

A QUEUEING MODEL WITH INPUT OF MPEG FRAME SEQUENCE AND INTERFERING TRAFFIC

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Abstract We study a queueing system having a mixture of a special semi-Markov process (SSMP) and a Poisson process as the arrival process, where the Poisson arrival is regarded as interfering traffic. It is shown by numerical examples that the SSMP customers receive worse treatment than Poisson customers, i.e., the mean waiting time of SSMP customers is longer than that of Poisson customers. We also propose a model of Moving Picture Experts Group (MPEG) frame arrivals as an SSMP batch arrival process. This model captures two features of the MPEG coding scheme: (i) the frequency of appearance of the I-, B-, and P-frames in a Group of Pictures (GOP), and (ii) the distinct distributions for the size of the three types of frames. The mean and variance of waiting time of ATM cells generated from the MPEG frames are evaluated in the numerical examples drawn from some real video data.

1. Introduction

Numerous models have been proposed that characterize the feature of traffic source on communication networks. For example, Poisson (M) and interrupted Poisson processes (IPP) have been used for audio traffic, and a Markov modulated Poisson process (MMPP) [3] for video traffic. In the multimedia environment such as B-ISDN the data compression is indispensable for sending huge amount of video data. A strong candidate for such compression scheme is the Moving Picture Experts Group (MPEG) [10]. Since most of the video will be encoded using the MPEG standard, there is a need for appropriate modeling of the video traffic generated by the MPEG coding scheme.

A Transform Expand Sample (TES) and a Markov chain have been used to characterize the traffic generated with MPEG. In a TES based modeling [15], each frame type I, B and P is modeled by a TES process and these frames are interleaved in the Group of Pictures (GOP) pattern like "IBBPBBPBBPBB" to faithfully model an MPEG video. The queueing model with these data as an input process is simulated. On the other hand, in a Markov chain based modeling [17], Markov chains are formed for the GOP as well as scene levels by avoiding the modeling of the exact frame pattern in every GOP.

The aim of this paper is to present an analytic model for evaluating the traffic characteristics of MPEG frames fed to a communication buffer together with other interfering traffic. The effects of interfering traffic have been studied by means of queues with mixed arrival processes in the past. The motivation for the queueing model with mixed arrivals is that, in the situation where many traffic sources are superposed, a tagged source is modeled closely while other sources can be regarded as interfering traffic all together. The GI+M/M/1 analyzed by Kuczura [7] is a queueing model having two types of the arrival process, a renewal process (GI) and a Poisson process. In [13, 14] the service time distribution is allowed to be general (i.e. GI+M/G/1) and GI and M customers may have different service time dis-

tributions. When there is a priority between GI and M we refer to [5]. Queueing models without any waiting room are analyzed in [9] and [22]. An overview of research on the single server queues with independent GI and M input streams is provided in [14]. These queueing systems operate in continuous time. A few studies [11, 12] deal with queueing systems that operate in discrete time such as GI+M^[X]/D/1/K ($K < \infty$ or $K = \infty$) and GI+M^[X]+B/D/1.

This paper consists of two parts. In the first part (Section 2), we study a queueing system having SSMP batch and Poisson arrivals combined as an input process, i.e. an SSMP^[X]+M/M/1 system, where the Poisson arrival is regarded as interfering traffic. The special semi-Markov process (SSMP) is a special case of the semi-Markov process such that the sojourn time distribution in each state depends only on that state. We assume that SSMP customers arrive in batch. Namely, each SSMP arrival point corresponds to the arrival of a batch of SSMP customers, where the batch size (the number of customers in a batch) may depend on the state of the SSMP immediately after the arrival. The SSMP was introduced by Ding and Decker [1] with the aim of modeling the video traffic with variable bit rate. It can be used as the arrival process of a wide class of traffic, because it fits any marginal distribution function for interarrival times, including GI and MMPP as special cases [2]. We extend Kuczura's approach for a GI+M/M/1 system [7] to analyze our SSMP^[X]+M/M/1 system, where the SSMP and Poisson arrivals have a common service time distribution function, namely exponential distribution with the same mean. We evaluate the waiting times of both SSMP and Poisson customers. Numerical results reveal the influence of Poisson customers on the waiting time of SSMP customers.

In the second part (Section 3), we propose an SSMP batch arrival process for the MPEG frame sequence in which major features of the MPEG coding are incorporated. We have not modeled the scene changes and the correlation among GOPs as done by Rose [17]. It is preferable to take these characteristics into account, but that would make our model too complicate to analyze. Hence we assume that the traffic feature is mainly affected by the coding scheme, and leave the modeling at the scene and GOP levels for the future work. The results in Section 2 are applied to the MPEG frame sequence as the SSMP^[X] arrival process. The Markov chain underlying the SSMP has three states corresponding to the I-, B-, and P-frames. The transition probabilities are determined according to the frequency of appearance of these frames in a GOP. The batch size accounts for the number of ATM cells in these frames. From the results of analysis, we can evaluate the waiting time of an arbitrary cell in the frame. Numerical examples are shown based on the data from three real video films.

2. SSMP^[X]+M/M/1 Queue

In this section, we start with definition of a special semi-Markov process (SSMP) and then analyze an SSMP^[X]+M/M/1 queueing system. In Section 2.1 an SSMP batch arrival process is introduced. The queue length in the SSMP^[X]+M/M/1 system is analyzed in Section 2.2. Then, the waiting time distributions for SSMP and Poisson customers are derived in Sections 2.3 and 2.4, respectively. A numerical example is shown in Section 2.5.

2.1. Special semi-Markov batch arrival process

The semi-Markov process with L ($< \infty$) states is a renewal process that passes through L states at successive renewal points according to a Markov chain with transition probability matrix $P = (p_{l,m}); l, m = 1, \dots, L$. The sojourn time spent in state l , given that the next state is m , has distribution function $A_{l,m}(t)$. For a given sequence of states visited, all

sojourn times are mutually independent. The special semi-Markov Process (SSMP) is a special case of the above semi-Markov process such that the sojourn time distribution in a given state depends only on that state [1, 2]. Hence the probability that the SSMP moves from state l to m in t time units is given by $p_{l,m}A_l(t)$, where $A_l(t)$ is the distribution function of the sojourn time in state l . Since P is a stochastic matrix, we have

$$\sum_{m=1}^L p_{l,m} = 1; \quad l = 1, \dots, L.$$

Let (π_1, \dots, π_L) be the stationary distribution of the Markov chain with transition probability matrix P . Then we have the following set of the balance equations and the normalizing condition:

$$\pi_m = \sum_{l=1}^L \pi_l p_{l,m}; \quad m = 1, \dots, L \quad ; \quad \sum_{l=1}^L \pi_l = 1. \tag{1}$$

We consider an SSMP with L states as a process governing the batch arrivals of customers (called SSMP customers) to a queue. In other words, every arrival of an SSMP batch corresponds to the state transition in the underlying Markov chain. See Figure 1 for the diagram of the SSMP arrival process, where A_l denotes the sojourn time in state l .

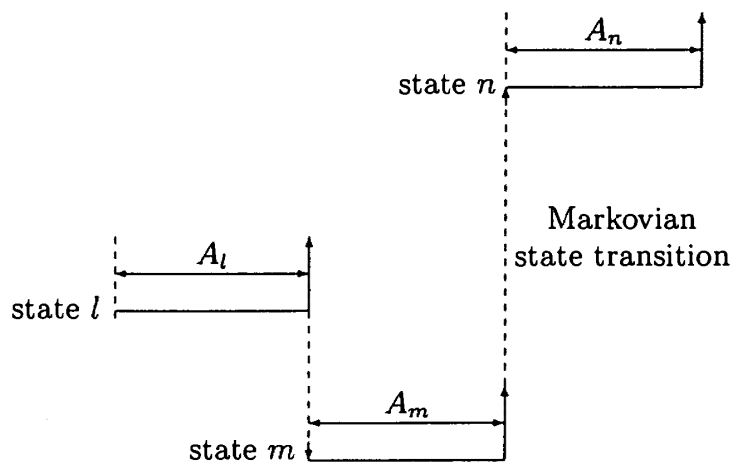


Figure 1: SSMP arrival process.

2.2. Queue length

In an $SSMP^{[X]}+M/M/1$ queueing system, the arrival process is a mixture of an SSMP batch and a Poisson process. Let $g_l(k)$ be the probability that the size of a batch that arrives to bring state l is k , where $k = 1, 2, \dots$. The arrival rate from the Poisson process is denoted by λ . The service times for the SSMP and Poisson customers are assumed to have common exponential distribution with mean $1/\mu$. Finally, it has a single server and an infinite-capacity waiting room.

