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AN ECONOMIC PREMIUM PRINCIPLE IN A CONTINUOUS-TIME ECONOMY

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Abstract This paper considers a continuous-time economic equilibrium model for deriving the economic premium principle of Bühlmann [2, 3] and Iwaki, Kijima and Morimoto [9]. In order to do this, we construct a continuous-time consumption/portfolio model, and consider an equilibrium in a pure-exchange economy. The state price density in equilibrium is obtained in terms of the Arrow–Pratt index of absolute risk aversion for a representative agent. As special cases, power and exponential utility functions are examined, and we derive an endogenously decided equilibrium insurance premium in explicit form.

1. Introduction

This is a companion paper to Iwaki, Kijima and Morimoto [9], in which they derived the economic premium principle of Bühlmann [2, 3] in a discrete-time multiperiod economic equilibrium model. In this paper, we show that a similar result holds in a continuous-time economic equilibrium model.

In the classic risk theory, insurance premiums are calculated based on the expected loss under the *physical* probability measure. However, if an insurance product is exposed to financial risk, i.e., the product or its derivative is tradable in a market,[1] then the premium should be calculated so as to reflect the financial risk. One approach to consider financial risk is to use an economic equilibrium model. The pioneering work was done by Bühlmann [2]. He considered a single-period consumption model in which each agent is characterized by his/her utility function and initial wealth, and the state price density[2] is determined so as to achieve an equilibrium. Since then, several extensions have been made based on his model. Examples of such extension are found in Iwaki, Kijima and Morimoto [9] and references therein.

Iwaki, Kijima and Morimoto [9] take the line of Bühlmann [2, 3] with some modifications and extensions. They consider a discrete-time multiperiod consumption/portfolio model in which each agent is characterized by his/her utility function and income and can *invest* his/her wealth in both an insurance market as well as a financial market so as to maximize the expected, discounted total utility from consumption. Even in this standard economic model, a similar result to Bühlmann [2, 3] is derived. In particular, the state price density in an economic equilibrium is obtained in terms of the Arrow–Pratt index of absolute risk aversion for a representative agent.

In this paper, we extend Bühlmann's model [2, 3] to a continuous-time setting with some modification. Since the risk to be insured is due to rare events which arrive only at discrete

[1]One of examples of such products is ART (Alternative Risk Transfer) products (see Iwaki, Kijima and Morimoto [9] for more details.).

[2]See, e.g., Huang and Litzenberger [7] for the definition of the state price density.

points in time, the insurance risk will be well modeled by a continuous-time/discrete-state process such as a Poisson process. On the other hand, security prices can might be considered to change continuously, as a first approximation, the risk of security price has been usually modeled by a continuous-time/continuous-state model. One of the disadvantages of the discrete-time model compared to the continuous-time model is such that the difference between the insurance risk and the security price risk becomes ambiguous. So, when considering an economy which includes both insurance and securities, the continuous-time model might be better. Another advantage of the continuous-time model over the discrete-time model is such that we can analytically derive an insurance premium in the equilibrium, and it helps us interpret its economic implication more easily and clearly. In our continuous-time model, we also obtain a similar result to Bühlmann [2, 3]. Furthermore, for power and exponential utility functions, closed form solutions of the state price density are given and we explicitly derive an endogenously decided insurance premium in the equilibrium.

Our model can be thought of as an extension of the ordinary continuous-time financial model to include insurance losses. In the finance literature, continuous-time consumption/portfolio models have been extensively used to obtain equilibrium security prices ever since a seminal paper by Merton [13]. In late '80s, an efficient approach called the *martingale method* to tackle optimal consumption/portfolio selection problem appeared in Karatzas et al. [12]. Instead of solving the HJB (Hamilton-Jacobi-Bellman) optimal equations of the dynamic programming, applying the martingale method to continuous-time or discrete-time multiperiod consumption/portfolio selection problems, we can solve multiperiod optimization problems just as if we solved single-period problems. Furthermore, we can interpret the economic implication of optimal solutions more easily. As to the martingale method, see Karatzas and Shreve [11] and references therein. In this paper, we also adopt the martingale method to solve a continuous-time consumption/portfolio selection problem, and obtain the state price density in the framework of a competitive equilibrium market of a pure exchange economy which includes insurance as well as financial securities.

This paper is organized as follows. In the next section we formally state our continuous-time model. Section 3 contains our main results. In this section, it is shown that, under some technical conditions, an optimal consumption/portfolio process for each agent exists and the state price density in equilibrium can be obtained under the market clearing condition. Based on the results, we derive our general economic premium principle. In Section 4 the special cases of power and exponential utility functions are examined. Finally, we state concluding remarks in Section 5.

Throughout this paper, all the random variables considered are bounded almost surely (a.s.) to avoid unnecessary technical difficulties. Equalities and inequalities for random variables hold in the sense of a.s.; however, we omit the notation a.s. for the sake of notational simplicity.

2. The Model

We consider a pure-exchange economy consisting of a finite number of agents, $i = 1, 2, \cdots, n$ say, who constitute buyers of insurance, insurance companies and reinsurance companies. In this economy it is assumed that every trade occurs continuously in the time interval $\mathcal{T} \stackrel{\text{def}}{=} [0, T]$, where $T > 0$. Resolution of uncertainty of the economy is assumed to be described by evolution of an m-dimensional standard Brownian motion $\boldsymbol{B} = \{\boldsymbol{B}(t); t \in \mathcal{T}\}$ and a Poisson process $N = \{N(t); t \in \mathcal{T}\}$ with intensity process $\lambda = \{\lambda(t); t \geq 0\}$ defined on a given probability space $(\Omega, \mathcal{F}, P)$. The Poisson process represents arrivals of some accidental events such as natural disasters, while the Brownian motion is the source of randomness other than such accidental events. For simplicity, we assume that $\boldsymbol{B}$ and N are independent.

