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TWO EFFICIENT ALGORITHMS FOR THE GENERALIZED MAXIMUM BALANCED FLOW PROBLEM

Akira Nakayama
Fukushima University

Chien Fei Su
Tokuritsu Co. Ltd.

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Abstract Minoux considered the *maximum balanced flow problem*, i.e. the problem of finding a maximum flow in a two-terminal network $\mathcal{N} = (V, A)$ with source s and sink t satisfying the constraint that any arc-flow of $\mathcal{N}$ is bounded by a fixed proportion of the total flow value from s to t, where V is vertex set and A is arc set. Several efficient algorithms, so far, have been proposed for this problem. As a natural generalization of this problem we focus on the problem of maximizing the total flow value of a generalized flow in a network $\mathcal{N} = (V, A)$ with gains $\gamma(a) > 0$ $(a \in A)$ satisfying any arc-flow of $\mathcal{N}$ is bounded by a fixed proportion of the total flow value from s to t, where $\gamma(a)f(a)$ units arrive at the vertex w for each arc-flow $f(a)$ $(a \equiv (v, w) \in A)$ entering vertex v in a generalized flow in $\mathcal{N}$. We call it the *generalized maximum balanced flow problem* and if $\gamma(a) = 1$ for any $a \in A$ then it is a maximum balanced flow problem. The authors believe that no algorithms have been shown for this generalized version. Our main results are to propose two polynomial algorithms for solving the generalized maximum balanced flow problem. The first algorithm runs in $O(m\mathrm{M}(n, m, B') \log B)$ time, where B is the maximum absolute value among integral values used by an instance of the problem, and $\mathrm{M}(n, m, B')$ denotes the complexity of solving a generalized maximum flow problem in a network with n vertices, and m arcs, and a rational instance expressed with integers between 1 and B'. In the second algorithm we combine a parameterized technique of Megiddo with one of algorithms for the generalized maximum flow problem, and show that it runs in $O(\{\mathrm{M}(n, m, B')\}^2)$ time.

1. Introduction

Minoux [7] considered the *maximum balanced flow problem*, i.e. the problem of finding a maximum flow in a two-terminal network $\mathcal{N} = (V, A)$ with source s and sink t satisfying the constraint that the value of any arc-flow of $\mathcal{N}$ is bounded by a fixed proportion of the total flow value from s to t, where V is vertex set and A is arc set. Such a constraint is described in terms of a *balancing rate* function $\alpha : A \to \mathbf{R}_+ - \{0\}$ with $\alpha(a) \leq 1$ $(a \in A)$, where $\mathbf{R}_+$ is the set of nonnegative reals. The maximum balanced flow problem is motivated by Minoux's research of reliability analysis of communication networks. If a flow from source s to sink t is balanced, then it is guaranteed that the value of the blocked arc-flow is at most the fixed proportion of the total flow value from s to t. So far, several algorithms ([1],[7],[8],[9],[12]) have been proposed for the maximum balanced flow problem. They contain a network simplex method without cycling, and a parameterized maximum flow algorithm, and a binary search method and so on. The latter two run in polynomial-time. For the problem of finding an integral maximum balanced flow, Zimmermann [12] showed this problem is *NP*-hard, where Cui [1] gave an efficient algorithm in the case when the balancing rate function is constant. On the other hand, Ichimori et al [5] considered the *weighted minimax flow problem* and proposed a couple of polynomial algorithms for this problem, and Fujishige et al. [3] have pointed out the equivalence of the maximum balanced flow problem and the weighted minimax flow problem.

By the way, we can study some directions for generalizing the maximum balanced flow problem. There is a problem of discussing the *kernel* or *null space* $\{x : \mathbf{Q}x = \mathbf{0}\}$ of any real matrix $\mathbf{Q}$ in place of the *circulation space* which is the kernel of the vertex-arc incidence matrix of the underlying graph with a new arc (t, s) attached. Zimmermann [12] treated another direction of generalization of the problem, i.e. the *maximum balanced submodular flow problen* over totally ordered commutative groups. In this paper we focus on *generalized maximum balanced flow problem*, i.e. the problem of maximizing the total flow value of a generalized flow in a network $\mathcal{N} = (V, A)$ with gains $\gamma(a) > 0$ $(a \in A)$ satisfying the value of any arc-flow of $\mathcal{N}$ is bounded by a fixed proportion of the total flow value from s to t, where $\gamma(a)f(a)$ units arrive at the vertex w for each arc-flow $f(a)$ $(a \equiv (v, w) \in A)$ entering vertex v in a generalized flow in $\mathcal{N}$. The generalized maximum balanced flow problem with gains $\gamma(a) = 1$ $(a \in A)$ is equivalent to the maximum balanced flow problem.

The objective of the present paper is to propose two efficient algorithms. The authors believe that no algorithms, so far, have been shown for the generalized maximum balanced flow problem. The complexity of the first algorithm is $O(n\mathrm{M}(n, m, B') \log B)$ time, where B is the maximum absolute value among integral values used by an instance of the generalized maximum balanced flow problem, and $\mathrm{M}(n, m, B')$ denotes the complexity of solving a generalized maximum flow problem in a network with n vertices, and m arcs, and integral/rational instance expressed with integers between 1 and B'. In the second algorithm we combine the parameterized technique of Megiddo with one of algorithms for the generalized maximum flow problem, and show that it runs in $O(\{\mathrm{M}(n, m, B')\}^2)$ time. Finally, we will touch our future studies in the concluding remarks.

2. Definitions and Preliminaries

Let $G = (V, A)$ be a connected directed graph with vertex set V and arc set A, where $|V| = n$ and $|A| = m$. We distinguish two special vertices : a *source* $s \in V$ and a *sink* $t \in V$. For simplicity, we assume that the directed graph contains no multiple arcs and self-loops. Moreover, we may assume that $(v, w) \in A$ implies $(w, v) \notin A$. For each arc $a \in A$, $\partial^+ a$ (resp. $\partial^- a$) is *tail* (resp. *head*) of a. Let $\gamma : A \to \mathbf{R}_+ - \{0\}$ be a *gain* function, $u : A \to \mathbf{R}_+$ a *capacity* function, $\beta : A \to \mathbf{R}$ a function, $\alpha : A \to \mathbf{R}_+ - \{0\}$ a *balancing rate* function with $\alpha(a) \leq 1$ $(a \in A)$ where $\mathbf{R}$ (resp. $\mathbf{R}_+$) is the set of reals (resp. nonnegative reals). Throughout this paper, we assume that u and β are integral, and that $\gamma(a)$ (resp. $\alpha(a)$) $(a \in A)$ is expressed as $\frac{\gamma_0(a)}{\gamma_1(a)}$ (resp. $\frac{\alpha_0(a)}{\alpha_1(a)}$) for some positive integers $\gamma_i(a)$ and $\alpha_i(a)$ $(i = 0, 1)$. For a function $f : A \to \mathbf{R}$ and a gain function $\gamma : A \to \mathbf{R}_+ - \{0\}$, *boundery* $\partial_\gamma f : V \to \mathbf{R}$ is defined by $\partial_\gamma f(v) \equiv \sum_{a \in \delta^+ v} f(a) - \sum_{a \in \delta^- v} \gamma(a) f(a)$, where $\delta^+ v = \{a \in A : \partial^+ a = v\}$ and $\delta^- v = \{a \in A : \partial^- a = v\}$. We also assume $\delta^+ t = \phi$. This assumption is without loss of generality.

