

THE DECISION PROCEDURE FOR PROFITABILITY OF INVESTMENT PROJECTS USING THE INTERNAL RATE OF RETURN OF SINGLE-PERIOD PROJECTS

Tadahiro Mizumachi Zentaro Nakamura
Keio University

(Received November 6, 2000; Revised December 3, 2001)

Abstract The internal rate of return (IRR) criterion is often used to evaluate profitability of investment projects. In this paper, we focus on a single-period project which consists of two types of cash flows; an investment at one period and a return at a succeeding period, and a financing at one period and a repayment at a succeeding period. We decompose the given investment project into a series of the single-period projects. From the viewpoint of the single-period project, we point out the applicability issue of the IRR criterion, namely the IRR criterion cannot be applied in which a project is composed of both investment type and financing type. Investigating the properties of a series of the single-period projects, we resolve the applicability issue of the IRR criterion and propose the decision procedure for profitability judgment toward any type of investment project based on the comparison between the IRR and the capital cost. We develop a new algorithm to obtain the value of the project investment rate (PIR) for the given project, which is a function of the capital cost, only using the standard IRR computing routine. This outcome is a theoretical breakthrough to widen the utilization of IRR in practical applications.

1. Introduction

The investment project is profitable if its net final value (NFV) is positive, and not profitable if its NFV is negative, where the entire series of net cash flows of the investment project and the capital cost are given. According to the internal rate of return (IRR) criterion, project profitability depends on the relation between IRR (denoted by r) and the capital cost (denoted by i). The project is profitable if $r > i$, and not profitable if $r < i$. Some of the investment projects with reinvestment during the project's life have multiple IRR values [3], [6]. In this case, the IRR criterion is not applicable.

In this paper, we focus on "a single-period project" in which cash inflow (or outflow) occurs at one period and inverse cash flow follows at the next (immediate succeeding) period. The single-period project can be classified into two types; "investment type" in which cash outflow (investment) occurs at one period and cash inflow (return) comes at the next period, and "financing type" in which cash inflow (finance) occurs at one period and cash outflow (repayment) follows at the next period. We decompose the investment project into the series of the single-period projects. If the total project is converted into the series of single-period projects which have the same value of IRR, the total project also has the equivalent value of IRR. However, the general total project is decomposed into the mixture of investment type and financing type. The former is profitable iff $r > i$ and the latter is profitable iff $r < i$. Therefore a simple judging criterion of IRR cannot be applied in such a case. The investment project with multiple IRR values is decomposed into the series which includes the financing type.

This paper proposes a procedure to convert a project which is a mixture of both investment type and financing type into a series of single-period projects comprised of only investment type, using properties of the single-period project. Then we can provide a procedure for judging

profitability of any investment project based on the comparison between the IRR and the capital cost.

2. Basic Assumptions and Notations

- 1) Let $A = [a_0, a_1, \dots, a_n]$ denote a project which generates net cash flows of a_t at the end of period t , where $t = 0, 1, \dots, n$. An investment project satisfies $a_0 < 0$ and $a_t > 0$ for some values of t .
- 2) Assume that the capital cost i is given. The domain of i is $i > -1$ when we consider i as the variable interest rate for mathematical examination.
- 3) The Net Final Value function of the project A at the end of period n is defined as follows:

$$S_n^A(i) = a_0(1+i)^n + a_1(1+i)^{n-1} + \dots + a_{n-1}(1+i) + a_n. \tag{2.1}$$

The project A is profitable if $S_n^A(i) > 0$, and not profitable if $S_n^A(i) < 0$.

- 4) The project balance of the project A at the end of period t is defined as follows:

$$\begin{aligned} S_0^A(i) &= a_0, \\ S_t^A(i) &= S_{t-1}^A(i)(1+i) + a_t, \quad t = 1, 2, \dots, n. \end{aligned} \tag{2.2}$$

The project balance at the end of period n is equivalent to the NFV.

- 5) The IRR of project A satisfies $S_n^A(r) = 0$ and is denoted by r .

3. Decomposition of Investment Project from a Viewpoint of Single-Period Projects

3.1. A single-period project

As an example, we shall consider an investment project $C = [-100, 120]$. The IRR of C is calculated to be $r = 20\%$ as the ratio of return 120 over the investment 100. Using r , C can be expressed by $C = -100[1, -(1+r)]$. Then the NFV of C under the capital cost i , denoted by $S_1^C(i)$, is given by $S_1^C(i) = -100(i - r)$. This equation shows that the relation between r and i determines the sign of NFV. For example, in the case of $i = 10\%$, NFV of C is positive since $r > i$. Then the project C is evaluated to be profitable.

Single-Period Project: We define a project whose cash flows are $a_{t-1} = c_t$, $a_t = -c_t(1+r)$, and $a_l = 0$ where $l \neq t-1, t$ to be the single-period project. The project denoted by $[0, \dots, 0, c_t, -c_t(1+r), 0, \dots, 0]$ is the single-period project at period t whose IRR is r . As shown in Figure 1, the case of $c_t < 0$ is called the single-period investment project, and the case of $c_t > 0$ is called the single-period financing project.