Let $\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{F}_t^N$, $t \in \mathcal{T}$, where $\mathcal{F}_t^N = \sigma\{N(s); s \leq t\}$ and $\mathcal{F}_t^B = \sigma\{\boldsymbol{B}(s); s \leq t\}$. The P-augmentations of filtrations are denoted by $\mathbb{F}^N = \{\mathcal{F}_t^N; t \in \mathcal{T}\}$, $\mathbb{F}^B = \{\mathcal{F}_t^B; t \in \mathcal{T}\}$, and $\mathbb{F} = \{\mathcal{F}_t; t \in \mathcal{T}\}$. Let $\tau_\ell = \inf\{t > 0; N(t) = \ell\}$, $\ell = 1, 2, \cdots$. Clearly, τ_ℓ, $\ell = 1, 2, \cdots$, are $\mathbb{F}^N$-stopping times, but not $\mathbb{F}^B$-stopping times. The conditional expectation operator given $\mathcal{F}_t$ is denoted by E_t with $E = E_0$. Furthermore, we assume that the process N is a Cox process. That is, given the intensity process λ and a stopping time $\tau_{\ell-1}$, the conditional survival probability of τ_ℓ is given by

$$P\{\tau_\ell > t | \lambda, \tau_{\ell-1}\} = \exp\left\{-\int_{\tau_{\ell-1}}^t \lambda(u) du\right\}, \quad t \in \mathcal{T}.^3 \tag{1}$$

In the economy, agent i is endowed $w_i(t)$ units of a single (perishable) commodity, and he/she encounters *risk* $X_i(t)$ measured in units of the commodity at time $t \in \mathcal{T}$. While the quantities $w_i(t)$ and $X_i(t)$ for $t \in (0, T]$ are assumed to be nonnegative random variables, $w_i(0)$ and $X_i(0)$ are nonnegative constants. We call $Z_i(t) \stackrel{\text{def}}{=} w_i(t) - X_i(t)$ the *income* for agent i at time $t \in \mathcal{T}$.

$$Z(t) \stackrel{\text{def}}{=} \sum_{i=1}^n Z_i(t), \qquad t \in \mathcal{T}. \tag{2}$$

is called the *aggregated income* at time t. We assume that evolution of the aggregated income process $\{Z(t); t \in \mathcal{T}\}$ is described by the SDE(Stochastic Differential Equation);

$$Z(0) = z; \quad dZ(t) = Z(t)\left[\nu(t)dt + \boldsymbol{\rho}(t)^\top d\boldsymbol{B}(t) + \upsilon(t)dN(t)\right], \quad t \in \mathcal{T}, \tag{3}$$

where z is a strictly positive constant, and $\boldsymbol{\rho} = \{\boldsymbol{\rho}(t) \stackrel{\text{def}}{=} (\rho_1(t), \cdots, \rho_m(t))^\top, t \in \mathcal{T}\}$, $\nu = \{\nu(t), t \in \mathcal{T}\}$ and $\upsilon = \{\upsilon(t), t \in \mathcal{T}\}$ are some $\mathbb{F}$-predictable processes. Here and hereafter, $\top$ denotes the transpose.

Next, we introduce an *insurance* to the economy. Let $p(t)$ denote the time-t premium-per-share of the insurance after paying sum insured at time t. We assume that the insurance pay $p(t)\delta(t)$ at time t if and only if an accidental event, which represented by the process N, occurs at time t, where $\delta = \{\delta(t); t \in \mathcal{T}\}$ is some $\mathbb{F}$-predictable, positive process. We assume that the insurance premium process $p = \{p(t); t \in \mathcal{T}\}$ follows the SDE;

$$p(0) = 1; \quad dp(t) = p(t)\alpha(t)dt, \quad t \in \mathcal{T}, \tag{4}$$

for some $\mathbb{F}$-predictable, positive process $\alpha = \{\alpha(t); t \in \mathcal{T}\}$. That is, we assume that once agents buy one unit of the insurance, agents are guaranteed capital accumulation at rate α, and that if and only if some accident occurs, sum insured which is proportional to the purchase price is paid.

One generalization of Bühlmann's model [2, 3] is to allow the agents to invest their wealth into a financial market consisting of the money market and m risky securities. We denote the time-t price of the money market account by $S_0(t)$ whereas the time-t price of risky security j by $S_j(t)$, $j = 1, 2, \cdots, m$. It is assumed that $S_0(t)$ satisfies

$$S_0(0) = 1; \qquad dS_0(t) = S_0(t)r(t)dt, \qquad t \in \mathcal{T}, \tag{5}$$

[3] As to some properties of the Cox process, see Iwaki, Kijima and Komoribayashi [8] and references therein.

where $r = \{r(t); t \geq 0\}$ is an $I\!F$-predictable, positive process which represents the *risk-free* interest rate, while the risky security prices $S_j(t)$ are defined by

$$S_j(0) = 1; \quad dS_j(t) = S_j(t)\left[\mu^{(j)}(t)dt + \boldsymbol{\sigma}^{(j)}(t)^{\top} d\boldsymbol{B}(t)\right], \quad t \in \mathcal{T}, \tag{6}$$

where $\boldsymbol{\sigma}^{(j)} = \{\boldsymbol{\sigma}^{(j)}(t) \stackrel{\text{def}}{=} (\sigma_1^{(j)}(t), \cdots, \sigma_m^{(j)}(t))^{\top}; t \in \mathcal{T}\}$ and $\mu^{(j)} = \{\mu^{(j)}(t); t \in \mathcal{T}\}$ are $I\!F$-predictable processes. In order to exclude arbitrage opportunities, we assume that $\alpha(t) < r(t)$ for each $t \in \mathcal{T}$.

It should be noted that prices $p(t)$ and $S_j(t)$, $j = 0, 1, 2, \cdots, m$, are also measured in units of the commodity. Formally, our continuous-time insurance/financial market is defined as follows. See Karatzas and Shreve [11] for the basic continuous-time financial market.

Definition 2.1 The continuous-time insurance/financial market consists of
the filtered probability space $(\Omega, \mathcal{F}, I\!F, P)$,
the m-dimensional standard Brownian motion $\boldsymbol{B}$,
the Poisson process N with intensity process λ,
the set of income processes Z_i, $i = 1, \cdots, n$ with coefficient processes ν, $\boldsymbol{\rho}$, and υ,
the risk-free rate process r,
the *paying-rate process* δ and *accumulation-rate process* α of the insurance,
the set of *mean rate of return processes* $\mu^{(j)}$, $j = 1, \cdots, m$,
and the set of *volatility processes* $\boldsymbol{\sigma}^{(j)}$, $j = 1, \cdots, m$.

