Journal of the Operations Research
Society of Japan
Vol. 45, No. 2, June 2002

MATHEMATICAL PROPERTIES OF DOMINANT AHP AND CONCURRENT CONVERGENCE METHOD

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(Received April 23, 2001; Final December 10, 2001)

Abstract This study discusses the mathematical structure of the dominant AHP and the concurrent convergence method which were originally developed by Kinoshita and Nakanishi. They introduced a new concept of a regulating alternative into an analyzing tool for a simple evaluation problem with a criterion set and an alternative set. Although the original idea of the dominant AHP and the concurrent convergence method is unique, the dominant AHP and the concurrent convergence method are not sufficiently analyzed in mathematical theory. This study shows that the dominant AHP consists of a pair of evaluation rules satisfying a certain property of overall evaluation vectors. This study also shows that the convergence of concurrent convergence method is guaranteed theoretically.

1. Introduction

AHP is a flexible decision making system that can deal with the subjective judgments of a decision maker. Numerously successful applications have been reported in this field [6]. In AHP, the decision maker identifies an ambiguous evaluation problem into a hierarchy structure within the evaluation goal, criteria and alternatives, each of which corresponds to a node of the hierarchy. The hierarchy with a top, middle and bottom structure usually consists of three levels, the goal, the criteria, and the alternatives, respectively. This study also discusses the three-level hierarchy. Directed arcs of the hierarchy form a parents-child relationship among the nodes and the existence of a pair of parents-child nodes means that the decision maker judges the relative importance of the child-nodes from the parents-node. That is, for example, directed arcs from the top level to the middle level indicate the decision maker's judgment on the relative importance of all criteria from the goal. Saaty [6] proposes that in this three-level hierarchy the decision maker firstly judges the relative importance of the criteria from the goal and secondarily judges that of the alternative from the criteria. Judgments of the relative importance are expressed numerically, which are called evaluation values. Let I and J be a set of alternatives and that of criteria, respectively, and denote their cardinalities by $|I|$ and $|J|$, respectively. Then we have a total of $|J| \times (|I| + 1)$ evaluation values in the three-level hierarchy. By plotting a set of evaluation values on the arcs of hierarchy, the hierarchy becomes a tree of a network with the directed arcs. In the original AHP, the evaluation value of a child-node from a parents-node is quantified under the assumption that the decision maker compares all pairs between distinct two children of the parents.

Kinoshita and Nakanishi [2] focus on the following empirical result: When the decision maker evaluates relative importance of the criteria from the goal, he/she focuses on a specific alternative and refers to relative importance of the criteria from the specific alternative. The specific alternative is called the regulating alternative. Kinoshita and Nakanishi [2] assume that if there exists exactly one regulating alternative then the relative importance of the criteria from the regulating alternative determines that from other alternatives. If there exists only one regulating alternative in the alternative set, then the regulating alternative is called the dominant one and they implement the assumption into the dominant AHP. The mathematical description of the dominant AHP is as follows:

Step0 : The decision maker selects a regulating alternative from the alternative set I. Let alternative k be the regulating alternative.

Step1 : From the viewpoint of every criterion $j \in J$, the decision maker evaluates the relative importance of all alternatives and quantifies the evaluation values of all alternatives. Let a_{ij} be the evaluation value of the alternative i from the criterion j and let A be an $|I| \times |J|$ evaluation matrix whose (i,j) element is a_{ij}.

Step2 : From the viewpoint of the regulating alternative k, the decision maker evaluates the relative importance of all criteria and quantifies the evaluation values of all criteria by such as the eigenvalue method for a pairwise comparison matrix of the criteria. Let $\boldsymbol{b}^k$ be a $|J|$-dimensional vector whose jth element is the evaluation value of the criterion j from the regulating alternative k.

Step 3: Let A_k be a $|J| \times |J|$ diagonal matrix whose (j,j) element is a_{kj}. Calculate $AA_k^{-1}\boldsymbol{b}^k$ and define the ith element of $AA_k^{-1}\boldsymbol{b}^k$ as the overall evaluation value of alternative i.

Suppose that the alternative k is the dominant one. Let $\hat{\boldsymbol{b}}^i$ be a $|J|$-dimensional vector whose jth element is the unknown evaluation value of the criterion j from the alternative $i \neq k$, and let A_i be a diagonal matrix whose (j,j) element is a_{ij}. Then, Kinoshita and Nakanishi [2] propose a following evaluation rule under their assumption:

$$\hat{\boldsymbol{b}}^i = \frac{A_i A_k^{-1}\boldsymbol{b}^k}{\boldsymbol{e}^\top A_i A_k^{-1}\boldsymbol{b}^k} \tag{1.1}$$

for all $i \in I \setminus \{k\}$, where $\boldsymbol{e}$ is all one vector and $^\top$ stands for the transpose operation. They define

$$AA_i^{-1}\hat{\boldsymbol{b}}^i \tag{1.2}$$

as the overall evaluation vector derived from the alternative i and they point out that $AA_i^{-1}\hat{\boldsymbol{b}}^i$ coincides (except for a scalar multiple) with $AA_k^{-1}\boldsymbol{b}^k$ for all $i \in I \setminus \{k\}$. Therefore, they assert that the overall evaluation vector $AA_k^{-1}\boldsymbol{b}^k$ is valid.

In order to deal with an additional data to A, they relax their assumption and extend single regulating alternative to multiple ones. Let K be an index set of regulating alternatives, then $\boldsymbol{b}^k$ of the regulating alternative $k \in K$ can be given by Step 2 and $|K|$ types of A, say $\left\{A^{(k)} | k \in K\right\}$, can be given by Step 1. They assume that the relative importance of criteria from every alternative is an aggregately relative importance of criteria from all regulating alternatives. Under the assumption they develop a two-stage procedure as follows: First, merge $\left\{A^{(k)} \,|\, k \in K\right\}$ into a positive matrix A by an appropriate method [2]. Next, apply the evaluation rule (1.1) to estimating $\hat{\boldsymbol{b}}^i$ from relative importance $\boldsymbol{b}^k$ of each regulating alternative $k \in K$. Hence, calculate

$$A_i A_k^{-1}\boldsymbol{b}^k \tag{1.3}$$

for all $i \in I$ and all $k \in K$ and generate $\hat{\boldsymbol{b}}^i$ from $\left\{A_i A_k^{-1}\boldsymbol{b}^k \,|\, k \in K\right\}$ by an iterative procedure [2], for every $i \in I$.

The two-stage procedure is called the concurrent convergence method in [2]. However, convergence of the iterative procedure in the second stage has not been guaranteed theoretically and it is still an open problem [14]. Kinoshita and Nakanishi [2] observe in a numerical example that the concurrent convergence method provides a limit point set $\left\{\hat{\boldsymbol{b}}^i \mid i \in I\right\}$ and that $AA_1^{-1}\hat{\boldsymbol{b}}^1$ coincides (except for a scalar multiple) with $AA_i^{-1}\hat{\boldsymbol{b}}^i$ for all $i \in I$. The latter observation arises in both the dominant AHP and the concurrent convergence method. It is called the consistency property.

The first aim of this study is to show that some pairs of evaluation rules satisfy the consistency property other than the pair of (1.1) and (1.2) in the dominant AHP. The second aim is to prove the convergence of the iterative procedure in the concurrent convergence method. This study contributes the mathematical foundations and generalizations of the dominant AHP and the concurrent convergence method.

This paper has five sections. Section 2 discusses the mathematical structure of the dominant AHP, especially the pair of evaluation rules (1.1) and (1.2). Section 3 shows the convergence and the consistency property of the concurrent convergence method. Section 4 provides a numerical example illustrating these properties of the concurrent convergence method. The final section is a brief conclusion. We outline some future extensions as well.