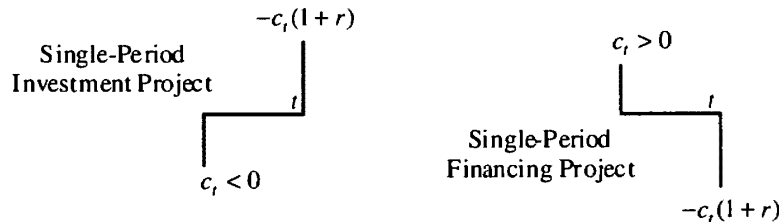


Figure 1: Single-period project

We shall hereafter denote a single-period project at period t with IRR r by $c_t e_t(r)$, using coefficient c_t and the unit project $e_t(r) = [0, \dots, 0, 1, -(1+r), 0, \dots, 0]$ whose cash flows are $a_{t-1} = 1$, $a_t = -(1+r)$, and $a_l = 0$ where $l \neq t-1, t$. The NFV of $c_t e_t(r)$ is given by

$$S_n^{c_t e_t(r)}(i) = \{c_t(1+i) - c_t(1+r)\}(1+i)^{n-t} = c_t(i-r)(1+i)^{n-t}. \tag{3.1}$$

From this equation, we obtain the next property.

Property 3.1. The NFV of the single-period project at period t with IRR r satisfies the next property.

The case of a single-period investment project ($c_t < 0$):

$$S_n^{c_t e_t(r)}(i) > 0, \text{ where } i < r.$$

$$S_n^{c_t e_t(r)}(i) = 0, \text{ where } i = r.$$

$$S_n^{c_t e_t(r)}(i) < 0 \text{ and } S_n^{c_t e_t(r)}(i) \text{ is a strictly decreasing function of } i^1, \text{ where } i > r.$$

The case of a single-period financing project ($c_t > 0$):

$$S_n^{c_t e_t(r)}(i) < 0, \text{ where } i < r.$$

$$S_n^{c_t e_t(r)}(i) = 0, \text{ where } i = r.$$

$$S_n^{c_t e_t(r)}(i) > 0 \text{ and } S_n^{c_t e_t(r)}(i) \text{ is a strictly increasing function of } i, \text{ where } i > r.$$

Then, we define a series of single-period projects as follows.

A Series of Single-Period Projects: Let $\bar{\mathbf{A}}(r)$ denote a project composed of a series of single-period projects $c_t e_t(r)$ at each period $t = 1, 2, \dots, n$. $\bar{\mathbf{A}}(r)$ is described as follows.

$$\bar{\mathbf{A}}(r) = \sum_{t=1}^n c_t e_t(r) = [c_1, -c_1(1+r) + c_2, \dots, -c_t(1+r) + c_{t+1}, \dots, -c_{n-1}(1+r) + c_n, -c_n(1+r)]. \quad (3.2)$$

We call $\bar{\mathbf{A}}(r)$ a series of single-period projects. The symbol (r) of $\bar{\mathbf{A}}(r)$ means that the IRR of the single-period project is r . We hereafter consider the case of $c_1 < 0$, that is, the project at period 1 is of the single-period investment project.

The NFV of $\bar{\mathbf{A}}(r)$ is calculated as the sum of the single-period projects as follows.

$$S_n^{\bar{\mathbf{A}}(r)}(i) = \sum_{t=1}^n S_n^{c_t e_t(r)}(i) \quad (3.3)$$

3.2. Decomposition of investment project into series of single-period projects

Let us consider an investment project $\mathbf{A} = [a_0, a_1, \dots, a_n]$ with the IRR of r . We shall below describe that \mathbf{A} can be decomposed into the series of single-period projects $\bar{\mathbf{A}}(r)$. The project balance at the end of each period of \mathbf{A} under the capital cost r can be expressed referring to (2.2) as follows.

$$S_0^{\mathbf{A}}(r) = a_0.$$

$$S_t^{\mathbf{A}}(r) = S_{t-1}^{\mathbf{A}}(r)(1+r) + a_t, \quad t = 1, 2, \dots, n-1. \quad (3.4)$$

$$S_n^{\mathbf{A}}(r) = S_{n-1}^{\mathbf{A}}(r)(1+r) + a_n = 0.$$

(3.4) immediately yields

$$a_0 = S_0^{\mathbf{A}}(r),$$

$$a_t = -S_{t-1}^{\mathbf{A}}(r)(1+r) + S_t^{\mathbf{A}}(r), \quad t = 1, 2, \dots, n-1, \text{ and} \quad (3.5)$$

$$a_n = -S_{n-1}^{\mathbf{A}}(r)(1+r).$$

Using (3.5), we can rewrite $\mathbf{A} = [a_0, a_1, \dots, a_t, \dots, a_n]$ as

$$\mathbf{A} = [S_0^{\mathbf{A}}(r), -S_0^{\mathbf{A}}(r)(1+r) + S_1^{\mathbf{A}}(r), \dots, -S_{t-1}^{\mathbf{A}}(r)(1+r) + S_t^{\mathbf{A}}(r), \dots, -S_{n-1}^{\mathbf{A}}(r)(1+r)]. \quad (3.6)$$

Referring to (3.2), using $e_t(r)$, \mathbf{A} is further transformed into

$$\mathbf{A} = \sum_{t=1}^n S_{t-1}^{\mathbf{A}}(r) e_t(r). \quad (3.7)$$

Therefore we obtain the next theorem.

Theorem 3.1. The investment project with the IRR r can be decomposed into the series of single-period projects whose IRR is r , namely, $\mathbf{A} = \bar{\mathbf{A}}(r) = \sum_{t=1}^n c_t \mathbf{e}_t(r)$ where $c_t = S_{t-1}^{\mathbf{A}}(r)$.