We analyze the queue length in the $SSMP^{[X]}+M/M/1$ system. The queue length $X(t)$ at time t is the number of both SSMP and Poisson customers, including those both waiting and in service, in the system at time t . We extend Kuczura's approach [7] for a $GI + M/M/1$ queueing system in order to analyze our $SSMP^{[X]}+M/M/1$ system. Notice that, between the successive batch arrival epochs of SSMP customers, the process $X(t)$ behaves exactly like the queue length in an $M/M/1$ system. Whereas the arrival points of GI customers in a $GI + M/M/1$ system are regeneration points of a piecewise Markov process [8],

the arrival points of SSMP customers are not regeneration points in the SSMP^[X]+M/M/1 system. Therefore we study the bivariate Markovian sequence $\{(X^{(n)}, S^{(n)}); n = 0, 1, 2, \dots\}$ embedded at the points of SSMP arrivals, where $X^{(n)}$ denotes the number of both SSMP and Poisson customers found in the system by the first customer in the n th arriving batch of SSMP customers, and $S^{(n)}$ denotes the state of the underlying Markov chain immediately after the n th SSMP arrival (Figure 2). Since the SSMP with a single state degenerates to the GI process, our SSMP^[X]+M/M/1 system is an extension of the GI + M/M/1 system studied in [7].

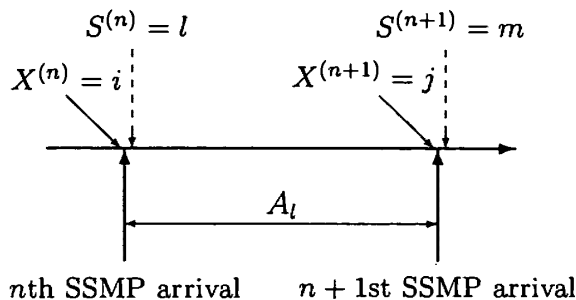


Figure 2: State transition in the Markov chain $\{(X^{(n)}, S^{(n)}); n = 0, 1, 2, \dots\}$.

Recall that the transition probability

$$P_{i,j}(t) := P\{X(t) = j | X(0) = i\}; \quad t > 0$$

in the birth-and-death process for the queue length of an M/M/1 system with arrival rate λ and service rate μ is given by [18, p.93]

$$P_{i,j}(t) = \rho^{\frac{1}{2}(j-i)} e^{-(\lambda+\mu)t} \left[I_{i-j} \left(2t\sqrt{\lambda\mu} \right) + \rho^{-\frac{1}{2}} I_{i+j+1} \left(2t\sqrt{\lambda\mu} \right) + (1-\rho) \sum_{k=1}^{\infty} \rho^{-\frac{1}{2}(k+1)} I_{i+j+k+1} \left(2t\sqrt{\lambda\mu} \right) \right], \quad (2)$$

where $\rho := \lambda/\mu$, and $I_i(t)$ is the modified Bessel function of the first kind of index i . For a nonnegative integer i , it is defined as

$$I_i(t) \equiv I_{-i}(t) := \left(\frac{t}{2}\right)^i \sum_{j=0}^{\infty} \frac{1}{j!(i+j)!} \left(\frac{t}{2}\right)^{2j}; \quad t \geq 0,$$

For the time-homogeneous Markov chain $\{(X^{(n)}, S^{(n)}); n = 0, 1, 2, \dots\}$, the state transition probability is given by

$$P\{X^{(n+1)} = j, S^{(n+1)} = m | X^{(n)} = i, S^{(n)} = l\} = p_{l,m} \sum_{k=1}^{\infty} g_l(k) \int_0^{\infty} P_{i+k,j}(t) dA_l(t) \quad i, j = 0, 1, 2, \dots; \quad l, m = 1, \dots, L. \quad (3)$$

Assuming that this Markov chain is ergodic, the limiting distribution

$$P(i, l) := \lim_{n \rightarrow \infty} P\{X^{(n)} = i, S^{(n)} = l\}; \quad i = 0, 1, 2, \dots; \quad l = 1, \dots, L \quad (4)$$

satisfies the balance equations

$$P(j, m) = \sum_{i=0}^{\infty} \sum_{l=1}^L \sum_{k=1}^{\infty} p_{l,m} g_l(k) P(i, l) \int_0^{\infty} P_{i+k,j}(t) dA_l(t); \quad j = 0, 1, 2, \dots; m = 1, \dots, L \quad (5)$$

and the normalization condition

$$\sum_{i=0}^{\infty} \sum_{l=1}^L P(i, l) = 1. \quad (6)$$

Let us introduce the generating function for $\{P(i, l); i = 0, 1, 2, \dots\}$ by

$$\Phi_l(z) := \sum_{i=0}^{\infty} P(i, l) z^i; \quad l = 1, \dots, L.$$

By definition, we must have $\Phi_l(1) = \pi_l; l = 1, \dots, L$. Multiplying (5) by z^j and summing over $j = 0, 1, 2, \dots$, we obtain

$$\Phi_m(z) = \sum_{i=0}^{\infty} \sum_{l=1}^L \sum_{k=1}^{\infty} p_{l,m} g_l(k) P(i, l) \int_0^{\infty} \Gamma_{i+k}(z, t) dA_l(t); \quad m = 1, \dots, L, \quad (7)$$

where

$$\Gamma_i(z, t) := \sum_{j=0}^{\infty} P_{i,j}(t) z^j; \quad i = 0, 1, 2, \dots$$

While this function is not simple, its Laplace transform is given by [18, p.89]

$$\gamma_i(z, s) := \int_0^{\infty} e^{-st} \Gamma_i(z, t) dt = \frac{z^{i+1} - (1-z)[\eta(s)]^{i+1} / [1-\eta(s)]}{zs - (1-z)(\mu - \lambda z)}, \quad (8)$$

where

$$\eta(s) := \frac{\lambda + \mu + s - \sqrt{(\lambda + \mu + s)^2 - 4\lambda\mu}}{2\lambda}.$$

Let us transform the real integral

$$\int_0^{\infty} \Gamma_{i+k}(z, t) dA_l(t)$$

appearing in (7) into a complex integral involving $\gamma_{i+k}(z, s)$ and $\alpha_l(s)$, the Laplace-Stieltjes transform (LST) of $A_l(t)$. To do so, note the inverse transform

$$\Gamma_{i+k}(z, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \gamma_{i+k}(z, s) ds,$$

where $c > 0$, $i := \sqrt{-1}$, and the integration path $\int_{c-i\infty}^{c+i\infty}$ is the *Bromwich integral*, being written as \int_{Br} hereafter. Furthermore, if $\alpha_l(t)$ denotes the LST of $A_l(t)$, we have

$$\int_0^{\infty} e^{st} dA_l(t) = \alpha_l(-s).$$

Thus we get

$$\int_0^{\infty} \Gamma_{i+k}(z, t) dA_l(t) = \frac{1}{2\pi i} \int_{Br} \gamma_{i+k}(z, s) \alpha_l(-s) ds. \quad (9)$$

Substituting (9) into (7), we obtain

$$\Phi_m(z) = \sum_{l=1}^L p_{l,m} \sum_{k=1}^{\infty} g_l(k) \sum_{i=0}^{\infty} P(i, l) \frac{1}{2\pi i} \int_{Br} \gamma_{i+k}(z, s) \alpha_l(-s) ds. \tag{10}$$

Changing the order of summation and integration, we get the following set of simultaneous equations for $\{\Phi_l(z); l = 1, \dots, L\}$:

$$\Phi_m(z) = \sum_{l=1}^L p_{l,m} \frac{1}{2\pi i} \int_{Br} \left[\frac{zG_l(z)\Phi_l(z) - (1-z)H_l(s)}{zs - (1-z)(\mu - \lambda z)} \right] \alpha_l(-s) ds; \quad m = 1, \dots, L, \tag{11}$$

where $G_l(z)$ is the generating function of $\{g_l(k); k = 1, 2, \dots\}$, and

$$H_l(s) := \frac{\eta(s)G_l[\eta(s)]\Phi_l[\eta(s)]}{1 - \eta(s)}; \quad l = 1, \dots, L. \tag{12}$$

Note that letting $z = 1$ in (11) recovers (1), because

$$\frac{1}{2\pi i} \int_{Br} \frac{\alpha_l(-s)}{s} ds = 1.$$

Following Kuczura [7], we may comment on the Bromwich integral in (11) as follows. Since $P_{i+k,j}(t)$ is the probability, its generating function $\Gamma_{i+k}(z, t)$ is uniformly convergent for $|z| \leq 1$, and $\gamma_{i+k}(z, s)$ is analytic for $|z| \leq 1$ and $\Re(s) > 0$. Hence the bracketed part of the integrand in (11) is analytic for $|z| \leq 1$ and $\Re(s) > 0$, since it is the convergent series of $\sum_{i=0}^{\infty} P(i, l)\gamma_{i+k}(z, s)$. On the other hand, since $A_l(t)$ is the distribution function, $\alpha_l(s)$ is analytic for $\Re(s) > 0$. For $\Re(s) < 0$, $\alpha_l(s)$ may or may not be analytic. However, $\alpha_l(s)$ is meromorphic for $\Re(s) < 0$ in many cases, including the cases in which the distribution of A_l is exponential, Erlang, and a linear combination thereof.