We refer to this market as

$$\mathcal{M} = \left(\{Z_i; \nu, \boldsymbol{\rho}, \upsilon\}_{i=1}^{n}, r, \{\mu^{(j)}, \boldsymbol{\sigma}^{(j)}\}_{j=1}^{m}, (\alpha, \delta)\right).$$

Let $Y_i(t)$, $\theta_i^{(0)}(t)$, and $\theta_i^{(j)}(t)$, $t \in \mathcal{T}$, denote the number of shares in insurance, the number of shares in the money market, and the number of shares in risky security j, respectively, held by agent i at time t in the market $\mathcal{M}$. Hereafter, we call $Y_i(t)$ a *risk exchange* and $(\boldsymbol{\theta}_i(t), Y_i(t))$ a *portfolio* of agent i at time $t \in \mathcal{T}$, where

$$\boldsymbol{\theta}_i(t) \stackrel{\text{def}}{=} \left(\theta_i^{(0)}(t), \theta_i^{(1)}(t), \cdots, \theta_i^{(m)}(t)\right)^{\top}.$$

Once the market $\mathcal{M}$ is given, each agent i chooses a non-negative *consumption process* $c = \{c_i(t); t \in \mathcal{T}\}$, and a portfolio process $(\boldsymbol{\theta}_i, Y_i) = \{(\boldsymbol{\theta}_i(t), Y_i(t)); t \in \mathcal{T}\}$ to maximize his/her expected discounted total utility from consumption.

Given a portfolio process $(\boldsymbol{\theta}_i, Y_i)$ and a *cumulative net-income process* $\left\{\int_0^t (Z_i(s) - c_i(s)ds; t \in \mathcal{T}\right\}$, the *wealth process* $\{W_i(t); t \in \mathcal{T}\}$ is defined by $W_i(0) = 0$ and

$$\begin{aligned} dW_i(t) &= (Z_i(t) - c_i(t))dt + p(t)Y_i(t)\left[\alpha(t)dt + \delta(t)dN(t)\right] \\ &\quad + \sum_{j=1}^{m} S_j(t)\theta_i^{(j)}(t)\left[\mu^{(j)}(t)dt + \boldsymbol{\sigma}^{(j)}(t)^{\top} d\boldsymbol{B}(t)\right] \\ &\quad + \left(W_i(t) - p(t)Y_i(t) - \sum_{j=1}^{m} S_j(t)\theta_i^{(j)}(t)\right) r(t)dt \\ &= (Z_i(t) - c_i(t))dt + r(t)W(t)dt + p(t)Y_i(t)\left[(\alpha(t) - r(t))dt + \delta(t)dN(t)\right] \\ &\quad + \boldsymbol{\vartheta}_i(t)^{\top}\left[(\boldsymbol{\mu}(t)dt - r(t)\mathbf{1}_m)dt + \boldsymbol{\Sigma}(t)d\boldsymbol{B}(t)\right], \quad t \in \mathcal{T}, \end{aligned} \tag{7}$$

where

$$\mathbf{\Sigma}(t) \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{\sigma}^{(1)}(t)^{\top} \\ \vdots \\ \boldsymbol{\sigma}^{(m)}(t)^{\top} \end{pmatrix}, \quad \boldsymbol{\mu}(t) \stackrel{\text{def}}{=} \begin{pmatrix} \mu^{(1)}(t) \\ \vdots \\ \mu^{(m)}(t) \end{pmatrix}, \quad \boldsymbol{\vartheta}_i(t) \stackrel{\text{def}}{=} \begin{pmatrix} S_1(t)\theta_i^{(1)}(t) \\ \vdots \\ S_m(t)\theta_i^{(m)}(t) \end{pmatrix},$$

and $\mathbf{1}_m$ denotes an m-dimensional unit vector. For simplicity, the volatility matrix $\mathbf{\Sigma}(t)$ is assumed to be invertible, i.e., there exists the inverse matrix $\mathbf{\Sigma}^{-1}(t)$ for all $t \in \mathcal{T}$. We note that this assumption implies that the security market is complete (see Theorem 1.6.6. of Karatzas and Shreve [11]). [4]

Definition 2.2 We define a process $\phi = \{\phi(t); t \in \mathcal{T}\}$ satisfying $\phi(0) = 1$, $0 < \phi(t) < \infty$, and for each $t \in \mathcal{T}$ and any $s > t$, $s \in \mathcal{T}$,

$$E_t[\phi(s)S_j(s)] = \phi(t)S_j(t), \quad j = 0, 1, \cdots, m, \tag{8}$$

and

$$p(t) = \frac{1}{\phi(t)} E_t\left[\phi(s)p(s) + \int_t^s \phi(u)\delta(u)p(u)dN(u)\right], \tag{9}$$

as the *state price density process.*

We note that for each $A \in \mathcal{F}_t$, $E[\phi(t)1_{\{A\}}]$ could be considered to be the time-0 price of a state contingent claim which pays one unit of the commodity at time t if and only if A occurs, where $1_{\{\cdot\}}$ denotes the indicator function. Once the state price density is given, an equivalent martingale measure P^* is given by $P^*(A) = E[\phi(T)S_0(T)1_{\{A\}}]$, $\forall A \in \mathcal{F}$. See, e.g., Chapter 7 of Huang and Litzenberger [7] for more details as to the state price density process.

Now, we show our first result which is simple application of the Girsanov theorem.

Lemma 2.1 (i) *The state price density is represented as*

$$\phi(t) = S_0(t)^{-1}\phi^B(t)\phi^N(t), \quad t \in \mathcal{T},$$

where

$$\phi^N(t) = \left(\prod_{k=1}^{\infty} \frac{\psi(\tau_k)}{\lambda(\tau_k)} 1_{\{\tau_k \le t\}} + 1_{\{\tau_1 > t\}}\right) e^{\int_0^t (\lambda(s) - \psi(s))ds}; \quad \psi(t) = \frac{r(t) - \alpha(t)}{\delta(t)}, \tag{10}$$

$$\phi^B(t) = \exp\left\{-\int_0^t \boldsymbol{\xi}(s)^{\top} d\boldsymbol{B}(s) - \frac{1}{2}\int_0^t \|\boldsymbol{\xi}(s)\|^2 ds\right\}; \quad \boldsymbol{\xi}(t) = \mathbf{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1}_m), \tag{11}$$

and where $\|\cdot\|$ *denotes the Euclidean norm in* $\mathbb{R}^m$. *Here,* $\psi = \{\psi(t), t \in \mathcal{T}\}$ *is a positive, predictable process with respect to* $\mathbb{F}$. (ii) *The process* ψ *represents the intensity process under the equivalent martingale measure* P^*.

[4]If the market $\mathcal{M}$ is incomplete (typical in insurance), then we may employ the *fictitious security* approach in the following analyses. The idea is to add fictitious securities to the market in such a way that the market becomes complete and to solve optimization problems with the constraint that no position is taken for the added securities. See Cuoco [4], Section 6.7 of Karatzas and Shreve [11] and Section 5.7 of Pliska [14] for details.