Given a network $\mathcal{N} = (G, u, \gamma, \alpha, \beta, s, t)$, the *generalized maximum balanced flow problem*, (**GMBF**) for short, is defined as follows:

(**GMBF**) : Maximize $\mathrm{val}_{\mathcal{N}}(f)$ subject to

$$\partial_\gamma f(v) = 0, \quad (v \in V - \{s, t\}), \tag{2.1}$$

$$0 \leq f(a) \leq u(a), \quad (a \in A), \tag{2.2}$$

$$f(a) \leq \alpha(a)\mathrm{val}_{\mathcal{N}}(f) + \beta(a), \, (a \in A), \tag{2.3}$$

where $\mathrm{val}_{\mathcal{N}}(f) = \sum_{a \in \delta^- t} \gamma(a) f(a)$. If problem (**GMBF**) with $\gamma(a) = 1$ for any $a \in A$, then it is called the *maximum balanced flow problem*. Given a network $\mathcal{N}' = (G, u, \gamma, s, t)$, the *generalized maximum flow problem*, **(GMF)** for short, is as follows.

(GMF): Maximize $\text{val}_{\mathcal{N}'}(f')$ subject to (2.1) $\sim$ (2.2),

where f in (2.1) $\sim$ (2.2) should be replaced by f'. Let $\mathcal{N}_z = (G = (V, A), u_z, \gamma, s, t)$ be the network $\mathcal{N}$ with a parameter $z \geq 0$, where $u_z(a) = \min\{u(a), \alpha(a)z + \beta(a)\}$ for each $a \in A$. Given network $\mathcal{N}_z$, consider the following parameterized problem $(\mathbf{GMF}(z))$.

$(\mathbf{GMF}(z))$: Maximize $\text{val}_{\mathcal{N}_z}(f_z)$ subject to (2.1) and

$$0 \leq f_z(a) \leq u_z(a), (a \in A), \tag{2.4}$$

where f in (2.1) should be replaced by f_z. A *generalized flow* f (resp. f') of $\mathcal{N}$ (resp. $\mathcal{N}'$) is a funcion $f : A \to \mathbf{R}$ (resp. $f' : A \to \mathbf{R}$) satisfying (2.1) $\sim$ (2.2). A generalized flow f is *balanced* in $\mathcal{N}$ if f also satisfies (2.3). If we consider a generalized flow in $\mathcal{N}_z$, then it is a funcion $f_z : A \to \mathbf{R}$ satisfying (2.1) and (2.4). The *value* of a generalized balanced flow f of $\mathcal{N}$ is $\text{val}_{\mathcal{N}}(f)$. The value of a generalized flow f_z (resp. f') of $\mathcal{N}_z$ (resp. $\mathcal{N}'$) is defined similarly. An *optimal* flow of $\mathcal{N}$ is a generalized balanced flow maximizing its value. An *optimal* flow f_z (resp. f') of $\mathcal{N}_z$ (resp. $\mathcal{N}'$) is a generalized flow maximizing the value. A *residual* network with respect to a generalized flow f_z of $\mathcal{N}_z$ is defined as $\mathcal{N}_z(f_z) = (G(f_z) = (V, A(f_z)), u_z^{f_z}, \gamma^{f_z}, s, t)$, where $A(f_z)$, $u_z^{f_z}$, and γ^{f_z} are defined as follows:

$$A(f_z) = A^+(f_z) \cup A^-(f_z), \tag{2.5}$$

$$A^+(f_z) = \{a \in A : f_z(a) < u_z(a)\}, \tag{2.6}$$

$$A^-(f_z) = \{a \equiv (j, i) : \bar{a} \equiv (i, j) \in A, f_z(\bar{a}) > 0\}, \tag{2.7}$$

$$u_z^{f_z}(a) = \begin{cases} u_z(a) - f_z(a), & (a \in A^+(f_z)), \\ \gamma(\bar{a}) f_z(\bar{a}), & (a \in A^-(f_z)), \end{cases} \tag{2.8}$$

$$\gamma^{f_z}(a) = \begin{cases} \gamma(a), & (a \in A^+(f_z)), \\ 1/\gamma(\bar{a}), & (a \in A^-(f_z)). \end{cases} \tag{2.9}$$

The dual problem $(\mathbf{DGMF}(z))$ for a primal problem $(\mathbf{GMF}(z))$ can be written as:
$(\mathbf{DGMF}(z))$: Minimize $\sum_{a \in A} u_z(a)\theta_z(a)$ subject to

$$\pi_z(\partial^+ a) - \gamma(a)\pi_z(\partial^- a) + \theta_z(a) \geq 0, (a \in A), \tag{2.10}$$

$$\theta_z(a) \geq 0, (a \in A), \tag{2.11}$$

$$\pi_z(v) \in \mathbf{R}, \ (v \in V), \tag{2.12}$$

where $\pi_z(s) = 0, \pi_z(t) = 1$. Note that if we let $\theta_z(a) = [-\pi_z(\partial^+ a) + \gamma(a)\pi_z(\partial^- a)]^+$ for any $\pi_z(v)$ $(v \in V)$ then (π_z, θ_z) is a dual feasible solution of $(\mathbf{DGMF}(z))$, where $[d]^+ \equiv \max\{0, d\}$ for $d \in \mathbf{R}$. Complementary slackness conditions imply that at optimality, for each $a \in A$,

$$f_z(a) > 0 \ \rightarrow \pi_z(\partial^+ a) - \gamma(a)\pi_z(\partial^- a) + \theta_z(a) = 0, \tag{2.13}$$

$$f_z(a) < u_z(a) \ \rightarrow \ \theta_z(a) = 0. \tag{2.14}$$

Define A_z,$C(\theta_z)$, and $D(\theta_z)$ by:

$$A_z = \{a \in A : u(a) > \alpha(a)z + \beta(a)\}, \tag{2.15}$$

$$C(\theta_z) = \sum_{a \in A_z} \alpha(a)\theta_z(a), \tag{2.16}$$

$$D(\theta_z) = \sum_{a \in A_z} \beta(a)\theta_z(a) + \sum_{a \in A - A_z} u(a)\theta_z(a). \tag{2.17}$$

If there is an optimal solution of (**DGMF**(z)), then the value is expressed as:

$$\mathrm{val}_{\mathcal{N}_z}(f_z^*) = C(\theta_z^*)z + D(\theta_z^*), \tag{2.18}$$

where f_z^* is an optimal flow in $\mathcal{N}_z$ and θ_z^* is a corresponding optimal solution of (**DGMF**(z)). We call a network *feasible* if there exists a feasible flow in it, i.e. a flow satisfying all the constraints given in the network. The following proposition shows fundamental relations as to an instance of network $\mathcal{N}$.