If we assume that $\alpha_l(s)$ is meromorphic for the left-half plane $\Re(s) < 0$, all the poles of $\alpha_l(-s)$ are in the right-half plane $\Re(s) > 0$. Hence the integrand in (11) is meromorphic in the right-half plane. Thus we can use the residue theorem to evaluate the integrand over the contour consisting of the line $(c + iR, c - iR)$ and a semicircle of radius R in the right-half plane which connects $c - iR$ with $c + iR$ counterclockwise. We can choose c and R such that all the poles of $\alpha_l(-s)$ are interior to this contour for all $l = 1, \dots, L$. Then the Bromwich integrals in (11) are evaluated only at the poles of $\alpha_l(-s)$'s. Here the terms resulting from $H_l(s)$ are simply constants. Therefore, (11) is not a set of integral equations but simply a set of linear equations for $\{\Phi_l(z); l = 1, \dots, L\}$ containing unknown constants as coefficients. These constants are determined from the condition that the generating function $\Phi_l(z)$ is analytic for $|z| \leq 1$ and that $\Phi_l(1) = \pi_l$ for $l = 1, \dots, L$. See the examples in Sections 2.5 and 3.2 for this procedure in the specific cases. For the moment, let us assume that $\Phi_l(z)$'s are obtained by solving (11) and determining the constants in this way.

The marginal distribution for the number of SSMP and Poisson customers found in the system by an SSMP batch arrival is denoted by

$$P(i) := \lim_{n \rightarrow \infty} P\{X^{(n)} = i\} = \sum_{l=1}^L P(i, l); \quad i = 0, 1, 2, \dots \tag{13}$$

The generating function for $\{P(i); i = 0, 1, 2, \dots\}$ is then given by

$$\Phi(z) := \sum_{i=0}^{\infty} P(i)z^i = \sum_{l=1}^L \Phi_l(z). \tag{14}$$

Substituting (11) into (14) yields

$$\Phi(z) = \sum_{l=1}^L \frac{1}{2\pi i} \int_{Br} \left[\frac{zG_l(z)\Phi_l(z) - (1-z)H_l(s)}{zs - (1-z)(\mu - \lambda z)} \right] \alpha_l(-s) ds,$$

from which we can confirm that $\Phi(1) = 1$.

2.3. Waiting times of an SSMP customer

We proceed to consider the waiting times of SSMP and Poisson customers. Let us start with the waiting time W of an SSMP customer, which is defined as the delay from the arrival instant of an arbitrary SSMP customer until the beginning of his service.

Let us consider a randomly chosen *tagged* SSMP customer included in a batch that arrives to bring state l . Recall that the probability generating function for the number of customers placed before the tagged customer in this batch is given by [20, p.45]

$$\hat{G}_l(z) = \frac{1 - G_l(z)}{g_l(1 - z)}, \tag{15}$$

where g_l is the mean batch size. Thus the LST $D_l(s)$ of the distribution function for the sum of the service times to those customers before the tagged customer in the batch is given by

$$D_l(s) = \hat{G}_l[B(s)] = \frac{1 - G_l[B(s)]}{g_l[1 - B(s)]}, \tag{16}$$

where $B(s) := \mu/(s + \mu)$.

If the service is given in the order of arrival, the waiting time of an arbitrary SSMP customer (tagged) in a batch consists of the waiting time of the first customer of that batch and the service times for the customers placed before the tagged customer in the batch. Therefore, the LST of distribution function for the waiting time of an arbitrary SSMP customer included in a batch that brings state l is given by

$$\Phi_l[B(s)]D_l(s).$$

Hence we get the LST of the distribution function for the waiting time W of an arbitrary SSMP customer as

$$\Omega(s) = \frac{1}{g} \sum_{l=1}^L g_l \Phi_l[B(s)]D_l(s) = \frac{1}{g[1 - B(s)]} \sum_{l=1}^L \Phi_l[B(s)]\{1 - G_l[B(s)]\}, \tag{17}$$

where

$$g := \sum_{l=1}^L \pi_l g_l$$

is the overall mean batch size. The mean $E[W]$ and the second moment $E[W^2]$ of the waiting time are then given by

$$E[W] = \frac{1}{g\mu} \left(\sum_{l=1}^L E_l[X]g_l + \frac{g^{(2)}}{2} \right), \tag{18}$$

$$E[W^2] = \frac{1}{g\mu^2} \left(\sum_{l=1}^L \{ (E_l[X] + E_l[X^2])g_l + E_l[X]g_l^{(2)} \} + g^{(2)} + \frac{g^{(3)}}{3} \right), \tag{19}$$

where

$$g_l^{(i)} = G_l^{(i)}(1); \quad g^{(i)} = \sum_{l=1}^L \pi_l g_l^{(i)}; \quad i = 2, 3$$

$$E_l[X] = \Phi_l^{(1)}(1); \quad E_l[X^2] = \Phi_l^{(2)}(1) + E_l[X].$$

2.4. Waiting time of a Poisson customer

We also consider the waiting time W^* of a Poisson customer. According to the PASTA (Poisson arrivals see time averages) property, the number of customers that an arriving Poisson customer finds has the same distribution as the number X^* of customers present in the system at an arbitrary time in steady state. Thus we will find the generating function $\Phi^*(z)$ for the probability distribution of X^* .

To do so, note that the interval between an arbitrary time and the preceding SSMP arrival time corresponds to the backward recurrence time in the Markov renewal process that counts the number of state transitions in the SSMP. The joint distribution for the backward recurrence time and the probability that the SSMP is in state l is given by

$$\hat{A}_l(t) = \frac{1}{E[A]} \int_0^t [1 - A_l(x)] dx; \quad t \geq 0, \quad (20)$$

where

$$E[A] := \sum_{l=1}^L \pi_l E[A_l]$$

is the mean interarrival time between the batches of SSMP customers. The LST $\hat{\alpha}_l(s)$ of $\hat{A}_l(t)$ is given by

$$\hat{\alpha}_l(s) = \frac{1 - \alpha_l(s)}{E[A]s}. \quad (21)$$

Conditioning on the number of customers and the state of the SSMP at the preceding arrival point, and integrating with the backward recurrence time distribution in (20), the steady-state distribution of X^* is given by

$$P(X^* = j) = \sum_{i=0}^{\infty} \sum_{l=1}^L P(i, l) \sum_{k=1}^{\infty} g_l(k) \int_0^{\infty} P_{i+k, j}(t) d\hat{A}_l(t); \quad j = 0, 1, 2, \dots, \quad (22)$$

which is transformed into

$$\Phi^*(z) := \sum_{j=0}^{\infty} P(X^* = j) z^j = \sum_{i=0}^{\infty} \sum_{l=1}^L P(i, l) \sum_{k=1}^{\infty} g_l(k) \int_0^{\infty} \Gamma_{i+k}(z, t) d\hat{A}_l(t). \quad (23)$$

Using the relation similar to (9), we obtain

$$\Phi^*(z) = \sum_{l=1}^L \frac{1}{2\pi i} \int_{Br} \left[\frac{zG_l(z)\Phi_l(z) - (1-z)H_l(s)}{zs - (1-z)(\mu - \lambda z)} \right] \hat{\alpha}_l(-s) ds, \quad (24)$$

where $H_l(s)$ and $\hat{\alpha}_l(s)$ are given in (12) and (21), respectively. Again, the Bromwich integrals are evaluated only at the poles of $\hat{\alpha}_l(-s)$'s in the right-half plane $\Re(s) > 0$ in most cases.

The LST $\Omega^*(s)$ of the distribution function for the waiting time W^* of a Poisson customer is expressed as

$$\Omega^*(s) = \Phi^*[B(s)]. \quad (25)$$

The mean $E[W^*]$ and the second moment $E[(W^*)^2]$ are then given by

$$E[W^*] = \frac{1}{\mu} E[X^*] \quad ; \quad E[(W^*)^2] = \frac{E[X^*] + E[(X^*)^2]}{\mu^2}, \quad (26)$$

respectively, where $E[X^*]$ and $E[(X^*)^2]$ are obtained from the generation function $\Phi^*(z)$.

