Journal of the Operations Research
Society of Japan
Vol. 45, No. 3, September 2002

ON GRÖTSCHEL-LOVÁSZ-SCHRIJVER'S RELAXATION OF STABLE SET POLYTOPES

Tetsuya Fujie
Kobe University of Commerce

Akihisa Tamura
Kyoto University

(Received October 3, 2001; Revised May 24, 2002)

Abstract Grötschel, Lovász and Schrijver introduced a convex set containing the stable set polytope of a graph. They proved that the set is a polytope if and only if the corresponding graph is perfect. In this paper, we give an alternative proof of the fact based on a new representation of the convex set described by infinitely many convex quadratic inequalities.

1. Introduction

In this paper, we consider a graph $G = (V, E)$ without loops and multiple edges. A subset $S \subseteq V$ of mutually nonadjacent vertices is called a *stable set*, and a subset $C \subseteq V$ of mutually adjacent vertices is called a *clique*. We denote the sets of all stable sets and of all cliques by $\mathcal{S}$ and $\mathcal{C}$, respectively. $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{R}^n$ the n-dimensional Euclidean space. We denote by $\mathbb{R}^V$ the set of all mappings from V to $\mathbb{R}$. If there is no confusion, we may identify $\mathbb{R}^V$ with $\mathbb{R}^{|V|}$.

Given a nonnegative weight vector $\boldsymbol{w} = (w_i \in \mathbb{R} \mid i \in V)$, the *maximum weight stable set problem* (MWSSP) is to find a stable set S of G, which maximizes the sum of its weights, $\sum_{i\in S} w_i$. The MWSSP is a well-known NP-hard problem. The *stable set polytope* is a polytope in $\mathbb{R}^V$, which is defined as

$$\mathrm{STAB}(G) = \mathrm{conv}\left(\{\boldsymbol{e}^S \mid S \in \mathcal{S}\}\right),$$

where $\boldsymbol{e}^S = (e_i^S \in \{0,1\} \mid i \in V)$ is the incidence vector of S, defined as $e_i^S = 1 \Leftrightarrow i \in S$, and $\mathrm{conv}(\cdot)$ denotes the convex hull. The MWSSP is equivalent to maximizing $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{x} = \sum_{i\in V} w_i x_i$ subject to $\boldsymbol{x} \in \mathrm{STAB}(G)$. Since the *nonnegativity constraints*

$$x_i \geq 0 \quad (i \in V) \tag{1}$$

and the *clique constraints*

$$\sum_{i\in C} x_i \leq 1 \quad (C \in \mathcal{C}) \tag{2}$$

are valid for $\mathrm{STAB}(G)$, the polytope $\mathrm{QSTAB}(G)$ defined by

$$\mathrm{QSTAB}(G) = \{\boldsymbol{x} \in \mathbb{R}^V \mid \boldsymbol{x} \text{ satisfies (1) and (2)}\}$$

contains $\mathrm{STAB}(G)$, *i.e.*, $\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G)$. Note that optimizing a linear function over $\mathrm{QSTAB}(G)$ is NP-hard in general [8].

Grötschel, Lovász and Schrijver [7] (see also [8]) introduced the convex set $\mathrm{TH}(G)$. An *orthonormal representation* of G is a set of vectors $\{\boldsymbol{u}_i \in \mathbb{R}^N \mid i \in V\}$ such that $||\boldsymbol{u}_i|| =$

$1\ (i \in V)$ and $\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{u}_j = 0\ (i \neq j,\ (i,j) \notin E)$, where N is a positive integer. The *orthonormal representation constraints*

$$\sum_{i \in V} (\boldsymbol{c}^{\mathrm{T}}\boldsymbol{u}_i)^2 x_i \leq 1 \quad \begin{pmatrix} \{\boldsymbol{u}_i \in \mathbb{R}^N \mid i \in V\} \ : \ \text{orthonormal representation of } G, \\ \boldsymbol{c} \in \mathbb{R}^N \text{ with } ||\boldsymbol{c}|| = 1 \end{pmatrix} \tag{3}$$

are valid for STAB(G) [11]. Hence, the set

$$\mathrm{TH}(G) = \left\{ \boldsymbol{x} \in \mathbb{R}^V \mid \boldsymbol{x} \text{ satisfies (1) and (3) } \right\}$$

contains STAB(G). We note that TH(G) is a convex set, not necessarily a polytope, since there are infinitely many orthonormal representation constraints (see Theorem 1.4). Nevertheless, TH(G) has the following quite nice properties.