Proposition 2.1: *Let $z \geq 0$. Then we have (i) $\sim$ (iii).*
(i) If f is a generalized balanced flow in $\mathcal{N}$, then $\mathrm{val}_{\mathcal{N}}(f) \leq \mathrm{val}_{\mathcal{N}'}(f') \leq nB'^2$ for some generalized maximum flow f' in $\mathcal{N}'$, where $B' \equiv \max_{a\in A}\{\max\{\gamma_0(a), \gamma_1(a), u(a)\}\}$.
(ii) $\mathcal{N}_z$ is feasible if and only if $z \geq [\max_{a\in A} \frac{-\beta(a)}{\alpha(a)}]^+$.
(iii) If $z \geq 2B^2$, then problem (**GMF**) *is identical to problem* (**GMF**(z)), *where $B \equiv \max\{B', B''\}$ and $B'' \equiv \max_{a\in A}\{\max\{\alpha_1(a), |\beta(a)|\}\}$.*

Proof: It is easy to see (i) and (ii). We only prove (iii). Let $J_1(\mathcal{N}) = [\max_{a\in A} \frac{-\beta(a)}{\alpha(a)}]^+$ and $J_2(\mathcal{N}) = [\max_{a\in A} \frac{u(a)-\beta(a)}{\alpha(a)}]^+$. If $z \geq J_2(\mathcal{N})$, then problem (**GMF**(z)) with an instance is identical to problem (**GMF**) with the instance without α and β. From $J_1(\mathcal{N}) \leq J_2(\mathcal{N}) \leq 2B^2$, we have (iii). □

A characterization as to the value of an optimal flow in $\mathcal{N}$ (if it exists) is given as follows.
Proposition 2.2: *If network $\mathcal{N}$ is feasible, the value of an optimal flow in $\mathcal{N}$ is the maximum z such that $z = \mathrm{val}_{\mathcal{N}_z}(f_z^*)$ for some generalized maximum flow f_z^* in $\mathcal{N}_z$.* □

Function $y = F(z)$ with a variable z satisfies the following property, where $F(z) = \mathrm{val}_{\mathcal{N}_z}(f_z^*)$.
Proposition 2.3: *Assume network $\mathcal{N}_z$ is feasible and let $F(z) = \mathrm{val}_{\mathcal{N}_z}(f_z^*)$ for some generalized maximum flow f_z^* in $\mathcal{N}_z$. Then $y = F(z)$ is nondecreasing, continuous, piecewise linear, and concave.* □

From (2.18), we have another characterization of the optimal value in $\mathcal{N}$.
Proposition 2.4: *The value z^* of an optimal flow in network $\mathcal{N}$ is*

$$z^* = \max\{z : C(\theta_z^*) \neq 1, z = \frac{D(\theta_z^*)}{1 - C(\theta_z^*)}\},$$

where θ_z^* is a corresponding dual optimal solution of (**DGMF**(z)). □

In the following, we give a criterion for a generalized flow f' in network $\mathcal{N}'$ to be maximum. For each $v \in V$, let P_v be a residual directed path from v to t in the network $\mathcal{N}'(f')$, where the path is *simple*. The *gain* $\gamma^{f'}(P_v)$ of P_v with respect to $\gamma^{f'}$ is $\gamma^{f'}(P_v) \equiv \Pi_{a\in A(P_v)}\gamma^{f'}(a)$, where $\gamma^{f'}$ is the gain function of $\mathcal{N}'(f')$ and $A(P_v)$ is the arc set of the path P_v. The highest gain path from v to t is a residual directed path P_v' such that $\gamma^{f'}(P_v') = \max_{P_v} \gamma^{f'}(P_v)$ where the maximum is taken over all the residual directed paths from v to t. For a residual directed simple cycle Q of $\mathcal{N}'(f')$, the gain of Q is defined similarly. A *flow-generating cycle* (resp. flow-absorbing cycle) is a directed residual simple cycle Q satisfying $\gamma^{f'}(Q) > 1$ (resp. $\gamma^{f'}(Q) < 1$). A *labeling* function with respect to $\mathcal{N}'(f')$ is a function $\mu : V \longrightarrow (\mathbf{R}_+ - \{0\}) \cup \{\infty\}$ such that $\mu(t) = 1$. The *relabeled gain* $\gamma_\mu^{f'}(a)$ of arc $a \in A(f')$ with respect to μ is defined by $\gamma_\mu^{f'}(a) = \gamma^{f'}(a)\mu(\partial^+a)/\mu(\partial^-a)$. The *canonical label* of $v \in V$ in $\mathcal{N}'(f')$ is the inverse of the gain of the highest gain residual path from v to t. If no such path exists, its label is ∞. The following theorem is a result of Wayne [11].

Theorem 2.5: *Let f' be a feasible generalized flow in network $\mathcal{N}'$. The generalized flow f' is maximum if and only if there exists a labeling function μ such that:*

$$\gamma_\mu^{f'}(a) \leq 1, \quad (a \in \mathcal{N}'(f')), \tag{2.19}$$

$$\mu(s) = \infty, \tag{2.20}$$

where $\mathcal{N}'(f')$ is the residual network with respect to f' of $\mathcal{N}'$. □

The above theorem shows that if f' is a generalized maximum flow there is no *generalized augmenting path* in $\mathcal{N}'(f')$, where this path is defined as a residual flow-generating cycle, together with a (possibly trivial) residual path from a vertex on the cycle to t. It also shows that there is no residual directed path from s to t.

3. Algorithms for Generalized Maximum Balanced Flows

In this section, we describe the first algorithm based on a binary search method. The binary search algorithm is composed of three parts (Steps 1 ~ 3). Step 1 is an initialization and determines an upper and a lower bounds for the binary search. The work of Step 2 is to repeat a binary routine until the difference between the upper and lower bounds is sufficiently small. Step 3 determines whether there exists an optimal flow in network $\mathcal{N}$ or not if we can not make the decision during Step 2. The detailed description of the algorithm is as follows. In the description, an '*optimal flow*' in network $\mathcal{N}_z$ means a generalized maximum flow for some specific value of z while one in network $\mathcal{N}$ is a generalized maximum balanced flow.