2. Mathematical Structure of the Dominant AHP Model

In this section, we discuss mathematical properties of the overall evaluation vector $AA_k^{-1}\boldsymbol{b}^k$ and alternative i's evaluation vector $\hat{\boldsymbol{b}}^i$ of the criteria that is estimated by regulating alternative k. (Note that A_i is well defined for all $i \in I$ since A is a positive matrix.) We only focus on the directions of the overall evaluation vectors, $AA_k^{-1}\boldsymbol{b}^k$ and $AA_i^{-1}\hat{\boldsymbol{b}}^i$, and the evaluation vectors of criteria, $\boldsymbol{b}^k$ and $\hat{\boldsymbol{b}}^i$. The overall evaluation vector is to indicate the relative importance of alternatives and its length has no information concerning alternatives. So, if a vector $\boldsymbol{a}$ coincides except for a scalar multiple with a vector $\boldsymbol{b}$, we say that $\boldsymbol{a}$ has the same direction as $\boldsymbol{b}$. In order to express the mathematical properties of the dominant AHP, we introduce two linear transformations, $B_i^k(\cdot)$ and $V_i(\cdot)$ as follows: For a $|J|$-dimensional vector $\boldsymbol{b}$, we define $V_i(\boldsymbol{b}) = AA_i^{-1}\boldsymbol{b}$ and $B_i^k(\boldsymbol{b}) = A_iA_k^{-1}\boldsymbol{b}$ for all $i, k \in I$. Then, the overall evaluation vector by the evaluation rule (1.2) is given by the function value $V_i(\hat{\boldsymbol{b}}^i)$.

Lemma 1 *Suppose that $\hat{\boldsymbol{b}}^i$ is defined by (1.1) for all $i \in I \setminus \{k\}$, then*

$$\hat{\boldsymbol{b}}^i = \frac{B_i^k(\boldsymbol{b}^k)}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k}. \tag{2.1}$$

That is, $\hat{\boldsymbol{b}}^i$ has the same direction as $B_i^k(\boldsymbol{b}^k)$ for all $i \in I \setminus \{k\}$.

Proof:

$$\hat{\boldsymbol{b}}^i = \frac{A_iA_k^{-1}\boldsymbol{b}^k}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k} = \frac{B_i^k(\boldsymbol{b}^k)}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k}$$

for all $i \in I \setminus \{k\}$. ■

Then we summarize the consistency property of the dominant AHP as follows:

Theorem 2 *Let $\boldsymbol{b}$ be a $|J|$-dimensional vector, then $V_k(\boldsymbol{b}) = V_i(B_i^k(\boldsymbol{b}))$ for all $i, k \in I$. Suppose that $\hat{\boldsymbol{b}}^i$ is defined by (1.1) for all $i \in I \setminus \{k\}$, then $V_k(\boldsymbol{b}^k)$ has the same direction as $V_i(\hat{\boldsymbol{b}}^i)$.*

Proof: For every $|J|$-dimensional vector $\boldsymbol{b}$,

$$V_i(B_i^k(\boldsymbol{b})) = AA_i^{-1}\left(A_iA_k^{-1}\boldsymbol{b}\right) = AA_k^{-1}\boldsymbol{b} = V_k(\boldsymbol{b}) \tag{2.2}$$

for all $i, k \in I$. It follows from Lemma 1 and (2.2) that

$$V_i(\hat{\boldsymbol{b}}^i) = V_i\left(\frac{B_i^k(\boldsymbol{b}^k)}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k}\right) = \frac{AA_i^{-1}B_i^k(\boldsymbol{b}^k)}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k} = \frac{V_i(B_i^k(\boldsymbol{b}^k))}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k} = \frac{V_k(\boldsymbol{b}^k)}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k}$$

for all $i \in I \setminus \{k\}$. ■

From Theorem 2, Kinoshita and Nakanishi [2] mention that the pair of evaluation rules (1.1) and (1.2) provides a consistent overall evaluation vector among all alternatives.

Under the assumption that alternative i has the evaluation vector $\hat{\boldsymbol{b}}^i$ of criteria, we apply (1.1) to estimating alternative k's evaluation vector of criteria from $\hat{\boldsymbol{b}}^i$ and then obtain $\boldsymbol{b}^k$. Hence, $B_i^k(\cdot)$ can be considered as an inverse function of $B_k^i(\cdot)$ in the sense as follows:

Theorem 3 *Let $\boldsymbol{b}$ be a $|J|$-dimensional vector, then $B_i^k(B_k^i(\boldsymbol{b})) = \boldsymbol{b}$ for all $i, k \in I$. Suppose that $\hat{\boldsymbol{b}}^i$ is defined by (1.1) for all $i \in I \setminus \{k\}$, then $B_k^i(\hat{\boldsymbol{b}}^i)$ has the same direction as $\boldsymbol{b}^k$.*

Proof: It follows from definitions of $B_k^i(\cdot)$ and $B_i^k(\cdot)$ that

$$B_i^k\left(B_k^i(\boldsymbol{b})\right) = B_i^k\left(A_kA_i^{-1}\boldsymbol{b}\right) = A_iA_k^{-1}A_kA_i^{-1}\boldsymbol{b} = \boldsymbol{b}. \tag{2.3}$$

Since

$$\hat{\boldsymbol{b}}^i = \frac{A_iA_k^{-1}\boldsymbol{b}^k}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k} = B_i^k\left(\frac{\boldsymbol{b}^k}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k}\right),$$

it follows from (2.3) that

$$B_k^i\left(\hat{\boldsymbol{b}}^i\right) = \frac{\boldsymbol{b}^k}{\boldsymbol{e}^\top A_iA_k^{-1}\boldsymbol{b}^k}.$$

for all $i \in I \setminus \{k\}$. ■

In the context of the dominant AHP, we now consider normalization of A and inner dependence among criteria, which are discussed in [7] and [9], respectively. Suppose that $\boldsymbol{b}$ is an evaluation vector of criteria from alternative 1 and that A is an evaluation matrix of alternatives from criteria then we might assume that $A_1^{-1}\boldsymbol{b}$ is an evaluation vector of criteria from the goal. Let N be a $|J| \times |J|$ diagonal matrix whose (j,j) element is $1/\sum_{i\in I} a_{ij}$, then AN is a column-normalized evaluation matrix of alternatives and $ANA_1^{-1}\boldsymbol{b}$ is an overall evaluation vector of alternatives for the normalized evaluation matrix AN. (For an evaluation vector $\boldsymbol{b}$ which is independent of a specific alternative's view point, Saaty [7] shows that $AN\boldsymbol{b}$ is an overall evaluation vector of alternatives). Suppose that the inner dependence relation among criteria is represented by a $|J| \times |J|$ matrix M and that $A_1^{-1}\boldsymbol{b}$ is an evaluation vector of criteria from the goal, then Saaty and Takizawa [9] shows that $MA_1^{-1}\boldsymbol{b}$ is an overall evaluation vector of criteria having the inner dependence. Therefore, $AMA_1^{-1}\boldsymbol{b}$ is an overall evaluation vector of alternatives and $A_iMA_1^{-1}\boldsymbol{b}$ could be an estimated evaluation vector of criteria from the viewpoint of alternative i. Though $AMA_1^{-1}\boldsymbol{b}$ is interpreted as $A(MA_1^{-1}\boldsymbol{b})$ according to [9], we could regard $AMA_1^{-1}\boldsymbol{b}$ as $(AM)A_1^{-1}\boldsymbol{b}$. Here, AM is called an adjusted evaluation matrix by the inner dependence. For the adjusted evaluation matrix AM we have $(AM)A_i^{-1}A_iA_1^{-1}\boldsymbol{b} = AMA_1^{-1}\boldsymbol{b}$. Therefore, $A_iA_1^{-1}\boldsymbol{b}$ might be an estimated evaluation vector of criteria from the viewpoint of alternative i. (For the diagonal matrix N, Saaty [7] gives the same interpretation $AN\boldsymbol{b} = (AN)\boldsymbol{b} = A(N\boldsymbol{b})$ as the above.)

For two $|J| \times |J|$ matrices M and N, we propose a new pair of linear transformations $B_i^k(\cdot)$ and $V_i(\cdot)$ as follows:

$$B_i^k(\boldsymbol{b}) = A_i M A_k^{-1}\boldsymbol{b} \tag{2.4}$$

and

$$V_i(\boldsymbol{b}) = ANA_i^{-1}\boldsymbol{b} \tag{2.5}$$

for all $i, k \in I$, then we have the following similar results to Theorems 2 and 3.