Theorem 1.1 ([7]) (i) STAB$(G) \subseteq$ TH$(G) \subseteq$ QSTAB(G), *and* (ii) *optimizing a linear function over* TH(G) *can be done in polynomial time.* ■

An important consequence of Theorem 1.1 is that we can conclude the polynomial time solvability of the MWSSP for "perfect graphs" defined as follows. Let $\omega(G)$ denote the maximum cardinality of a clique of a graph G. A k-coloration of G is a partition of V into k stable sets of G, and the minimum integer k for which G admits a k-coloration is denoted by $\chi(G)$. It is clear that $\omega(G) \leq \chi(G)$ holds in general. A graph G is said to be *perfect* if $\omega(G') = \chi(G')$ holds for any vertex-induced subgraph G' of G. See [6, 8] for a comprehensive treatment of perfect graphs. The polynomial time solvability for perfect graphs is now proved by the following polyhedral characterization of perfect graphs.

Theorem 1.2 ([2, 5]) STAB(G) = QSTAB(G) *if and only if* G *is perfect.* ■

In this paper, we propose a new representation of TH(G) and, using the representation, we provide alternative proofs of Theorem 1.1 and the following Theorems 1.3 and 1.4 concerning geometric properties of TH(G). To state Theorem 1.3, we need some definitions. Let $K \subseteq \mathbb{R}^n$ be a closed convex set. For our purpose, we have only to consider the case where K has dimension n. $F \subseteq K$ is called a *facet* of K if F has dimension $n-1$ and there is an inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq b$ valid for K such that $F = K \cap \{\boldsymbol{x} \mid \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} = b\}$. We also say that the inequality $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq b$ defines facet F of K.

Theorem 1.3 ([7]) *A facet of* TH(G) *is defined by a positive multiple of one of the nonnegativity constraints or of one of the clique constraints.* ■

Theorem 1.4 ([7]) TH(G) *is a polytope if and only if* G *is perfect.* ■

Our representation of TH(G) is Theorem 1.5 below, which is obtained by applying the general representation theorem (Theorem 2.2) to the Lovász and Schrijver's representation [12]. The Lovász and Schrijver's representation of TH(G) is a projection of a feasible set of a certain semidefinite programming relaxation problem for the MWSSP (see Theorem 2.1). Let $\mathbb{S}^n$ be the set of all $n \times n$ symmetric matrices. For a graph $G = (V, E)$, $\mathbb{S}^V$ is the set of all $|V| \times |V|$ symmetric matrices whose rows and columns are indexed by V. The inequality $\boldsymbol{A} \succeq \boldsymbol{O}$ means $\boldsymbol{A}$ is a symmetric and positive semidefinite matrix, where $\boldsymbol{O}$ denotes the zero matrix.

Theorem 1.5

$$\mathrm{TH}(G) = \left\{ \boldsymbol{x} \in \mathbb{R}^V \mid f(\boldsymbol{M}, \boldsymbol{x}) \le 0 \quad (\boldsymbol{M} \in \mathcal{W}(G)) \right\}, \tag{4}$$

where

$$\begin{aligned} f(\boldsymbol{M}, \boldsymbol{x}) &= \boldsymbol{x}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{x} - \sum_{i \in V} M_{ii} x_i, \\ \mathcal{W}(G) &= \left\{ \boldsymbol{M} \in \mathbb{S}^V \mid M_{ij} = 0 \quad (i \ne j,\ (i,j) \notin E), \quad \boldsymbol{M} \succeq \boldsymbol{O} \right\}. \end{aligned}$$ ■

See Section 2 for proofs of Theorems 1.5 and 1.1.

Section 3 is devoted to the alternative proof of Theorem 1.3. Our proof is based on Theorem 3.1 concerning sets represented by convex quadratic inequalities.

The proof of Theorem 1.4 proceeds as follows. By Theorem 1.2, it is sufficient to show that TH(G) is a polytope if and only if STAB(G) = TH(G) = QSTAB(G). Thus, the "if" part is trivial. To prove the "only if" part, suppose TH(G) is a polytope. Then, by Theorem 1.3, TH(G) = QSTAB(G). Our contribution is to provide a new proof of the fact that TH(G) = STAB(G). To this end, we will prove the following theorem in Section 4.