Input: $\mathcal{N} = (G, u, \gamma, \alpha, \beta, s, t)$
Output: An optimal flow in $\mathcal{N}$ if it exists(or we decide that none exists.)
Step 1: Initialization
Set $B' = \max_{a \in A}\{\max\{\gamma_0(a), \gamma_1(a), u(a)\}\}$ and $B'' = \max_{a \in A}\{\max\{\alpha_1(a), |\beta(a)|\}\}$.
Set $U \leftarrow \min\{nB'^2, 2B^2\}$, where $B = \max\{B', B''\}$ and U is an upper bound.
Find an optimal flow f_U of $\mathcal{N}_U$.
If $\mathrm{val}_{\mathcal{N}_U}(f_U) \geq U$ **then** stop (an optimal flow in $\mathcal{N}$ is f_z with $z \equiv \mathrm{val}_{\mathcal{N}_U}(f_U)$.).
Set $L \leftarrow [\max_{a \in A} \frac{-\beta(a)\alpha_1(a)}{\alpha_0(a)}]^+$, where L is a lower bound.
Find an optimal flow f_L of $\mathcal{N}_L$.
If $\mathrm{val}_{\mathcal{N}_L}(f_L) < L$ and $C(\theta_L^*) \leq 1$ **then** stop ($\mathcal{N}$ is infeasible.).
If $\mathrm{val}_{\mathcal{N}_L}(f_L) = L$ and $C(\theta_L^*) < 1$ **then** stop (an optimal flow in $\mathcal{N}$ is f_L.).
Step 2: Binary search
repeat
  Set $z \leftarrow \frac{U+L}{2}$
  If $\mathrm{val}_{\mathcal{N}_z}(f_z) = z$ **then**
    begin
      If $C(\theta_z^*) < 1$ **then** stop(f_z is optimal in $\mathcal{N}$.).
      Otherwise, set $L \leftarrow z$.
    end
  else if $\mathrm{val}_{\mathcal{N}_z}(f_z) < z$ **then**
      begin
        If $C(\theta_z^*) = 1$ **then** stop ($\mathcal{N}$ is infeasible.).
        If $C(\theta_z^*) < 1$ **then** set $U \leftarrow z$ **else** set $L \leftarrow z$.
      end
    else set $L \leftarrow z$.

(22) **until** $U - L < \frac{1}{B^{8m}}$.
Step 3: Decision of an optimal flow in $\mathcal{N}$ if it exists
(23) Set $z \leftarrow \frac{D(\theta_U^*)}{1-C(\theta_U^*)}$.
(24) Find an optimal flow f_z of $\mathcal{N}_z$.
(25) **If** $\mathrm{val}_{\mathcal{N}_z}(f_z) = z$, **then** stop (an optimal flow f_z in $\mathcal{N}$ is obtained.).
(26) **Otherwise**, stop ($\mathcal{N}$ is infeasible.). □

Finding an optimal flow f_z of $\mathcal{N}_z$ for a specific value of z in steps 1 $\sim$ 3, we use one of efficient algorithms for the generalized maximum flow problem. Moreover, in computing $C(\theta_z^*)$ and/or $D(\theta_z^*)$ we need to know the values of π_z^*, which can be obtained by one shortest path computation. We explain the way to calculate π_z^* in detail in the following section.

4. Analysis

In this section, we mainly analyse the correctness of the above algorithm, where at the end of this section we will outline the second algorithm combining the parameterized technique of Megiddo with one of algorithms for the generalized maximum flow problem, and show that it runs in $O(\{\mathrm{M}(n, m, B')\}^2)$ time, where $B' \equiv \max_{a \in A}\{\max\{\gamma_0(a), \gamma_1(a), u(a)\}\}$.

Proposition 4.1: *While the algorithm continues in Steps 1 and 2, it maintains the following two invariants that*
(i) an invariant with respect to U : $C(\theta_U^*) < 1$ *and* $\mathrm{val}_{\mathcal{N}_U}(f_U) < U$,
(ii) an invariant with respect to L : *one of the following three*
(a) $C(\theta_L^*) > 1$ *and* $\mathrm{val}_{\mathcal{N}_L}(f_L) < L$,
(b) $\mathrm{val}_{\mathcal{N}_L}(f_L) > L$,
(c) $C(\theta_L^*) \geq 1$ *and* $\mathrm{val}_{\mathcal{N}_L}(f_L) = L$.

Proof: Figure 1 shows two cases (i)+(ii)(b) and (i)+(ii)(c), where $C(\theta_L^*)$ and $C(\theta_U^*)$ are the slopes of two lines l_L with a point $\mathrm{P}_1(L, \mathrm{val}_{\mathcal{N}_L}(f_L))$ and l_U with a point $\mathrm{P}_2(U, \mathrm{val}_{\mathcal{N}_U}(f_U))$, where (i)+(ii)(a) will be seen in Figure 2. We only prove (i). Initially, suppose that the algorithm does not stop during Step 1. Then the invariant (i) is satisfied from (4) of Step 1 and proposition 2.1. Note that we may assume $C(\theta_U^*) = 0$, though $C(\theta_U^*)$ may not be able to be determined uniquely. While the repeat statement (9)$\sim$(22) continues in Step 2, U is updated at (19) only. So, we have (i). □

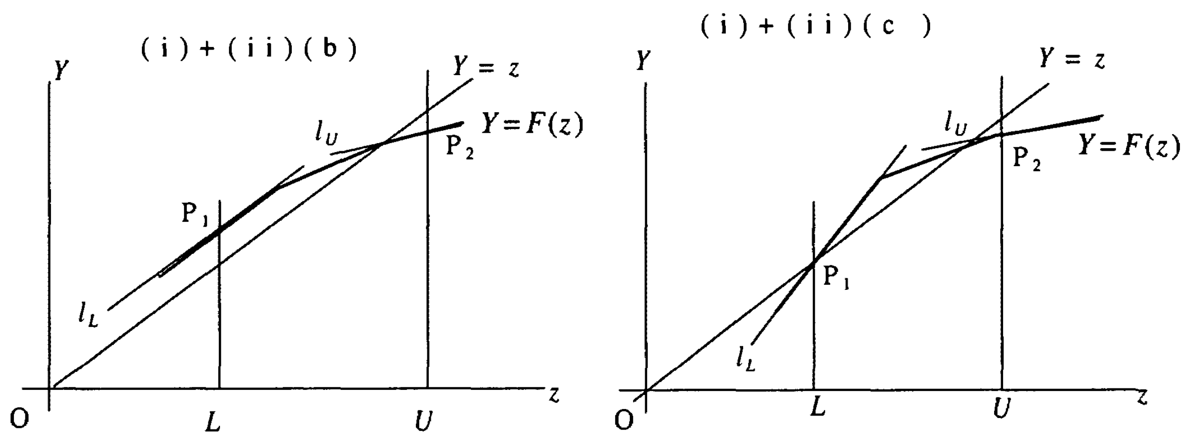

Figure 1: Cases (i)+(ii)(b) and (i)+(ii)(c) of Proposition 4.1

Next, we will show that each interval $[L, U]$ is cut in half. Suppose that there exists an optimal flow in network $\mathcal{N}$. If the algorithm does not stop during Step 1 then the value of the flow is contained in $[L, U]$ at the end of the step. This observation is preserved during Step 2 while the length of interval $[L, U]$ is cut in half.

