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ON VARGAS'S PROOF OF CONSISTENCY TEST FOR 3 × 3 COMPARISON MATRICES IN AHP[1]

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Abstract In this paper we have raised, discussed, and solved the problems relating the Vargas's verification of consistency test for 3 × 3 comparison matrices in the Analytic Hierarchy Process (AHP). In the AHP community, assessing attribute weights by using a consistency ratio of 0.1 or less has been widely practiced. Though Vargas did study the problem, the method he adopted might lead to certain questionable results. By trying to clarify such a critical situation, we pointed out the problems arose form the solutions of two inequality systems within the proof process of Vargas. We proved that the solution for the first system was an interval, not a point as he had indicated. After we solved the first inequality system, we found that the criterion for the solution interval of the first system would satisfy the second inequality system. We have then justified the inherent problem in Vargas's verification procedure.

1. Introduction

Numerous books and papers concerning the theory and applications of the Analytic Hierarchy Process (AHP) have been written since Saaty [10] first presented this decision model. Due to the practical nature of the method, which is suitable for solving complicated and elusive decision problems, it has been widely applied in diverse areas. Many researchers have made their efforts to improve the theoretical foundations of AHP. Among the most interesting discussions is the verification of consistency of pairwise comparison matrices. Since uncertainties in our judgements and in many real-world cases are unlikely to avoid, many studies have been made toward this direction. Sajjad's study [13] has shown how such uncertainties could be incorporated within the framework of AHP and how the resulting uncertainties in the related priorities of the decision alternatives could be computed. Aupetit and Genest [1] mentioned that some useful properties of the Perron eigenvalue of a positive reciprocal matrix could be applied in the context of AHP. For the problem of retrieving weights from judgment matrices, the entries are the related important ratios of alternatives. Cook, Golany [6] and Barzilai [2] argued that the only solution satisfying consistency axioms would be the geometric mean. For the problem of retrieving weights from inconsistent additive judgement matrices, Barzilai and Golany [3] proved that the only solution satisfying consistency axioms would be the arithmetic mean. Some researches, such as Ra [9], Finan and Hurley [7], Monsuur [8], or Xu and Wei [22], only to mention some of them, raised and tried to modified AHP in eliciting and evaluating subjective judgments. The judgments include the constant-sum measurement scale (1–99 scale) for comparing two elements, the logarithmic least squares method for computing normalized values, the sum of inverse column, the sums for measuring the degree of inconsistency, and the sensitivity

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analysis of pairwise comparison matrices.

When making decisions on several alternatives of a matter, we consider good properties of acceptable solutions for the problem of deriving weights from pairwise comparisons should include the immunity to rank reversals, the independence of description of the problem, the independence of scale inversion, the left-right eigenvector asymmetry, the uniqueness, the independence of order of operations and inter-level consistency, the preservation of algebraic structure of the problem, the extensibility to the additive case, the related optimization models and the related error measures (Barzilai [2]).

However, several researchers, like Watson and Freeling [20] and [21], Belton and Gear [4] and [5], Schoner and Wedley [15], Schoner Wedley and Choo [16] and others, have criticized AHP in various respects. Among them, the ambiguity of the questions to be answered by decision-makers, the scale used to measure the intensity of preference, the principle of synthesizing priorities, rank reversal, etc. These criticisms make AHP theory not only more mature, but more applicable as well. In 1994, besides the reiterated fundamentals of AHP, Saaty [12] also raised topics regarding the scales of measurement, judgments, consistency and the eigenvector. In this paper, we have focused the discussion on the consistency test for assessing attribute weights.

Zahedi [23] pointed out that, after Saaty's developing of the eigenvalue method for estimating local relative weights, several techniques for estimating the attribute weights have been suggested. The techniques can be roughly grouped into two categories:

(1) A deterministic approach: The method has an underlying assumption that the preferences would be known with certainty and the only source of error would be inconsistency. In this paradigm, the error exists only in the elicitation of the preferences. In other words, inconsistency is due to measurement errors and can be easily reduced.

(2) A statistical approach: The method is inherently probabilistic. In other words, the preferences probably contain errors. Therefore, no consistency checks existed to avoid the happening of errors. Under the circumstances, developing a measure of inconsistency is meaningless. The major effort in this paradigm will be in identifying what statistical model could be the best explanation of the variability in the input data.