Theorem 1.6 *Let $\widehat{\boldsymbol{x}}$ be a vertex of* QSTAB(G). *If $\widehat{\boldsymbol{x}}$ is non-integral, then $\widehat{\boldsymbol{x}} \notin$* TH($G$). ■

Since we have TH(G) = QSTAB(G), Theorem 1.6 implies that vertices of TH(G) consist of integral vertices of QSTAB(G). Since conv(QSTAB(G) $\cap \{0,1\}^V$) = STAB(G) is a well-known fact, TH(G) = STAB(G) is proved.

We note that the original proofs of Theorems 1.3 and 1.4 utilized the concepts of the antiblocker of a down-monotone set and the complement of an undirected graph, while our alternative proofs utilizes neither of them: Our proofs relies on the representation (4) and geometrical properties of convex sets described by convex quadratic inequalities. The following facts are merits of our approach. Using (4), we can obtain interesting properties such as Theorem 1.6 which holds for not only perfect graphs but also imperfect graphs. Moreover, our proof technique can be extended to more general case. For example, Theorems 1.1, 1.3 and 1.4 can be extended to a bidirected graph case [4]. In the bidirected graph case, the antiblocker and the complement are meaningless, and hence, the original proofs may not be extended to the case.

2. A Representation of TH(G) by Convex Quadratic Inequalities

In this section, we first prove Theorem 1.5 and then prove Theorem 1.1 using Theorem 1.5. Theorem 1.5 is derived by Theorems 2.1 and 2.2 below.

Theorem 2.1 ([12])

$$\mathrm{TH}(G) = \left\{ \boldsymbol{x} \in \mathbb{R}^V \;\middle|\; \begin{array}{lll} {}^\exists \boldsymbol{X} \in \mathbb{S}^V \text{ s.t.} & X_{ij} = 0 & ((i,j) \in E), \\ & X_{ii} - x_i = 0 & (i \in V), \\ & \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \succeq \boldsymbol{O} & \end{array} \right\}.$$ ■

Note that, by using the Schur complements (see, *e.g.* [9]), we have $\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \succeq \boldsymbol{O} \Leftrightarrow \begin{pmatrix} 1 & \boldsymbol{x}^{\mathrm{T}} \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \succeq \boldsymbol{O}$. Hence, Theorem 2.1 states that TH(G) is a feasible set of a semidefinite programming relaxation of the MWSSP, which is obtained in the following way:

$$\begin{aligned}
\{e^S \mid S \in \mathcal{S}\} &= \left\{ x \in \mathbb{R}^V \,\middle|\, \begin{array}{ll} x_i x_j = 0 & ((i,j) \in E), \\ x_i \in \{0,1\} & (i \in V) \end{array} \right\} \\
&= \left\{ x \in \mathbb{R}^V \,\middle|\, \begin{array}{ll} x_i x_j = 0 & ((i,j) \in E), \\ x_i^2 - x_i = 0 & (i \in V) \end{array} \right\} \\
&= \left\{ x \in \mathbb{R}^V \,\middle|\, \begin{array}{ll} X_{ij} = 0 & ((i,j) \in E), \\ X_{ii} - x_i = 0 & (i \in V), \\ \boldsymbol{X} - xx^{\mathrm{T}} = \boldsymbol{O} & \end{array} \right\} \\
&\subseteq \left\{ x \in \mathbb{R}^V \,\middle|\, \begin{array}{lll} {}^{\exists}\boldsymbol{X} \in \mathbb{S}^V \text{ s.t.} & X_{ij} = 0 & ((i,j) \in E), \\ & X_{ii} - x_i = 0 & (i \in V), \\ & \boldsymbol{X} - xx^{\mathrm{T}} \succeq \boldsymbol{O} & \end{array} \right\}. \qquad (5)
\end{aligned}$$