Proposition 4.2: *Let L_i and U_i be a lower and an upper bounds at the beginning of i-th repetition of the repeat statement (9)∼(22) of Step 2, where L_1 and U_1 are the values obtained at the end of Step 1. Then we have for each i*
(i) $U_{i+1} - L_{i+1} = \frac{U_i - L_i}{2}$,
(ii) If there is the value z^* of an optimal flow in $\mathcal{N}$, then $z^* \in [L_i, U_i]$.

Proof: When we prove (ii), we show that $z^* \in [L_{i+1}, U_{i+1}]$ assuming $z^* \in [L_i, U_i]$. We have seen that (ii) holds for $i = 1$. Let $z \equiv z_i = \frac{U_i + L_i}{2}$ at the beginning of i-th repetition of the repeat statement. First, we consider the case when $\text{val}_{\mathcal{N}_{z_i}}(f_{z_i}) = z_i$. We devide this case into the following two cases:
Case 1.1 ($C(\theta^*_{z_i}) < 1$) : From proposition 2.3, we see that z_i is optimal.
Case 1.2 ($C(\theta^*_{z_i}) \geq 1$) : From proposition 2.3 and (14) of Step 2, we have $z^* \in [z_i, U_i] = [L_{i+1}, U_{i+1}]$. Hence, we have $U_{i+1} - L_{i+1} = U_i - z_i = \frac{U_i - L_i}{2}$. See Figure 2, where $\text{P}'(z_i, \text{val}_{\mathcal{N}_{z_i}}(f_{z_i}))$ is the intersection point of $Y = z$ and $Y = F(z)$.

Next we consider the case when $\text{val}_{\mathcal{N}_{z_i}}(f_{z_i}) \neq z_i$. Moreover, devide this case into the following two cases:
Case 2.1 ($\text{val}_{\mathcal{N}_{z_i}}(f_{z_i}) < z_i$) : From proposition 2.3 and (19) of Step 2, if $C(\theta^*_{z_i}) < 1$ and there is an optimal value z^* then $z^* \in [L_i, z_i] = [L_{i+1}, U_{i+1}]$, which implies (i). For $C(\theta^*_{z_i}) = 1$, we have no optimal flow in $\mathcal{N}$. Otherwise($C(\theta^*_{z_i}) > 1$), an optimal flow in $\mathcal{N}$, if exists, can not belong to $[L_i, z_i]$, i.e. $z^* \in [L_{i+1}, U_{i+1}]$. We also have (i) in this case.
Case 2.2 ($\text{val}_{\mathcal{N}_{z_i}}(f_{z_i}) > z_i$) : We can see (i) and (ii) similarly. □

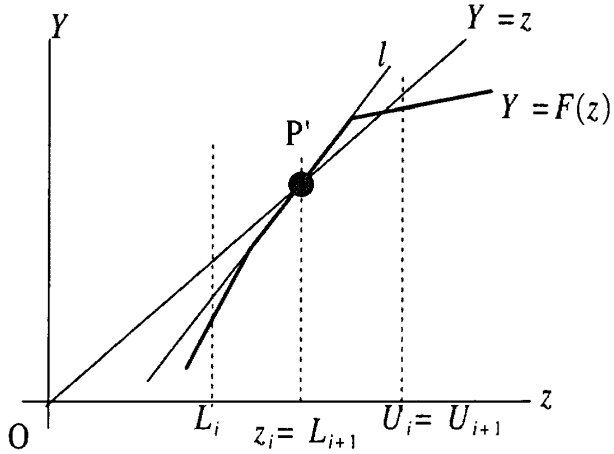

Figure 2: Case 1.2 in the proof of Proposition 4.2

In the following we describe the way to calculate potentials $\pi^*_z(v)$ ($v \in V$). Given network $\mathcal{N}_z$ with a specific value z, find a generalized maximum flow f_z at first. Then let T_z be a subgraph of the residual graph $G(f_z)$ induced by the vertices that can reach the sink t by using the (residual) arcs of $A(f_z)$. From theorem 2.5, $G(f_z)$ has no generalized augmenting paths. So, there exist no flow-generating cycles in $G(f_z)$. Consequently, the canonical labels $\mu_z(v)$ ($v \in V$) are well-defined and can be computed by a Bellman-Ford shortest path algorithm with length $w_z(a) \equiv -\log \gamma^{f_z}(a)$ ($a \in A(T_z)$). Let P_v be such a shortest path from v to t for each $v \in V(T_z)$. Then we have $\mu_z(v) = 2^{w_z(P_v)}$ $(v \in V(T_z))$, where $w_z(P_v) = \sum_{a \in A(P_v)} w_z(a)$. After the shortest path computation, we have $\mu_z(v)$ $(v \in V)$ satisfying that $\mu_z(\partial^- a) \geq \gamma^{f_z}(a)\mu_z(\partial^+ a)$ for each $a \in A(f_z)$, where $\mu_z(v)$ is defined as ∞ for each $v \in V - V(T_z)$. The following proposition shows a relation between μ_z and π_z^*.

Proposition 4.3: *The potentials π_z^* can be obtained by $\pi_z^*(v) = \frac{1}{\mu_z(v)}$ $(v \in V)$, where we define $\pi_z^*(v) = 0$ for each v with $\mu_z(v) = \infty$.* □

The following proposition states that the denominator of each nonzero rational potential $\pi_z^*(v)$ $(v \in V - \{s,t\})$ might be as big as Γ_1, where $\Gamma_1 \equiv \Pi_{a \in A}(\gamma_0(a)\gamma_1(a))$.