Proof. First, from (10), we have

$$\ln \phi^N(t) = \int_0^t \left(\ln \frac{\psi(u)}{\lambda(u)}\right) dN(u) + \int_0^t (\lambda(u) - \psi(u))du,$$

or, in the differential form,

$$d \ln \phi^N(t) = \left(\ln \frac{\psi(t)}{\lambda(t)}\right) dN(t) + (\lambda(t) - \psi(t))dt, \tag{12}$$

where $\ln x$ denotes the natural logarithm of x. By applying a version of Ito's formula, we then obtain

$$\begin{aligned} d\phi^N(t) &= \phi^N(t) \left[(\lambda(t) - \psi(t))dt + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\ln \frac{\psi(t)}{\lambda(t)}\right)^n dN(t)\right] \\ &= \phi^N(t) \left(\frac{\psi(t)}{\lambda(t)} - 1\right) [dN(t) - \lambda(t)dt]. \end{aligned}$$

It follows that

$$\frac{d(\phi(t)S_0(t))}{\phi(t)S_0(t)} = -\boldsymbol{\xi}(t)^\top d\boldsymbol{B}(t) + \left(\frac{\psi(t)}{\lambda(t)} - 1\right) [dN(t) - \lambda(t)dt], \tag{13}$$

so that the process $\{\phi(t)S_0(t), t \in \mathcal{T}\}$ is an exponential martingale under P. The requirement (8) and (9) can then be verified at once, and thus the first statement is proved. The second statement is established by Theorem in Brémaud [1], p.166 and the definition of equivalent martingale measures (see Harrison and Kreps [5] and Harrison and Pliska [6]). □

We note that, for each $t \in \mathcal{T}$, the expectation $E[S_0(t)^{-1}\phi^N(t)1_{\{A^N\}}]$ for any $A^N \in \mathcal{F}_t^N$ ($E[S_0(t)^{-1}\phi^B(t)1_{\{A^B\}}]$, $A^B \in \mathcal{F}_t^B$, resp.) can be considered as the time-0 price of the contingent claim that pays one unit of the commodity at time t if and only if the event A^N (A^B) occurs.

Hereafter, from Lemma 2.1(ii), we refer to ψ as a *risk-neutral intensity process.* It is readily seen from (7) and Lemma 2.1 that for $t \in \mathcal{T}$,

$$\begin{aligned} \frac{d\phi(t)W_i(t)}{\phi(t)} &= (Z_i(t) - c_i(t))dt \\ &\quad + (\boldsymbol{\Sigma}(t)^\top \boldsymbol{\vartheta}(t) - W_i(t)\boldsymbol{\xi}(t))^\top d\boldsymbol{B}(t) \\ &\quad + \left(p(t)Y_i(t)\delta(t)\left(\frac{\psi(t)}{\lambda(t)}\right) + W_i(t)\left(\frac{\psi(t)}{\lambda(t)} - 1\right)\right)(dN(t) - \lambda(t)dt). \end{aligned} \tag{14}$$

The next definition is similar to the one given in Karatzas and Shreve [11].

Definition 2.3 A consumption/portfolio process $(c_i, (\boldsymbol{\theta}_i, Y_i))$ is *admissible* for agent i if the corresponding wealth process satisfies

$$\phi(t)W_i(t) + E_t\left[\int_t^T \phi(s)Z_i(s)ds\right] \geq 0, \qquad t \in \mathcal{T}. \tag{15}$$

The class of admissible processes is denoted by $\mathcal{A}_i$.

In Definition 2.3, Equation (15) says that the sum of the current wealth and the present value of the total income in the future is nonnegative at any time $t \in \mathcal{T}$. Hence, if a consumption/portfolio process is admissible, we can exclude the possibility of bankruptcy in the following analyses. Also, from (14) and (15), if $(c_i, (\boldsymbol{\theta}_i, Y_i))$ is admissible, then the consumption process c_i must satisfy the *budget constraint*,

$$E\left[\int_0^T \phi(t)c_i(t)dt\right] \le E\left[\int_0^T \phi(t)Z_i(t)dt\right], \tag{16}$$

for each agent i.

Now, suppose that, while all the agents have a common *discount process* $\beta = \{\beta(t); t \in \mathcal{T}\}$, agent i has a *utility function* $U_i : (0,\infty) \to I\!R$ which is strictly increasing, strictly concave and twice continuously differentiable with the properties $U_i'(\infty) \equiv \lim_{x\to\infty} U_i'(x) = 0$ and $U_i'(0+) \equiv \lim_{x\downarrow 0} U_i'(x) = \infty$. The problem that each agent faces in the market $\mathcal{M}$ is as follows.

(MP) Find an optimal consumption/portfolio process $(\hat{c}_i, (\hat{\boldsymbol{\theta}}_i, \hat{Y}_i))$ to maximize the expected total, discounted utility from consumption

$$E\left[\int_0^T e^{-\int_0^t \beta(u)du} U_i(c_i(t))dt\right]$$

over the admissible consumption/portfolio processes $(c_i, (\boldsymbol{\theta}_i, Y_i)) \in \mathcal{A}_i$ that satisfy

$$E\left[\int_0^T e^{-\int_0^t \beta(u)du} \min\{0, U_i(c_i(t))\}dt\right] > -\infty. \tag{17}$$

We note that, in problem (MP), the condition (17) is just technical to guarantee the existence of an optimal solution. Also, the factor $\beta(t)$ discounts the utility from consumption at time t and can be considered to represent the time preference of the agents. Of course, it can be the risk-free discount factor, in which case we define $\beta(t) = r(t)$.

3. Main Results

In order to solve the problem (MP) efficiently, we apply the martingale method (see, e.g., Karatzas and Shreve [11]) to our model.

Lemma 3.1 *In the market $\mathcal{M}$, consider a non-negative consumption process c_i that satisfies*

$$E\left[\int_0^T \phi(t)c_i(t)dt\right] = E\left[\int_0^T \phi(t)Z_i(t)dt\right].$$

There exists then a unique portfolio process $(\boldsymbol{\theta}_i, Y_i)$, such that $(c_i, (\boldsymbol{\theta}_i, Y_i)) \in \mathcal{A}_i$ and

$$W_i(t) = \frac{1}{\phi(t)} E_t\left[\int_t^T \phi(s)(c_i(s) - Z_i(s))ds\right], \qquad t \in \mathcal{T}. \tag{18}$$

