *separately*. For the former inference part, we utilize the continuous Bayesian updating formula of Liptser-Shiryaev [28]. The latter dynamic control part is essentially the same as the one in Section 2. By Separation Principle and the Bayesian formula, many authors treated the portfolio selection problem under incomplete information [10, 8, 13, 9, 11, 27].

3.1. The inference part

As we described in Remark 1, investors know the diffusion parameter $\Sigma\Sigma'$ exactly, but do not know the $\mathcal{F}_\mu$-measurable drift parameter $\boldsymbol{\mu}$. Hereafter, as noted in Remark 2, we consider the sequence of probability measure $\{\mathcal{P}^{(k)}, k \in \mathbf{Z}\}$ on the filtered probability space $(\{\mathcal{F}_t; t \geq 0\}, \Omega)$, where the prior distribution for $\boldsymbol{\mu}$ follows $\nu^{(k)}$. Conditioned on the information $\mathcal{G}_t$, the investors estimate $\boldsymbol{\mu}$ as follows:

$$\begin{aligned} \boldsymbol{m}_t^{(k)} &\triangleq E^{(k)}\left[\boldsymbol{\mu} \mid \mathcal{G}_t\right] , \\ \boldsymbol{\Gamma}_t^{(k)} &\triangleq E^{(k)}\left[\left(\boldsymbol{\mu} - \boldsymbol{m}_t^{(k)}\right)\left(\boldsymbol{\mu} - \boldsymbol{m}_t^{(k)}\right)' \,\middle|\, \mathcal{G}_t\right] , \end{aligned}$$

where $E^{(k)}[\cdot]$ denotes the expectation under $\mathcal{P}^{(k)}$, $\boldsymbol{m}_t^{(k)}$ is the estimation for $\boldsymbol{\mu}$, and $\boldsymbol{\Gamma}_t^{(k)}$ is its estimation error. By assumption, we have $E^{(k)}[|\boldsymbol{\mu}|^4] = E^{(k)}\left[|\boldsymbol{\mu}|^4 | \mathcal{G}_0\right] < \infty$. Using infinitesimal observations $d\boldsymbol{S}_t$, we can improve the estimation for $\boldsymbol{\mu}$ and its error by Theorem 12.7 in [28]:

$$d\boldsymbol{m}_t^{(k)} = \boldsymbol{\Gamma}_t^{(k)}\left(\Sigma\Sigma'\right)^{-1}\left[(\mathrm{diag}(\boldsymbol{S}_t))^{-1} d\boldsymbol{S}_t - \boldsymbol{m}_t^{(k)} dt\right] , \tag{9}$$

$$d\boldsymbol{\Gamma}_t^{(k)} = -\boldsymbol{\Gamma}_t^{(k)}\left(\Sigma\Sigma'\right)^{-1}\boldsymbol{\Gamma}_t^{(k)} . \tag{10}$$

The continuous Bayesian updating formula for the estimation of $\boldsymbol{\mu}$ is given by Theorem 12.8 in [28]:

$$\begin{aligned} \boldsymbol{m}_t^{(k)} &= \left[\mathbf{I} + \boldsymbol{\Gamma}_0^{(k)}\left(\Sigma\Sigma'\right)^{-1} t\right]^{-1} \cdot \left[\boldsymbol{m}_0^{(k)} + \boldsymbol{\Gamma}_0^{(k)}\left(\Sigma\Sigma'\right)^{-1}\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1} d\boldsymbol{S}_u\right] \qquad (11) \\ &= \left[\frac{1}{k}\boldsymbol{\Gamma}_0^{-1} + \left(\Sigma\Sigma'\right)^{-1} t\right]^{-1} \cdot \left[\frac{1}{k}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{m}_0 + \left(\Sigma\Sigma'\right)^{-1}\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1} d\boldsymbol{S}_u\right] , \end{aligned}$$