Proposition 4.4: *Each nonzero value of $\pi_z^*(v)$ $(v \in V(T_z))$ is an integral multiple of $\Gamma_1{}^{-1}$ and bounded by Γ_1, where T_z is defined above. So are $\theta_z^*(a)$ $(a \in A)$.*

Proof: For each $v \in V(T_z)$, let P_v be a simple shortest (directed) path from v to t of the residual graph $G(f_z)$ with length $w_z(a) \equiv -\log \gamma^{f_z}(a)$ $(a \in A(f_z))$. We can find such a simple path efficiently. Note that P_z is also a highest gain residual path from v to t of $G(f_z)$ with a gain $\gamma^{f_z}(a)$ $(a \bullet A(f_z))$. From proposition 4.3, $\pi_z^*(v)$ is expressed as

$$\Pi_{a \in A(P_v)}\gamma^{f_z}(a) = \frac{\Pi_{a \in A(P_v): a \in A}\gamma(a)}{\Pi_{a \in A(P_v): \bar{a} \in A}\gamma(\bar{a})} = \frac{\Pi_{a \in A(P_v): a \in A}\gamma_0(a)\Pi_{a \in A(P_v): \bar{a} \in A}\gamma_1(\bar{a})}{\Pi_{a \in A(P_v): a \in A}\gamma_1(a)\Pi_{a \in A(P_v): \bar{a} \in A}\gamma_0(\bar{a})}, \tag{4.1}$$

where $\bar{a}$ is the reverse arc of a. This means that $\pi_z^*(v)$ is an integral multiple of Γ_1^{-1} and bounded by Γ_1. Next, we determine the values of $\theta_z^*(a)$ $(a \in A)$. Choose any arc $a \in A$ and assume $u_z(a) > 0$. If $a \in A(f_z)$, then from $a \notin A(P_{\partial^- a})$, $\theta_z^*(a) \equiv [-\pi_z^*(\partial^+ a) + \frac{\gamma_0(a)}{\gamma_1(a)}\pi_z^*(\partial^- a)]^+$ and $\pi_z^* \geq 0$, $\theta_z^*(a)$ is an integral multiple of Γ_1^{-1} and bounded by Γ_1 for $\theta_z^*(a) > 0$. Otherwise, $\bar{a} \in A(f_z)$. From theorem 2.5 we have $\frac{\gamma_1(a)\pi_z^*(\partial^- \bar{a})}{\gamma_0(a)\pi_z^*(\partial^+ \bar{a})} \leq 1$, which implies $-\pi_z^*(\partial^+ a) + \frac{\gamma_0(a)}{\gamma_1(a)}\pi_z^*(\partial^- a) \geq 0$. Hence we have this proposition. □

From proposition 4.4 and integrality of u and β we have

Proposition 4.5: *For any $z \geq 0$, $D(\theta_z^*)$ (resp. $C(\theta_z^*)$) is an integral multiple of $\Gamma_1{}^{-1}$ (resp. $(\Gamma_1\Gamma_2)^{-1}$) where $\Gamma_2 \equiv \Pi_{a \in A}\alpha_1(a)$.* □

Let L_i and U_i be a lower and an upper bounds at the beginning of i-th repetition of the repeat statement of Step 2 for $i \geq 1$. Consider an intersection between $Y = z$ and a line with a slope $C(\theta_{U_i}^*)$ passing through a point $(U_i, \mathrm{val}_{N_{U_i}}(f_{U_i}))$. The line is $F(z) = C(\theta_{U_i}^*)z + D(\theta_{U_i}^*)$. From proposition 4.1, the intersection is well-defined and is (z_i', z_i') with $z_i' = \frac{D(\theta_{U_i}^*)}{1 - C(\theta_{U_i}^*)}$. Let $H(z_i') \equiv z_i' - \mathrm{val}_{\mathcal{N}_{z_i'}}(f_{z_i'})$.

Proposition 4.6: *If $H(z_i') > 0$ and any one of the following conditions are satisfied during step 2, then we have $U_i - L_i \geq \frac{1}{B^{8m}}$.*
(i) $L_i < \mathrm{val}_{\mathcal{N}_{L_i}}(f_{L_i})$,
(ii) $L_i = \mathrm{val}_{\mathcal{N}_{L_i}}(f_{L_i})$ and $C(\theta_{L_i}^*) \geq 1$,
(iii) $L_i > \mathrm{val}_{\mathcal{N}_{L_i}}(f_{L_i})$, $C(\theta_{L_i}^*) > 1$, $z_i' \geq L_i$ and $H(z_i') \leq U_i - L_i$.

Proof: Note that these conditions (i)~(iii) correspond to (ii)(b), (ii)(c), and (ii)(a) in proposition 4.1, respectively. The left graph (I) of Figure 3 shows a situation of (iii), where $P'_1(L_i, \mathrm{val}_{\mathcal{N}_{L_i}}(f_{L_i}))$, $P'_2(U_i, \mathrm{val}_{\mathcal{N}_{U_i}}(f_{U_i}))$, and $P'_4(z_i', \mathrm{val}_{\mathcal{N}_{z_i'}}(f_{z_i'}))$ are points on $Y = F(z)$. Moreover, we have $\overline{P'_3P'_4} = H(z_i')$. From $z_i' = \frac{D(\theta_{U_i}^*)}{1 - C(\theta_{U_i}^*)}$ and $\mathrm{val}_{\mathcal{N}_{z_i'}}(f_{z_i'}) = C(\theta_{z_i'}^*)z' + D(\theta_{z_i'}^*)$, we have

$$H(z_i') = z_i' - C(\theta^*_{z_i'})z_i' - D(\theta^*_{z_i'}) = (1 - C(\theta^*_{z_i'}))\frac{D(\theta^*_{U_i})}{1 - C(\theta^*_{U_i})} - D(\theta^*_{z_i'}) > 0. \quad (4.2)$$

From proposition 4.5, $H(z_i')$ is an integral multiple of $(\Gamma_1^2\Gamma_2(\Gamma_1\Gamma_2 - J))^{-1}$ for some integer J $(0 \leq J < \Gamma_1\Gamma_2)$. Any case of the above conditions (i)~(iii) satisfies $U_i - L_i \geq H(z_i')$. Note $H(z_i') \leq z_i' - L_i$ for cases (i) and (ii), because $\mathrm{val}_{\mathcal{N}_{z_i'}}(f_{z_i'}) = z_i' - H(z_i')$ and $Y = F(z)$ is nondecreasing. So, we have $U_i - L_i \geq (\Gamma_1^3\Gamma_2^2)^{-1} \geq \frac{1}{B^{8m}}$. □

Suppose that we have reached Step 3. If $H(z') > 0$, then from proposition 4.6 we only have the invariant (i)+(ii)(a) with $z' < L$ or $H(z') > U - L$. If $H(z') = 0$, then we have an optimal flow $f_{z'}$ in $\mathcal{N}$ from proposition 4.1. Assume $H(z') > 0$. If $z' < L$, then we see that $\mathcal{N}$ is infeasible. The remaining case is as follows: $H(z') > U - L$, $L > \mathrm{val}_{\mathcal{N}_L}(f_L)$, $C(\theta_L^*) > 1$ and $z' \geq L$.