Some considered that the upper bound of consistency index should be verified by using an actual sample concerning subjective issues, based on the size of consistency index (CI) with regard to the size of pairwise comparison matrix and on rank reversal problem. However, we have concentrated ourselves on the problems mentioned in the first category instead. If there is an evaluator existed to imply the actual relative weights of n judgements, the matrix of the pairwise comparisons would be

$$A = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ \vdots & \vdots & \vdots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix} \tag{1.1}$$

In this case, the relative weights could be trivially obtained from each of the n rows of the matrix A. In other words, matrix A has rank 1; and the following holds:

$$AW = nW \tag{1.2}$$

where $W = (w_1, w_2, \ldots, w_n)^T$ is the vector of actual relative weights, and n is the number of elements. In linear algebra, n and W in (1.2) are called the eigenvalue and right eigenvector of matrix A respectively.

AHP posits that the evaluator does not know W and, therefore, is not able to produce the pairwise relative weights of A accurately. Thus, the observed matrix A contains inconsistencies. The estimation of W (denoted as $\widetilde{W}$) could be obtained, similar to (1.2),

$$\widetilde{A}\widetilde{W} = \lambda_{\max}\widetilde{W}. \tag{1.3}$$

If $\widetilde{A}$ is the observed matrix of pairwise comparisons, $\lambda_{\max}$ is the largest eigenvalue of $\widetilde{A}$, and $\widetilde{W}$ is its right eigenvector. $\widetilde{W}$ contains the estimation of W.

In (1.3), $\lambda_{\max}$ may be considered as the estimation of n in (1.2). Saaty [10] has shown that $\lambda_{\max}$ is always greater than or equal to n. The closer the value of computed $\lambda_{\max}$ is to n, the more consistent are the observed values of $\widetilde{A}$. This property has led to the construction of the consistency index μ as

$$\mu = \frac{\lambda_{\max} - n}{n-1}, \tag{1.4}$$

and the consistency ratio θ as

$$\theta = \frac{\mu}{X_{\overline{\mu}}} \tag{1.5}$$

that $X_{\overline{\mu}}$ is the distribution of the average index of randomly generated weights for $n \times n$ matrices (Saaty [11]). As a rule of thumb, a consistency ratio of 0.1 or less is considered acceptable. Otherwise, it is recommended that $\widetilde{A}$ should be reconstructed to resolve inconsistencies in pairwise comparisons. Salo and Hämäläinen [14] have demonstrated that the election of the threshold level for the consistency ratio (such as 0.1) is contingent on the choice of the 1-to-9 scales, even though many other scales have also been proposed. It casts doubts on the utilization of the 0.1 levels, and suggests that a more robust, scale-independent consistency measures would be preferred.

Vargas [19] tried to verify the consistency ratio θ that should be no more than 0.1. He made the verification with the help of 3×3 matrices. The threshold is widely accepted by many practitioners of AHP method. Since he is able to validate that the consistency ratio valued at 0.1 is reasonable, his achievement is beyond doubt and his verification procedure deserves more careful examination.

In order to prove that the consistency ratio θ of a comparison matrix should be no more than 0.1, we intend to show that, under certain circumstances, the threshold can be, for example, 0.12. Therefore, when assessing attribute weights, the ratios under 0.1 are consistent, and the ratios between 0.1 and 0.12 rarely meet the needs. Hence, those matrices with the ratios between 0.1 and 0.12 become negligible.

A sound proof of the consistency test should derive from an interval between 0 and 0.12. We then have a good reason to take the threshold value as 0.1. The comparison matrices with consistency ratio less than 0.1 can be accepted as consistent, and not too many comparison matrices with consistency ratio between 0.1 and 0.12 are rejected.

Based on the supposition above, we found out that there are problems contained in Vargas's verification of the consistency test. The problems are listed as follows:

(1) The solution of an inequality system in the proof is an interval instead of a point as he indicated, and

(2) His procedure of testing the reliability contains a structural problem.