where $\mathbf{I}$ is an identity matrix. Next transform Eq. (1) into entirely observable s.d.e. :

$$(\mathrm{diag}(\boldsymbol{S}_t))^{-1} d\boldsymbol{S}_t = \boldsymbol{m}_t^{(k)} dt + \Sigma d\tilde{\mathbf{W}}_t^{(k)} , \tag{12}$$

where

$$\tilde{\mathbf{W}}_t^{(k)} \triangleq \Sigma^{-1}\left[\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1} d\boldsymbol{S}_u - \int_0^t \boldsymbol{m}_u^{(k)} du\right] = \Sigma^{-1}\int_0^t (\boldsymbol{\mu} - \boldsymbol{m}_u^{(k)}) du + \mathbf{W}_t .$$

Then $\tilde{\mathbf{W}}_t^{(k)}$ is $\mathcal{G}_t$-measurable, and

$$E^{(k)}\left[\tilde{\mathbf{W}}_t^{(k)} - \tilde{\mathbf{W}}_s^{(k)} \middle| \mathcal{G}_s\right] = E^{(k)}\left[\Sigma^{-1}\int_s^t (\boldsymbol{\mu} - \boldsymbol{m}_u^{(k)}) du \middle| \mathcal{G}_s\right] = 0 .$$

Furthermore, $\left\langle \tilde{\mathbf{W}}^{(k)} \right\rangle_t = \langle \mathbf{W} \rangle_t = t$. Hence $\tilde{\mathbf{W}}_t^{(k)}$ is $\mathcal{G}_t$ standard Brownian motion owing to Lévy's theorem under the probability measure $\mathcal{P}^{(k)}$. At the end of this subsection, we give the asymptotic form of the estimation (11), with respect to $\mathcal{P}^{(k)}$, as :

$$\lim_{k\to\infty} \boldsymbol{m}_t^{(k)} = \frac{1}{t}\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1} d\boldsymbol{S}_u \stackrel{\triangle}{=} \tilde{\boldsymbol{\mu}}_t . \tag{13}$$

3.2. The dynamic control part

After deriving the continuous Bayesian updating formula for $\boldsymbol{\mu}$, we can treat Problem $\mathbf{P}_2$ as Problem $\mathbf{P}_1$ with Complete Information 1, owing to Separation Principle. The instantaneous return of the portfolio value process is now given as

$$\frac{dV_t}{V_t} = \left\{ \boldsymbol{b}_t' \left(\boldsymbol{m}_t^{(k)} - r_0 \mathbf{1} \right) + r_0 \right\} dt + \boldsymbol{b}_t' \Sigma d\tilde{\mathbf{W}}_t^{(k)} .$$

Define then the value function as follows:

$$\tilde{J}(V_t, \boldsymbol{m}_t^{(k)}, t) \stackrel{\triangle}{=} \max_{\{\boldsymbol{b}_{u;t\le u\le T}\}} E_t^{(k)}[u(V_T(\boldsymbol{b}_\bullet))] . \tag{14}$$

The HJB equation is:

$$0 = \underset{\boldsymbol{b}_t^{(k)}}{\mathrm{maximize}}\ J_V V_t \left\{ \boldsymbol{b}_t^{(k)'} (\boldsymbol{m}_t^{(k)} - r_0 \mathbf{1}) + r_0 \right\} + J_t + V_t \boldsymbol{b}_t^{(k)'} \boldsymbol{\Gamma}_t^{(k)'} J_{Vm}$$
$$+ \frac{1}{2} J_{VV} V_t^2 \boldsymbol{b}_t^{(k)'} \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}_t^{(k)} + \frac{1}{2} tr \left(J_{mm} \boldsymbol{\Gamma}_t^{(k)} (\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1} \boldsymbol{\Gamma}_t^{(k)'} \right) , \tag{15}$$

where $tr(\cdot)$ is the trace of matrices. Then by the first order condition, the optimal portfolio has the form of

$$\boldsymbol{b}_t^{(k)*} = -\frac{J_V}{J_{VV} V_t} (\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1} \left(\boldsymbol{m}_t^{(k)} - r_0 \mathbf{1} \right) - \frac{1}{J_{VV} V_t} (\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1} \boldsymbol{\Gamma}_t^{(k)'} J_{Vm} . \tag{16}$$