2.5. Numerical example: SSMP+M/M/1

We illustrate the results of analysis in the preceding sections numerically by assuming that the arrivals are single and that the sojourn time in state l follows exponential distribution with mean $1/\alpha_l$:

$$G_l(z) = z \quad ; \quad A_l(t) = 1 - e^{-\alpha_l t}; \quad t \geq 0, \quad l = 1, \dots, L.$$

Then the complex integral in (9) reduces to

$$\int_0^\infty \Gamma_{i+1}(z, t) dA_l(t) = \alpha_l \gamma_{i+1}(z, \alpha_l), \tag{27}$$

which is free from the Bromwich integral. In this case, the equations in (11) become

$$\Phi_m(z) = \sum_{l=1}^L \frac{p_{l,m} \alpha_l [z^2 \Phi_l(z) - (1-z)H_l]}{\alpha_l z - (1-z)(\mu - \lambda z)}; \quad m = 1, \dots, L, \tag{28}$$

where

$$H_l := \frac{[\eta(\alpha_l)]^2 \Phi_l[\eta(\alpha_l)]}{1 - \eta(\alpha_l)}; \quad l = 1, \dots, L \tag{29}$$

are constants to be determined. Similarly, from (24) we have

$$\Phi^*(z) = \frac{1}{E[A]} \sum_{l=1}^L \frac{z^2 \Phi_l(z) - (1-z)H_l}{\alpha_l z - (1-z)(\mu - \lambda z)}. \tag{30}$$

In the case $L = 2$, let us assume that the transition probability matrix of the underlying two-state Markov chain is given by

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix},$$

where $0 < p, q < 1$. The stationary distribution of this Markov chain is

$$\pi_1 = \frac{q}{p+q} \quad ; \quad \pi_2 = \frac{p}{p+q}.$$

We solve the simultaneous equations (28) with $L = 2$ for $\Phi_1(z)$ and $\Phi_2(z)$, and get

$$\Phi_1(z) = \frac{H_1 \alpha_1 [\alpha_2 z (1-p-\kappa z) - (1-p)(1-z)(\mu - \lambda z)] + H_2 \alpha_2 q [\alpha_1 z - (1-z)(\mu - \lambda z)]}{T(z)},$$

$$\Phi_2(z) = \frac{H_1 \alpha_1 p [\alpha_2 z - (1-z)(\mu - \lambda z)] + H_2 \alpha_2 [\alpha_1 z (1-q-\kappa z) - (1-q)(1-z)(\mu - \lambda z)]}{T(z)}.$$

where $\kappa := 1 - p - q$, and

$$T(z) := (\mu - \lambda z) [\alpha_2 z (1 - z + qz) - (1 - z)(\mu - \lambda z)] - \alpha_1 z [\alpha_2 z (1 - \kappa z) - (1 - z + pz)(\mu - \lambda z)].$$

Adding $\Phi_1(z)$ and $\Phi_2(z)$ yields

$$\Phi(z) = \frac{H_1 \alpha_1 [\alpha_2 z (1 - \kappa z) - (1 - z)(\mu - \lambda z)] + H_2 \alpha_2 [\alpha_1 z (1 - \kappa z) - (1 - z)(\mu - \lambda z)]}{T(z)}. \tag{31}$$

These expressions contain two unknown constants H_1 and H_2 . They can be determined by first applying the normalization condition $\Phi(1) = 1$ and then forcing the zero of the numerator in the unit circle to coincide with the zero of the denominator in (31), since $\Phi(z)$ is analytic in $|z| \leq 1$. Clearly, $T(0) = -\mu^2 < 0$, and

$$T(1) = (\mu - \lambda)(\alpha_1 p + \alpha_2 q) - \alpha_1 \alpha_2 (p + q) = (\alpha_1 p + \alpha_2 q)(\mu - \alpha - \lambda),$$

where

$$\alpha := \frac{1}{E[A]} = \frac{1}{\pi_1/\alpha_1 + \pi_2/\alpha_2} = \frac{\alpha_1 \alpha_2 (p + q)}{\alpha_1 p + \alpha_2 q} \quad (32)$$

is the overall arrival rate of SSMP customers. Therefore, if the condition

$$\alpha + \lambda < \mu \quad (33)$$

is satisfied, we have $T(1) > 0$. Hence $T(z)$ has a (real) zero, say z_1 , such that $0 < z_1 < 1$. Using the normalization condition $\Phi(1) = 1$ and requiring that the numerator in (31) be zero at $z = z_1$, we find H_1 and H_2 as

$$H_1 = \frac{[\mu - z_1(\lambda + \mu + \alpha_1) + z_1^2(\lambda + \kappa\alpha_1)]\{q\alpha_2(\mu - \lambda) + \alpha_1[p(\mu - \lambda) - \alpha_2(1 - \kappa)]\}}{(1 - \kappa)(1 - z_1)(\lambda z_1 - \mu)\alpha_1(\alpha_1 - \alpha_2)}, \quad (34)$$

$$H_2 = \frac{[\mu - z_1(\lambda + \mu + \alpha_2) + z_1^2(\lambda + \kappa\alpha_2)]\{q\alpha_2(\mu - \lambda) + \alpha_1[p(\mu - \lambda) - \alpha_2(1 - \kappa)]\}}{(1 - \kappa)(1 - z_1)(\lambda z_1 - \mu)\alpha_2(\alpha_2 - \alpha_1)}. \quad (35)$$

These constants also make $\Phi_1(z)$ and $\Phi_2(z)$ analytic in $|z| \leq 1$. Note that the left-hand side of the inequality in (33) is the sum of the arrival rates of SSMP and Poisson customers and that the right-hand side is the service rate. Hence, (33) is a sufficient condition for the stability of the present system.

We can now calculate the mean and variance for the waiting times of SSMP and Poisson customers. For this purpose, the following set of parameters is assumed:

$$p = \frac{2}{25}, \quad q = \frac{3}{25}, \quad \alpha_2 = \frac{1}{2}\alpha_1, \quad \mu = 10.0.$$

Let us plot the performance values by changing the arrival rate α of SSMP customers. We show the results for several values of Poisson arrival rate λ in the same figures in order to observe the influence of the interfering traffic. In Figures 3 and 4 the mean and variance of the waiting time of SSMP and Poisson customers are shown respectively. We observe the influence of the interfering traffic on the waiting time of SSMP customers. Given the interfering traffic rate λ , they increase as α gets big until the condition (33) is reached. On the other hand, in the limit $\alpha \rightarrow 0$, the waiting time for both SSMP and Poisson customers approaches that of the M/M/1 queue as

$$\lim_{\alpha \rightarrow 0} E[W] = \lim_{\alpha \rightarrow 0} E[W^*] = \frac{\lambda}{\mu(\mu - \lambda)}; \quad \lim_{\alpha \rightarrow 0} E[W^2] = \lim_{\alpha \rightarrow 0} E[(W^*)^2] = \frac{2\lambda}{\mu(\mu - \lambda)^2}.$$

It is also observed that SSMP customers always receive worse treatment, i.e., bigger mean and variance of the waiting time, than Poisson customers. This is because the coefficient of variation (c.v.) of the interarrival times for the SSMP is bigger than that of the Poisson process which is unity. Kuczura's [7] reports that the arrival process having bigger c.v. receives worse treatment than that with smaller c.v., which agrees with the present result.

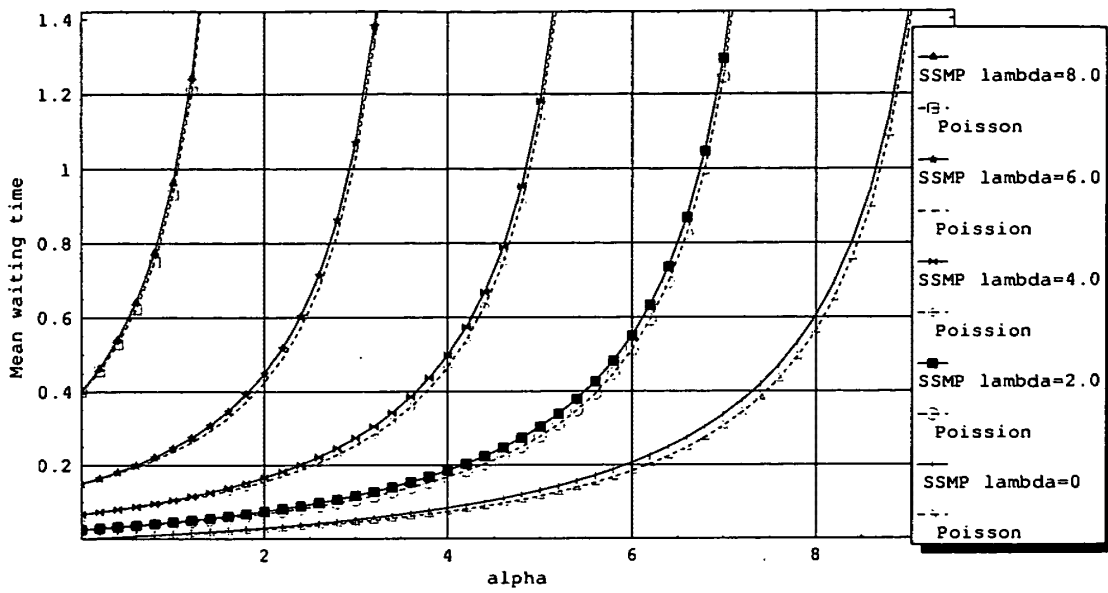


Figure 3: Mean waiting time for SSMP and Poisson customers.