Proof. We introduce the martingale

$$M(t) \stackrel{\text{def}}{=} E_t\left[\int_0^T \phi(s)(c_i(s) - Z_i(s))ds\right], \qquad t \in \mathcal{T},$$

and invoke the martingale representation theorem (see, e.g., Karatzas and Shreve [10] and Brémaud [1]) to write it as a stochastic integral with respect to the Brownian motion and the Poisson process,

$$M(t) \;=\; \int_0^t \boldsymbol{\eta}(s)^\top d\boldsymbol{B}(s) + \int_0^t \pi(s)(dN(s) - \lambda(s)ds), \qquad t \in \mathcal{T},$$

for a unique $I\!F^B$-progressively measurable $\boldsymbol{\eta} : [0,T] \times \Omega \to I\!R^m$ and a unique $I\!F^N$-predictable $\pi : [0,T] \times \Omega \to I\!R$ with $\int_0^t ||\boldsymbol{\eta}(s)||^2 ds < \infty$ and $\int_0^t |\pi(s)| ds < \infty$. Then, from (7), we can identify

$$W_i(t) \stackrel{\text{def}}{=} \frac{1}{\phi(t)} \left[M(t) - \int_0^t \phi(s)(c_i(s) - Z_i(s))ds \right], \qquad t \in \mathcal{T},$$

as a wealth process with the consumption process c_i and a portfolio process defined by

$$\boldsymbol{\vartheta}_i(t) \;=\; (\boldsymbol{\Sigma}^{-1}(t))^\top \left[W_i(t)\boldsymbol{\xi}(t) + \frac{1}{\phi(t)}\boldsymbol{\eta}(t) \right], \tag{19}$$

$$Y_i(t) \;=\; \frac{\frac{\pi(t)\lambda(t)}{\phi(t)\psi(t)} + \left(\frac{\lambda(t)}{\psi(t)} - 1\right) W_i(t)}{\delta(t)p(t)}, \tag{20}$$

$$\theta_i^{(0)}(t) \;=\; \frac{1}{S_0(t)}[W_i(t) - \boldsymbol{\vartheta}_i(t)^\top \mathbf{1}_m - p(t)Y_i(t)], \qquad t \in \mathcal{T}, \tag{21}$$

whence the result follows by the definition of the admissibility of the consumption/portfolio process. □

For every utility function U_i, we shall denote by I_i the inverse of the derivative U_i'. Under the assumptions stated above, the inverse I_i is also continuous, strictly decreasing, and maps $(0, \infty)$ onto itself with the properties $I_i(0+) = U_i'(0+) = \infty$ and $I_i(\infty) = U_i'(\infty) = 0$. The next theorem provides an optimal consumption/portfolio rule for each agent. We omit the proof since it is similar to that of Theorem 4.4.5 in Karatzas and Shreve [11].

Theorem 3.1 *Under the conditions stated above, a unique optimal consumption process $\hat{c}_i$ for agent i and the corresponding wealth process $\hat{W}_i$ in the market $\mathcal{M}$ are given, respectively, by*

$$\hat{c}_i(t) = I_i\left(y_i e^{\int_0^t \beta(u)du}\phi(t)\right), \qquad t \in \mathcal{T}, \tag{22}$$

where y_i is a solution of the equation;

$$E\left[\int_0^T \phi(t)\left\{I_i\left(y_i\, e^{\int_0^t \beta(u)du}\phi(t)\right) - Z_i(t)\right\} dt\right] = 0, \tag{23}$$

and

$$\hat{W}_i(t) = \frac{1}{\phi(t)} E_t\left[\int_t^T \phi(s)(\hat{c}_i(s) - Z_i(s))ds\right], \qquad t \in \mathcal{T}, \tag{24}$$

where $\hat{W}_i(T) = 0$ for all agents. A unique optimal portfolio $(\hat{\boldsymbol{\theta}}_i(t), \hat{Y}_i(t))$ is given by (19) – (21) *with $W_i(t)$ being replaced by $\hat{W}_i(t)$.*

We note that the solution of the problem (MP), which is a result of Theorem 3.1, coincides with that of the following optimization problem:

$$(\mathbf{MP'}) \quad \left| \begin{array}{ll} \max & E\left[\int_0^T e^{-\int_0^t \beta(u)du} U_i(c_i(t))dt\right] \\ \text{s.to} & E\left[\int_0^T \phi(t)c_i(t)dt\right] = E\left[\int_0^T \phi(t)Z_i(t)dt\right]. \end{array} \right.$$

We also note that y_i which appearing in (22) and (23) corresponds to the Lagrangian multiplier with respect to the linear constraint in (MP$'$).

We are now in a position to develop our economic premium principle. To this end, we formally state the notion of equilibrium market.

Definition 3.1 Given the income processes Z_i and utility functions U_i, $i = 1, 2, \cdots, n$, as well as the discount process β, we say that $\mathcal{M}$ is an equilibrium market, if the following conditions hold:

(1) Clearing of the commodity market:

$$\sum_{i=1}^{n} \hat{c}_i(t) = Z(t), \qquad t \in \mathcal{T}. \tag{25}$$

(2) Clearing of the insurance market:

$$\sum_{i=1}^{n} \hat{Y}_i(t) = 0, \qquad t \in \mathcal{T}. \tag{26}$$

(3) Clearing of the securities market:

$$\sum_{i=1}^{n} \hat{\theta}_i^{(j)}(t) = 0, \quad j = 1, 2 \cdots, m, \quad t \in \mathcal{T}. \tag{27}$$

(4) Clearing of the money market:

$$\sum_{i=1}^{n} \hat{\theta}_i^{(0)}(t) = 0, \qquad t \in \mathcal{T}. \tag{28}$$

Here $\hat{c}_i(t)$, $\hat{\theta}_i^{(j)}(t)$, $\hat{Y}_i(t)$ are the optimal solutions for problem (MP).

The next theorem characterizes the equilibrium market. The proof is similar to that of Theorem 4.5.2 in Karatzas and Shreve [11] and omitted.

Theorem 3.2 *If $\mathcal{M}$ is an equilibrium market, then*

$$Z(t) = \sum_{i=1}^{n} I_i \left(y_i \, e^{\int_0^t \beta(u)du} \phi(t) \right), \qquad t \in \mathcal{T}, \tag{29}$$

where $y_1, \cdots, y_n$ are solutions of (23).

Conversely, if $\mathcal{M}$ is a market for which the state price density ϕ satisfies (29) *and*

$$E\left[\int_0^T \phi(t) \left\{ I_i \left(y_i \, e^{\int_0^t \beta(u)du} \phi(t) \right) - Z_i(t) \right\} dt \right] = 0, \qquad i = 1, 2, \cdots, n, \tag{30}$$

for some $(y_1, \cdots, y_n) \in (0, \infty)^n$, then $\mathcal{M}$ is an equilibrium market.