This relaxation method can be applied to more general sets arising from nonconvex quadratic programming problems. Let

$$F = \{x \in \mathbb{R}^n \mid x^{\mathrm{T}} \boldsymbol{Q}_i x + q_i^{\mathrm{T}} x + \pi_i \le 0 \ \ (i = 1, \ldots, m)\},$$

where $\boldsymbol{Q}_i \in \mathbb{S}^n, q_i \in \mathbb{R}^n$ and $\pi_i \in \mathbb{R}$ for $i = 1, \ldots, m$. The notation $\boldsymbol{A} \bullet \boldsymbol{B}$ stands for the inner product of $n \times n$ symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$: $\boldsymbol{A} \bullet \boldsymbol{B} = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij}$. Then we have

$$\begin{aligned}
F &= \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \boldsymbol{Q}_i \bullet \boldsymbol{X} + q_i^{\mathrm{T}} x + \pi_i \le 0 \quad (i = 1, \ldots, m), \\ \boldsymbol{X} - xx^{\mathrm{T}} = \boldsymbol{O} \end{array} \right\} \\
&\subseteq \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{ll} {}^{\exists}\boldsymbol{X} \in \mathbb{S}^n \text{ s.t.} & \boldsymbol{Q}_i \bullet \boldsymbol{X} + q_i^{\mathrm{T}} x + \pi_i \le 0 \quad (i = 1, \ldots, m), \\ & \boldsymbol{X} - xx^{\mathrm{T}} \succeq \boldsymbol{O} \end{array} \right\} =: N_+(F).
\end{aligned}$$

$N_+(F)$ is studied in [3, 10] etc. In particular, it is shown that $N_+(F)$ is equivalent to a set defined by convex quadratic inequalities.

Theorem 2.2 ([3, 10])

$$N_+(F) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^m t_i \left(x^{\mathrm{T}} \boldsymbol{Q}_i x + q_i^{\mathrm{T}} x + \pi_i \right) \le 0 \quad \left(\begin{array}{l} t_i \ge 0 \ \ (i = 1, \ldots, m) \\ \text{s.t. } \sum_{i=1}^m t_i \boldsymbol{Q}_i \succeq \boldsymbol{O} \end{array} \right) \right\}. \qquad \blacksquare$$

Fujie and Kojima [3] proved the theorem under a certain condition. Kojima and Tunçel [10] later showed that the equality holds without such a condition.

Proof of Theorem 1.5. Let $\boldsymbol{E}_{ij} \in \mathbb{S}^V$ denote the matrix where (i,j) and (j,i) elements are equal to 1 and other elements are equal to 0. Then

$$\mathrm{TH}(G) = \left\{ x \in \mathbb{R}^V \,\middle|\, \begin{array}{lll} {}^{\exists}\boldsymbol{X} \in \mathbb{S}^V \text{ s.t.} & X_{ij} = 0 & ((i,j) \in E), \\ & X_{ii} - x_i = 0 & (i \in V), \\ & \boldsymbol{X} - xx^{\mathrm{T}} \succeq \boldsymbol{O} & \end{array} \right\}$$

$$
= \left\{ \boldsymbol{x} \in \mathbb{R}^V \,\middle|\, \begin{array}{lll} {}^\exists \boldsymbol{X} \in \mathbb{S}^V \text{ s.t.} & \boldsymbol{E}_{ij} \bullet \boldsymbol{X} & \leq 0 \quad ((i,j) \in E), \\ & -\boldsymbol{E}_{ij} \bullet \boldsymbol{X} & \leq 0 \quad ((i,j) \in E), \\ & \boldsymbol{E}_{ii} \bullet \boldsymbol{X} - x_i & \leq 0 \quad (i \in V), \\ & -\boldsymbol{E}_{ii} \bullet \boldsymbol{X} + x_i & \leq 0 \quad (i \in V), \\ & \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \succeq \boldsymbol{O} & \end{array} \right\}
$$

$$
= \left\{ \boldsymbol{x} \in \mathbb{R}^V \,\middle|\, \begin{array}{l} \sum_{(i,j) \in E} \left(t_{ij}^+ - t_{ij}^-\right) \boldsymbol{x}^{\mathrm{T}} \boldsymbol{E}_{ij} \boldsymbol{x} + \sum_{i \in V} \left(t_{ii}^+ - t_{ii}^-\right) \left(\boldsymbol{x}^{\mathrm{T}} \boldsymbol{E}_{ii} \boldsymbol{x} - x_i\right) \leq 0 \\ \left(\begin{array}{l} t_{ij}^+,\ t_{ij}^- \geq 0 \quad ((i,j) \in E), \\ t_{ii}^+,\ t_{ii}^- \geq 0 \quad (i \in V), \\ \sum_{(i,j) \in E} \left(t_{ij}^+ - t_{ij}^-\right) \boldsymbol{E}_{ij} + \sum_{i \in V} \left(t_{ii}^+ - t_{ii}^-\right) \boldsymbol{E}_{ii} \succeq \boldsymbol{O} \end{array} \right) \end{array} \right\}.
$$