Theorem 4 *Consider that $B_i^k(\cdot)$ and $V_i(\cdot)$ are defined by (2.4) and (2.5),respectively. Suppose that there exists a nonzero scalar λ such that $NM = \lambda N$. Let $\boldsymbol{b}$ be a $|J|$-dimensional vector, then $V_k(\boldsymbol{b})$ has the same direction as $V_i(B_i^k(\boldsymbol{b}))$ for all $i, k \in I$. Moreover, if M^2 is a multiplication of the unit matrix, then $B_i^k\,(B_k^i(\boldsymbol{b}))$ has the same direction as $\boldsymbol{b}$.*

Proof: Since $NM = \lambda N$, we have

$$V_i\left(B_i^k(\boldsymbol{b})\right) = ANA_i^{-1}B_i^k(\boldsymbol{b}) = ANA_i^{-1}A_iMA_k^{-1}\boldsymbol{b} = ANMA_k^{-1}\boldsymbol{b} = \lambda ANA_k^{-1}\boldsymbol{b} = \lambda V_k(\boldsymbol{b}).$$

Let E be the unit matrix, then there exists a scalar μ such that $M^2 = \mu E$. Therefore, we have

$$B_i^k\left(B_k^i(\boldsymbol{b})\right) = A_iMA_k^{-1}B_i^k(\boldsymbol{b}) = A_iMA_k^{-1}A_kMA_i^{-1}\boldsymbol{b} = A_iM^2A_i^{-1}\boldsymbol{b} = \mu\boldsymbol{b}.$$

■

Corollary 5 *Suppose that $B_i^k(\cdot)$, $V_i(\cdot)$, N, M and λ are the same as Theorem 4 and let $\hat{\boldsymbol{b}}^i = B_i^k(\boldsymbol{b}^k)$, then $V_i(\hat{\boldsymbol{b}}^i)$ has the same direction as $V_k(\boldsymbol{b}^k)$. Moreover, if M^2 is a multiplication of the unit matrix, then $B_k^i(\hat{\boldsymbol{b}}^i)$ has the same direction as $\boldsymbol{b}^k$.*

Proof: It is directly from Theorem 4. ■

Introducing a nonnegative parameter α into a matrix M of (2.4), we consider a $|J| \times |J|$ matrix as follows:

$$M(\alpha) = \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha \\ \alpha & \cdots & \alpha & 1 \end{bmatrix}.$$

Then $M(0)$ is the unit matrix. We consider a variation of the pair of (2.4) and (2.5) as follows:

$$B_i^k(\boldsymbol{b}) = A_iM(\alpha)A_k^{-1}\boldsymbol{b}, \tag{2.6}$$

and

$$V_i(\boldsymbol{b}) = AM(\alpha)A_i^{-1}\boldsymbol{b} \tag{2.7}$$

for all $i, k \in I$.

Corollary 6 *Let $B_i^k(\cdot)$ and $V_i(\cdot)$ be defined by (2.6) and (2.7), respectively. Let $\boldsymbol{b}$ be a $|J|$-dimensional vector and suppose that $\alpha = 0$ or 1, then $V_k(\boldsymbol{b})$ has the same direction as $V_i\left(B_i^k(\boldsymbol{b})\right)$ for all $i \in I \setminus \{k\}$.*

Proof: Since $M(0)^2$ is the unit matrix and $M(1)^2 = |J|M(1)$, it follows immediately from Theorem 4. ■

Suppose that $(1/|J|)M(1)$ is an evaluation matrix representing inner dependence among criteria and that $\boldsymbol{b}$ is an evaluation vector of criteria from alternative k. We assume that

$AM(1)$ is the adjusted evaluation matrix and $A_iA_1^{-1}\boldsymbol{b}$ is an estimated evaluation vector of criteria from the viewpoint of alternative i. By applying the inner dependence matrix $M(1)$ and the unit matrix E to N of (2.5) and M of (2.4), respectively, it follows from Theorem 4 that $V_k(\boldsymbol{b})$ has the same direction as $V_i\left(B_i^k(\boldsymbol{b})\right)$ for every $|J|$-dimensional vector $\boldsymbol{b}$ and all $i \in I \setminus \{k\}$. By decomposing the inner dependence matrix $(1/|J|)M(1)$ into $(1/|J|)M(1)$ of (2.7) and $(1/|J|)M(1)$ of (2.6), we have

$$\frac{1}{|J|}AM(1)A_k^{-1}\boldsymbol{b} = A\frac{1}{|J|}M(1)A_i^{-1}A_i\frac{1}{|J|}M(1)A_k^{-1}\boldsymbol{b} = V_i(B_i^k(\boldsymbol{b})),$$

where $V_k(\boldsymbol{b}) = (1/|J|)AM(1)A_k^{-1}\boldsymbol{b}$ and $B_i^k(\boldsymbol{b}) = (1/|J|)A_iM(1)A_k^{-1}\boldsymbol{b}$. This also implies from Corollary 6 that $V_k(\boldsymbol{b})$ has the same direction as $V_i\left(B_i^k(\boldsymbol{b})\right)$ for every $|J|$-dimensional vector $\boldsymbol{b}$ and all $i \in I \setminus \{k\}$.

The following theorem asserts that there exists an evaluation vector of criteria such that the overall evaluation vector by the pair of (2.6) and (2.7) is independent of α.

Theorem 7 *Suppose that a $|J|$-dimensional vector $\boldsymbol{b}$ satisfies $A_k^{-1}\boldsymbol{b} = \lambda\boldsymbol{e}$ for some nonzero scalar λ. Let $B_i^k(\cdot)$ and $V_i(\cdot)$ be defined by (2.6) and (2.7), respectively. For every positive α, then both $V_k(\boldsymbol{b})$ and $V_i(B_i^k(\boldsymbol{b}))$ have the same direction as $A\boldsymbol{e}$.*

Proof: It follows from $M(\alpha)\boldsymbol{e} = (1+\alpha(|J|-1))\boldsymbol{e}$ that

$$V_k(\boldsymbol{b}) = AM(\alpha)A_k^{-1}\boldsymbol{b} = \lambda AM(\alpha)\boldsymbol{e} = \lambda(1+\alpha(|J|-1))A\boldsymbol{e}.$$

In a similar way to the above, we have for every $i \in I$

$$\begin{aligned} V_i(B_i^k(\boldsymbol{b})) &= AM(\alpha)A_i^{-1}A_iM(\alpha)A_k^{-1}\boldsymbol{b} = AM(\alpha)^2A_k^{-1}\boldsymbol{b} \\ &= \lambda(1+\alpha(|J|-1))AM(\alpha)\boldsymbol{e} = \lambda(1+\alpha(|J|-1))^2A\boldsymbol{e}. \end{aligned}$$

■

Next we give an illustration to make the dominant AHP model more readable.

Example 1 *Schenkerman[10] gives an example for criticizing Saaty's AHP that the order of alternatives can be in reverse by normalizing the judgment value of an alternative from a criterion. In his example, he considers how to evaluate the circumferences of three rectangles of farmland by the lengths and the breaths of the rectangles. Here the alternatives are farmland. The set of alternatives is $I = \{1,2,3\}$. The criteria are the length and the breadth. The set of criteria is $J = \{1,2\}$. The lengths and the breadth (column) of the three rectangles (row) is listed as*

$$A = \begin{bmatrix} 100 & 800 \\ 200 & 100 \\ 200 & 600 \end{bmatrix}.$$

Generally, the length is as the same important as the breadth for measuring the circumference of a rectangle. Then the evaluation vector of the criteria from the goal is given in [10] by $\boldsymbol{b} = [0.5, 0.5]^\top$. Let alternative 1 be a regulating alternative. We obtain easily

$$A_1 = \begin{bmatrix} 100 & 0 \\ 0 & 800 \end{bmatrix}, \quad AM(\alpha)A_1^{-1} = \begin{bmatrix} 1+8\alpha & 1+\frac{1}{8}\alpha \\ 2+\alpha & \frac{1}{8}+\frac{1}{4}\alpha \\ 2+6\alpha & \frac{3}{4}+\frac{1}{4}\alpha \end{bmatrix}.$$

Consider two vectors $\boldsymbol{b}$ and $\boldsymbol{b}_1^1 = [1/9, 8/9]^\top$ as regulating alternative 1's evaluation vectors of two criteria. The overall evaluation vectors by the pair of (2.6) and (2.7) with several