Both of the projects \mathbf{D} and \mathbf{E} shown in Figure 2 have the same IRR $r = 25\%$. \mathbf{D} and \mathbf{E} can be decomposed into the series $\bar{\mathbf{D}}(r)$ and $\bar{\mathbf{E}}(r)$ as follows respectively.

$$\mathbf{D} = [-120, 70, 60, 50] = -120\mathbf{e}_1(r) - 80\mathbf{e}_2(r) - 40\mathbf{e}_3(r) = \bar{\mathbf{D}}(r)$$

$$\mathbf{E} = [-100, 165, -110, 75] = -100\mathbf{e}_1(r) + 40\mathbf{e}_2(r) - 60\mathbf{e}_3(r) = \bar{\mathbf{E}}(r)$$

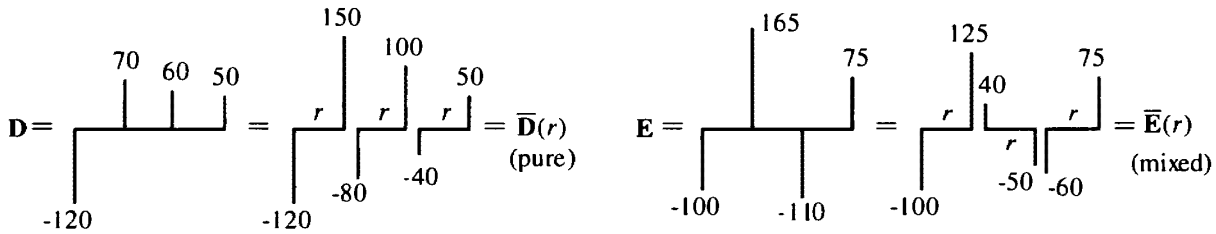


Figure 2: Decomposition of investment project

We shall classify the series of single-period projects into two categories as follows.

Pure Investment Series: The series composed only of the single-period investment projects. It means that $c_t \leq 0$ holds for all $t = 1, 2, \dots, n$. $\bar{\mathbf{D}}(r)$ is an example of a pure investment series.

Mixed Investment Series: The series composed both of the single-period investment projects and the single-period financing projects. There exist c_t of both signs. $\bar{\mathbf{E}}(r)$ is an example of a mixed investment series.

From property 3.1 and (3.3), we obtain the next property about the pure investment series.

Property 3.2. When $\bar{\mathbf{A}}(r)$ is a pure investment series (shortly described as $\bar{\mathbf{A}}(r)$ is pure hereafter), the NFV of $\bar{\mathbf{A}}(r)$ satisfies the next property.

$$S_n^{\bar{\mathbf{A}}(r)}(i) > 0, \text{ where } i < r.$$

$$S_n^{\bar{\mathbf{A}}(r)}(i) = 0, \text{ where } i = r.$$

$$S_n^{\bar{\mathbf{A}}(r)}(i) < 0 \text{ and } S_n^{\bar{\mathbf{A}}(r)}(i) \text{ is strictly decreasing function of } i, \text{ where } i > r.$$

3.3. Problem of the IRR criterion and idea for the solution

Property 3.2 tells us that the IRR criterion is applicable to the pure investment series. In the case of $r > i$, the entire series is profitable since all single-period investment projects in the series are profitable. In the case of $r < i$, all single-period investment projects in the series are not profitable, so that the series is judged to be not profitable. Property 3.2 also tells us that the pure investment series doesn't have multiple IRRs. It implies that the investment project with multiple IRRs is decomposed into the mixed investment series.

We consider $\bar{\mathbf{E}}(r)$ as an example of a mixed investment series as shown in Figure 2. In the case of $r > i$, the single-period investment projects at period 1 and 3 are profitable, and the single-period financing project at period 2 is not profitable. On the contrary if $r < i$, the single-period investment projects at period 1 and 3 are not profitable, and the single-period financing project at period 2 is profitable. It follows that in both cases of $r > i$ and $r < i$, there co-exist profitable single-period projects and not profitable ones. Therefore, in case of mixed investment series, the profitability of the entire project cannot be determined based on the profitability judgment using the IRR criterion for each single-period project.

In order to resolve the above problem inherent to the mixed investment series, we decompose the whole investment project into the series of single-period project, regarding the capital cost i as the value of IRR for the single-period financing project. For example, in the case of $i = 10\%$,

project **E** can be decomposed into the series of single-period projects as follows (refer to Figure 3).

$$E = [-100, 165, -110, 75] = -100e_1(r) + 43.8e_2(i) - 61.8e_3(r).$$

The IRR of the single-period investment project in this example becomes $r = 21.2\%$.

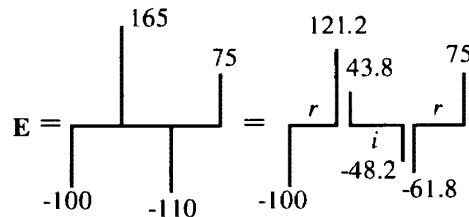


Figure 3: Decomposition into series of single-period projects with two rates r and i

The single-period financing project does not influence the profitability of the whole project because the IRR of the single-period financing project equals to the capital cost. Therefore the profitability of the single-period investment project determines the profitability of the entire project. If $r > i$, the whole project is profitable and if $r < i$, it is not profitable. Project **E** in Figure 3 is profitable since $r = 21.2\% > i = 10\%$.