If we define $\boldsymbol{M} = \sum_{(i,j)\in E} \left(t_{ij}^+ - t_{ij}^-\right) \boldsymbol{E}_{ij} + \sum_{i\in V} \left(t_{ii}^+ - t_{ii}^-\right) \boldsymbol{E}_{ii}$, then we have (4). ■

Proof of Theorem 1.1. Since we have shown that optimizing a linear function over TH(G) is a semidefinite programming problem, it can be solved in polynomial time [1]. This proves (ii). To prove (i), we note that (5) shows STAB(G) $\subseteq$ TH(G). Moreover, since $\boldsymbol{E}_{ii} \in \mathcal{W}(G)$,

$$f(\boldsymbol{E}_{ii}, \boldsymbol{x}) = x_i(x_i - 1) \leq 0$$

is valid for TH(G). Hence, the nonnegativity constraint $x_i \geq 0$ is valid for TH(G). Similarly, since $(\boldsymbol{e}^C)(\boldsymbol{e}^C)^{\mathrm{T}} \in \mathcal{W}(G)$ for $C \in \mathcal{C}$,

$$f((\boldsymbol{e}^C)(\boldsymbol{e}^C)^{\mathrm{T}}, \boldsymbol{x}) = \sum_{i \in C} x_i \left(\sum_{i \in C} x_i - 1 \right) \leq 0$$

is valid for TH(G). Hence, the clique constraint $\sum_{i\in C} x_i \leq 1$ is valid for TH(G). Therefore, we have TH(G) $\subseteq$ QSTAB(G). ■

3. Alternative Proof of Theorem 1.3

Let us consider a set

$$K = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \boldsymbol{x}^{\mathrm{T}} \boldsymbol{Q} \boldsymbol{x} - \boldsymbol{q}^{\mathrm{T}} \boldsymbol{x} + \pi \leq 0 \quad \left(\begin{pmatrix} \pi & \boldsymbol{q}^{\mathrm{T}}/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \in \mathcal{P} \cap \mathcal{Q}_+ \right) \right\}, \tag{6}$$

where $\mathcal{P}$ is a closed convex cone in $\mathbb{S}^{1+n}$ and

$$\mathcal{Q}_+ = \left\{ \begin{pmatrix} \pi & \boldsymbol{q}^{\mathrm{T}}/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \,\middle|\, \boldsymbol{Q} \in \mathbb{S}^n,\ \boldsymbol{Q} \succeq \boldsymbol{O} \right\} \subseteq \mathbb{S}^{1+n}.$$

K is a closed convex set defined by infinitely many convex quadratic inequalities. This description of K using $\mathcal{P}$ and $\mathcal{Q}_+$ is due to Kojima and Tunçel [10]. TH(G) in Theorem 1.3 is described in the form of (6) if we take

$$\mathcal{P} = \text{cone} \left(\left\{ \begin{pmatrix} \pi & \boldsymbol{q}^{\mathrm{T}}/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \,\middle|\, \begin{array}{ll} \pi = 0, & \\ q_i = Q_{ii} & (i \in V), \\ Q_{ij} = 0 & (i \neq j,\ (i,j) \notin E) \end{array} \right\} \right),$$

where cone(P) denotes the cone generated by P. The following theorem is concerned with a characterization of "facet defining" convex quadratic inequalities of K.

Theorem 3.1 *Assume that K defined by (6) has dimension n and that F is a facet of K.*

(i) *There is* $\begin{pmatrix} \pi & \boldsymbol{q}^{\mathrm{T}}/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \in \mathcal{P} \cap \mathcal{Q}_+$ *such that*

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x} + \pi = 0 \quad ({}^{\forall}\boldsymbol{x} \in F).$$

(ii) $\operatorname{rank} \boldsymbol{Q} \leq 1$, *that is,* $\boldsymbol{Q} = \boldsymbol{O}$ *or* $\boldsymbol{Q} = \boldsymbol{p}\boldsymbol{p}^{\mathrm{T}}$ *for some* $\boldsymbol{p} \in \mathbb{R}^n$.