Table 1: The overall evaluation vectors in the dominant AHPs

α	*Overall evaluation vectors by* $\boldsymbol{b}$	*Overall evaluation vectors by* $\boldsymbol{b}_1^1$
$\alpha = 0$	$[0.29, 0.31, 0.40]^\top$	$[0.45, 0.15, 0.40]^\top$
$\alpha = 1/2$	$[0.41, 0.19, 0.40]^\top$	$[0.45, 0.15, 0.40]^\top$
$\alpha = 1$	$[0.45, 0.15, 0.40]^\top$	$[0.45, 0.15, 0.40]^\top$
$\alpha = 10$	$[0.51, 0.09, 0.40]^\top$	$[0.45, 0.15, 0.40]^\top$
Saaty's AHP	$[0.37, 0.23, 0.40]_{\#}^\top$	$[0.50, 0.10, 0.40]_{\$}^\top$
The real rate	$[0.45, 0.15, 0.40]^\top$	

#: See [10]. \$: Replacing $\boldsymbol{b}$ *with* $\boldsymbol{b}_1^1$*, we can obtain \$ by the same way as that of #.*

αs are shown in Table 1. Note that $\boldsymbol{b}_1^1 = \lambda [a_{11}, a_{12}]^\top$ for some λ. Then for any $\alpha \geq 0$ the overall evaluation vector by $\boldsymbol{b}_1^1$ is equal to the percentages of the real circumferences of the three rectangles. Moreover, for $\alpha = 1$ the overall evaluation vector by any positive vector $\boldsymbol{b}$ yields the percentages of the real circumferences of the three rectangles.

3. Mathematical Structure of the Concurrent Convergence Method

In this section, we consider the case that there exist several regulating alternatives, that is the case of $|K| \geq 2$. The concurrent convergence method begins with merging $\left\{A^{(k)} \mid k \in K\right\}$ to generate a common evaluation matrix A for all alternatives. This is the first stage of the concurrent convergence method. Then, we go to the following initial step of the second stage:

Algorithm 0

Step 0: For a given set of the evaluation vectors of criteria, $\left\{\boldsymbol{b}^k \mid k \in K\right\}$, in the first stage, let

$$\boldsymbol{b}_0^k := \boldsymbol{b}^k \tag{3.1}$$

for all $k \in K$. Let $t := 0$ and go to **Step 1**.

Step 1: Let

$$\boldsymbol{b}_{t+1}^i := \frac{1}{|K|} \sum_{k \in K} \frac{A_i A_k^{-1} \boldsymbol{b}_t^k}{\boldsymbol{e}^\top A_i A_k^{-1} \boldsymbol{b}_t^k} \tag{3.2}$$

for all $i \in I$.

Step 2: If $\max_{i \in I} \|\boldsymbol{b}_{t+1}^i - \boldsymbol{b}_t^i\| = 0$ then set $\bar{\boldsymbol{b}}^i = \boldsymbol{b}_{t+1}^i$ for all $i \in I$ and stop. Otherwise, update $t := t + 1$ and go to **Step 1**.

Note that all $\boldsymbol{b}^k$ of Step 0 are normalized in the first stage, that is, $\boldsymbol{e}^\top \boldsymbol{b}^k = 1$ for all $k \in K$.

Kinoshita and Nakanishi [2] observe in some numerical experiments that Algorithm 0 has a limit point set $\left\{\bar{\boldsymbol{b}}^i \mid i \in I\right\}$ such that $AA_i^{-1}\bar{\boldsymbol{b}}^i$ has the same direction as $AA_l^{-1}\bar{\boldsymbol{b}}^l$ for all $i, l \in I$.

To prove the observation above, we consider the following Algorithms 1 which is simplified from Algorithm 0.

Algorithm 1

Step 0: For all $k \in K$, let

$$\boldsymbol{p}_0^k := A_k^{-1} \boldsymbol{b}^k. \tag{3.3}$$

Set $t := 0$ and go to **Step 1**.

Step 1: For all $i \in I$, let

$$\boldsymbol{p}_{t+1}^{i} := \frac{1}{|K|} \sum_{k \in K} \frac{\boldsymbol{p}_t^k}{\boldsymbol{e}^\top A_i \boldsymbol{p}_t^k}. \tag{3.4}$$

Step 2: If $\max_{k \in K} \left\| \boldsymbol{p}_{t+1}^k - \boldsymbol{p}_t^k \right\| = 0$, then let $\bar{\boldsymbol{p}}^k := \boldsymbol{p}_{t+1}^k$ for all $k \in K$ and stop.
Otherwise, set $t := t + 1$ and go to **Step 1**.

We can correspond $\left\{ \boldsymbol{p}_t^k \,|\, t = 0, 1, 2, \ldots \right\}$ of Algorithm 1 to $\left\{ \boldsymbol{b}_t^k \,|\, t = 0, 1, 2, \ldots \right\}$ of Algorithm 0 as follows:

Lemma 8 *The equation*

$$A_k^{-1} \boldsymbol{b}_t^k = \boldsymbol{p}_t^k \tag{3.5}$$

holds for all $k \in K$ and $t = 0, 1, \ldots$ and

$$\boldsymbol{b}_t^i = \frac{1}{|K|} \sum_{l \in K} \frac{A_i \boldsymbol{p}_t^l}{\boldsymbol{e}^\top A_i \boldsymbol{p}_t^l}$$

holds for all $i \in I$ and $t = 1, 2 \ldots$. If Algorithm 0 is convergent finitely , then so is Algorithm 1, and vice versa.

Proof: We will show (3.5) by induction of iteration t. At iteration $t = 0$, $A_k^{-1} \boldsymbol{b}_0^k = \boldsymbol{p}_0^k$ follows directly from (3.1) and (3.3) for all $k \in K$. We assume that at iteration s $A_k^{-1} \boldsymbol{b}_s^k = \boldsymbol{p}_s^k$ holds for all $k \in K$. Then it follows from (3.2) and (3.4) that

$$A_k^{-1} \boldsymbol{b}_{s+1}^k = A_k^{-1} \frac{1}{|K|} \sum_{l \in K} \frac{A_k A_l^{-1} \boldsymbol{b}_s^l}{\boldsymbol{e}^\top A_k A_l^{-1} \boldsymbol{b}_s^l} = \frac{1}{|K|} \sum_{l \in K} \frac{\boldsymbol{p}_s^l}{\boldsymbol{e}^\top A_k \boldsymbol{p}_s^l} = \boldsymbol{p}_{s+1}^k$$

for all $k \in K$. Therefore, we complete (3.5) for all $k \in K$ and $t = 0, 1, \ldots$. This means from (3.2) that

$$\boldsymbol{b}_t^i = \frac{1}{|K|} \sum_{l \in K} \frac{A_i \boldsymbol{p}_t^l}{\boldsymbol{e}^\top A_i \boldsymbol{p}_t^l}$$

for all $i \in I$ and $t = 1, 2, \ldots$.

When Algorithm 0 stops at iteration t, we have $\boldsymbol{b}_{t+1}^i = \boldsymbol{b}_t^i$ for all $i \in I$ and hence, $A_k^{-1} \boldsymbol{b}_{t+1}^k = A_k^{-1} \boldsymbol{b}_t^k$ for all $k \in K$. Therefore, Algorithm 1 stops at iteration t. On the contrary, if Algorithm 1 stops at iteration t, we have $A_k^{-1} \boldsymbol{b}_{t+1}^k = A_k^{-1} \boldsymbol{b}_t^k$. This implies that $\boldsymbol{b}_{t+1}^k = \boldsymbol{b}_t^k$. ∎

From Lemma 8 we discuss the convergence of Algorithm 1 instead of Algorithm 0. Furthermore, Lemma 8 means that it is sufficient to calculate (3.4) for all $k \in K \subseteq I$. Therefore, we consider Algorithm 1 replacing I with K in the sequel.