This $\boldsymbol{b}_t^{(k)*}$ will be called the *optimal Bayes portfolio*, hereafter. The first term corresponds to the *tangency portfolio* which is also selected in the single-period problem and the second term is so-called the *hedge portfolio* against changes in the opportunity set.

Furthermore we are able to derive the optimal Bayes portfolio in closed-form, by the virtue of log-function. Also its asymptotic form is obtained.

Theorem 2 (The asymptotically optimal Bayes portfolio)
Under Incomplete Information 2 and with the observable s.d.e. (12), the optimal Bayes portfolio under $\mathcal{P}^{(k)}$ is given by

$$\boldsymbol{b}_t^{(k)*} = (\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1} \left(\boldsymbol{m}_t^{(k)} - r_0 \mathbf{1} \right) . \tag{17}$$

Furthermore, $\boldsymbol{b}_t^{(k)*}$ converges to the asymptotically optimal Bayes portfolio $\boldsymbol{b}_t^*$ as $k \to \infty$:

$$\lim_{k\to\infty} \boldsymbol{b}_t^{(k)*} = (\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1} (\tilde{\boldsymbol{\mu}}_t - r_0 \mathbf{1}) \stackrel{\triangle}{=} \boldsymbol{b}_t^* . \tag{18}$$

Proof. As in [31, 6], the value function of Eq. (14) is conjectured as

$$\tilde{J}\left(V_t, \boldsymbol{m}_t^{(k)}, t\right) = f(t)\log V_t + g(\boldsymbol{m}_t^{(k)}, t) \ . \tag{19}$$

$f(t)$ and $g(\boldsymbol{m}_t^{(k)}, t)$ are given as solutions to the p.d.e. of Eq. (15), in which $\boldsymbol{b}_t^{(k)*}$ of Eq. (16) is substituted for $\boldsymbol{b}_t^{(k)}$ and Eq. (19) for the value function. The boundary condition is $f(T) = 1$ and $g(\boldsymbol{m}_T^{(k)}, T) = 0$. Then we have $\frac{J_V}{J_{VV}V_t} = -1$ and $J_{Vm} = 0$ in Eq. (16) to obtain Eq. (17). Furthermore, since $\boldsymbol{m}_t^{(k)} \to \tilde{\boldsymbol{\mu}}_t$ (as $k \to \infty$) from Eq. (13), $\boldsymbol{b}^{(k)*}$ converges to $\boldsymbol{b}_t^*$ with probability one. □

Seeing Eq. (17), the second term in Eq. (16) disappears. The optimal Bayes portfolio $\boldsymbol{b}^{(k)*}$ is the optimal portfolio $\boldsymbol{b}^*$ under Complete Information 1, with $\boldsymbol{m}_t^{(k)}$ is plugged-in for the true drift $\boldsymbol{\mu}$. Hence the log-utility investors have no interest in hedging, that is, they use the estimated drift $\boldsymbol{m}_t$ as if it were the completely known drift.

4. Continuous-time Universal Portfolios

In this section, we derive the universal portfolio in the continuous-time framework under Incomplete Information 2. We establish an interpretation of the universal portfolio in the Bayes' sense, which is discussed in the previous section. First we define the notion of *universality* posed by Cover [4].

Definition 1 (Universality)
Under Incomplete Information 2, $\mathcal{G}_t$-predictable portfolio $\boldsymbol{b}_\bullet$ is said to be universal, if it has the property such that:

$$\lim_{T\to\infty} \sup \frac{1}{T} \log \frac{V_T(\boldsymbol{b}^*)}{V_T(\boldsymbol{b}_\bullet)} = 0 \ , \tag{20}$$

where $\boldsymbol{b}^*$ is the optimal portfolio with Complete Information 1, given by Eq. (6).