(iii) *If* $\boldsymbol{Q} \neq \boldsymbol{O}$ *then*

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x} + \pi = \left(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} - \alpha\right)\left(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} - \beta + \alpha\right) \tag{7}$$

for some $(\boldsymbol{p}, \alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$.

Proof : Since F is a facet of K, there exist n affinely independent points $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n$ in F.

(i) Let $\widehat{\boldsymbol{x}} = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{x}^j$. We have $\widehat{\boldsymbol{x}} \in F$ because F is convex. Since $\widehat{\boldsymbol{x}}$ is a boundary point of K and $\mathcal{P} \cap \mathcal{Q}_+$ is a closed set, we can prove that there exists $\begin{pmatrix} \pi & \boldsymbol{q}^{\mathrm{T}}/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \in \mathcal{P} \cap \mathcal{Q}_+$ such that $g(\widehat{\boldsymbol{x}}) := \widehat{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{Q}\widehat{\boldsymbol{x}} - \boldsymbol{q}^{\mathrm{T}}\widehat{\boldsymbol{x}} + \pi = 0$. Since g is a convex function and $g(\boldsymbol{x}) \leq 0$ is valid for F, we have $0 = g(\widehat{\boldsymbol{x}}) \leq \sum_{j=1}^{n} \frac{1}{n} g(\boldsymbol{x}^j) \leq 0$. Thus, $g(\boldsymbol{x}^j) = 0$ holds for $j = 1, \ldots, n$. Analogously, we can prove $g(\boldsymbol{x}) = 0$ holds for any $\boldsymbol{x} \in F$.

(ii) From (i) and the convexity of F, we have $g(\frac{1}{2}(\boldsymbol{x}^j + \boldsymbol{x}^1)) = \frac{1}{2}g(\boldsymbol{x}^j) + \frac{1}{2}g(\boldsymbol{x}^1)\ (= 0)$ for $j = 2, \ldots, n$. This implies $(\boldsymbol{x}^j - \boldsymbol{x}^1)^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{x}^j - \boldsymbol{x}^1) = 0$ for $j = 2, \ldots, n$. Since $\boldsymbol{Q} \succeq \boldsymbol{O}$, there is an $n \times m$ matrix $\boldsymbol{V}$ with $\boldsymbol{Q} = \boldsymbol{V}\boldsymbol{V}^{\mathrm{T}}$. Hence, $\boldsymbol{V}^{\mathrm{T}}(\boldsymbol{x}^j - \boldsymbol{x}^1) = 0$ for $j = 2, \ldots, n$. Moreover, $\boldsymbol{x}^2 - \boldsymbol{x}^1, \ldots, \boldsymbol{x}^n - \boldsymbol{x}^1$ are linearly independent. Therefore, $\operatorname{rank} \boldsymbol{Q} = \operatorname{rank} \boldsymbol{V} \leq 1$.

(iii) From (ii), $\boldsymbol{V} = \boldsymbol{p}$ for some $\boldsymbol{p} \in \mathbb{R}^n$ with $\boldsymbol{p} \neq \boldsymbol{0}$, where $\boldsymbol{0}$ denotes the zero vector. Hence $\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x}^j - \boldsymbol{x}^1) = 0$ $(j = 2, \ldots, n)$. Let $\alpha = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{x}^1 (= \boldsymbol{p}^{\mathrm{T}}\boldsymbol{x}^2 = \cdots = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{x}^n)$. Then $F = K \cap \{\boldsymbol{x} \mid \boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} = \alpha\}$. On the other hand, $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + \pi = (\boldsymbol{p}^{\mathrm{T}}\boldsymbol{x})^2 + \pi = \alpha^2 + \pi$ for any $\boldsymbol{x} \in F$. Hence there is β such that $(\boldsymbol{q}, \alpha^2 + \pi) = \beta(\boldsymbol{p}, \alpha)$. Then we can show that $(\boldsymbol{p}, \alpha, \beta)$ satisfies (7). ■