Lemma 9 *The vector $\boldsymbol{p}_t^k$ is a positive vector for every $k \in K$ and $t = 0, 1, \ldots$.*

Proof: Note that $\boldsymbol{b}_0^k$ is a positive vector for all $k \in K$. Since every diagonal element of the diagonal matrix A_k is positive, it follows from Lemma 8 that $\boldsymbol{p}_0^k$ is also a positive vector for all $k \in K$. Assume that $\boldsymbol{p}_t^k$ is a positive vector for all $k \in K$, then it follows from (3.4) that $\boldsymbol{p}_{t+1}^k$ is positive for all $k \in K$. ∎

Lemma 10 *For every $k \in K$ and $t = 0, 1, \ldots$, the following equation holds*

$$\boldsymbol{e}^\top A_k \boldsymbol{p}_t^k = 1. \tag{3.6}$$

Proof: From (3.3) it is easy to see that $e^\top A_k p_0^k = e^\top b^k = 1$. For $t = 0, 1, \ldots$, we have from (3.4) that

$$e^\top A_k p_{t+1}^k = \frac{1}{|K|} \sum_{l \in K} \frac{e^\top A_k p_t^l}{e^\top A_k p_t^l} = 1.$$

It is the assertion. ■

Lemmas 9 and 10 imply that $\left\{p_t^k \mid t = 0, 1, \ldots\right\}$ is a bounded set in the positive orthant for all $k \in K$.

Lemma 11 *For all $k \in K$ and $t = 0, 1, \ldots$,*

$$\frac{p_{t+1}^k}{e^\top A_k p_{t+1}^k} = p_{t+1}^k.$$

Proof: The assertion is directly from Lemma 10. ■

Consider the following convex cone

$$\text{Cone}\left(\left\{p_{t+1}^k | k \in K\right\}\right) = \left\{ p \;\middle|\; p = \sum_{k \in K} \mu_k p_{t+1}^k \text{ and } \mu_k \geq 0 \text{ for all } k \in K \right\}, \tag{3.7}$$

which is generated by the vectors $\left\{p_{t+1}^k \mid k \in K\right\}$. For a set D we denote the relative interior and relative boundary of D by ri$D$ and bd$D$, respectively.

Lemma 12 *Let R be an extreme ray set of* Cone$\left(\left\{p_t^k | k \in K\right\}\right)$. *If* $\dim R = 1$, *then Algorithm 1 stops. Otherwise, for every $k \in K$ and $t = 0, 1, \ldots$,*

$$p_{t+1}^k \in \text{riCone}\left(\left\{p_t^k | k \in K\right\}\right) \tag{3.8}$$

$$p_{t+1}^k \notin R. \tag{3.9}$$

Proof: Note that Cone(R) = Cone$\left(\left\{p_t^k | k \in K\right\}\right)$.

Firstly we consider the case of $\dim R = 1$. Since $p_t^k \in \text{Cone}(R)$ for all $k \in K$, there exist positive numbers μ_{lk} for all $k, l \in K$ such that $p_t^l = \mu_{lk} p_t^k$. This means from (3.4) and Lemma 11 that

$$p_{t+1}^k = \frac{1}{|K|} \sum_{l \in K} \frac{p_t^l}{e^\top A_k p_t^l} = \frac{1}{|K|} \sum_{l \in K} \frac{\mu_{lk} p_t^k}{\mu_{lk} e^\top A_k p_t^k} = \frac{1}{|K|} \sum_{l \in K} \frac{p_t^k}{e^\top A_k p_t^k} = \frac{p_t^k}{e^\top A_k p_t^k} = p_t^k$$

for all $k \in K$. This satisfies the stopping condition of Step 2.

Next, we consider $\dim R \geq 2$. Then, it follows that riCone$(R) \subset$ Cone(R) and that riCone$(R) \cap R = \emptyset$. Note that if $\nu_k > 0$ for all $k \in K$ then $(\sum_{k \in K} \nu_k p_t^k) \in \text{riCone}(R)$. It follows from Lemma 9 that $e^\top A_k p_t^l > 0$ for all $k, l \in K$. This means from (3.4) that

$$p_{t+1}^k \in \text{riCone}(R)$$

for all $k \in K$. This means that $p_{t+1}^k \notin R$ for all $k \in K$. ■

Lemma 13

$$\text{Cone}\left(\left\{p_{t+1}^k \mid k \in K\right\}\right) \subset \text{Cone}\left(\left\{p_t^k \mid k \in K\right\}\right) \text{ and}$$
$$\text{Cone}\left(\left\{p_{t+1}^k \mid k \in K\right\}\right) \cap \text{bdCone}\left(\left\{p_t^k \mid k \in K\right\}\right) = \{ \text{ the origin } \}$$

for $t = 1, 2, \ldots$.

Proof: It is directly from Lemma 12. ■

Lemma 13 means that $\mathrm{Cone}\left(\left\{\boldsymbol{p}_t^k \mid k \in K\right\}\right)$ shrinks monotonically for $t = 0, 1, \ldots$. We assume that Algorithm 1 generates an infinite sequence of points $\left\{\boldsymbol{p}_t^k \mid k \in K\right\}$. Let $S^k = \left\{A_k^{-1}\boldsymbol{b} \mid \boldsymbol{b} \geq 0 \text{ and } \boldsymbol{e}^\top \boldsymbol{b} = 1\right\}$ for all $k \in K$, then S^k is compact and the product set $\prod S^k$ is also compact. This implies the following lemma.

Lemma 14 *For all $k \in K$ and $t = 0, 1, \ldots,$*

$$\boldsymbol{p}_t^k \in S^k.$$

Moreover, there exist an index set $T \subseteq \{1, 2, \ldots\}$ and an accumulation point $\hat{\boldsymbol{p}}_k$ for all $k \in K$ such that

$$\lim_{t \in T, t \to \infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k.$$

Proof: It follows from Lemma 10 that $\boldsymbol{p}_t^k \in S^k$ for all $k \in K$ and $t = 0, 1, \ldots$. Since the set S^k is compact for all $k \in K$, the product set $\prod S^k$ is also compact. Therefore, $\lim_{t \in T, t \to \infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k$ for all $k \in K$. ■

Lemma 15 *Suppose that an index set $T \subseteq \{1, 2, \ldots\}$ and $|K|$ points $\left\{\hat{\boldsymbol{p}}^k \mid k \in K\right\}$ satisfy $\lim_{t \in T, t \to \infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k$ for all $k \in K$, then*

$$\hat{\boldsymbol{p}}^k \in S^k \tag{3.10}$$

for all $k \in K$ and

$$\mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \mid k \in K\right\}\right) \setminus \mathrm{Cone}\left(\left\{\boldsymbol{p}_t^k \mid k \in K\right\}\right) = \emptyset \tag{3.11}$$

for $t = 0, 1, \ldots$.

Proof: For all $k \in K$ and every $t = 0, 1, 2, \ldots$, $\boldsymbol{p}_t^k$ is included in the compact set S^k. Therefore, $\hat{\boldsymbol{p}}^k \in S^k$ for all $k \in K$.

Note that $\mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \mid k \in K\right\}\right) \subset \mathrm{Cone}\left(\left\{\boldsymbol{p}_s^k \mid k \in K\right\}\right)$ for every $s \in T$, that is,

$$\mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \mid k \in K\right\}\right) \setminus \mathrm{Cone}\left(\left\{\boldsymbol{p}_s^k \mid k \in K\right\}\right) = \emptyset.$$

For every $t = 0, 1, \ldots$, there exists an index $s \in T$ such that $s \geq t$. It follows from Lemma 13 that $\mathrm{Cone}\left(\left\{\boldsymbol{p}_s^k \mid k \in K\right\}\right) \subset \mathrm{Cone}\left(\left\{\boldsymbol{p}_t^k \mid k \in K\right\}\right)$. Therefore, we have

$$\mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \mid k \in K\right\}\right) \setminus \mathrm{Cone}\left(\left\{\boldsymbol{p}_t^k \mid k \in K\right\}\right) = \emptyset$$

for every $t = 0, 1, \ldots$. ■

Lemma 16 *Suppose that an index set $T \subseteq \{1, 2, \ldots\}$ and $|K|$ points $\left\{\hat{\boldsymbol{p}}^k \mid k \in K\right\}$ satisfy*

$$\lim_{t \in T, t \to \infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k \tag{3.12}$$

for all $k \in K$, then $\mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \mid k \in K\right\}\right)$ is a half-line.