We will discuss the main points of the out-finding in the following sections: In the next section, we will describe the origin of the problem. In section 3, we will deal with mathematical formulation, in which a series of lemmas and main theorem are proven. In the concluding section, we will summarize the contributions of this paper.

2. Discussion on Related Work

AHP provides a useful analytic criterion for the measurement of the consistency test for a comparison matrix. The reliability of a reciprocal matrix is measured by taking the ratio of the consistency index of the matrix, μ, to the average random consistency $\overline{\mu}(n)$. If the ratio is less than or equal to a threshold θ, we will accept the results as being sufficiently consistent. Let us now review the procedure of the Vargas's and examine how he proved that the threshold value θ should be 0.1:

Firstly, Vargas rewrote a 3×3 reciprocal matrix X and, then decomposed it as follows:

$$X = \begin{bmatrix} 1 & x_{12} & x_{13} \\ x_{21} & 1 & x_{23} \\ x_{31} & x_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{w_1}{w_2}a & \frac{w_1}{w_3}\frac{1}{a} \\ \frac{w_2}{w_1}\frac{1}{a} & 1 & \frac{w_2}{w_3}a \\ \frac{w_3}{w_1}a & \frac{w_3}{w_2}\frac{1}{a} & 1 \end{bmatrix} \tag{2.1}$$

if $a = \left(\frac{x_{12}x_{23}}{x_{13}}\right)^{\frac{1}{3}}$, $w_2 = a\frac{w_1}{x_{12}}$, $w_3 = \frac{1}{a}\frac{w_1}{w_{13}}$ and w_1 is any positive number. When $a = 1$, it shows that $x_{12}x_{23} = x_{13}$ and X is a consistent matrix.

Vargas multiplied the first column of X by x_{13} and the second column by x_{23}, then he obtained the matrix

$$X^* = \begin{bmatrix} x_{13} & \frac{x_{23}}{x_{21}} & x_{13} \\ \frac{x_{13}}{x_{12}} & x_{23} & x_{23} \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{w_1}{w_3}\frac{1}{a} & \frac{w_1}{w_2}a^2 & \frac{w_1}{w_3}\frac{1}{a} \\ \frac{w_2}{w_3}\frac{1}{a^2} & \frac{w_2}{w_3}a & \frac{w_2}{w_3}a \\ 1 & 1 & 1 \end{bmatrix}. \tag{2.2}$$

Vargas found that the geometric mean of the rows of X^* is the vector

$$\left(\frac{w_1}{w_3}, \frac{w_2}{w_3}, 1\right)^T. \tag{2.3}$$

He also found out the arithmetic mean of the rows of X^* is the vector

$$\left(\frac{1}{3}\frac{w_1}{w_3}\left(\frac{2+a^3}{a}\right), \frac{1}{3}\frac{w_2}{w_3}\left(\frac{1+2a^3}{a^2}\right), 1\right)^T. \tag{2.4}$$

Vargas compared (2.3) and (2.4), then he wanted to solve the following inequalities

$$\left|\frac{1}{3}\left(\frac{2+a^3}{a}\right) - 1\right| < \theta \text{ and } \left|\frac{1}{3}\left(\frac{1+2a^3}{a^2}\right) - 1\right| < \theta \tag{2.5}$$

as the principal eigenvalue of the reciprocal 3×3 matrix, $\lambda_{\max}$, satisfied $\lambda_{\max} = 1 + a + \frac{1}{a}$. (Note that Tummala and Wan [8] has a close-form solution for the largest eigenvalue of a 3×3-matrix.) Moreover, the consistency index, μ defined as $\mu = \frac{\lambda_{\max} - 3}{2}$, and an average consistency index, $X_{\overline{\mu}}$, as a truncated normal distribution $N\left(\overline{\mu(3)}, \frac{\sigma(3)}{\sqrt{500}}\right)$. Using the result of Vargas [19], it indicated that $\overline{\mu(3)} = 0.5381$ and $\frac{\sigma(3)}{\sqrt{500}} = 0.0433$. If we construct a