In general, decomposing the investment project into the series, using rate r for investment project and i for financing project, enables the profitability determination of the whole project according to the relation between r and i . We consider in the next section, the way of calculating the value of r in the series where the capital cost i is given.

4. Analysis of Series of Single-Period Projects

4.1. Property of pure investment series

The next theorem is a property of the pure investment series that will frequently be referred to in later investigations.

Theorem 4.1. We shall consider a project **B** represented by

$$B = \bar{A}(r) + [0, \dots, 0, c] \tag{4.1}$$

where $\bar{A}(r)$ is the pure investment series and c is a constant value at the end of period n . Let r^B denote the IRR of **B**. Then the following statements hold.

- 1) In the case $c = 0$, $r^B = r$.
- 2) In the case $c < 0$, there doesn't exist r^B which satisfies $r^B > r$.
- 3) In the case $c > 0$, there exists a unique r^B which satisfies $r^B > r$.

Proof. From (4.1), the NFV of **B** is given by

$$S_n^B(i) = S_n^{\bar{A}(r)}(i) + c. \tag{4.2}$$

From property 3.2, the NFV of $\bar{A}(r)$ satisfies the next statement.

$$S_n^{\bar{A}(r)}(i) < 0 \text{ and } S_n^{\bar{A}(r)}(i) \text{ is strictly decreasing, where } i > r. \tag{4.3}$$

- 1) The case of $c = 0$: Since $B = \bar{A}(r)$, it is clear that $r^B = r$.
- 2) The case of $c < 0$: Referring to the right hand of (4.2), (4.3) and $c < 0$, it follows

$$S_n^B(i) < 0, \text{ where } i > r. \tag{4.4}$$

This tells us that $S_n^B(i)$ doesn't have r^B that satisfies $S_n^B(r^B) = 0$ where $i > r$.

- 3) The case of $c > 0$: From property 3.2, the NFV of $\bar{A}(r)$ also satisfies the next statement.

$$S_n^{\bar{A}(r)}(r) = 0. \tag{4.5}$$

From (2.1) and $a_0 < 0$, we have

$$\lim_{i \rightarrow \infty} S_n^{\bar{A}(r)}(i) = -\infty. \tag{4.6}$$

From (4.3), (4.5) and (4.6), there must exist a unique i^* under $i > r$ that satisfies the next equation.

$$S_n^{\bar{A}(r)}(i^*) = -c < 0. \tag{4.7}$$

Considering i^* as r^B , from (4.2),

$$S_n^B(r^B) = S_n^{\bar{A}(r)}(r^B) + c = 0. \tag{4.8}$$

Therefore **B** has a unique IRR r^B where $i > r$.

Q.E.D.

Let us consider truncated project $A_t = [a_0, a_1, \dots, a_t]$ which consists of cash flows of project **A** at the end of periods 0 through t ($t < n$). Let r_t and r_n denote the IRR of A_t and **A** respectively. When **A** is decomposed into $\bar{A}(r_n)$, referring to (3.2), A_t can be expressed by

$$A_t = \bar{A}_t(r_n) + [0, \dots, 0, c_{t+1}]. \tag{4.9}$$

$\bar{A}_t(r_n)$ is a partial series of single-period projects at periods 1 through t of $\bar{A}(r_n)$. c_{t+1} at the end of period t is the value of coefficient of the single-period project of $\bar{A}(r_n)$ at period $t+1$ (see Figure 4). We must note that the structure of (4.9) is equivalent to that of (4.1) in theorem 4.1. Then we obtain the next theorem.

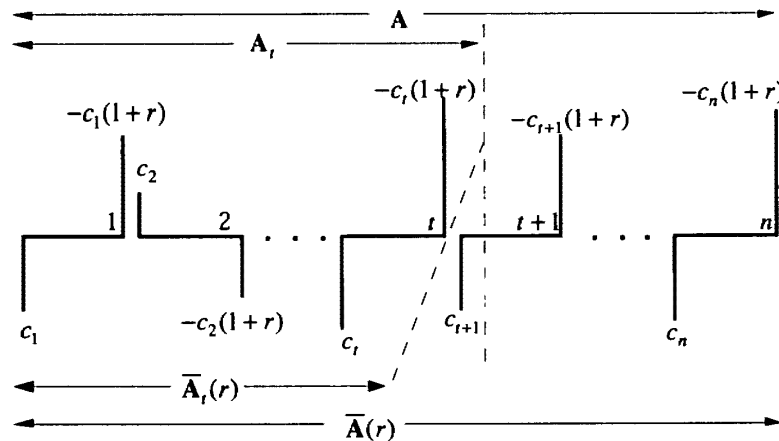


Figure 4: Relation among **A**, A_t , $\bar{A}(r_n)$ and $\bar{A}_t(r_n)$

Theorem 4.2. The investment project **A** can be decomposed into the pure investment series $\bar{A}(r_n)$ iff $\max_{1 \leq t \leq n} r_t = r_n$.

Proof. 1) Necessity: When **A** is decomposed into pure investment series $\bar{A}(r_n)$, $\bar{A}_t(r_n)$ is also pure and $c_{t+1} \leq 0$ holds in (4.9). Then from theorem 4.1, A_t doesn't have IRR r_t which satisfies $r_t > r_n$. This holds for all $t = 1, 2, \dots, n-1$, so that $\max_{1 \leq t \leq n} r_t = r_n$.