Eq. (20) asserts the gap in the growth rate of the portfolio value between $\boldsymbol{b}^*$ and $\boldsymbol{b}_\bullet$ will vanish asymptotically. Since the pioneering work by Cover [4], various portfolios possessing universality are proposed [14, 20]. In this paper, we adopt the definition for universal portfolios given by Cover [4] and Cover-Ordentlich [5].

Definition 2 (Universal portfolios)
Define the universal portfolio at time t as

$$\hat{\boldsymbol{b}}_t \triangleq \frac{\int_{\boldsymbol{b}\in\mathbf{R}^n} \boldsymbol{b} V_t(\boldsymbol{b}) d\boldsymbol{b}}{\int_{\boldsymbol{b}\in\mathbf{R}^n} V_t(\boldsymbol{b}) d\boldsymbol{b}} = \int_{\boldsymbol{b}\in\mathbf{R}^n} \boldsymbol{b} w_t(\boldsymbol{b}) d\boldsymbol{b} \ , \tag{21}$$

where $\boldsymbol{b}$ is the portfolio constant through time and

$$w_t(\boldsymbol{b}) \triangleq \frac{V_t(\boldsymbol{b})}{\int_{\boldsymbol{b}\in\mathbf{R}^n} V_t(\boldsymbol{b}) d\boldsymbol{b}} \ , \tag{22}$$

will be called the weighting density function.

Remark 3
Since $V_t(\boldsymbol{b}) \geq 0$, then $w_t(\boldsymbol{b}) \geq 0$. From $\int_{\boldsymbol{b}\in\mathbf{R}^n} w_t(\boldsymbol{b}) d\boldsymbol{b} = 1$, $w_t(\boldsymbol{b})$ can be regarded as a density. Also $w_t(\boldsymbol{b})$ weights more to the constant portfolio which has achieved high portfolio growth by time t. As a result, the universal portfolio $\hat{\boldsymbol{b}}_t$ can be interpreted as the average of constant portfolios, weighted by $w_t(\boldsymbol{b})$.

The universal portfolio holds the property of Eq. (20) for very general (discrete) asset price processes [4, 5]. In contrast, we conduct an analysis to grasp the weighting density function, $w_t(\boldsymbol{b})$, in the familiar continuous-time framework. As a result, the universal portfolio can be obtained as the expectation of constant portfolios, $\boldsymbol{b}$, with respect to the measure $w_t(\boldsymbol{b})d\boldsymbol{b}$. From this viewpoint, we state the following theorem:

Theorem 3 (The continuous-time universal portfolio)
Under Incomplete Information 2, the universal portfolio coincides with the asymptotically optimal Bayes portfolio :

$$\hat{\boldsymbol{b}}_t = \boldsymbol{b}_t^* = (\Sigma\Sigma')^{-1}(\tilde{\boldsymbol{\mu}}_t - r_0\mathbf{1}) \ .$$

Proof. Given V_0, the value of portfolio constant through time is given as

$$V_t(\boldsymbol{b}) = V_0 \exp\left[\left\{r_0 + \boldsymbol{b}'(\tilde{\boldsymbol{\mu}}_t - r_0\mathbf{1}) - \frac{1}{2}\boldsymbol{b}'\Sigma\Sigma'\boldsymbol{b}\right\} t\right] \ , \tag{23}$$

where $\tilde{\boldsymbol{\mu}}_t$ is given by Eq. (13). Then the weighting density function is

$$\begin{aligned} w_t(\boldsymbol{b}) &\propto V_t(\boldsymbol{b}) & (24)\\ &\propto \exp\left[-\frac{1}{2}\left\{\boldsymbol{b} - (\Sigma\Sigma')^{-1}(\tilde{\boldsymbol{\mu}}_t - r_0\mathbf{1})\right\}'(\Sigma\Sigma' t)\left\{\boldsymbol{b} - (\Sigma\Sigma')^{-1}(\tilde{\boldsymbol{\mu}}_t - r_0\mathbf{1})\right\}\right] & (25)\end{aligned}$$