Proof: It follows from lemmas 15 and 13 that $\hat{\boldsymbol{p}}^k \in S^k \cap \text{Cone}\left(\left\{\boldsymbol{p}_1^l \,|\, l \in K\right\}\right)$. Since $S^k \cap \text{Cone}\left(\left\{\boldsymbol{p}_1^l \,|\, l \in K\right\}\right)$ is included in the positive orthant, so is $\hat{\boldsymbol{p}}^k$ and hence,

$$\boldsymbol{e}^\top A_l \hat{\boldsymbol{p}}^k > 0 \tag{3.13}$$

for all $l, k \in K$. We can well define

$$\epsilon_{lk} = \min\left\{\boldsymbol{e}^\top A_l \boldsymbol{p} \,|\, \boldsymbol{p} \in \text{Cone}\left\{\boldsymbol{p}_1^i \,|\, i \in K\right\} \cap S^k\right\}$$

and $\epsilon_{lk} > 0$ for all $l, k \in K$. Notice Lemma 13 asserting that $\boldsymbol{p}_t^k \in \left(\text{Cone}\{\boldsymbol{p}_t^i \,|\, i \in K\} \cap S^k\right) \subseteq \left(\text{Cone}\{\boldsymbol{p}_1^i \,|\, i \in K\} \cap S^k\right)$ for all $k \in K$ and every $t = 1, 2, \ldots$, and hence $\boldsymbol{e}^\top A_l \boldsymbol{p}_t^k \geq \epsilon_{lk}$ for all $l, k \in K$ and every $t = 1, 2, \ldots$. Let $\epsilon = \min\{\epsilon_{lk} \,|\, l, k \in K\}$. We have $\boldsymbol{e}^\top A_l \boldsymbol{p}_t^k \geq \epsilon$ for all $l, k \in K$ and every $t = 1, 2, \ldots$. It follows from (3.4) and (3.13) that for every $l \in K$

$$\lim_{t \in T, t \to \infty} \boldsymbol{p}_{t+1}^l = \frac{1}{|K|} \lim_{t \in T, t \to \infty} \sum_{k \in K} \frac{\boldsymbol{p}_t^k}{\boldsymbol{e}^\top A_l \boldsymbol{p}_t^k} = \frac{1}{|K|} \sum_{k \in K} \frac{\hat{\boldsymbol{p}}^k}{\boldsymbol{e}^\top A_l \hat{\boldsymbol{p}}^k}. \tag{3.14}$$

Let $\tilde{\boldsymbol{p}}^l = \lim_{t \in T, t \to \infty} \boldsymbol{p}_{t+1}^l$, then it implies from (3.14) that

$$\tilde{\boldsymbol{p}}^l \in \text{riCone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right) \tag{3.15}$$

for all $l \in K$. This means that $\text{Cone}\left(\left\{\tilde{\boldsymbol{p}}^l \,|\, l \in K\right\}\right) \subseteq \text{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right)$.

Now we are going to show $\text{Cone}\left(\left\{\tilde{\boldsymbol{p}}^l \,|\, l \in K\right\}\right) = \text{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right)$. Suppose that $\text{Cone}\left(\left\{\tilde{\boldsymbol{p}}^l \,|\, l \in K\right\}\right) \subset \text{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right)$ and let

$$B_\epsilon(\boldsymbol{x}) = \{\, \boldsymbol{y} \,|\, \|\boldsymbol{x} - \boldsymbol{y}\| \leq \epsilon \,\} \text{ and } G(\epsilon) = \text{Cone}\left(\cup_{l \in K} B_\epsilon(\tilde{\boldsymbol{p}}^l)\right).$$

Because $\text{Cone}\left(\left\{\tilde{\boldsymbol{p}}^l \,|\, l \in K\right\}\right)$ is a closed set, there exists a positive scalar $\bar{\epsilon}$ such that

$$\text{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right) \setminus G(\bar{\epsilon}) \neq \emptyset. \tag{3.16}$$

It follows from (3.15) that there exists an index $t_1 \in T$ such that $\text{Cone}\left(\left\{\boldsymbol{p}_{t_1+1}^k \,|\, k \in K\right\}\right) \subseteq G(\bar{\epsilon})$. Therefore, suppose that $t_2 \geq t_1 + 1$ and that $t_2 \in T$, then it follows from Lemma 13 that $\text{Cone}\left(\left\{\boldsymbol{p}_{t_2}^k \,|\, k \in K\right\}\right) \subset \text{Cone}\left(\left\{\boldsymbol{p}_{t_1+1}^k \,|\, k \in K\right\}\right) \subseteq G(\bar{\epsilon})$. It follows from (3.16) that $\text{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right) \setminus \text{Cone}\left(\left\{\boldsymbol{p}_{t_2}^k \,|\, k \in K\right\}\right) \neq \emptyset$. This is a contradiction for (3.11) of Lemma 15. Therefore, $\text{Cone}\left(\left\{\tilde{\boldsymbol{p}}^l \,|\, l \in K\right\}\right) = \text{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right)$ and hence, it follows from (3.15) that $\text{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right)$ is a half-line. ■

The following lemma guarantees the existence of a limit point of the infinite sequence $\{\boldsymbol{p}_t^k \,|\, t = 0, 1, \ldots\}$ for all $k \in K$.

Lemma 17 *If Algorithm 1 repeats infinitely, there exist a half-line* $\boldsymbol{H}$ *and a limit point* $\hat{\boldsymbol{p}}^k$ *of* $\left\{\boldsymbol{p}_t^k \,|\, t = 0, 1, \ldots\right\}$ *for all* $k \in K$ *such that*

$$\hat{\boldsymbol{p}}^k \in \boldsymbol{H}. \tag{3.17}$$

Hence, $\hat{\boldsymbol{p}}^k$ *has the same direction as* $\hat{\boldsymbol{p}}^l$ *for all* $l \in K$.

Proof: Suppose that an index set $T \subseteq \{1,2,\dots\}$ and $|K|$ points $\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}$ satisfy $\lim_{t\in T, t\to\infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k$ for all $k \in K$, then it follows from lemmas 14 and 16 that there exists a half-line $\boldsymbol{H}$ such that $\hat{\boldsymbol{p}}^k \in H$ for all $k \in K$. Let $C(\epsilon) = \cap_{k\in K}\text{Cone}\left(\left\{\boldsymbol{x} \in S^k \,\middle|\, \|\boldsymbol{x} - \hat{\boldsymbol{p}}^k\| \le \epsilon\right\}\right)$, then for every positive scalar ϵ we have $C(\epsilon) \cap S^k \setminus \boldsymbol{H} \ne \emptyset$ and

$$C(\epsilon) \cap S^k \subseteq \left(\text{Cone}\left(\left\{\boldsymbol{x} \in S^k \,\middle|\, \|\boldsymbol{x} - \hat{\boldsymbol{p}}^k\| \le \epsilon\right\}\right) \cap S^k\right) = \left\{\boldsymbol{x} \in S^k \,\middle|\, \|\boldsymbol{x} - \hat{\boldsymbol{p}}^k\| \le \epsilon\right\} \quad (3.18)$$

for all $k \in K$.

Choose a positive scalar ϵ arbitrarily. It follows from Lemma 15 that for all $k \in K$ there exists $t_\epsilon^k \in T$ such that $\boldsymbol{p}_t^k \in C(\epsilon) \cap S^k$ for every $t \in \{s \in T \,|\, s \ge t_\epsilon^k\}$. Let $\hat{t}_\epsilon = \max\{t_\epsilon^k \,|\, k \in K\}$, then we have $\text{Cone}\left(\left\{\boldsymbol{p}_t^l \,|\, l \in K\right\}\right) \cap S^k \subseteq C(\epsilon) \cap S^k$ for every $t \in \{s \in T \,|\, s \ge \hat{t}_\epsilon\}$ and all $k \in K$. This implies from Lemma 13 that $\text{Cone}\left(\left\{\boldsymbol{p}_t^k \,|\, k \in K\right\}\right) \subseteq C(\epsilon)$ for every $t \ge \hat{t}_\epsilon$. Hence, it follows from (3.18) that $\boldsymbol{p}_t^k \in \text{Cone}\left(\left\{\boldsymbol{p}_t^l \,|\, l \in K\right\}\right) \cap S^k \subseteq C(\epsilon) \cap S^k \subseteq \left\{\boldsymbol{x} \in S^k \,\middle|\, \|\boldsymbol{x} - \hat{\boldsymbol{p}}^k\| \le \epsilon\right\}$ for every $t \ge \hat{t}_\epsilon$ and all $k \in K$. ■

From the viewpoint of set convergence [5], Lemma 17 implies that $\text{Cone}\left(\left\{\boldsymbol{p}_t^k \,|\, k \in K\right\}\right)$ converges on a half line of the positive orthant. When Algorithm 1 converges within finitely many iterations, the point set $\{\bar{\boldsymbol{p}}^k \,|\, k \in K\}$ of Step 2 has the same property as stated in Lemma 17.