Proposition 4.7: *If $H(z') > U - L$, $L > \mathrm{val}_{\mathcal{N}_L}(f_L)$, $C(\theta_L^*) > 1$ and $z' \geq L$ after (24) of Step 3, then we have no optimal flows in network $\mathcal{N}$.*

Proof: From $H(z') > U - L$ and $U > z'$ we have $z' - L < H(z')$. Consider the right graph (II) of Figure 3, where we have $\overline{P_3P_4} = \overline{P_4P_5} = H(z')$. We can not find any optimal solutions in the region of a triangle $P_3P_4P_5$. □

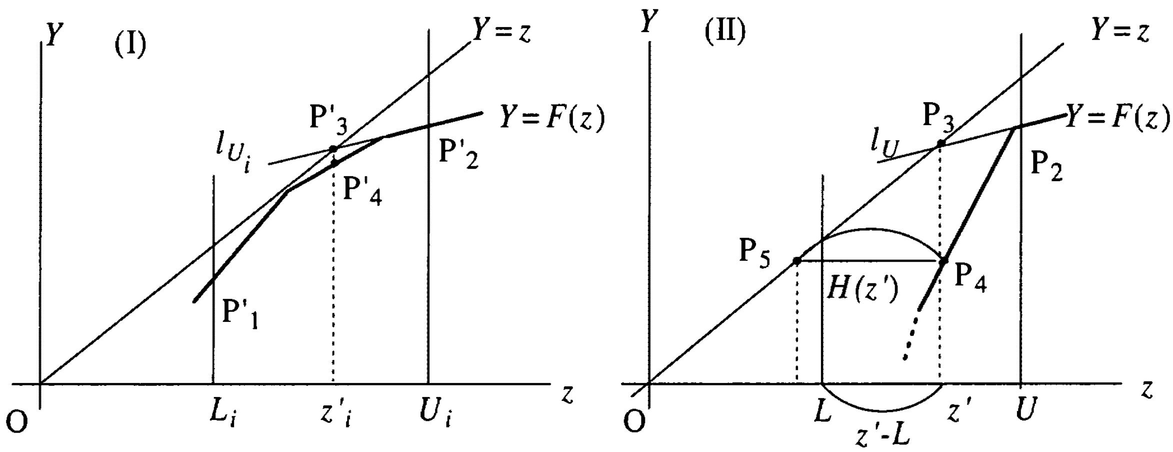

Figure 3: Situations (I) and (II) of Propositions 4.6 and 4.7

In order to compute an optimal flow in $\mathcal{N}_z$ in Steps 1~3 we will use one of efficient algorithms for the generalized maximum flow problem. Such two examples are *Iterative Fat-Path Algorithm* by Radzik [10] and *Highest-Gain Augmenting Path Algorithm* due to Goldfarb, Jin, and Orlin [4]. The former is a best known combinatorial approximate algorithm, while the latter is a best known combinatorial exact one with $O(m^2(m + n\log n)\log B')$, where B' is the largest integer used to represent the gains and capacities in the network. We take up the former here. His algorithm repeatedly cancels flow-generating cycles for finding an ϵ-optimal flow in network $\mathcal{N}' = (G, u, \gamma, s, t)$ with a given ϵ $(1 \geq \epsilon > 0)$. A generalized flow f in $\mathcal{N}'$ is *ϵ-optimal* if $\mathrm{val}_{\mathcal{N}'}(f') - \mathrm{val}_{\mathcal{N}'}(f) \leq \epsilon\ \mathrm{val}_{\mathcal{N}'}(f')$ for some generalized maximum flow f' in $\mathcal{N}'$. The next lemma from [11] indicates that if a generalized flow is ϵ-optimal for sufficiently small ϵ, then it is essentially optimal.

Lemma 4.8: *Let f be a B'^{-3m}-optimal flow in network $\mathcal{N}'$. Then we can compute an optimal flow in $\mathcal{N}'$ in $\tilde{O}(mn\min\{m, n\log B'\})$ time, where $\tilde{O}$ is the complexity deleting a factor polylogarithmic in n and* $B' = \max_{a\in A}\{\max\{\gamma_0(a), \gamma_1(a), u(a)\}\}$. □

The description of his algorithm is as follows where we wll omit the details here.

Iterative_Fat-Path_Algorithm$(\mathcal{N}', \epsilon)$:
$\Delta \leftarrow \Delta_0$;
$f \leftarrow 0$, where 0 is zero flow in $\mathcal{N}'$;
while $\Delta > \epsilon \ \mathrm{val}_{\mathcal{N}'}(f)$ **do**
begin
$(h, \mu) \leftarrow$ **Cancel_Cycles** $(\mathcal{N}'(f), \frac{\epsilon}{25n^4})$;
$f \leftarrow f + h$;
$f \leftarrow$ **Remove_Cycles**(f);
$\tilde{\gamma}(a) \leftarrow \frac{\gamma(a)}{1+\frac{\epsilon}{25n^3}}$ for each (residual) arc a of $\mathcal{N}'(f)$;
$\tilde{g} \leftarrow$ **Fat_Augmentation**$((G(f), u^f, \tilde{\gamma}), \frac{\Delta}{4m})$;
$g \leftarrow$ **Interpretation**$(\tilde{g})$;
$f \leftarrow f + g$;
$\Delta \leftarrow \frac{\Delta}{2}$;
end;
return f; □

In this algorithm, we can choose Δ_0 as any value satisfying $\frac{\Delta_0}{n} \leq \mathrm{val}_{\mathcal{N}'}(f') \leq \Delta_0$ for some optimal flow f' in $\mathcal{N}'$. Moreover, procedures not defined are as follows. **Cancel_Cycles**$(\mathcal{N}', \epsilon)$ returns a pair (h, μ) of a pseudoflow h with $\partial_\gamma h(v) \leq 0$ $(v \in V)$ and a labeling μ such that $\gamma_\mu(a) \leq 1 + \epsilon$ for each a of $\mathcal{N}'(h)$ where a *pseudoflow* h in $\mathcal{N}'$ is a function $h : A \to \mathbf{R}_+$ satisfying (2.2). **Remove_Cycles**(f) repeatedly finds and deletes a flow-generating cycle flow from f, and returns the final pseudoflow without such cycle flows. **Fat_Augmentation**$((G(f), u^f, \tilde{\gamma}), \frac{\Delta}{4m})$ finds a pseudoflow $\tilde{g}$ in $(G(f), u^f, \tilde{\gamma})$ such that $|\mathrm{val}_{(G(f),u^f,\tilde{\gamma})}(g') - \mathrm{val}_{(G(f),u^f,\tilde{\gamma})}(\tilde{g})| \leq \frac{\Delta}{4}$ for some optimal flow g' in $(G(f), u^f, \tilde{\gamma})$. **Interpretation**$(\tilde{g})$ returns a function $g : A \to \mathbf{R}_+$ such that g is equal to $\tilde{g}$ as a function restricted on $A(f)$ but is interpreted as a pseudoflow in $\mathcal{N}'$. Concerning the complexity of the above algorithm, the following facts (i) and (ii) are known from [10]: (i) The running time of one iteration of the while statement is $O(m(m + n\log n\log(\frac{n}{\epsilon}\log B')))$ where $B' = \max_{a\in A}\{\max\{\gamma_0(a), \gamma_1(a), u(a)\}\}$; (ii) There are at most $O(\log\frac{n}{\epsilon})$ iterations. We summarize the total complexity as a theorem.