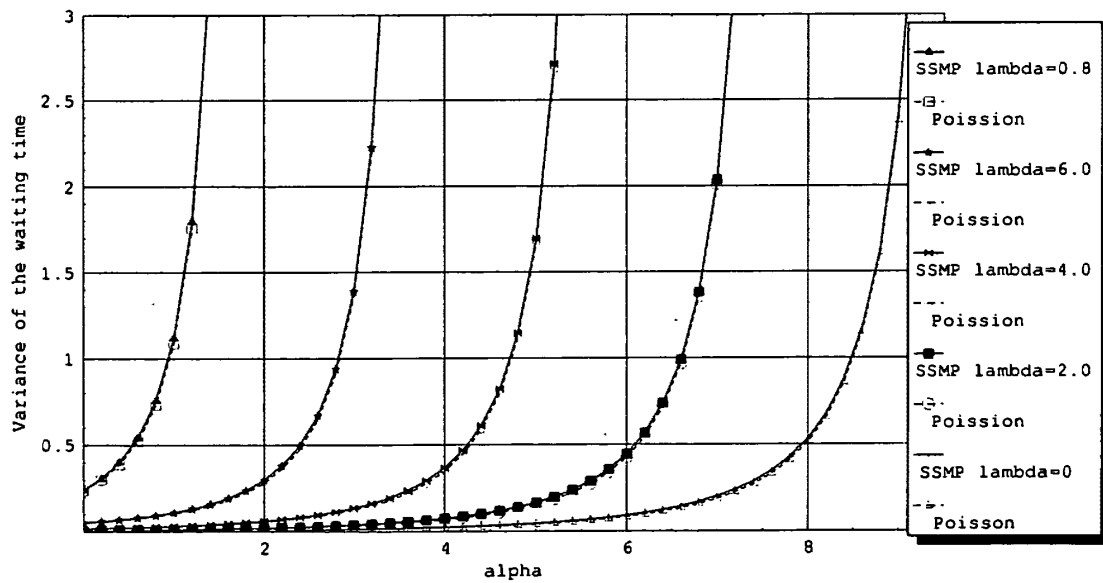


Figure 4: Variance of the waiting time for SSMP and Poisson customers.

3. Modeling of MPEG Video Traffic

In this section we present a queueing model for evaluating the waiting time of an arbitrary ATM cell generated from the frames of MPEG sequence in the presence of interfering traffic. An SSMP batch arrival process is assumed such that the underlying Markov chain has three states corresponding to the I-, B-, and P-frames, and that the batch accounts for a group of ATM cells generated from each frame. In Section 3.1 a brief description of the MPEG coding scheme is given. Analysis of the SSMP^[X] + M/M/1 queueing system is done in Section 2. In Section 3.2 we determine the state transition probabilities of the Markov chain underlying the SSMP with three states as mentioned above. Assuming that the frame arrival process is Poisson we can obtain the formulas for evaluating the waiting time of an arbitrary ATM cell in the frame. Numerical examples using the statistics from real video films are presented in Section 3.3.

3.1. MPEG video coding scheme

In the MPEG coding [10], a video traffic is compressed using the following three types of frames.

- I-frames are generated independently of B- or P-frames and inserted periodically.
- P-frames are encoded for the motion compensation with respect to the previous I- or P-frame.
- B-frames are similar to P-frames, except that the motion compensation can be done with respect to the previous I- or P-frame, the next I- or P-frame, or the interpolation between them.

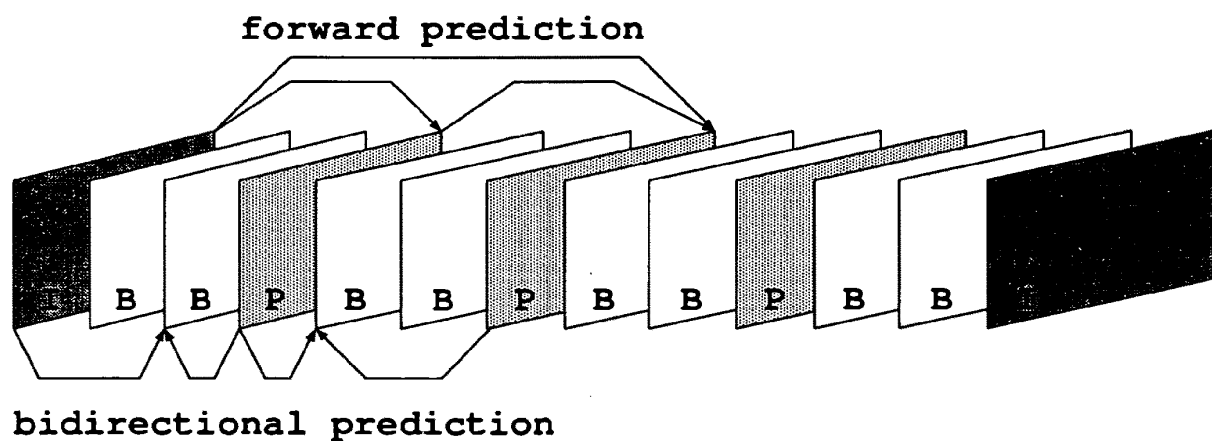


Figure 5: Group of pictures (GOP) of an MPEG stream [10].

These frames are arranged in a deterministic sequence “IBBPBBPBBPBB” as shown in Figure 5, which is called a Group of Pictures (GOP). The length of the GOP in Figure 5 is 12 frames. It is expected that this coding scheme leads to the statistical properties that are typical for MPEG video traffic streams. We utilize the MPEG frame traces for the Jurassic Park (dino), the Soccer World Cup Final 1994 Brazil-Italy (soccer), and the Star Wars (starwars). These data were prepared by Rose [17], and are now available from the web site <http://nero.informatik.uni-wuerzburg.de/MPEG/>.

Table 1 contains the statistics for the number of ATM cells in each frame (frame size) that have been calculated by assuming that every frame is divided into a group of cells each with a payload of 48 bytes. Clear difference can be observed in the frame size distribution

Table 1: Statistics for the frame size in ATM cells calculated from the MPEG traces in the web site <http://nero.informatik.uni-wuerzburg.de/MPEG/>.

video	I-frame			B-frame			P-frame		
	mean	var	c.v.	mean	var	c.v.	mean	var	c.v.
dino	143.427	918.704	0.211	19.033	135.021	0.612	37.659	632.568	0.667
soccer	206.101	4321.554	0.319	33.809	268.692	0.485	123.719	3584.561	0.484
starwars	114.615	1355.569	0.321	12.158	123.430	0.918	26.379	618.138	0.942

among the three types of frames I, B, and P. Namely, the I-frames require much more cells than the P-frames. The B-frames have the lowest cell requirement. The coefficients of variation (c.v.) are also different. Thus the traffic stream generated by the MPEG coding is mainly characterized by two features, (i) deterministic frame pattern in the GOP, and (ii) distinguishable frame size distributions for the three types of frames (I, B and P). In the following section, we propose a traffic model containing these two features of MPEG coding. However, we do not take into account the correlation of the frame size between successive frames.

3.2. Traffic model for MPEG frame sequence

We are now in a position to apply the analysis results of an SSMP^[X]+M/M/1 system in Section 2 to the queueing model with MPEG frame sequence and interfering traffic. In this model, the Markov chain underlying the SSMP has three states denoted by I, B, and P corresponding to the I-, B-, and P-frames, respectively. We determine each element of the transition probability matrix P of the Markov chain so as to match the frequency of frame appearance in a GOP. It is evident from Figure 5 that an I-frame is always followed by a B-frame and that a P-frame by a B-frame. Thus we set the transition probability from state I to state B to 1 and to any other state to 0. The transition probabilities from state P are the same as those from state I. We also observe that B-frames are followed by I-, B-, and P-frames. Taking the frequency of transitions from B-frames into account, we determine the transition probability matrix for the underlying Markov chain as follows:

$$P = \begin{matrix} & \begin{matrix} \text{I} & \text{B} & \text{P} \end{matrix} \\ \begin{matrix} \text{I} \\ \text{B} \\ \text{P} \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 1 & 0 \end{bmatrix} \end{matrix} \quad (36)$$

The stationary distribution of this Markov chain is given by

$$\pi_I = \frac{1}{12} \quad ; \quad \pi_B = \frac{2}{3} \quad ; \quad \pi_P = \frac{1}{4}.$$

For the sake of simplicity in the expressions, we assume that the arrival process of the frames is Poisson with rate α as a (very) special case of the SSMP. Let $G_I(z)$, $G_B(z)$, and $G_P(z)$ be the probability generating functions for the number of ATM cells generated from the I-, B-, and P-frames, respectively. Equations in (11) become

$$\Phi_m(z) = \frac{\alpha}{q(z)} \sum_{l=I,B,P} p_{l,m} [zG_l(z)\Phi_l(z) - (1-z)H_l]; \quad m = I, B, P, \quad (37)$$

where

$$q(z) := \alpha z - (1 - z)(\mu - \lambda z),$$

and H_l , $l = I, B, P$, are constants to be determined. Solving this set of equations we get

$$\Phi_I(z) = \frac{\Phi_P(z)}{3} = \frac{\alpha(1 - z)[(H_I + H_P)\alpha z G_B(z) + H_B q(z)]}{T(z)}$$

and

$$\Phi_B(z) = \frac{\alpha(1 - z)\{8(H_I + H_P)q(z) + H_B[4q(z) + \alpha z[G_I(z) + 3G_P(z)]]\}}{T(z)},$$

where

$$T(z) := \alpha z G_B(z)\{4q(z) + \alpha z[G_I(z) + 3G_P(z)]\} - 8[q(z)]^2. \quad (38)$$

Note that $T(1) = 0$. From the condition $\Phi_I(1) = \pi_I$, we get the relation

$$H_I + H_B + H_P = \frac{C}{\alpha},$$

where

$$C := \mu - \alpha g - \lambda \quad ; \quad g := \frac{1}{12}g_I + \frac{2}{3}g_B + \frac{1}{4}g_P.$$