Lemma 18 *Suppose that Algorithm 1 stops within finitely many iterations and let $\bar{\boldsymbol{p}}^k$ be defined by Step 2 of Algorithm 1 for all $k \in K$, then $\bar{\boldsymbol{p}}^k$ has the same direction as $\bar{\boldsymbol{p}}^l$ for all $k, l \in K$.*

Proof: Suppose that Algorithm 1 stops at iteration t, then we have

$$\boldsymbol{p}_{t+1}^k = \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k \quad (3.19)$$

for all $k \in K$. It follows from (3.4), Lemma 11 and (3.19) that

$$\begin{aligned}\hat{\boldsymbol{p}}^k &= \boldsymbol{p}_{t+1}^k = \frac{1}{|K|}\sum_{l\in K}\frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^\top A_k \boldsymbol{p}_t^l} = \frac{1}{|K|}\sum_{l\ne k, l\in K}\frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^\top A_k \boldsymbol{p}_t^l} + \frac{1}{|K|}\frac{\boldsymbol{p}_t^k}{\boldsymbol{e}^\top A_k \boldsymbol{p}_t^k} \\ &= \frac{1}{|K|}\sum_{l\ne k, l\in K}\frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^\top A_k \boldsymbol{p}_t^l} + \frac{1}{|K|}\boldsymbol{p}_t^k = \frac{1}{|K|}\sum_{l\ne k, l\in K}\frac{\hat{\boldsymbol{p}}^l}{\boldsymbol{e}^\top A_k \hat{\boldsymbol{p}}^l} + \frac{1}{|K|}\hat{\boldsymbol{p}}^k\end{aligned}$$

for all $k \in K$. Therefore, we have

$$(|K|-1)\hat{\boldsymbol{p}}^k = \sum_{l\ne k, l\in K}\frac{\hat{\boldsymbol{p}}^l}{\boldsymbol{e}^\top A_k \hat{\boldsymbol{p}}^l}$$

which means that $\hat{\boldsymbol{p}}^k \in \text{Cone}\left(\left\{\hat{\boldsymbol{p}}^l \,|\, l \in K, l \ne k\right\}\right)$. This implies that $\dim \text{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \,|\, k \in K\right\}\right) = 1$. Hence, $\hat{\boldsymbol{p}}^k$ has the same direction as $\hat{\boldsymbol{p}}^l$ for all $k, l \in K$. ■

By the above lemmas we can summarize the mathematical properties of the concurrent convergence method as follows:

Theorem 19 *The concurrent convergence method has a limit point set $\left\{\bar{\boldsymbol{b}}^i \,|\, i \in I\right\}$. Let $AA_i^{-1}\bar{\boldsymbol{b}}^i$ be the overall evaluation vector of alternative i, then the overall evaluation vector of alternative i has the same direction as that of alternative l for all $i, l \in I$.*

Proof: The assertion follows directly from Lemma 8, 17 and 18. ■

4. A Numerical Example

In this section, we give a numerical illustration for the concurrent convergence method. Consider an evaluation matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1/2 & 2 \\ 2/3 & 3/4 \\ 1 & 2 \end{bmatrix}$$

and suppose that $K = \{1,2,4\}$ and evaluation value vectors $\boldsymbol{b}_1 = [1/5, 4/5]^\top$, $\boldsymbol{b}_2 = [3/5, 2/5]^\top$ and $\boldsymbol{b}_4 = [1/3, 2/3]^\top$. Then,

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 2/3 & 0 \\ 0 & 3/4 \end{bmatrix}, \; A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

One can calculate $\boldsymbol{p}_0^k$ at Step 0 of Algorithm 1 and obtain

$$\boldsymbol{p}_0^1 = [1/10, 4/5]^\top, \; \boldsymbol{p}_0^2 = [6/5, 1/5]^\top, \; \boldsymbol{p}_0^4 = [1/3, 1/3]^\top$$

and

$$\boldsymbol{e}^\top A_1 = [2,1], \; \boldsymbol{e}^\top A_1 \boldsymbol{p}_0^1 = 1/5 + 4/5 = 1, \; \boldsymbol{e}^\top A_1 \boldsymbol{p}_0^2 = 12/5 + 1/5 = 13/5, \; \boldsymbol{e}^\top A_1 \boldsymbol{p}_0^4 = 2/3 + 1/3 = 1.$$

Therefore,

$$\boldsymbol{p}_1^1 = \frac{1}{|K|}\left(\frac{\boldsymbol{p}_0^1}{\boldsymbol{e}^\top A_1 \boldsymbol{p}_0^1} + \frac{\boldsymbol{p}_0^2}{\boldsymbol{e}^\top A_1 \boldsymbol{p}_0^2} + \frac{\boldsymbol{p}_0^4}{\boldsymbol{e}^\top A_1 \boldsymbol{p}_0^4}\right) = \begin{bmatrix} 1/30 \\ 4/15 \end{bmatrix} + 5/13 \begin{bmatrix} 6/15 \\ 1/15 \end{bmatrix} + \begin{bmatrix} 1/9 \\ 1/9 \end{bmatrix} = \begin{bmatrix} 0.298291 \\ 0.403419 \end{bmatrix}.$$

Similarly, one obtains

$$\boldsymbol{p}_1^2 = \begin{bmatrix} 0.553535 \\ 0.361616 \end{bmatrix}, \; \boldsymbol{p}_1^4 = \begin{bmatrix} 0.380719 \\ 0.309641 \end{bmatrix}.$$

Following the steps of the algorithm, one has

$$\boldsymbol{p}_2^1 = \begin{bmatrix} 0.343545 \\ 0.312909 \end{bmatrix}, \; \boldsymbol{p}_2^2 = \begin{bmatrix} 0.445264 \\ 0.388684 \end{bmatrix}, \; \boldsymbol{p}_2^4 = \begin{bmatrix} 0.361393 \\ 0.319304 \end{bmatrix}$$

for $t = 2$. And for $t = 3$, One can obtain

$$\boldsymbol{p}_3^1 = \begin{bmatrix} 0.346139 \\ 0.307721 \end{bmatrix}, \; \boldsymbol{p}_3^2 = \begin{bmatrix} 0.439030 \\ 0.390243 \end{bmatrix}, \; \boldsymbol{p}_3^4 = \begin{bmatrix} 0.359993 \\ 0.320003 \end{bmatrix}.$$

At iteration 4, i.e., $t = 4$, Algorithm 1 yields that

$$\boldsymbol{p}_4^1 = \begin{bmatrix} 0.346149 \\ 0.307703 \end{bmatrix}, \; \boldsymbol{p}_4^2 = \begin{bmatrix} 0.439008 \\ 0.390248 \end{bmatrix}, \; \boldsymbol{p}_4^4 = \begin{bmatrix} 0.359989 \\ 0.320006 \end{bmatrix}.$$

Here,

$$\boldsymbol{p}_4^1 \approx 0.788480 \boldsymbol{p}_4^2, \quad \boldsymbol{p}_4^1 \approx 0.961554 \boldsymbol{p}_4^4, \quad \boldsymbol{p}_4^2 \approx 1.219504 \boldsymbol{p}_4^4.$$

Overlooking a tolerance 10^{-6}, we consider it is a concrete example of Theorem 19. Moreover, Figure 1 portrays Lemma 10, i.e., for a given k, $\boldsymbol{e}^\top A_k \boldsymbol{p}_t^k = 1$ holds for each t, and Lemma 13, i.e., $\mathrm{Cone}\left(\left\{\boldsymbol{p}_{t+1}^k \mid k \in K\right\}\right) \subset \mathrm{Cone}\left(\left\{\boldsymbol{p}_t^k \mid k \in K\right\}\right)$.