Proof of Theorem 1.3. It is well-known that $\mathrm{STAB}(G)$ has dimension $|V|$. Indeed, $\boldsymbol{0} = \boldsymbol{e}^{\emptyset} \in \mathrm{STAB}(G)$ and $\boldsymbol{e}^{\{i\}} \in \mathrm{STAB}(G)$ for $i \in V$. Hence, $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G)$ implies $\mathrm{TH}(G)$ has dimension $|V|$. Let F be a facet of $\mathrm{TH}(G)$. From Theorem 3.1 (i), there is $\boldsymbol{M} \in \mathcal{W}(G)$ such that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x} - \sum_{i \in V} M_{ii}x_i = 0$ for any $\boldsymbol{x} \in F$. We have $\boldsymbol{M} \neq \boldsymbol{O}$ since otherwise $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x} - \sum_{i \in V} M_{ii}x_i = 0$ holds for any $\boldsymbol{x} \in \mathbb{R}^V$. Hence, by Theorem 3.1 (iii), we have a representation

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x} - \sum_{i \in V} M_{ii}x_i = \left(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} - \alpha\right)\left(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} - \beta + \alpha\right) = 0 \quad ({}^{\forall}\boldsymbol{x} \in F). \tag{8}$$

From (8), we have

$$(M_{ii} =)\ p_i^2 = \beta p_i \quad (i \in V), \tag{9}$$

$$\alpha(\beta - \alpha) = 0. \tag{10}$$

(9) implies $p_i = 0$ or β for each $i \in V$. We have $\beta \neq 0$ since otherwise we have $\boldsymbol{p} = \boldsymbol{0}$ and thus $\boldsymbol{M} = \boldsymbol{p}\boldsymbol{p}^{\mathrm{T}} = \boldsymbol{O}$. Let $C = \{i \in V \mid p_i = \beta\}$. For any $i, j \in C$, we have $M_{ij} = p_i p_j = \beta^2 \neq 0$.

Hence, by the definition of $\mathcal{W}(G)$, C must be a clique of G. By the definition of C, the inequality, valid for $\mathrm{TH}(G)$,

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x} - \sum_{i\in V} M_{ii}x_i = \left(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} - \alpha\right)\left(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} - \beta + \alpha\right) \le 0 \tag{11}$$

is equivalent to

$$\beta - \alpha \le \sum_{i\in C} \beta x_i \le \alpha \tag{12}$$

or

$$\alpha \le \sum_{i\in C} \beta x_i \le \beta - \alpha. \tag{13}$$

Since (10) implies $\alpha = \beta$ or $\alpha = 0$, (12) and (13) are equivalent. Here we only consider (12). If $\alpha = \beta$ then (12) becomes

$$0 \le \sum_{i\in C} \beta x_i \le \beta, \tag{14}$$

while if $\alpha = 0$ then (12) becomes

$$\beta \le \sum_{i\in C} \beta x_i \le 0. \tag{15}$$

(14) and (15) are also equivalent because (14) implies $\beta > 0$ and (15) implies $\beta < 0$. In summary, (11) is equivalent to

$$0 \le \sum_{i\in C} x_i \le 1,$$

where C is a clique of G. If $\sum_{i\in C} x_i \ge 0$ defines a facet of $\mathrm{TH}(G)$ then it must be the nonnegativity constraint for some element of V. Therefore, a facet defining inequality of $\mathrm{TH}(G)$ is a positive multiple of one of the nonnegativity constraints or of one of the clique constraints. ■

4. Proof of Theorem 1.6

In this section, we prove Theorem 1.6. Our proof technique is closely related to the description of Section 3 of [3] and Lemma 4.1 (iii) of [10].

Proof of Theorem 1.6 : Let $\widehat{\boldsymbol{x}}$ be a non-integral vertex of $\mathrm{QSTAB}(G)$. Since $\widehat{\boldsymbol{x}}$ is a vertex of $\mathrm{QSTAB}(G)$, there are $i_1, \dots, i_\ell \in V$ and $C_{\ell+1}, \dots, C_{|V|} \in \mathcal{C}$ such that $\widehat{\boldsymbol{x}}$ is a unique solution of the system :

$$x_{i_k} = 0 \quad (k = 1, \dots, \ell), \quad \sum_{i\in C_k} x_i = 1 \quad (k = \ell+1, \dots, |V|).$$

Let us denote the system by $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and define

$$\boldsymbol{M} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \sum_{k=1}^{\ell} \boldsymbol{E}_{i_k i_k} + \sum_{k=\ell+1}^{|V|} (\boldsymbol{e}^{C_k})(\boldsymbol{e}^{C_k})^{\mathrm{T}}.$$