$(1-\alpha)\%$ confidence interval, then the acceptance threshold equals to 0.4532, with $\alpha = 5\%$. Vargas examined the reason why θ should be 10%. For a given reciprocal matrix, one could accept the hypothesis that the entries of the given reciprocal matrix were reliable if $\frac{\mu}{X_{\overline{\mu}}} \leq \theta$. Through solving the inequalities of (2.5), Vargas proclaimed $a \approx 1 + \frac{1}{3}$, so $\lambda_{\max} = 1 + a + \frac{1}{a} = 3.0833$ and $\mu = \frac{\lambda_{\max} - 3}{2} = 0.0417$. Thus, he obtained

$$\frac{\mu}{X_{\overline{\mu}}} = \frac{0.0417}{0.4532} \approx 0.0919 < \theta \; (= 0.1 \text{ by Vargas}). \tag{2.6}$$

Vargas implicitly predicted that taking $\theta = 0.1$ was valid. His procedure was the only existing analytic method to discuss the verification of the consistency ratio. Hence, this particular method deserves a more careful examination. In the following we will

(1) Show that there are omissions in the solution of Vargas for the inequalities of (2.5);
(2) Prove the claim of Vargas for $\theta = 0.1$ through analytical technique;
(3) Solve the inequalities of (2.5) through analytical method for abstract θ;
(4) Offer a criterion such as the solution interval of (2.5) will satisfy (2.6);
(5) Indicate that the structural weakness within the procedure of Vargas for testing the reliability of consistency ratio.

3. Mathematical Formulation

Recall (2.5), let $f(t) = \frac{1}{3}\left(\frac{2+t^3}{t}\right) - 1$ for $t > 0$ and $g(s) = \frac{1}{3}\left(\frac{1+2s^3}{s^2}\right) - 1$ for $s > 0$. Given $\theta = 0.1$, then we need to solve $|f(t)| < 0.1$ for $t > 0$ and $|g(s)| < 0.1$ for $s > 0$.[2]

Here, we give the outline of our proof procedure. We need to display the solution for the system of two inequalities (2.5), and test the solution for the inequality (2.6). Since the system of (2.5) contains two inequalities, we will offer the individual solutions, which are intervals, and find their intersection. Because certain relations among boundary points of the solution intervals exist, we express the solution of (2.5) in a compact form. For the second inequality (2.6), we discover the criterion for determining whether or not inequality (2.6) holds. Hence, the structural weakness in the paper of Vargas becomes obvious.

Firstly, using the first and second derivatives of the respective functions and $f(1) = g(1) = 0$, we can prove that the solution for the inequality system is an interval, but not a point.

Lemma 1.
(a) The solution of $|f(t)| < 0.1$ is an interval, say (t_1, t_2), with $0 < t_1 < 1 < t_2$.
(b) The solution of $|g(s)| < 0.1$ is an interval, say (s_1, s_2) with $0 < s_1 < 1 < s_2$.

From Lemma 1, we conclude that the solution for those two inequalities is the intersection of two intervals, $(t_1, t_2) \cap (s_1, s_2)$, not a point of value 0.0919 as asserted by Vargas [19]. For further investigation, we are going to abstract the value of θ from the specified value 0.1 to any positive value θ

Lemma 2. Given a positive value θ, the solution of $|f(t)| < \theta$ is an interval, say $(t_1(\theta), t_2(\theta))$, with $f(t_1(\theta)) = \theta = f(t_2(\theta))$ and $0 < t_1(\theta) < 1 < t_2(\theta)$. Moreover, the solution of $|g(s)| < \theta$ is also an interval, say $(s_1(\theta), s_2(\theta))$, with $f(s_1(\theta)) = \theta = f(s_2(\theta))$ and $0 < s_1(\theta) < 1 < s_2(\theta)$.

[2]Since $f(t) = \left(\frac{2+t}{3t}\right)(t-1)^2$ and $g(s) = \left(\frac{2s+1}{3}\right)\left(\frac{s-1}{s}\right)^2$, both $f(t)$ and $g(s)$ are positive and it is, in fact, unnecessary to consider their absolute values.

Now, we will show some relations among t_1, t_2, s_1 and s_2.

Lemma 3. Given a positive value θ, then $s_1(\theta)t_2(\theta) = s_2(\theta)t_1(\theta) = 1$, $t_1(\theta)t_2(\theta) < 1$, $s_1(\theta)s_2(\theta) > 1$, $s_1(\theta) > t_1(\theta)$ and $t_2(\theta) < s_2(\theta)$.