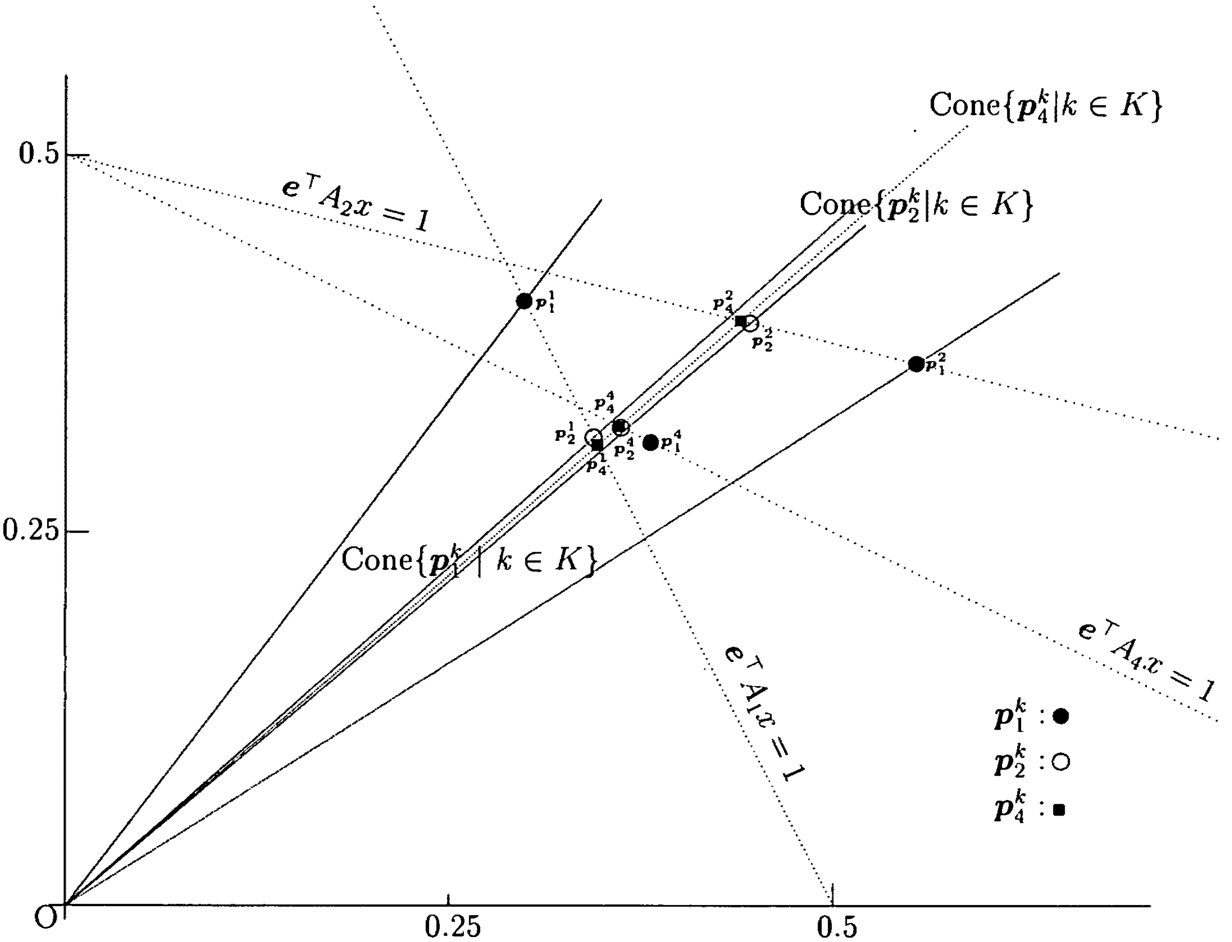

Figure 1: An illustration for the concurrent convergence method

5. Conclusion

This paper develops the mathematical foundations of the dominant AHP and a mechanism for the convergence of the concurrent convergence method. Hence, we show the mathematical description that the dominant AHP consists of a pair of simple evaluation rules (1.1) and (1.2) and that the pair of the rules provides the consistency property between regulating alternative's overall evaluation vector and other alternative's ones. Furthermore we discuss an extension of the evaluation rules (1.1) and (1.2) without violating the property. As stated in Example 1 in Section 2, one can apply the proposed evaluation rules to sensitive analysis for the overall evaluation vector.

This paper shows the convergence of the concurrent convergence method whose p_{t+1}^i is fixed as the non-weighted average of $\{p_t^l/(e^{\top}A_i p_t^l) \mid l \in K\}$ in Step 1. By the same way as the proofs from Lemma 9 to Theorem 19 in Section 3, we can guarantee the convergence of a variant concurrent convergence method whose p_{t+1}^i is given by a weighted average of $\{p_t^l/(e^{\top}A_i p_t^l) \mid l \in K\}$. Exploiting the convergence, we can extend the dominant AHP into an analyzing tool for an evaluation problem with a complex network structure [3, 12], interval AHP [1] and group AHP [4, 16]. We outline each of them briefly as follows:

An evaluation matrix of a complex network structure includes an evaluation sub-matrix whose element is hard to be quantified uniquely by the decision maker. (Sekitani [11] illustrates a mutual evaluation system which has multiple evaluation values of the criterion from the alternative.) This sub-matrix is called an unstable evaluation matrix in [2, 3, 15]. For example, $[b^1, \ldots, b^{|K|}]$ is an unstable evaluation matrix when $K = \{1, \ldots, |K|\}$. We can stabilize the sub-matrix by using the concurrent convergence method (see [3] for details) and then apply Sekitani and Takahashi's algorithm [12] to the whole evaluation matrix.

In the group AHP, there exist multiple evaluation vectors of criteria from an alternative and multiple evaluation matrices of alternatives from criteria. By the same way as Step 1 of the concurrent convergence method, we can unify an evaluation matrix A for multiple evaluation matrices of alternatives from criteria. Suppose that there exist m evaluation vectors $\{\boldsymbol{b}^i_{(1)}, \ldots, \boldsymbol{b}^i_{(m)}\}$ of criteria from alternative i and let $B^{(i)} = [\boldsymbol{b}^i_{(1)}, \ldots, \boldsymbol{b}^i_{(m)}]$ for all $i \in I$. Let $A^{(i)}$ be an $m \times |J|$ matrix whose every row vector is $\boldsymbol{e}^\top A_i$ for all $i \in I$, then we can apply the concurrent convergence method with input of $\{A^{(i)} \mid i \in I\}$ and $\{B^{(i)} | i \in I\}$ to the group AHP.

In the interval AHP, there exists a feasible region $\mathcal{B}^i$ of criterion's evaluation vector from alternative i for all $i \in I$. Suppose that $\cap_{i\in I}\{A_i^{-1}\boldsymbol{x} | \boldsymbol{x} \in \mathcal{B}^i\} = \emptyset$ and that $\hat{\boldsymbol{p}}^i$ is an output vector of the concurrent convergence method for all $i \in I$, then there exists an alternative $k \in I$ such that $A_k^{-1}\hat{\boldsymbol{p}}^k \notin \mathcal{B}^k$ by any choice of an input vector $\boldsymbol{p}^i$ of the concurrent convergence method from $\{A_i^{-1}\boldsymbol{x} | \boldsymbol{x} \in \mathcal{B}^i\}$ for all $i \in I$. Hence, the concurrent convergence method can not provide any acceptable overall evaluation vector if $\cap_{i\in I}\{A_i^{-1}\boldsymbol{x} | \boldsymbol{x} \in \mathcal{B}^i\} = \emptyset$.

Finally, we have three important points in the further research of the dominant AHP and the concurrent convergence method as follows:

1. To develop of a mathematical model for the concurrent convergence method. (Geometric means method and Eigenvalue method correspond to a statics model and an optimization model [13], respectively.)
2. To discuss an evaluation value 0 and to develop an evaluation method under incomplete information.(In the case of evaluation value $a_{ij} = 0$, A_i is singular.)
3. Case studies for evaluation problems by the dominant AHP and the concurrent convergence method.

Acknowledgments

The authors are grateful to the anonymous referees for their helpful comments.

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