2) Sufficiency: When $\max_{1 \leq t \leq n} r_t = r_n$ holds, from theorem 3.1, **A** is decomposed into $\bar{A}(r_n)$. We assume that there exist single-period financing projects among $\bar{A}(r_n)$. Then there must exist the value of t which satisfies followings; a) the series is pure until period t , and b) a single-period financing project exists at period $t+1$ for the first time. Namely, $\bar{A}_t(r_n)$ is pure and $c_{t+1} > 0$ holds in (4.9). From theorem 4.1, A_t has the IRR r_t which satisfies $r_t > r_n$. This contradicts to $\max_{1 \leq t \leq n} r_t = r_n$. Then the assumption is fault, namely, there is no single-period financing project among $\bar{A}(r_n)$. Therefore $\bar{A}(r_n)$ is pure. Q.E.D.

Applying theorem 4.2 to the project **B** in theorem 4.1, we get the next theorem.

Theorem 4.3. The investment project **B** which is obtained by adding a positive constant c to the cash flow of pure investment series $\bar{\mathbf{A}}(r_n^{\mathbf{A}})$ only at the end of period n can be decomposed into the pure investment series $\bar{\mathbf{B}}(r_n^{\mathbf{B}})$ using IRR $r_n^{\mathbf{B}}$ that satisfies $r_n^{\mathbf{B}} > r_n^{\mathbf{A}}$. Namely, the next statement is satisfied. (Numerical example is shown in Figure 5.)

$$\mathbf{B} = \bar{\mathbf{A}}(r_n^{\mathbf{A}}) + [0, \dots, 0, c] = \bar{\mathbf{B}}(r_n^{\mathbf{B}}). \tag{4.10}$$

Proof. Since the cash flows at the end of periods 0 through $n - 1$ of **B** are identical with those of $\bar{\mathbf{A}}(r_n^{\mathbf{A}})$, $\max_{1 \leq t \leq n-1} r_t^{\mathbf{B}} = \max_{1 \leq t \leq n-1} r_t^{\mathbf{A}}$ holds. Since $\bar{\mathbf{A}}(r_n^{\mathbf{A}})$ is pure, theorem 4.2 yields $\max_{1 \leq t \leq n} r_t^{\mathbf{A}} = r_n^{\mathbf{A}}$. From the case of $c > 0$ in theorem 4.1, **B** has the IRR $r_n^{\mathbf{B}}$ that satisfies $r_n^{\mathbf{B}} > r_n^{\mathbf{A}}$. It follows

$$\max_{1 \leq t \leq n-1} r_t^{\mathbf{B}} = \max_{1 \leq t \leq n-1} r_t^{\mathbf{A}} \leq \max_{1 \leq t \leq n} r_t^{\mathbf{A}} = r_n^{\mathbf{A}} < r_n^{\mathbf{B}} \tag{4.11}$$

Then we obtain $\max_{1 \leq t \leq n} r_t^{\mathbf{B}} = r_n^{\mathbf{B}}$. From theorem 4.2, **B** is decomposed into the pure investment series $\bar{\mathbf{B}}(r_n^{\mathbf{B}})$ with $r_n^{\mathbf{B}}$ that satisfies $r_n^{\mathbf{B}} > r_n^{\mathbf{A}}$. Q.E.D.

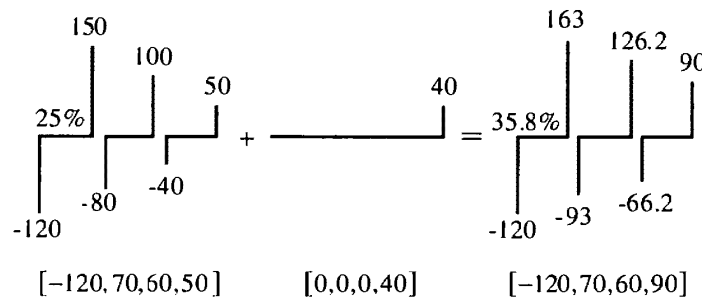


Figure 5: Numerical example for Theorem 4.3

4.2. Conversion of mixed investment series into pure investment series

In the case that $\max_{1 \leq t \leq n} r_t = r_n$ holds, from theorem 4.2, the investment project **A** can be decomposed into the pure investment series. Therefore, another case, that is the case in which $\max_{1 \leq t \leq n} r_t = r_m$, $m < n$ holds, is an issue to be investigated. Let $\bar{\mathbf{A}}(r, i)$ denote a series of single-period projects composed of single-period investment projects whose IRR is r and single-period financing projects whose IRR is i . We consider a decomposition of the investment project **A** into $\bar{\mathbf{A}}(r, i)$. In general, $\bar{\mathbf{A}}(r, i)$ can be expressed by

$$\bar{\mathbf{A}}(r, i) = \sum_{t=1}^n c_t e_t(x_t), \quad x_t = \begin{cases} r, & c_t \leq 0, \\ i, & c_t > 0. \end{cases} \tag{4.12}$$

The single-period project at period 1 is of an investment type. However, the value of r is unknown and whether the single-period project of each period is either an investment type or a financing one is not determined. The next theorem describes that a specified period, at which a single-period financing project exists, can be determined.

Theorem 4.4. The single-period project at period $m + 1$ of $\bar{\mathbf{A}}(r, i)$ is of a financing type under the condition that

$$\max_{1 \leq t \leq n} r_t = r_m, \quad m < n \tag{4.13}$$

holds for $\mathbf{A} = \bar{\mathbf{A}}(r, i)$.