Proof: We reconsider the equation $f(t) = \theta$, as $t^3 - 3(1+\theta)t + 2 = 0$. Let $F(t,\theta) = t^3 - 3(1+\theta)t + 2$, for $\theta > 0$ and $-\infty < t < \infty$. We know that $F(t,\theta)$ is a cubic polynomial in t and evaluate $\lim_{t\to\infty} F(t,\theta) = \infty$, $F(1,\theta) = -3\theta < 0$, $F(0,\theta) = 2 > 0$, and $\lim_{t\to-\infty} F(t,\theta) = -\infty$. By the Intermediate Value Theorem [17], without loss of generality, we suppose that $F(t,\theta) = 0$ has three solutions: $t_0(\theta)$, $t_1(\theta)$ and $t_2(\theta)$, such that $t_0(\theta) < 0 < t_1(\theta) < 1 < t_2(\theta)$.

Here, we will show $t_0(\theta) < -2$ and $t_1(\theta)t_2(\theta) < 1$. Since $F(-2,\theta) = 6\theta > 0$, we obtain $t_0(\theta) < -2$.

Using $F(t,\theta) = (t - t_0(\theta))(t - t_1(\theta))(t - t_2(\theta))$ and comparing the constant terms on both sides, we induce $2 = -t_0(\theta)t_1(\theta)t_2(\theta)$, from $t_0(\theta) < -2$. It follows that

$$t_1(\theta)t_2(\theta) < 1. \tag{3.1}$$

In the same fashion mentioned previously, from $g(s) = \theta$, we suppose that for $-\infty < s < \infty$ and $\theta > 0$, $G(s,\theta) = 2s^3 - 3(1+\theta)s^2 + 1$, there are three roots for $G(s,\theta)$ that are $s_0(\theta)$, $s_1(\theta)$ and $s_2(\theta)$ with $s_0(\theta) < 0 < s_1(\theta) < 1 < s_2(\theta)$.

In the remaining proof of Lemma 3, θ is a fixed positive number. To simplify the notation, we use t_1, t_2, s_1 and s_2 to denote $t_1(\theta), t_2(\theta), s_1(\theta)$ and $s_2(\theta)$.

Using $f(t_1) = \theta$ and $\theta = g(s_2)$, we have $\frac{1}{3}\left(\frac{2+t_1^3}{t_1}\right) - 1 = \frac{1}{3}\left(\frac{1+2s_2^3}{s_2^2}\right) - 1$. It follows

$$\frac{2}{t_1} + t_1^2 = \frac{1}{s_2^2} + 2s_2. \tag{3.2}$$

Consequently, from (3.2) we obtain $2\left(\frac{1-t_1s_2}{t_1}\right) = \left(\frac{1-t_1s_2}{s_2}\right)\left(\frac{1}{s_2} + t_1\right)$, then

$$(1 - t_1s_2)\left[\frac{2}{t_1} - \frac{1}{s_2}\left(\frac{1}{s_2} + t_1\right)\right] = 0. \tag{3.3}$$

From $0 < t_1 < 1 < s_2$, we ascertain

$$\frac{2}{t_1} > 2 > \frac{2}{s_2} > \frac{1}{s_2}\left(\frac{1}{s_2} + t_1\right). \tag{3.4}$$

From (3.3) and (3.4), we conclude $t_1s_2 = 1$. Similarly, we can deduce $s_1t_2 = 1$. Therefore,

$$t_1s_2 = 1 = s_1t_2. \tag{3.5}$$

Putting (3.1) and (3.5) together, we obtain $t_1 < \frac{1}{t_2} = s_1$. We have shown

$$t_1 < s_1. \tag{3.6}$$

Using the same method, we obtain

$$t_2 < s_2. \tag{3.7}$$

Here, we have derived the desired properties among the boundary points for the simultaneous system of inequalities. **Q.E.D.**

Since $(t_1(\theta), t_2(\theta)) \cap (s_1(\theta), s_2(\theta)) = (\max\{t_1(\theta), s_1(\theta)\}, \min\{t_2(\theta), s_2(\theta)\}) = (s_1(\theta), t_2(\theta))$, we have the next lemma.