Given $\mathbf{\Gamma} = (\gamma_1, \cdots, \gamma_n) \in (0, \infty)^n$, let

$$I(y; \mathbf{\Gamma}) = \sum_{i=1}^{n} I_i \left(\frac{y}{\gamma_i} \right), \qquad 0 < y < \infty. \tag{31}$$

Then, we can rewrite (29) as

$$Z(t) = I\left(e^{\int_0^t \beta(u)du}\phi(t); \mathbf{\Gamma}^*\right) \tag{32}$$

with $\mathbf{\Gamma}^* = (y_1^{-1}, \cdots, y_n^{-1})$, where $y_1, \cdots, y_n$ are given by (23). The function $I(y; \mathbf{\Gamma})$ is continuous and strictly decreasing with respect to y, and maps $(0, \infty)$ onto itself with the properties $I(0+; \mathbf{\Gamma}) = \infty$ and $I(\infty; \mathbf{\Gamma}) = 0$. Therefore, it has a continuous, strictly decreasing inverse $\mathcal{H}(\cdot; \mathbf{\Gamma}) : (0, \infty) \overset{\text{onto}}{\to} (0, \infty)$ with the properties $\mathcal{H}(0+; \mathbf{\Gamma}) = \infty$ and $\mathcal{H}(\infty; \mathbf{\Gamma}) = 0$. That is,

$$I(\mathcal{H}(x; \mathbf{\Gamma}); \mathbf{\Gamma}) = x, \qquad \forall\, x \in (0, \infty). \tag{33}$$

It follows from (32) that, if the market $\mathcal{M}$ is in equilibrium, then the state price density $\phi(t)$ and the aggregated income $Z(t)$ are connected through

$$\phi(t) = e^{-\int_0^t \beta(u)du}\mathcal{H}(Z(t); \mathbf{\Gamma}^*), \qquad t \in \mathcal{T}. \tag{34}$$

Also, from (22) and (34), the optimal consumption process of agent i is given by

$$\hat{c}_i(t) = I_i\left(\frac{\mathcal{H}(Z(t); \mathbf{\Gamma}^*)}{\gamma_i^*}\right), \qquad t \in \mathcal{T}. \tag{35}$$

We can characterize the function $\mathcal{H}(y; \mathbf{\Gamma})$ using a utility function of a *representative agent*, which is defined by

$$U(c; \mathbf{\Gamma}) \overset{\text{def}}{=} \max\left\{\sum_{i=1}^n \gamma_i U_i(c_i);\ c_i > 0, i = 1, \cdots, n,\ \sum_{i=1}^n c_i = c\right\}, \tag{36}$$
$$0 < c < \infty.$$

The next theorem is well known. For the proof, see, e.g., Karatzas and Shreve [11].

Theorem 3.3 *Let $\mathbf{\Gamma} \in (0, \infty)^n$ be given and let U_i be of class $C^3(0, \infty)$. Then the function $U(\cdot; \mathbf{\Gamma})$ is of class $C^3(0, \infty)$, strictly increasing, and strictly concave with*

$$U'(c; \mathbf{\Gamma}) = \mathcal{H}(c; \mathbf{\Gamma}), \qquad 0 < c < \infty. \tag{37}$$

Now, we can obtain the expression for the state price density $\phi(t)$, which is similar to the one given by Bühlmann [2, 3] and Iwaki, Kijima and Morimoto [9]. Given a utility function $U(x; \mathbf{\Gamma})$, we call

$$\kappa(x; \mathbf{\Gamma}) \overset{\text{def}}{=} -\frac{U''(x; \mathbf{\Gamma})}{U'(x; \mathbf{\Gamma})} \tag{38}$$

the Arrow–Pratt index of absolute risk aversion.

Theorem 3.4 *If the market $\mathcal{M}$ is in equilibrium, then the state price density $\phi(t)$ is given by*

$$\phi(t) = \frac{\exp\left\{-\int_0^t \beta(u)du - \int_0^{Z(t)} \kappa(x; \mathbf{\Gamma}^*)\, dx\right\}}{E\left[S_0(t)\exp\left\{-\int_0^t \beta(u)du - \int_0^{Z(t)} \kappa(x; \mathbf{\Gamma}^*)\, dx\right\}\right]}, \qquad t \in \mathcal{T}. \tag{39}$$

Proof. From (34) and (37), we have

$$\phi(t) = e^{-\int_0^t \beta(u)du} U'\left(Z(t); \mathbf{\Gamma}^*\right).$$

Solving (38) with respect to U', we obtain

$$U'(x; \mathbf{\Gamma}) = K \exp\left\{-\int_0^x \kappa\left(u; \mathbf{\Gamma}\right) du\right\},$$

where K is the normalizing constant. Since $E[\phi(t)S_0(t)] = 1$, the theorem follows. □

If, in particular, $\beta(t) = r(t)$ so that $S_0(t) = e^{\int_0^t \beta(u)du}$, then we have the following.

Corollary 3.1 *Suppose that $\beta(t) = r(t)$, $t \in \mathcal{T}$. Then, under the conditions of Theorem 3.3, we have*

$$\phi(t) = \frac{\exp\left\{-\int_0^{Z(t)} \kappa\left(x; \mathbf{\Gamma}^*\right) dx\right\}}{S_0(t) E\left[\exp\left\{-\int_0^{Z(t)} \kappa\left(x; \mathbf{\Gamma}^*\right) dx\right\}\right]}. \tag{40}$$

We note that if $S_0(t) = 1$, $t \in \mathcal{T}$, then Corollary 3.1 agrees with the result of Bühlmann [3]. Also, the expression (40) suggests us consider $\phi(t)S_0(t)$ rather than the state price density $\phi(t)$ itself. Namely, we can define a new probability measure P^* whose conditional expectation, given $\mathcal{F}_t$, is defined by

$$E_t^*[X] = \frac{1}{\phi(t)S_0(t)} E_t\left[\phi(T)S_0(T)X\right], \qquad t \in \mathcal{T}, \tag{41}$$

for any random variable X. For any price $S(t)$, we define the *relative* price with respect to the money market account $S_0(t)$ by

$$S^*(t) = \frac{S(t)}{S_0(t)}.$$

The next result can be easily proved using (8) and (9). The new probability measure P^* may be called a *risk-neutral* measure.