Proof. We assume that the single-period project at period $m + 1$ of $\bar{\mathbf{A}}(r, i)$ is of an investment type. As the same manner as (4.9), the truncated project \mathbf{A}_m can be expressed by

$$\mathbf{A}_m = \bar{\mathbf{A}}_m(r, i) + [0, \dots, 0, c_{m+1}]. \tag{4.14}$$

By assumption, $c_{m+1} \leq 0$ holds. From (4.13), $\max_{1 \leq t \leq m} r_t = r_m$ holds, so that, from theorem 4.2, \mathbf{A}_m can be decomposed into the pure investment series $\bar{\mathbf{A}}_m(r_m)$. Then we can rewrite (4.14) as

$$\bar{\mathbf{A}}_m(r_m) = \bar{\mathbf{A}}_m(r, i) + [0, \dots, 0, c_{m+1}]. \tag{4.15}$$

The statement (4.15) yields

$$\bar{\mathbf{A}}_m(r, i) = \bar{\mathbf{A}}_m(r_m) + [0, \dots, 0, -c_{m+1}]. \tag{4.16}$$

Since $-c_{m+1} \geq 0$, from theorem 4.3, $\bar{\mathbf{A}}_m(r, i)$ should be the pure investment series $\bar{\mathbf{A}}_m(r)$ and $r \geq r_m$ holds. It also follows from the assumption, $\bar{\mathbf{A}}_{m+1}(r, i)$ should be pure.

1) The case of $m + 1 = n$: The entire $\bar{\mathbf{A}}(r, i)$ is pure. This contradicts to (4.13).

2) The case of $m + 1 < n$: We already discussed the case when the entire $\bar{\mathbf{A}}(r, i)$ is pure above. Let l stand for the first period at which a single-period project is of a financing type in $\bar{\mathbf{A}}_n(r, i)$ under the condition $m + 1 < l$. Then $\bar{\mathbf{A}}_{l-1}(r, i)$ is the pure investment series $\bar{\mathbf{A}}_{l-1}(r)$ and \mathbf{A}_{l-1} can be expressed as follows:

$$\mathbf{A}_{l-1} = \bar{\mathbf{A}}_{l-1}(r) + [0, \dots, 0, c_l]. \tag{4.17}$$

$c_l > 0$ holds since the single-period project at period l is of a financing type. From theorem 4.3, \mathbf{A}_{l-1} has the IRR η_{l-1} that satisfies $\eta_{l-1} > r$. Then we have $\eta_{l-1} > r \geq r_m$. This contradicts to (4.13).

From 1) and 2), the assumption is fault, and therefore the single-period project at period $m + 1$ is of a financing type. Q.E.D.

When the single-period project at period $m + 1$ of $\mathbf{A} = \bar{\mathbf{A}}(r, i)$ is of a financing type, $\bar{\mathbf{A}}(r, i)$ can be expressed by picking up the single-period financing project $c_{m+1}\mathbf{e}_{m+1}(i)$ as follows (refer to Figure 6).

$$\begin{aligned} \mathbf{A} = \bar{\mathbf{A}}(r, i) &= \sum_{t=1}^m c_t \mathbf{e}_t(x_t) + c_{m+1} \mathbf{e}_{m+1}(i) + \sum_{t=m+2}^n c_t \mathbf{e}_t(x_t) \\ &= \sum_{t=1}^m c_t \mathbf{e}_t(x_t) + [0, \dots, 0, c_{m+1}, -c_{m+1}(1+i), 0, \dots, 0] + \sum_{t=m+2}^n c_t \mathbf{e}_t(x_t). \end{aligned} \tag{4.18}$$