Theorem 4.9: *[10] Iterative Fat-Path algorithm computes a generalized maximum flow of network $\mathcal{N}'$ in $O(m\log\frac{n}{\epsilon}(m + n\log n\log(\frac{n}{\epsilon}\log B'))$ time.* □

From theorem 4.9 and lemma 4.8, we can find an optimal flow in network $\mathcal{N}$ in $O(m^2(m + mn\log n\log B')\log B')$ time. Now we show the total running time of our algorithm:

Theorem 4.10: *Our binary search algorithm runs in $O(m\log B\ \mathrm{M}(n, m, B'))$ time, where $B = \max\{B', B''\}$ for B', B'' in our algorithm, and $\mathrm{M}(n, m, B')$ denotes the complexity of solving a generalized maximum flow problem in a network with n vertices, and m arcs, and nonnegative capacities and gains expressed as integers between 1 and B'.*

Proof: We use Bellman-Ford shortest path algorithm to compute π_z^*. Such a shortest path computation takes $O(nm)$. From $O(nm + \mathrm{M}(n, m, B')) = O(\mathrm{M}(n, m, B'))$, Steps 1 and 3 run in $O(\mathrm{M}(n, m, B'))$. The bottleneck operations are in Step 2. From $\Gamma_1^3\Gamma_2^2 \leq B^{8m}$, the number

k of repetitions of the repeat statement in Step 2 satisfies $\frac{2B^2}{2^k} \leq \frac{1}{B^{8m}}$. Each repetition takes $\mathrm{M}(n, m, B')$ time. From $B \geq B'$, we have this theorem. □

We present an example in the following.

Example: An instace $\mathcal{N}$ has G with $V = \{s, x, y, t\}$ and $A = \{(s,x), (s,y), (x,y), (x,t), (y,t)\}$. A triple $(u(a), \gamma(a), \alpha(a))$ for each $a \in A$ is given by $(s,x) : (8, \frac{1}{2}, \frac{4}{5})$, $(s,y) : (5, \frac{1}{2}, \frac{1}{2})$, $(x,y) : (2, \frac{1}{3}, \frac{4}{5})$, $(x,t) : (1, \frac{3}{2}, \frac{1}{2})$, $(y,t) : (4, 2, \frac{3}{5})$ where $\beta = 0$. Then we have $B = B' = 8$. Apply our algorithm for this instance. At the end of Step 1, we have $U = 2 \times 64 = 128$ and $L = 0$ and go to Step 2. After the fifth iteration of the repeat statement in Step 2, we have $L = 4$ and $U = 8$. Continuing these processes, we have $L = \frac{14}{3} - \frac{1}{3 \times 2^{121}}$ and $U = \frac{14}{3} + \frac{1}{3 \times 2^{120}}$ after Step 2. Finally, we get the optimal value $z^* = \frac{14}{3}$ during Step 3. ●

The second algorithm uses the parameterized search technique of Megiddo instead of our binary search in addition to one of efficient algorithms for finding generalized maximum flows. The reasons for proposing the second algorithm are as follows. If B' is sufficiently smaller than B'', then it is possible that the second algorithm is faster than the first one where B', B'' are defined in Step 1 of the first algorithm. Moreover, it can be regarded as a generalization of a parameterized algorithm given by Zimmermann [12]. We briefly explain the second algorithm because the analysis is quite similar to Zimmermann's. Choose an efficient algorithm $\mathcal{A}$ for the generalized maximum flow problem. Note that it is easy to test feasibility of the generalized maximum flow problem, because lower capacities are zero in our model. Each step of the algorithm $\mathcal{A}$ consists of additions, scalar multiplications, and comparisons. We use the algorithm $\mathcal{A}$ to obtain a generalized maximum flow in network $\mathcal{N}_z$, where this generalized maximum flow contains z as a parameter. Each scalar value p considered by the algorithm corresponds to a linear function $p + zq$ for some scalar q in the parameterized problem. Instead of adding $p + p'$ for another scalar p', we add linear functions $(p + zq) + (p' + zq')$, where q' is a scalar. If we compare $p + zq$ with $p' + zq'$, then we need to know the critical value z'' determined by $p + zq = p' + zq'$ unless $q = q'$. Then we can decide whether $z \geq z''$ or $z < z''$ by running the algorithm $\mathcal{A}$ for network $\mathcal{N}_{z''}$. In our case, add a super sink t' and an additional arc (t, t') to network $\mathcal{N}_z$. Define $u_z(t, t') = z$ and $\gamma(t, t') = \alpha(t, t') = 1$. Let $\mathcal{N}_z^*$ be the enlarged network. Suppose that network $\mathcal{N}_{z''}^*$ is feasible. If (t, t') is saturated then we have $z \geq z''$. Otherwise, we have $z < z''$. We say that (t, t') is *saturated* if the flow value of (t, t') is equal to $u_z(t, t')$. In the case when $\mathcal{N}_{z''}^*$ is infeasible, we have $z > z''$. With that information in hand, we can work the algorithm $\mathcal{A}$ for the parameterized problem $\mathcal{N}_z^*$. Finally, we get a generalized maximum flow f_z in network $\mathcal{N}_z^*$. If (t, t') is not saturated, then there exists no generalized maximum balanced flow in $\mathcal{N}$. Otherwise, the optimal value is $z^* = \min_{a \in A} z_a$, where $z_a = \max\{z : 0 \leq f_z(a) \leq u_z(a)\}$ for each $a \in A$.

If we use a highest-gain augmenting path method in the second algorithm, then we have

Theorem 4.11: *The second algorithm runs in* $O(\{m^2(m + n \log n) \log B'\}^2)$ *time, where* $B' = \max_{a \in A}\{\max\{\gamma_0(a), \gamma_1(a), u(a)\}\}$. □

5. Conclusions

As a new generalization of the maximum balanced flow problem considered by Minoux, we gave the generalized maximum balanced flow problem and proposed two efficient algorithms. One of future researches is to consider our model with nonzero lower capacities $l(a)$ $(a \in A)$. Though we can analyse the model with $l(a)$ $(a \in A)$ in the same way as in this paper, we must solve a feasibility problem, i.e. a problem of testing whether there is a feasible generalized flow in the underlying network with $l(a)$ $(a \in A)$. The authors believe that this feasibility problem is open. On making our algorithms strongly polynomial even if $l(a) = 0$ $(a \in A)$, there is an obstacle which is another open question of answering whether it is possible to solve the generalized maximum flow problem in strongly polynomial time.

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Akira Nakayama
Faculty of Administration and Social Sciences
Fukushima University
1 Kanayagawa, Fukushima, 960-1296 Japan
E-mail: nakayama@ads.fukushima-u.ac.jp

Chien Fei Su
President, Tokuritsu Co. Ltd.
318-1 Nishiyajima, Ohta, Gumma,
373-0823 Japan
E-mail: su-chan@geocities.co.jp