Recall that g is the mean size of a frame. We can then write

$$\Phi_I(z) = \frac{\Phi_P(z)}{3} = \frac{\alpha(1 - z)[(C - \alpha H_B)z G_B(z) + H_B q(z)]}{T(z)}, \quad (39)$$

$$\Phi_B(z) = \frac{(1 - z)\{4(2C - \alpha H_B)q(z) + \alpha^2 z H_B[G_I(z) + 3G_P(z)]\}}{T(z)}, \quad (40)$$

and

$$\Phi(z) = \frac{(1 - z)\{4C[\alpha z G_B(z) + 2q(z)] + \alpha^2 z H_B[G_I(z) - 4G_B(z) + 3G_P(z)]\}}{T(z)}. \quad (41)$$

It is shown in the Appendix that there are two zeros of $T(z)$ in $|z| \leq 1$ under the condition

$$\alpha g + \lambda < \mu, \quad (42)$$

one of which is $z = 1$. Let z_1 be the other one. By forcing the zero in the numerator of (39) to coincide with z_1 , we determine H_B as

$$H_B = \frac{C z_1 G_B(z_1)}{\alpha z_1 G_B(z_1) - q(z_1)}. \quad (43)$$

This choice also makes $\Phi_B(z)$ and $\Phi(z)$ analytic in $|z| \leq 1$. The inequality in (42) is a sufficient condition for the stability of the system. This completes the determination of parameters in the model.

Table 2: Parameters of the negative binomial distributions for the frame size in Table 1.

video	I-frame		B-frame		P-frame	
	n_I	p_I	n_B	p_B	n_P	p_P
dino	19	0.132	2	0.105	2	0.053
soccer	9	0.044	4	0.118	4	0.032
starwars	9	0.079	1	0.082	1	0.038

3.3. Numerical examples

Let us evaluate the waiting time of an arbitrary ATM cell in the model with MPEG frame sequence and interfering traffic. We need to assume some distribution function for the number of cells in each frame (frame size) so that we can calculate the value of z_1 numerically. Frey and Nguen-Quang [4] and Sarkar et al. [19] propose the gamma distribution for the frame size. As a discrete version of the gamma distribution, let us assume that the distribution of the frame size is *negative binomial* whose parameters are determined from the mean and variance of the actual data given in Table 1. Thus the probability generation functions for the frame size are given by

$$G_l(z) = \left(\frac{p_l z}{1 - q_l z} \right)^{n_l}; \quad q_l := 1 - p_l; \quad l = I, B, P$$

with parameters given in Table 2. We also assume that cells are transmitted on a 10 Mbps channel, which corresponds to $\mu = 2,350$ cells/sec.

Figures 6 and 7 show the mean and the variance of the waiting time of an arbitrary ATM cell in the MPEG frames for the Jurassic Park in the presence of interfering traffic. It is observed that at low arrival rate α (frames/sec) both the mean and variance are flat, while at high load they increase rapidly with α . We can also observe the influence of the interfering traffic, where its rate λ is given in the unit of cells/sec. Figures 8 and 9 are for the World Cup Final 1994, and Figures 10 and 11 for Star Wars. In the limit $\alpha \rightarrow 0$, the waiting time for both SSMP and Poisson customers approaches that of the corresponding $M^{[X]}/M/1$ queue as

$$\lim_{\alpha \rightarrow 0} E[W] = \lim_{\alpha \rightarrow 0} E[W^*] = \frac{g^{(2)}}{2\mu g} + \frac{\lambda}{\mu(\mu - \lambda)},$$

$$\lim_{\alpha \rightarrow 0} E[W^2] = \lim_{\alpha \rightarrow 0} E[(W^*)^2] = \frac{1}{g\mu^2} \left(g^{(2)} + \frac{g^{(3)}}{3} \right) + \frac{2\lambda}{\mu(\mu - \lambda)^2}.$$

4. Concluding Remarks

In this paper we have studied a queueing system having a mixture of an SSMP batch and a Poisson process as the arrival process, where the Poisson arrival is regarded as interfering traffic. It is shown by numerical examples that the SSMP customers receive worse treatment than Poisson customers, i.e., the mean waiting time of SSMP customers is longer than that of Poisson customers.

We have also proposed a model of the MPEG frame arrivals as an SSMP batch arrival process. This model captures two major features of the MPEG coding scheme: (i) the frequency of appearance of the I-, B-, and P-frames in a GOP, and (ii) the distinct distributions for the size of the three types of frames. In the numerical examples, the waiting time

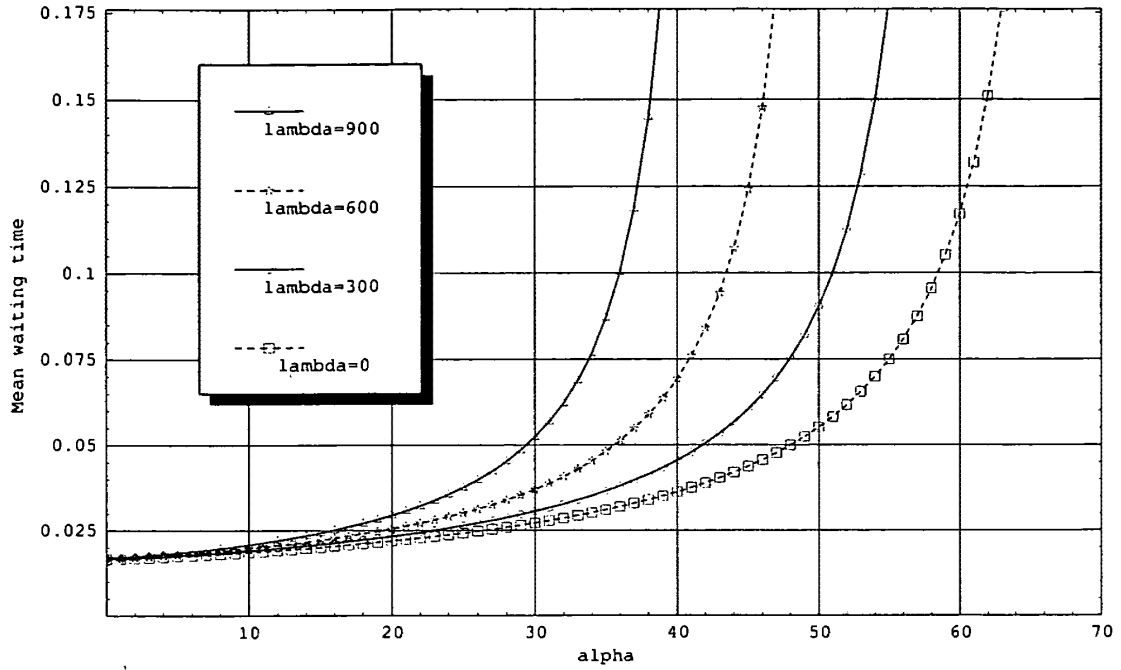


Figure 6: Mean waiting time for an arbitrary cell [sec] (dino).

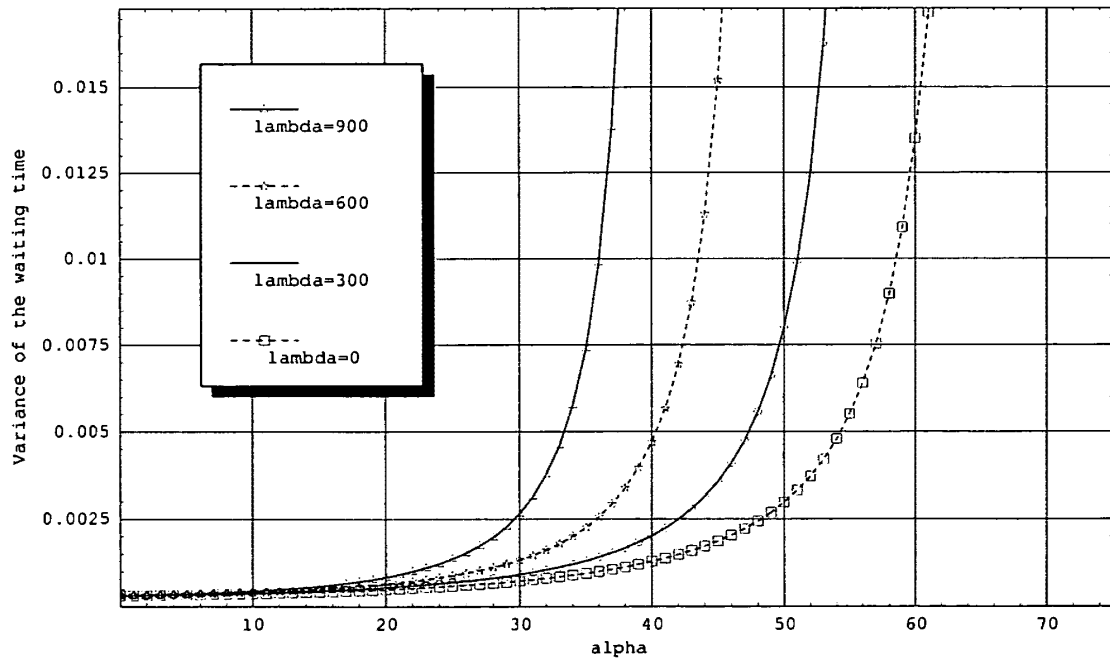


Figure 7: Variance of the waiting time for an arbitrary cell (dino).