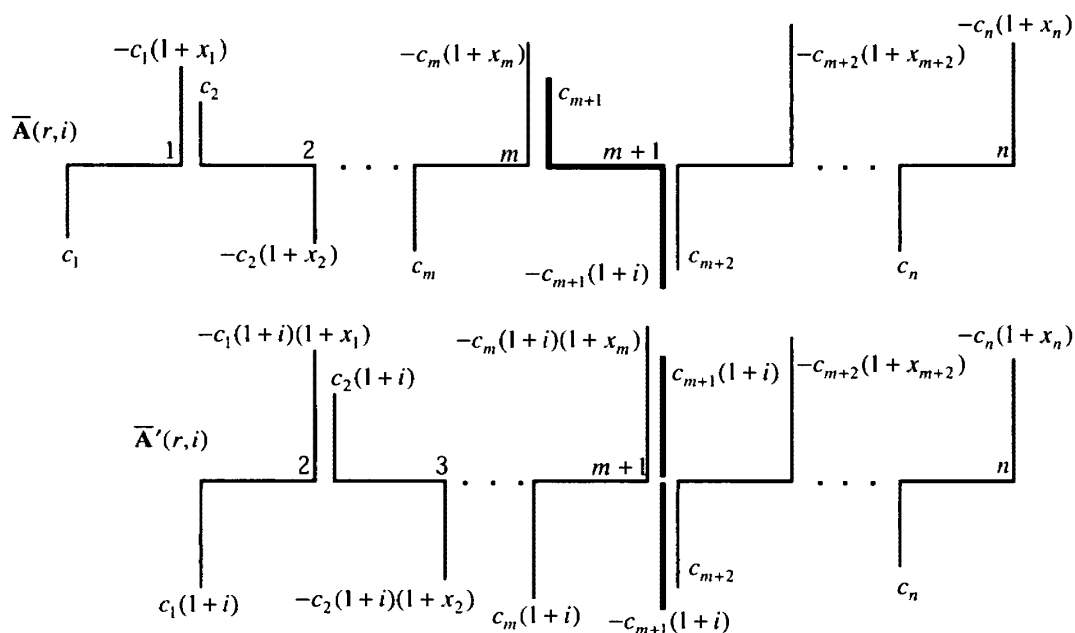


Figure 6: Conversion of $\bar{\mathbf{A}}(r, i)$ into $\bar{\mathbf{A}}'(r, i)$

Here, we define a cash flow conversion of $\mathbf{A} = [a_0, a_1, \dots, a_n]$ into $\mathbf{A}' = [a'_1, a'_2, \dots, a'_n]$ as follows:

$$\begin{cases} a'_t = a_{t-1}(1+i), & t = 1, 2, \dots, m. \\ a'_{m+1} = a_m(1+i) + a_{m+1}. \\ a'_t = a_t, & t = m+2, m+3, \dots, n. \end{cases} \quad (4.19)$$

Multiplying the cash flows at the end of periods 0 through m by $(1+i)$, and shifting each of them to one period later respectively, \mathbf{A} is converted into a investment project \mathbf{A}' whose life is $n-1$ periods from the end of period 1 through n . From the viewpoint of the series of single-period projects, referring to (4.18), $\bar{\mathbf{A}}(r, i)$ is converted into $\bar{\mathbf{A}}'(r, i)$ as follows:

$$\begin{aligned} \mathbf{A}' = \bar{\mathbf{A}}'(r, i) &= \sum_{t=1}^m c_t(1+i)\mathbf{e}_{t+1}(x_t) + [0, \dots, 0, c_{m+1}(1+i) - c_{m+1}(1+i), 0, \dots, 0] + \sum_{t=m+2}^n c_t\mathbf{e}_t(x_t) \\ &= \sum_{t=1}^m c_t(1+i)\mathbf{e}_{t+1}(x_t) + \sum_{t=m+2}^n c_t\mathbf{e}_t(x_t). \end{aligned} \quad (4.20)$$

Comparing (4.18) and (4.20), we can describe the property of the conversion as follows.

- 1) The single-period financing project $c_{m+1}\mathbf{e}_{m+1}(i)$ which existed in $\bar{\mathbf{A}}(r, i)$ is eliminated.
- 2) All single-period projects $c_t\mathbf{e}_t(x_t)$ converted from $\bar{\mathbf{A}}(r, i)$ to $\bar{\mathbf{A}}'(r, i)$ change neither the types of investment or financing nor the value of IRR.

In the case that $\bar{\mathbf{A}}'(r, i)$ is pure, namely $\max_{1 \leq t \leq n} r_t = r_n$ holds for \mathbf{A}' , we can obtain the value of r of $\bar{\mathbf{A}}(r, i)$ as the value of IRR of \mathbf{A}' . In another case, namely $\max_{1 \leq t \leq n} r_t = r_m$ holds for \mathbf{A}' , $\bar{\mathbf{A}}'(r, i)$ should be further converted into $\bar{\mathbf{A}}''(r, i)$. Thus we can eliminate all of the single-period financing projects in the series and can obtain the pure investment series.

We conclude that the value of r of $\mathbf{A} = \bar{\mathbf{A}}(r, i)$ can be calculated as the IRR of $\mathbf{A}^{(j)} = \bar{\mathbf{A}}^{(j)}(r, i)$ which is the pure investment series obtained after j -th conversion to eliminate the single-period financing project.

5. The Procedure for Judging Profitability for A Given Investment Project

The procedure for judging profitability for a given investment project $\mathbf{A} = [a_0, a_1, \dots, a_n]$ can be described as follows. We repeat Step 1 through Step 3 until we obtain the pure investment series.

Step 1: Find the period t which satisfies $a_t > 0$ and $a_{t+1} < 0$, and calculate r_t . Repeat the same procedure for all t which satisfy $a_t > 0$ and $a_{t+1} < 0$. Calculate r_n when $a_n > 0$ ².

Step 2: 1) If $\max_{1 \leq t \leq n} r_t = r_n$, then go to Step 4.

2) If $\max_{1 \leq t \leq n} r_t = r_m, m < n$, then go to Step 3.

Step 3: Convert the cash flows using the capital cost i as follows, and go to Step 1 with the project obtained here.

$$\begin{cases} a'_t = a_{t-1}(1+i), & t = 1, 2, \dots, m. \\ a'_{m+1} = a_m(1+i) + a_{m+1}. \\ a'_t = a_t, & t = m+2, m+3, \dots, n. \end{cases}$$

In the case that all a'_t become negative, \mathbf{A} is not profitable³ and the procedure is terminated.

Step 4: If $r_n > i$, \mathbf{A} is profitable. If $r_n < i$, \mathbf{A} is not profitable.

As a numerical example, consider a project $\mathbf{A} = [-100, 220, -140, 40, 110, -180, 200]$, where the capital cost is given as $i = 10\%$. The conversion of the original cash flow \mathbf{A} to \mathbf{A}' and then to \mathbf{A}'' is illustrated in Table 1.