Then $\boldsymbol{M} \in \mathcal{W}(G)$. Since $\widehat{\boldsymbol{x}}$ is assumed to be a vertex of $\mathrm{QSTAB}(G)$, $\boldsymbol{A}$ is nonsingular, and hence $\boldsymbol{M}$ is positive definite. Moreover, we have

$$\begin{aligned} f(\boldsymbol{M}, \widehat{\boldsymbol{x}}) &= \widehat{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{M}\widehat{\boldsymbol{x}} - \sum_{i\in V} M_{ii}\widehat{x}_i \\ &= \sum_{k=1}^{\ell} \widehat{x}_{i_k}(\widehat{x}_{i_k} - 1) + \sum_{k=\ell+1}^{|V|} \left\{ \sum_{i\in C_k} \widehat{x}_i \left(\sum_{i\in C_k} \widehat{x}_i - 1 \right) \right\} = 0. \end{aligned} \tag{16}$$

Since $\widehat{\boldsymbol{x}}$ is assumed to be non-integral, there is $j \in V$ with $0 < \widehat{x}_j < 1$. Since $\boldsymbol{M}$ is positive definite, $\boldsymbol{M}' = \boldsymbol{M} - \varepsilon \boldsymbol{E}_{jj}$ is positive semidefinite for sufficiently small $\varepsilon > 0$. Hence we have $\boldsymbol{M}' \in \mathcal{W}(G)$. On the other hand, by (16) and $\widehat{x}_j(\widehat{x}_j - 1) < 0$,

$$f(\boldsymbol{M}', \widehat{\boldsymbol{x}}) = f(\boldsymbol{M}, \widehat{\boldsymbol{x}}) - \varepsilon \widehat{x}_j(\widehat{x}_j - 1) = -\varepsilon \widehat{x}_j(\widehat{x}_j - 1) > 0.$$

Therefore, $\widehat{\boldsymbol{x}} \notin \mathrm{TH}(G)$. ∎

Acknowledgements

This research is supported by a Grant-in-Aid of the Ministry of Education, Culture, Sports, Science and Technology of Japan. The authors are grateful to anonymous referees for many corrections and helpful suggestions on the first draft of the paper.

References

[1] W. F. Alizadeh: Interior point methods in semidefinite programming with application to combinatorial optimization. *SIAM Journal on Optimization*, **5** (1995) 13–51.

[2] V. Chvátal: On certain polytopes associated with graphs. *Journal of Combinatorial Theory, Series B*, **18** (1975) 138–154.

[3] T. Fujie and M. Kojima: Semidefinite programming relaxation for nonconvex quadratic programs. *Journal of Global Optimization*, **10** (1997) 367–380.

[4] T. Fujie and A. Tamura: A semidefinite programming relaxation for the generalized stable set problem. Working Paper #181, Kobe University of Commerce, 2000.

[5] D. R. Fulkerson: On the perfect graph theorem. In T. C. Hu and S. Robinson (eds.): *Mathematical Programming* (Academic Press, New York, 1973), 69–77.

[6] M. Golumbic: *Algorithmic Graph Theory and Perfect Graphs* (Academic Press, New York, 1980).

[7] M. Grötschel, L. Lovász and A. Schrijver: Relaxations of vertex packing. *Journal of Combinatorial Theory, Series B*, **40** (1986) 330–343.

[8] M. Grötschel, L. Lovász, and A. Schrijver: *Geometric Algorithms and Combinatorial Optimization* (Springer-Verlag, New York, 1988).

[9] R. A. Horn and C. R. Johnson: *Matrix Analysis* (Cambridge University Press, Cambridge, 1985).

[10] M. Kojima and L. Tunçel: Cones of matrices and successive convex relaxations of nonconvex sets. *SIAM Journal on Optimization*, **10** (2000) 750–778.

[11] L. Lovász: On the Shannon capacity of a graph. *IEEE Transactions on Information Theory*, **25** (1979) 1–7.

[12] L. Lovász and A. Schrijver: Cones of matrices and set functions and 0-1 optimization. *SIAM Journal on Optimization*, **1** (1991) 166–190.

Tetsuya Fujie
Department of Management Science
Kobe University of Commerce
8-2-1 Gakuen-nishimachi, Nishiku,
Kobe 651-2197, JAPAN
E-mail: fujie@kobeuc.ac.jp