Lemma 4. The solution for the simultaneous system of inequalities $|f(t)| < \theta$ and $|g(s)| < \theta$ is $(s_1(\theta), t_2(\theta))$.

We are now in the place to go back to the special case of $\theta = 0.1$. After we learned the solutions for the inequalities, say $(s_1(0.1), t_1(0.1))$, we could answer the following question: *If* $x \in (s_1(0.1), t_2(0.1))$ and $\mu(x) = \dfrac{\lambda_{\max}(x) - 3}{2}$, *with* $\lambda_{\max}(x) = 1 + x + \dfrac{1}{x}$, *then is* $\dfrac{\mu(x)}{0.4532}$ *less than or equal to 0.1?*

Lemma 5. Each $x \in (s_1(0.1), t_2(0.1))$ satisfies $\dfrac{\mu(x)}{0.4532} \le 0.1$ if and only if $\dfrac{\mu(t_2(0.1))}{0.4532} \le 0.1$. Moreover, we obtain $\dfrac{\mu(t_2(0.1))}{0.4532} \le 0.1$.

Proof: To simplify the notation, let $c = 0.4532$, $s_1 = s_1(0.1)$, and $t_2 = t_2(0.1)$. From $\lambda_{\max}(x)$ is concave-up with an absolute minimum point at $x = 1$, then

$$\frac{\mu(x)}{c} \le 0.1, \text{ for } x \in (s_1, t_2) \Leftrightarrow \frac{\mu(s_1)}{c} \le 0.1 \text{ and } \frac{\mu(t_2)}{c} \le 0.1. \tag{3.8}$$

By (3.5), we know $s_1 t_2 = 1$, then $\mu(s_1) = \mu(t_2)$. Hence, we can reduce (3.8) as

$$\frac{\mu(x)}{c} \le 0.1, \text{ for } x \in (s_1, t_2) \Leftrightarrow \frac{\mu(t_2)}{c} \le 0.1. \tag{3.9}$$

Recall $f(t_2) = 0.1$, then

$$(t_2 - 1)^2 = \frac{3t_2}{10(2 + t_2)}. \tag{3.10}$$

Using (3.10), we compute

$$\frac{\mu(t_2)}{c} \le 0.1 \Leftrightarrow 1 + t_2 + \frac{1}{t_2} - 3 \le \frac{2c}{10} \Leftrightarrow (t_2 - 1)^2 \le \frac{2c}{10} t_2 \Leftrightarrow \frac{3}{2c} - 2 \le t_2. \tag{3.11}$$

With the help of a computer program, we get

$$f(1.34) = -3.954 \times 10^{-3} \text{ and } f(1.35) = 1.327 \times 10^{-3}.$$

We know $1.34 < t_2 < 1.35$, with $\dfrac{3}{2c} - 2 = 1.3098$. Hence, using (3.11), we prove $\dfrac{\mu(t_2)}{c} \le 0.1$. **Q.E.D.**

Combining with Lemma 4 and Lemma 5, we conclude that using the solutions for the two inequalities under the restriction of $\theta = 0.1$, the ratio between the consistency index and the average consistency index will be less than 0.1. In other words, we have proven the claim of Vargas for $\theta = 0.1$ through analytical technique.

Finally, we reconsider the structure of verification by Vargas. We will point out the structural weakness within the procedure of Vargas for testing the reliability. If we replace 0.1 by a given θ, we can show that

Lemma 6. $t_2(\theta)$ is a strictly increasing function from 1 to ∞ and there exists a unique point, $\theta_{\#}$, with $\dfrac{3}{2c} - 2 = t_2(\theta_{\#})$, where $c = 0.4532$.