This implies that the weighting density function, $w_t(\boldsymbol{b})$, is proportional to the density of a multivariate normal distribution:

$$N\left((\Sigma\Sigma')^{-1}(\tilde{\boldsymbol{\mu}}_t - r_0\mathbf{1}),\ \frac{1}{t}(\Sigma\Sigma')^{-1}\right) \ . \tag{26}$$

Hence the universal portfolio $\hat{\boldsymbol{b}}_t$ is given by its mean:

$$\hat{\boldsymbol{b}}_t = \int_{\boldsymbol{b}\in\mathbf{R}^n} \boldsymbol{b} w_t(\boldsymbol{b}) d\boldsymbol{b} = (\Sigma\Sigma')^{-1}(\tilde{\boldsymbol{\mu}}_t - r_0\mathbf{1}) \ . \tag{27}$$

This coincides with Eq. (18). □

This theorem implies the universal portfolio simultaneously estimates the drift parameter and controls the portfolio choice continuously, as the asymptotically optimal Bayes portfolio.

Since the weighting density function, Eq. (22), can be regarded as the p.d.f. of a multivariate normal distribution, its mean and mode coincide with each other. Then taking the expectation with $w_t(\boldsymbol{b})d\boldsymbol{b}$ to obtain the universal portfolio is equivalent to maximizing $w_t(\boldsymbol{b})$ to obtain its mode. This portfolio also can be obtained by maximizing the constant portfolio value of Eq. (23). This is called the *sample path-wise optimal portfolio (SPOP)* proposed by Ishijima-Shirakawa [18]. One can easily check that SPOP is given by the asymptotically optimal Bayes portfolio $\boldsymbol{b}_t^*$ of Eq. (18). This result holds in the problem setting with more constraints on portfolio weights [18].

We proceed to analyze the asymptotic behavior of the universal portfolio. The purpose is to show the universality defined as Eq. (20). The following theorem assures that the universal portfolio converges to the optimal portfolio under Complete Information 1, and also the gap in growth rate between these two portfolios vanishes in the long run.

Theorem 4 (Convergence of the universal portfolio)
In the long run $T \to \infty$, the universal portfolio with Incomplete Information 2 converges to the optimal portfolio with Complete Information 1. That is :

$$\lim_{T\to\infty} \hat{\boldsymbol{b}}_T = \boldsymbol{b}^* \ , a.s. \tag{28}$$

Moreover, the gap in growth rate between $\hat{\boldsymbol{b}}$ and $\boldsymbol{b}^*$ vanishes as $T \to \infty$:

$$\lim_{T\to\infty} \frac{1}{T} \log V_T(\hat{\boldsymbol{b}}_\bullet) = \lim_{T\to\infty} \frac{1}{T} \log V_T(\boldsymbol{b}^*) = \boldsymbol{b}^{*'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* \ . \tag{29}$$

Proof. By the strong law of large numbers ([22], p.104), the estimates for the drift in asymptotic form of Eq. (13) converges to

$$\lim_{T\to\infty} \tilde{\boldsymbol{\mu}}_T = \boldsymbol{\mu} + \lim_{T\to\infty} \boldsymbol{\Sigma}\left(\frac{1}{T}\mathbf{W}_T\right) = \boldsymbol{\mu} \ , \ a.s.$$

Then the universal portfolio converges to

$$\lim_{T\to\infty} \hat{\boldsymbol{b}}_T = (\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}(\boldsymbol{\mu} - r_0\mathbf{1}) = \boldsymbol{b}^* \ .$$