Table 1: The cash flow conversion for the numerical example (* means the maximum r_t)

t	0	1	2	3	4	5	6
A	-100	220	-140	40	110	-180	200
r_t		*120.0%			73.7%		66.4%
conversion		-100×1.1	220×1.1 -140	↓	↓	↓	↓
A'		-110	102	40	110	-180	200
r_t					*56.7%		49.3%
conversion			-110×1.1	-102×1.1	40×1.1	110×1.1 -180	↓
A''			-121	112.2	44	-59	200
r_t					22.4%		*47.0%

Step 1: We have $r_1 = 120.0\%$, $r_4 = 73.7\%$ and $r_6 = 66.4\%$ for **A**.

Step 2: $\max_{1 \leq t \leq 6} r_t = r_1$ holds.

Step 3: We obtain **A'** by conversion with $m = 1$.

Step 1: We have $r_4 = 56.7\%$ and $r_6 = 49.3\%$ for **A'**.

Step 2: $\max_{2 \leq t \leq 6} r_t = r_4$ holds.

Step 3: We obtain **A''** by conversion with $m = 4$.

Step 1: We have $r_4 = 22.4\%$ and $r_6 = 47.0\%$ for **A''**.

Step 2: $\max_{3 \leq t \leq 6} r_t = r_6$ holds. Then we go to Step 4.

Step 4: Since $r_6 = 47\% > 10\% = i$, **A** is judged to be profitable.

6. Conclusion

This paper analyzed the problem of judging profitability of a general investment project using internal rate of return (IRR) criterion. A general project, in which cash inflow and outflow are mixed during the life comprised of n periods, can be described as a combination of "the single-period project." The single-period project can be categorized into two types; the investment type and the financing type, where the former is profitable under the condition of $r > i$ while the latter is not profitable under the same condition. Therefore, the existence of the single-period financing project among the whole project disturbs the utilization of IRR criterion.

Toward this problem, we have presented a conversion procedure utilizing the capital cost i toward the single-period financing pattern. Then, we can eliminate the constituent that is not profitable where $r > i$. It follows that the IRR criterion, namely the comparison of the IRR r and the capital cost i , for judging the profitability of the whole project, can be applied to any type of project. Thus this paper extended the utilization of IRR criterion to the general type of cash flow patterns.

This problem was discussed by D. Teichrow, A. A. Robichek, and M. Montalbano [4], [5]. The IRR r of $\bar{A}(r, i)$ is equivalent to the "project investment rate (PIR)" $r(i)$ ⁴ they proposed. It can be said that we have developed a new algorithm to compute the PIR directly only with the standard IRR computing routine.

Endnotes

1. From (3.1), we obtain $\frac{d}{di} S_n^A(i) = c_t \{ (1+i)^{n-t} + (i-r)(n-t)(1+i)^{n-t-1} \}$. Therefore, the coefficient c_t determines the sign of the derivative where $i > r$.
2. If r_t is the maximum rate in Step 2, A_t should be decomposed into a pure investment series. If $a_t < 0$, A_t cannot be decomposed into a pure investment series. In the case of $a_t > 0$ and $a_{t+1} > 0$, r_t should not be the maximum rate, because when A_t is pure, there exists r_{t+1} that satisfies $r_{t+1} > r_t$, and when A_t is not pure, there exists η that satisfies $\eta > r_t$ for $l < t$.
3. In this case, project A cannot be decomposed into $\bar{A}(r, i)$ under the condition of the given capital cost i . However, the conversion doesn't change the NFV of the project under the given capital cost. If all a'_t are negative, then the NFV of A' is negative. Therefore the NFV of the original project A is negative and A is not profitable.
4. $r(i)$ can be computed by $r = r(k)$ which satisfies that the final value function $F_n(r, k) = 0$. The function $r = r(k)$ is analytically investigated in many literatures, but only the projects with a life of two periods are discussed [1], [2], [5]. As a trial and error approach, the computer codes using Newton-Raphson method for calculating the value of $r(k)$ are proposed [7].

Acknowledgement

We would like to express our heartfelt gratitude to Professor Hirokazu Kono of Keio University for his valuable advices and helpful discussions.

References

- [1] J. C. T. Mao: *Quantitative Analysis of Financial Decisions* (Macmillan, 1969).
- [2] C. S. Park, G. P. Sharp-bette: *Advanced Engineering Economics* (Wiley, 1990).
- [3] S.Senju, T.Fushimi, S.Fujita: *Profitability Analysis* (Asian Productivity Organization, 1989).
- [4] D. Teichroew, A. A. Robichek, M. Montalbano: Mathematical analysis of rates of return under certainty. *Management Science*, **11**-3 (1965) 395-403.
- [5] D. Teichroew, A. A. Robichek, M. Montalbano: An analysis of criteria for investment and financing decisions under certainty. *Management Science*, **12**-3 (1965) 151-179.
- [6] H.G.Thuesen, W.J.Fabrycky, G.J.Thuesen: *Engineering Economy, 4th Edition* (Prentice-Hall, 1971).
- [7] T. L. Ward: Computer rate of return analysis of mixed capital projects. *The Engineering Economist*, **39**-2 (1994) 165-176.

Tadahiro Mizumachi
 Department of Administration Engineering
 Faculty of Science and Technology, Keio University
 3-14-1 Hiyoshi, Kouhoku, Yokohama
 223-8522, Japan
 E-mail: machi@ae.keio.ac.jp