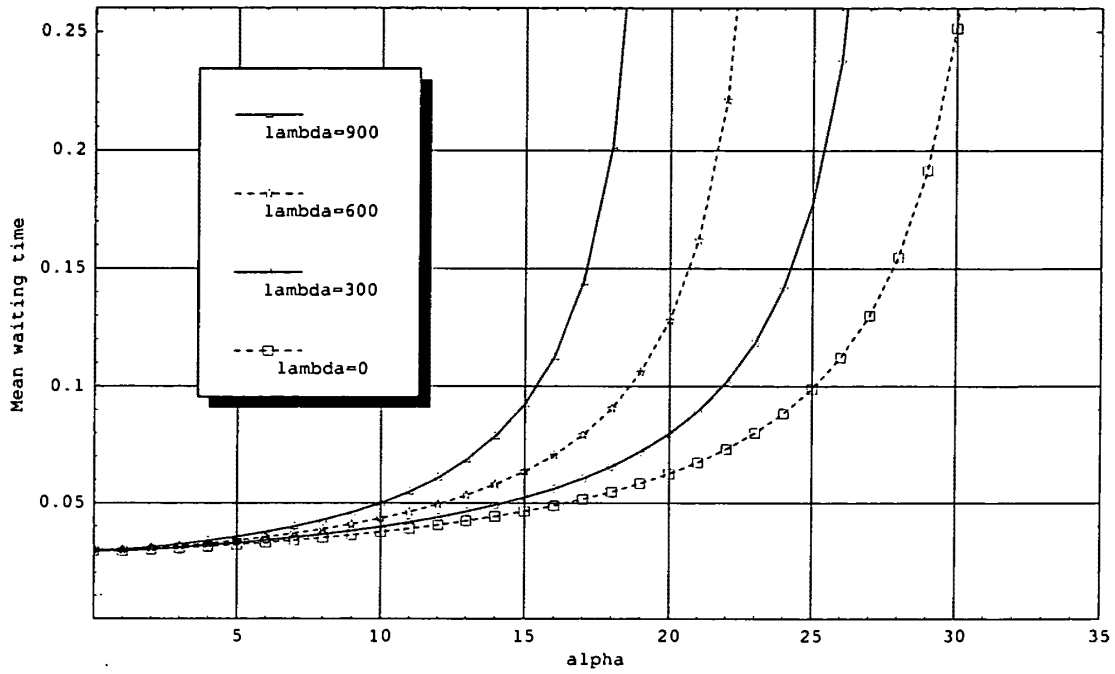


Figure 8: Mean waiting time for an arbitrary cell [sec] (soccer).

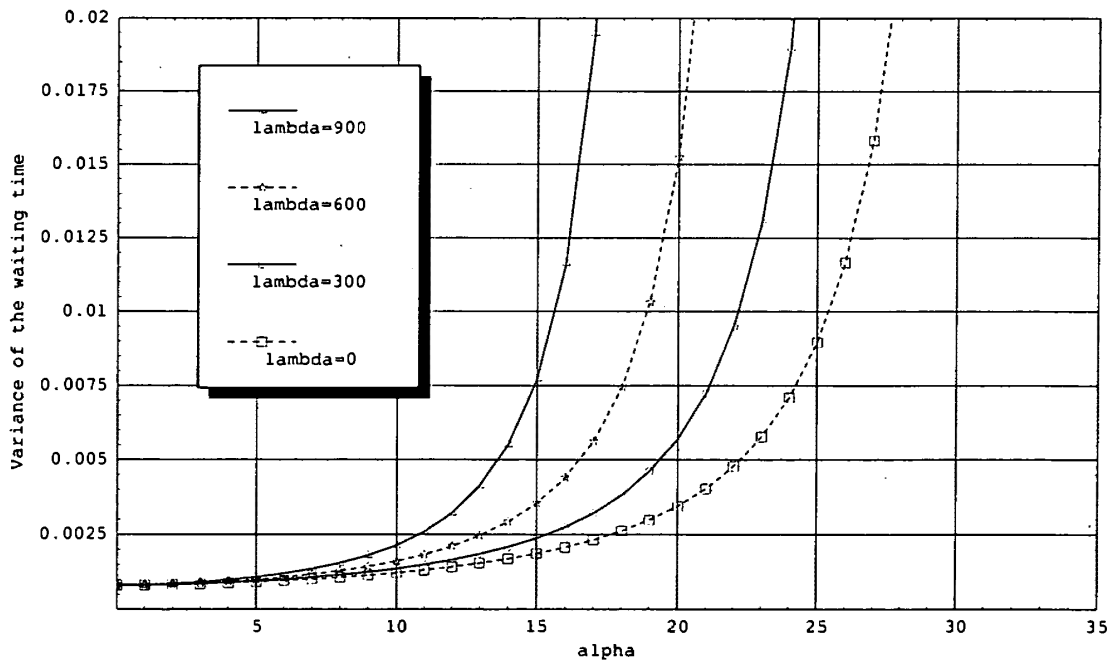


Figure 9: Variance of the waiting time for an arbitrary cell (soccer).

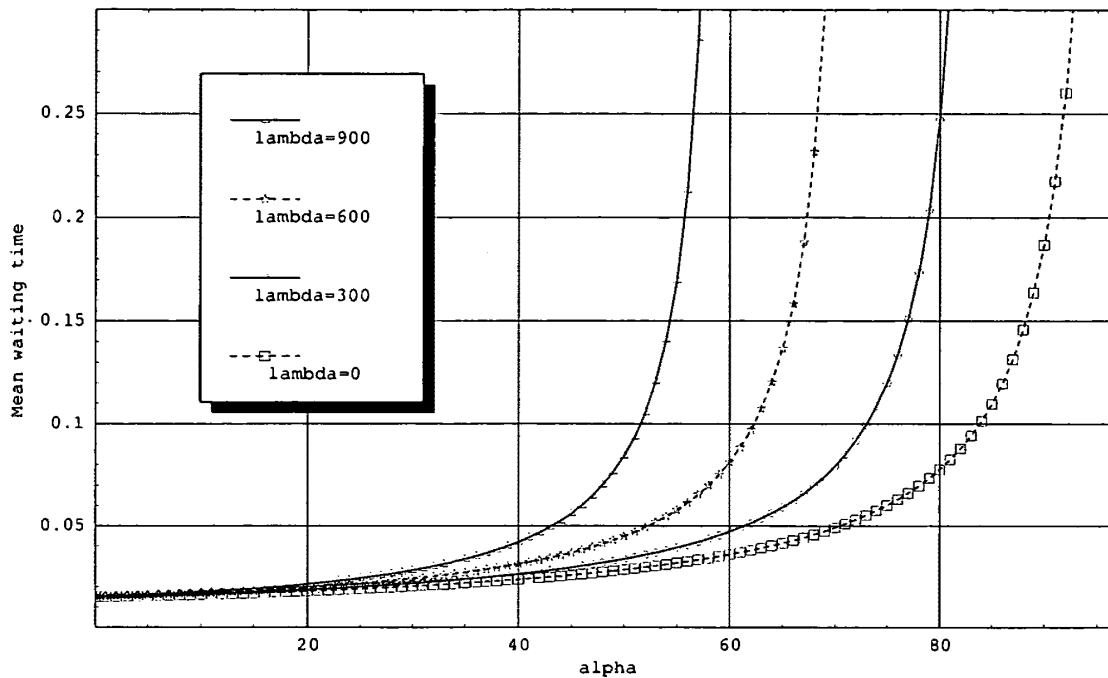


Figure 10: Mean waiting time for an arbitrary cell [sec] (starwars).