Theorem 3.5 *The relative insurance premium with cumulative sum insured, $p^*(t) + \int_0^t \delta(s) p^*(s)dN(s)$, as well as the relative security prices $S_j^*(t)$ is a martingale under P^*.*

For risk $X(t)$ at future time t, the insurance premium $\mathcal{P}(X(t))$ at time 0 is given by the economic premium principle

$$\mathcal{P}(X(t)) = E[X(t)\phi(t)] = E^*[X^*(t)]. \tag{42}$$

Here the second equality follows from (41) with $t = 0$ and $T = t$. That is, the economic premium principle agrees with the risk-neutral valuation in finance, which calculates the expectation of relative price under the risk-neutral measure. We here note that our economic premium principle possesses the same properties examined in Theorem 4.1 and Corollary 4.1 of Iwaki, Kijima and Morimoto [9] although we do not repeat them here.

Finally, we characterize the risk-free interest rate process r, the risk-neutral intensity process ψ and the process of the market price of risk $\boldsymbol{\xi}$ in the equilibrium market $\mathcal{M}$ by the utility function of the representative agent.

Theorem 3.6 *Suppose that the utility function of the representative agent is analytic, i.e., infinitely differentiable and that*

$$\frac{U^{(k)}(x+\epsilon y)}{k!}y^k \to 0 \quad (k\to\infty)$$

for all $x>0$, $\epsilon\in(0,1)$ *and* $y\in I\!R$, *where* $U^{(k)}(\cdot)$ *denotes the k-th derive function of* $U(\cdot)$. *Then, in the equilibrium of the market* $\mathcal{M}$, *the risk-free rate process* r, *the risk-neutral intensity process* ψ, *and the process of the market price of risk* $\boldsymbol{\xi}$ *are given, respectively, by*

$$\begin{aligned} r(t) &= \beta(t)+\lambda(t)-\psi(t) \\ &\quad -\frac{1}{U'(Z(t);\boldsymbol{\Gamma}^*)}\left\{U''(Z(t);\boldsymbol{\Gamma}^*)Z(t)\nu(t)+\frac{1}{2}U'''(Z(t);\boldsymbol{\Gamma}^*)Z(t)^2\|\boldsymbol{\rho}\|^2\right\}, \end{aligned} \tag{43}$$

$$\psi(t)=\lambda(t)\frac{U'(Z(t)(1+v(t));\boldsymbol{\Gamma}^*)}{U'(Z(t);\boldsymbol{\Gamma}^*)}, \tag{44}$$

and

$$\boldsymbol{\xi}(t)=J(Z(t);\boldsymbol{\Gamma}^*)\boldsymbol{\rho}(t), \quad t\in\mathcal{T}, \tag{45}$$

where $J(x;\boldsymbol{\Gamma}) \stackrel{\text{def}}{=} -\frac{xU''(x;\boldsymbol{\Gamma})}{U'(x;\boldsymbol{\Gamma})}$ *is the index of relative risk-aversion for the representative agent's utility function.*

Proof. Let us consider the process

$$h=\left\{h(t)\stackrel{\text{def}}{=}\mathcal{H}(Z(t);\boldsymbol{\Gamma}^*)=U'(Z(t);\boldsymbol{\Gamma}^*);\quad t\in\mathcal{T}\right\}. \tag{46}$$

An application of Ito's rule gives

$$\begin{aligned} h(t) &= U'(Z(0);\boldsymbol{\Gamma}^*) \\ &\quad +\int_0^t\left\{U''(Z(s);\boldsymbol{\Gamma}^*)Z(s)\nu(s)+\frac{1}{2}U'''(Z(s);\boldsymbol{\Gamma}^*)Z(t)^2\|\boldsymbol{\rho}\|^2\right\}ds \\ &\quad +\int_0^t U''(Z(s);\boldsymbol{\Gamma}^*)Z(s)\boldsymbol{\rho}^\top(s)d\boldsymbol{B}(s) \\ &\quad +\int_0^t\sum_{k=1}^{\infty}\frac{1}{k!}U^{(k+1)}(Z(s);\boldsymbol{\Gamma}^*)\left(Z(s)v(s)\right)^k dN(s), \quad t\in\mathcal{T}. \end{aligned} \tag{47}$$

On the other hand, the process

$$\zeta=\left\{\zeta(t)\stackrel{\text{def}}{=}\phi(t)e^{\int_0^t\beta(s)ds};t\in\mathcal{T}\right\} \tag{48}$$

satisfies the equation

$$\begin{aligned} \zeta(t) &= 1+\int_0^t\zeta(s)(\beta(s)-r(s)+\lambda(s)-\psi(s))ds \\ &\quad -\int_0^t\zeta(s)\boldsymbol{\xi}^\top(s)d\boldsymbol{B}(s) \\ &\quad +\int_0^t\zeta(s)\left(\frac{\psi(s)}{\lambda(s)}-1\right)dN(s), \quad t\in\mathcal{T}. \end{aligned} \tag{49}$$

Now (34) and (37) hold if and only if $\zeta \equiv h$. Therefore, comparing the coefficients of (47) and (49), we obtain

$$U'(Z(0);\boldsymbol{\Gamma}^*) = 1 \tag{50}$$

and the desired results. □

We note that from (10) and Theorem 3.6, for any given paying-rate process δ, the accumulation-rate process α of the insurance is in equilibrium given by

$$\alpha(t) = r(t) - \delta(t)\psi(t), \quad t \in \mathcal{T}, \tag{51}$$

with $r(t)$ and $\psi(t)$ given by (43) and (44). Similarly, for any volatility process $\boldsymbol{\Sigma}$, the mean rate of return process $\boldsymbol{\mu}$ of the securities in the equilibrium is given by

$$\boldsymbol{\mu}(t) = r(t)\mathbf{1}_m + \boldsymbol{\Sigma}(t)\boldsymbol{\xi}(t), \quad t \in \mathcal{T}, \tag{52}$$

with $r(t)$ and $\boldsymbol{\xi}$ given by (43) and (45). Note that, from (34), (37) and (49), we obtain

$$\frac{dU'(Z(t);\boldsymbol{\Gamma}^*)}{U'(Z(t);\boldsymbol{\Gamma}^*)} = (\beta(t) - r(t))dt - \boldsymbol{\xi}^\top(t)d\boldsymbol{B}(t) + \left(\frac{\psi(t)}{\lambda(t)} - 1\right)(dN(t) - \lambda(t)dt). \tag{53}$$

(53) says that the rate of increase in the marginal utility of the representative agent, i.e., the rate of change in the time-preference of the representative agent in the market $\mathcal{M}$, consists of 3 factors, a deterministic discount-factor in real terms, $\beta - r$, a premium for the risk represented by the Brownian motion $\boldsymbol{B}$ and a premium for the risk represented by the Poisson process N. This interpretation of (53) might help us understand a meaning of the results of Theorem 3.6.