We then evaluate the asymptotic growth rate of the universal portfolio:

$$\lim_{T\to\infty} \frac{1}{T}\log V_T(\hat{\boldsymbol{b}}_\bullet) = \lim_{T\to\infty}\left\{\left(\frac{1}{T}\int_0^T \hat{\boldsymbol{b}}_t dt\right)'\boldsymbol{\mu} - \frac{1}{2}\left(\frac{1}{T}\int_0^T \hat{\boldsymbol{b}}_t'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\hat{\boldsymbol{b}}_t dt\right) + \frac{1}{T}\int_0^T \hat{\boldsymbol{b}}_t'\boldsymbol{\Sigma}d\mathbf{W}_t\right\} \ .$$

Since $\lim_{T\to\infty} \hat{\boldsymbol{b}}_T = \boldsymbol{b}^*$, $\lim_{T\to\infty}\frac{1}{T}\int_0^T \hat{\boldsymbol{b}}_t dt = \boldsymbol{b}^*$ and $\lim_{T\to\infty}\frac{1}{T}\int_0^T \hat{\boldsymbol{b}}_t'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\hat{\boldsymbol{b}}_t dt = \boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^*$. Taking the time change into consideration, we get $\int_0^T \hat{\boldsymbol{b}}_t'\boldsymbol{\Sigma}d\mathbf{W}_t = \widehat{W}_{\tau_T}$ $a.s.$, where $\widehat{W}_t$ is a standard Brownian motion, and $\tau_T = \int_0^T \hat{\boldsymbol{b}}_t'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\hat{\boldsymbol{b}}_t dt$. Also from $\lim_{T\to\infty}\hat{\boldsymbol{b}}_T = \boldsymbol{b}^* \neq \mathbf{0}$, $\lim_{T\to\infty}\tau_T = +\infty$ $a.s.$. Then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \hat{\boldsymbol{b}}_t'\boldsymbol{\Sigma}d\mathbf{W}_t = \lim_{T\to\infty}\frac{\tau_T}{T}\cdot\frac{1}{\tau_T}\widehat{W}_{\tau_T} = \lim_{T\to\infty}\frac{\tau_T}{T}\lim_{T'\to\infty}\frac{1}{T'}\widehat{W}_{T'} = \boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* 0 = 0 \ .$$

Hence,

$$\lim_{T\to\infty}\frac{1}{T}\log V_T(\hat{\boldsymbol{b}}_\bullet) = \boldsymbol{b}^{*'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* \ .$$

On the other hand, the growth rate of $\boldsymbol{b}^*$ is

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\log V_T(\boldsymbol{b}^*) &= \lim_{T\to\infty}\frac{1}{T}\left\{\left(\boldsymbol{b}^{*'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^*\right)T + \boldsymbol{b}^*\boldsymbol{\Sigma}\mathbf{W}_T\right\} \\
&= \boldsymbol{b}^{*'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* + \lim_{T\to\infty}\boldsymbol{b}^{*'}\frac{1}{T}\boldsymbol{\Sigma}\mathbf{W}_T \\
&= \boldsymbol{b}^{*'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* \ .
\end{aligned}$$

This completes the proof. □

Compared to the universality of Definition 1, this theorem explicitly guarantees that the universal portfolio asymptotically learns the optimal portfolio with Complete Information 1. Concerning the asymptotically optimal Bayes portfolio, it coincides with the universal portfolio from Theorem 3, it also has the universal property of Theorem 4 in our model.

5. Conclusion

Under the continuous-time framework with incomplete information on asset price processes, we have shown that the universal portfolio can be interpreted in the Bayes' sense and analyzed its asymptotic behavior. The universal portfolio is originated in the heart of information theory [5]. On the other hand, the numerous works on dynamic portfolio selection problem under incomplete information has been studied intensively in the financial economics literature. Hence we have addressed the question of how two theories are connected, and given an answer for this to some extent.

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