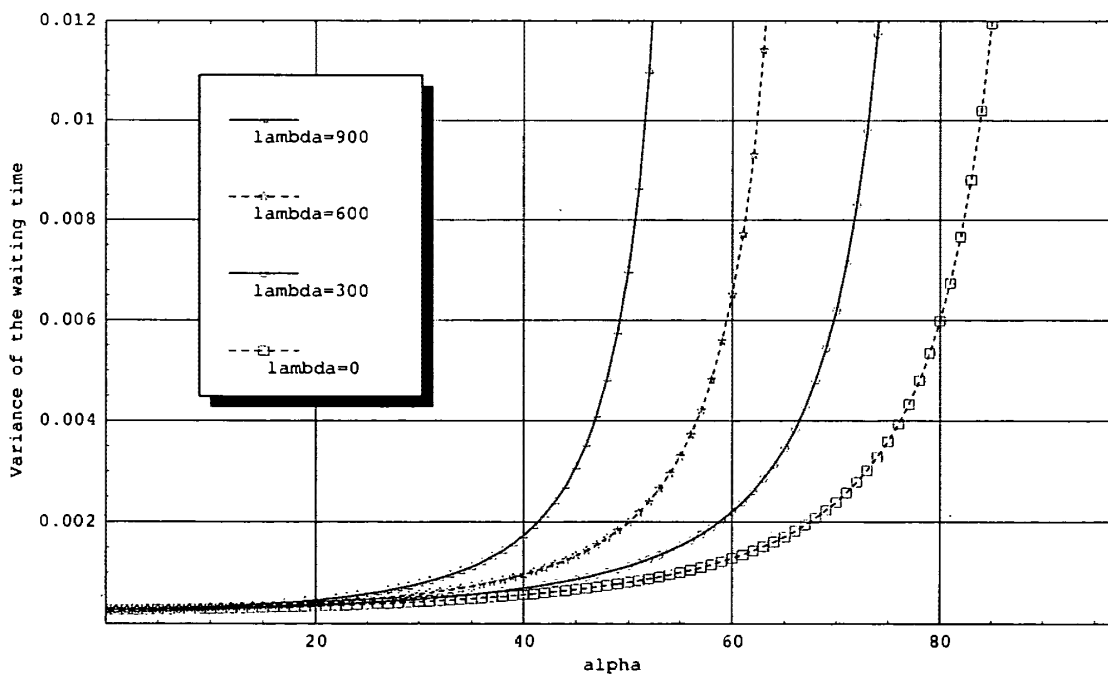


Figure 11: Variance of the waiting time for an arbitrary cell (starwars).

of each ATM cell generated from the MPEG frames is evaluated. It is observed that both the mean and variance are flat when the arrival rate changes at low levels, but that they increase rapidly at high levels of the arrival rate.

For modeling the MPEG frame sequence, we have assumed an underlying Markov chain with only three states representing the I-, B-, P-frames, and replaced the deterministic pattern "IBBPBBPBBPBB" with probabilistic transitions among the three states. However, it is possible to construct a Markov chain with 12 states exactly modeling the deterministic pattern "IBBPBBPBBPBB". Then there will be 12 constants in (37) to be determined from the same number of conditions derived from the zeros of the denominator in the expression for $\Phi_i(z)$.

From queueing theoretic point of view, it is straightforward to generalize the present analysis of the SSMP^[X]+M/M/1 system to an SMP^[X]+M/M/1 system, i.e., the one with a mixture of a batch semi-Markov process and a Poisson process as the arrival process. Further extension to an SMP^[X]+M/M/c system is also possible. These can be done again by the application of the theory of piecewise Markov processes found in [8].

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Appendix: Number of Zeros of $T(z)$ in (38) in $|z| \leq 1$

The denominator in the expression for $\Phi_l(z)$ in Section 3.2 is

$$\begin{aligned} T(z) &:= \alpha z G_B(z) \{4q(z) + \alpha z [G_I(z) + 3G_P(z)]\} - 8[q(z)]^2 \\ &= 4q(z) [\alpha z G_B(z) - 2q(z)] + \alpha^2 z^2 G_B(z) [G_I(z) + 3G_P(z)], \end{aligned} \quad (\text{A.1})$$

where

$$q(z) := \alpha z - (1 - z)(\mu - \lambda z). \quad (\text{A.2})$$

We show that $T(z)$ has exactly two zeros on the unit disk $|z| \leq 1$, one of which is $z = 1$, if the condition

$$\alpha g + \lambda < \mu \quad (\text{A.3})$$

is satisfied. The proof is based on Rouché's theorem [21, p.116]: *If $f(z)$ and $h(z)$ are analytic functions of z inside and on a closed contour C , and $|h(z)| < |f(z)|$ on C , then $f(z)$ and $f(z) + h(z)$ have the same number of zeros inside C .* We prove the above claim in a way similar to that in [6].

Let

$$\begin{aligned} f(z) &:= 4q(z)[\alpha z G_B(z) - 2q(z)], \\ h(z) &:= \alpha^2 z^2 G_B(z)[G_I(z) + 3G_P(z)]. \end{aligned} \tag{A.4}$$

Then $T(z) = f(z) + h(z)$. Let us choose a closed contour C so as to include $z = 1$ as an internal point, which is obviously a zero of $T(z)$. In particular, we choose C as

$$C := \{z = e^{i\theta}; 0 < \theta < 2\pi\} \cup \lim_{\varepsilon \rightarrow 0} C_\varepsilon, \tag{A.5}$$

where

$$C_\varepsilon := \left\{ z = 1 + \varepsilon e^{i\varphi}; -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \right\} \tag{A.6}$$

is a semicircle centered at $z = 1$ with radius $\varepsilon > 0$. The functions $f(z)$ and $h(z)$ are analytic inside and on the contour C .

We now compare $|f(z)|$ and $|h(z)|$ on C . First, we look at z on the unit circle $|z| = 1$. Since $q(z) = (\alpha + \lambda + \mu)z - (\lambda z^2 + \mu)$, we see that

$$|q(z)| \geq \alpha + \lambda + \mu - (\lambda + \mu) = \alpha$$

on $|z| = 1$. Hence, for $|z| = 1$, $z \neq 1$, it holds that

$$|f(z)| \geq 4\alpha(2\alpha - \alpha) = 4\alpha^2 \quad ; \quad |h(z)| < 4\alpha^2,$$

because $G_l(z) < 1$ for $l = I, B$, and P . Thus, $|h(z)| < |f(z)|$ for $|z| = 1$, $z \neq 1$.

We next look at $z = 1 + \varepsilon e^{i\varphi}$ on C_ε , for which

$$q(z) = (\alpha + \lambda + \mu)(1 + \varepsilon e^{i\varphi}) - \lambda(1 + \varepsilon e^{i\varphi})^2 - \mu = \alpha + (\mu + \alpha - \lambda)\varepsilon e^{i\varphi} + o(\varepsilon).$$

It follows that

$$\begin{aligned} |f(z)|^2 &= 16 \left| \alpha + (\mu + \alpha - \lambda)\varepsilon e^{i\varphi} + o(\varepsilon) \right|^2 \\ &\times \left| \alpha(1 + \varepsilon e^{i\varphi})[1 + g_B \varepsilon e^{i\varphi} + o(\varepsilon)] - 2[\alpha + (\mu + \alpha - \lambda)\varepsilon e^{i\varphi} + o(\varepsilon)] \right|^2 \\ &= 16 \left| \alpha^2 - \alpha(3\mu - 3\lambda + 2\alpha - \alpha g_B)\varepsilon e^{i\varphi} + o(\varepsilon) \right|^2 \\ &= 16\alpha^4 + 32\alpha^3(3\mu - 3\lambda + 2\alpha - \alpha g_B)\varepsilon \cos \varphi + o(\varepsilon), \end{aligned} \tag{A.7}$$

where $3\mu - 3\lambda + 2\alpha - \alpha g_B > 0$ if (A.3) holds, and $\cos \varphi \geq 0$ for $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$. We also have

$$\begin{aligned} |h(z)|^2 &= \left| \alpha^2(1 + \varepsilon e^{i\varphi})^2[1 + g_B \varepsilon e^{i\varphi} + o(\varepsilon)][1 + g_I \varepsilon e^{i\varphi} + 3(1 + g_P \varepsilon e^{i\varphi}) + o(\varepsilon)] \right|^2 \\ &= \alpha^4 \left| 4 + (8 + g_I + 4g_B + 3g_P)\varepsilon e^{i\varphi} + o(\varepsilon) \right|^2 \\ &= 16\alpha^4 + 8\alpha^4(8 + g_I + 4g_B + 3g_P)\varepsilon \cos \varphi + o(\varepsilon). \end{aligned} \tag{A.8}$$

Therefore, if (A.3) holds, we see that $|h(z)|^2 < |f(z)|^2$ (thus $|h(z)| < |f(z)|$) on C_ε for a sufficiently small value of ε . Hence we have shown that $|h(z)| < |f(z)|$ on the entire contour C . Thus the functions $f(z)$ and $h(z)$ satisfy the condition of Rouché's theorem with contour C . It follows that $f(z)$ and $f(z) + h(z) = T(z)$ have the same number of zeros inside C .

Finally, we consider the number of zeros of $f(z) = 4q(z) \cdot [\alpha z G_B(z) - 2q(z)]$ inside C . Clearly there is one zero of $q(z)$ inside C , which is

$$z = \left[\alpha + \lambda + \mu - \sqrt{(\alpha + \lambda + \mu)^2 - 4\lambda\mu} \right] / 2\lambda. \quad (\text{A.9})$$

We again apply Rouché's theorem to $\alpha z G_B(z) - 2q(z)$ with contour C given in (A.5). We then see that $|\alpha z G_B(z)| < |q(z)|$ ($\leq |-2q(z)|$) on C . Thus $\alpha z G_B(z) - 2q(z)$ and $-2q(z)$ have the same number of zeros inside C . The latter has a single zero inside C as given in (A.9). Thus $f(z)$ has two zeros inside C . Hence we conclude that $T(z)$ has two zeros inside C . *q.e.d.*

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