4. Some Special Cases

In this section, we consider an equilibrium market $\mathcal{M}$ with some special utility functions for the agents. Namely, we study the cases of power and exponential utility functions, and show that the state price density $\phi(t)$ can be expressed in terms of the aggregated income $Z(t)$, the discount function $\beta(t)$ and the parameter of the utility function. Recall that the general form (49) in Theorem 3.4 includes the unknown values γ_i^*. However, in the special cases, these parameters are expressed by the initial aggregated income in equilibrium through (32) or (50).

4.1. Power utility functions

First, we consider the case in which each agent has a utility function defined by

$$U_i(x) = \iota_i^{1-\varrho}\frac{x^\varrho}{\varrho}, \quad 0 < x < \infty, \ \iota_i > 0, \ \varrho \in (-\infty, 1), \tag{54}$$

where ϱ is termed as the *shape parameter*.[5] Note that every agent has the common shape parameter ϱ.

Now, it is easily seen that the inverse of the marginal utility is given by

$$I_i(y) = \iota_i y^{-1/(1-\varrho)}, \qquad y \in (0, \infty).$$

[5]The case $\varrho = 0$ corresponds to the logarithmic utility function.

It follows from (31) that

$$I(y;\boldsymbol{\Gamma}) = \sum_{i=1}^{n} I_i\left(\frac{y}{\gamma_i}\right) = \left(\sum_{i=1}^{n} \iota_i \gamma_i^{1/(1-\varrho)}\right) y^{-1/(1-\varrho)}.$$

Note that, in equilibrium, we have from (32) with $t=0$ that

$$Z(0) = I(1;\boldsymbol{\Gamma}^*) = \sum_{i=1}^{n} \iota_i (\gamma_i^*)^{1/(1-\varrho)}.$$

Since $\mathcal{H}(x;\boldsymbol{\Gamma})$ is the inverse of $I(y;\boldsymbol{\Gamma})$, we obtain

$$U'(x;\boldsymbol{\Gamma}^*) = \mathcal{H}(x;\boldsymbol{\Gamma}^*) = \left(\frac{x}{Z(0)}\right)^{-(1-\varrho)}, \qquad x \in (0,\infty).$$

It follows from (34) that

$$\phi(t) = e^{-\int_0^t \beta(u)du}\left(\frac{Z(t)}{Z(0)}\right)^{-(1-\varrho)}, \qquad t \in \mathcal{T}.$$

Especially, in the case that $S_0(t) = e^{\int_0^t \beta(u)du}$ as in Corollary 3.1, we have

$$\phi(t)S_0(t) = \left(\frac{Z(t)}{Z(0)}\right)^{-(1-\varrho)}, \qquad t \in \mathcal{T}.$$

Since $E[\phi(t)S_0(t)] = 1$, it follows that

$$\phi(t)S_0(t) = \frac{Z(t)^{-(1-\varrho)}}{E[Z(t)^{-(1-\varrho)}]}, \qquad t \in \mathcal{T}.$$

The economic premium principle for this case is given, from (42), by

$$\mathcal{P}(X(t)) = \frac{E[X^*(t)Z(t)^{-(1-\varrho)}]}{E[Z(t)^{-(1-\varrho)}]}, \qquad t \in \mathcal{T}, \tag{55}$$

where $X^*(t) = X(t)/S_0(t)$ is the relative price of $X(t)$. From Theorem 3.6, the risk-neutral intensity process ψ is given by

$$\psi(t) = \lambda(t)(1+v(t))^{-(1-\varrho)}, \quad t \in \mathcal{T}. \tag{56}$$

The premium-per-share of the insurance in equilibrium is given, from (4), by

$$p(t) = e^{\int_0^t \alpha(s)ds}$$

with the accumulation-rate process α given by (51).

4.2. Exponential utility functions

We next consider the case in which each agent has a utility function defined by

$$U_i(x) = \mathcal{K}\frac{1 - e^{-\varrho_i x}}{\varrho_i}, \qquad 0 < x < \infty, \tag{57}$$

where ϱ_i and $\mathcal{K}$ are positive constants. Note that the parameter ϱ_i represents the index of absolute risk aversion. We choose $\mathcal{K}$ in such way that $\lim_{x\downarrow 0} U_i'(x)$ becomes sufficiently large so that the probability of negative consumption becomes negligible. Using similar arguments to the power utility case, it is not difficult to derive

$$I(y;\mathbf{\Gamma}^*) = -\sum_{i=1}^{n} \frac{\log y}{\varrho_i} + Z(0).$$

Letting ϱ be such that

$$\frac{1}{\varrho} = \sum_{i=1}^{n} \frac{1}{\varrho_i},$$

we then obtain

$$\mathcal{H}(x;\mathbf{\Gamma}^*) = e^{-\varrho(x - Z(0))}, \qquad x \in (0,\infty).$$

It follows that

$$\phi(t) = \exp\left\{-\int_0^t \beta(u)du - \varrho(Z(t) - Z(0))\right\}, \qquad t \in \mathcal{T}.$$

Especially, in the case that $S_0(t) = e^{\int_0^t \beta(u)du}$, we have

$$\phi(t)S_0(t) = \frac{e^{-\varrho Z(t)}}{E[e^{-\varrho Z(t)}]}, \qquad t \in \mathcal{T}.$$

The economic premium principle for this case is given by

$$\mathcal{P}(X(t)) = \frac{E[X^*(t)e^{-\varrho Z(t)}]}{E[e^{-\varrho Z(t)}]}, \qquad t \in \mathcal{T}.$$

Note that if $X^*(t) = Z(t)$, then

$$\mathcal{P}(S_0(t)Z(t)) = \frac{E[Z(t)e^{-\varrho Z(t)}]}{E[e^{-\varrho Z(t)}]}, \qquad t \in \mathcal{T},$$

which is called the *Esscher principle*. Finally, the risk-neutral intensity process ψ, is given by

$$\psi(t) = \lambda(t)e^{-\varrho v(t)Z(t)}, \quad t \in \mathcal{T}.$$

5. Concluding Remarks

This paper considers a continuous-time economic equilibrium model for deriving the economic premium principle of Bühlmann [2, 3] and Iwaki, Kijima and Morimoto [9]. The state price density in equilibrium is obtained in terms of the Arrow–Pratt index of absolute risk aversion for a representative agent. As special cases, power and exponential utility functions are examined, and we derive an endogenously decided equilibrium insurance premium in explicit form. However, in this paper, we implicitly assume that the market is complete, both of insurance and security markets may be actually incomplete. Extension of our economic premium principle to incomplete market models will be treated in the near future.

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