Proof: Recall $F(t_2(\theta),\theta)=0$, we have

$$[t_2(\theta)]^3 - 3(1+\theta)t_2(\theta) + 2 = 0 \tag{3.12}$$

with $t_2(\theta) > 1$. Using the implicit differentiation, we obtain that

$$([t_2(\theta)]^2 - (1+\theta))\frac{d}{d\theta}t_2(\theta) = t_2(\theta).$$

Since $\theta > 0$, $t_2(\theta) > 1$ and (3.12), we know

$$[t_2(\theta)]^3 - (1+\theta)t_2(\theta) = 2(1+\theta)t_2(\theta) - 2 > 0, \tag{3.13}$$

by (3.13), so

$$t_2(\theta) > \sqrt{1+\theta} \tag{3.14}$$

From (3.14), we know $\frac{d}{d\theta}t_2(\theta) = \frac{t_2(\theta)}{[t_2(\theta)]^2-(1+\theta)} > 0$. Using (3.12), $\lim_{\theta\to+0} t_2(\theta)$ is bounded (for example, $\lim_{\theta\to+0} t_2(\theta) < 3$). We take the limit of $\theta \to +0$ on (3.12), then

$$\left(\lim_{\theta\to+0}[t_2(\theta)]\right)^3 - 3\lim_{\theta\to+0} t_2(\theta) + 2 = \left(\lim_{\theta\to+0} t_2(\theta) - 1\right)^2 \left(\lim_{\theta\to+0} t_2(\theta) + 2\right) = 0. \tag{3.15}$$

Therefore, we compute $\lim_{\theta\to+0} t_2(\theta) = 1$ based on (3.15). Again, by (3.14), we obtain $\lim_{\theta\to+\infty} t_2(\theta) = \infty$. Hence, for $\theta \in (0,\infty)$, $t_2(\theta)$ is a strictly increasing function from 1 to ∞. So, there is a unique point, say $\theta_{\#}$, with $\frac{3}{2c} - 2 = t_2(\theta_{\#})$. **Q.E.D.**

We extend Lemma 5 from $\theta = 0.1$ to arbitrary positive number to derive our final result, Theorem 1.

Theorem 1. Every $x \in (s_1(\theta), t_2(\theta))$ satisfies $\frac{\mu(x)}{c} \le \theta$ if and only if $\frac{3}{2c} - 2 \le t_2(\theta)$, where $c = 0.4532$.

Proof: By the same method used to derive (3.8), (3.9), (3.10) and (3.11), we get

$$\frac{\mu(x)}{c} \le \theta, \text{ for } x \in (s_1(\theta), t_2(\theta)) \Leftrightarrow \frac{3}{2c} - 2 \le t_2(\theta).$$ **Q.E.D.**

By Lemma 6, if $0 < \theta < \theta_{\#}$, then $\frac{3}{2c} - 2 > t_2(\theta)$, and if $\theta > \theta_{\#}$, then $\frac{3}{2c} - 2 < t_2(\theta)$. That means that using Theorem 1, $\theta_{\#}$ is the greatest lower bound for those θ such that the solutions of (3.12) will satisfy (2.5). With the help of a mathematical program, we can estimate $\theta_{\#} \approx 0.08084$.

Therefore, even if we change θ from 0.1 to any positive number, θ, with $\theta > \theta_{\#}$, the procedure of Vargas still verify that θ is a proper choice for testing the consistency of the reciprocal matrix. We prove that the criterion of Theorem 1 will generate an interval such as $(\theta_{\#}, \infty)$ not as it was expected to be (0, portal-value) with 0.1 < portal-value and 0.1 is very close to the portal-value. So, the verification structure of Vargas to prove $\theta = 0.1$ indeed has some inherent faults.

4. Conclusion

Since assessing attribute weights is often practiced in AHP community, the proof of the assertion that the consistency ratio should be no more than 0.1 is of great importance. In this paper, we have pointed out the problems existing in the verification of Vargas. Because the solution to the inequality system (2.5) constitutes the essential part of the proof, we have shown that the solution is an interval, not a point as Vargas indicated. We have also pointed out an anomalous phenomenon in his analytic structure: based on his proof structure, we can extend the threshold θ from 0.1 to any positive value greater than 0.1 that leads to a serious problem.

One of the slight imperfections in AHP studies is that quite a few statistical verifications of the upper bound of consistency index have been done since Vargas's paper was published 20 years ago. Therefore, it is meaningful to apply AHP to experiments concerning subjective issues and verify the upper bound of consistency index in our future study.

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