Journal of the Operations Research
Society of Japan
Vol. 45, No. 2, June 2002

VALUING CORPORATE DEBT: THE EFFECT OF CROSS-HOLDINGS OF STOCK AND DEBT

Teruyoshi Suzuki
NLI Research Institute and Kyoto University

(Received July 12, 1999; Final January 11, 2001)

Abstract We have developed a simple approach to valuing risky corporate debt when corporations own securities issued by other corporations. We assume that corporate debt can be valued as an option on corporate *business* asset value, and derive payoff functions when there exist cross-holdings of stock or debt between two firms. Next we show that payoff functions with multiple cross-holdings can be solved by the contraction principle. The payoff functions which we derive provide a number of insights about the risk structure of company cross-holdings. First, the Modigliani-Miller theorem can obtain when there exist cross-holdings between firms. Second, by establishing cross-shareholdings each of stock holders distributes a part of its payoff values to the bond holder of the other's firm, so that both firms can decrease credit risks by cross-shareholdings. In the numerical examples, we show that the correlation in firms can be a critical condition for reducing credit risk by cross-holdings of stock using Monte Carlo simulation. Moreover, we show we can calculate the default spread easily when complicated cross-holdings exist in many firms by a simulation that includes the contraction principle method. By using our method, we can reevaluate which cross-held shares are beneficial or disadvantageous for the cross-holding firms.

1. Introduction

The cross-holding of stock among companies (also called cross-shareholding and reciprocal ownership of shares) is a common practice in Japan, Korea and Europe. Its economic significance in post-war Japan is apparent from the vast cross-shareholding patterns that reflect the structure of the pre-war zaibatsu, and from the implications for corporate finance, which has heavily depended on bank financing. Yonezawa[17] proposes four hypothesis about the role of cross-holding as follows: (1) protection from takeover, (2) benefits of cooperation, (3) reduction of agency cost, (4) reduction of default risk.

In the early 1990s, various papers focused on the effect of cross-holding on stock markets, particularly since the market capitalization of the Tokyo Stock Exchange at that time was the highest in the world. Bierman and Noyes[1] point out that merely aggregating each company's market value to evaluate market capitalization is misleading because cross-holdings cause double counting. Ikeda[9] shows why Japanese firms have had considerably higher price/earning ratios than U.S. firms. Numerous other articles focused on the effect of cross-holding on the stock market, including Bierman and Noyes[2], Brioschi, Buzzacchi and Massimo[4], Feguson and Hitzig[6], McDonald[13] and Sinha[16]. There are also many Japanese papers on cross-holding written in the 1980s and early 1990s that examine the effect of cross-holding on corporate governance and the stock market.

In the late 1990s, many Japanese companies began unwinding their cross-shareholdings at market value amid growing difficulties in the harsh business environment and poor performance of their shareholdings. As a result, public interest in cross-shareholding resurged in Japan. For the same reasons, the valuation of corporate debt became an important concern among institutional investors. We thus need to develop a method for valuing corporate debt when stock cross-holdings exist.

Few studies on this problem are found in the literature. The valuation of corporate debt incorporating the effect of capital structures is investigated by Merton[14], Black and Cox[3], Geske[7], Longstaff and Schwartz[12] Briys and Varenne[5], and Kijima and Suzuki[10]. However, they do not consider cross-holding because it does not exist in the U.S.

From the preceding two facts – that most Japanese papers on cross-holding focus on the stock market, and that cross-holding does not exist in U.S. – it is clear that the effect of cross-holding on the corporate debt market has not been sufficiently investigated.

This paper develops a simple new approach to valuing corporate debt that considers cross-holding by extending the Merton model in two steps. First, we define cross-holding in the context of the corporate capital structure. Second, we show the payoff functions of corporate securities when cross-holdings exist among two firms. When there exist complicated cross-holdings among $N > 3$ firms, we cannot obtain payoff functions by closed-form expressions as in the two-firm case. But we can solve this problem by the contraction method.

In Japanese seminal papers, Kurasawa[11] discusses the rationality of cross-holdings, and Yonezawa[17] investigates the role of cross-holding in Japanese society. Their model describes cross-holdings by using corporate cash flows and considering how firms can be in default. Our approach is distinct from their model in that we use balance sheets to model cross-holdings. By using balance sheets, we can use an option approach model that is widely developed in the literature. In addition, by using the payoff functions that we derive, we provide a number of insights on the effect of cross-holdings on corporate debt.

The remainder of this article is organized as follows: Section 2 discusses corporate capital structures with respect to cross-holdings, and defines cross-holdings of stock and cross-holdings of debt. Section 3 derives a payoff function for corporate securities that incorporates both cross-holdings of stock and cross-holdings of debt. Section 4 shows that the Modigliani-Miller theorem obtains when there exist cross-shareholdings between firms. Section 5 presents the results of the theoretical analysis by using simple examples. Section 6 shows numerical examples that provide insights into the unwinding of cross-holdings. Section 7 summarizes our findings and makes concluding remarks.

2. Corporate Capital Structure

To provide a corporate capital structure when firms own stocks and bonds of other firms, we make the following assumptions.

Assumption 1 (Corporate Liabilities) *Firm $i (i = 1, \cdots, N)$ has two classes of claims: a single homogeneous class of debt and the residual claim, equity with continuous dividend payment per unit time, d_i. Each debt issue is promised a continuous coupon payment per unit time, c_i, which continues until the common maturity date of the bonds, T. Also, each firm i promises to pay a total of B_i dollars to bondholders at the maturity T.*

Assumption 2 (Business Assets) *Let $V_k(t)$ denote the value of business assets at time t such that $V_k(t) \geq 0, 0 \leq t \leq T$. Assume that business asset values are independently distributed. Assume further that each firm $i (i = 1, \cdots, N)$ has common opportunities to invest in $V_k(t)$ for business asset $k = 1, \cdots, M$.*

Assumption 3 (Corporate Assets) *Let $F_i(t)$ denote the debt value of firm i at time t and $S_i(t)$ denote the equity value of firm i at time t. Then the total asset value $A_i(t)$ of firm* *i is given by:*

$$A_i(t) = \sum_{k=1}^{M} v_{ik} V_k(t) + \sum_{j=1}^{N} s_{ij} S_j(t) + \sum_{j=1}^{N} f_{ij} F_j(t), \quad i = 1, \cdots, N, \quad 0 \le t \le T, \tag{1}$$

where v_{ik} is the ratio of firm i's investment value in business asset V_k, f_{ij} is the ratio of the debt value held by firm i to the total debt value of firm j, and s_{ij} is the ratio of the equity value held by firm i to the total equity value of firm j.

Remark 2.1 We specify the firm i by $v_{i\cdot}, s_{i\cdot}, f_{i\cdot}$ in portfolio A_i. Although we cannot distinguish firm i from another firm $j \ne i$ having the same portfolio as firm i, this condition is adequate for valuing corporate debt.

Assumption 4 *The ratios s_{ij}, f_{ij} satisfy the equations,*

$$0 \le \sum_{i=1}^{N} s_{ij} \le 1, \quad 0 \le \sum_{i=1}^{N} f_{ij} \le 1, \quad s_{jj} = f_{jj} = 0, \quad 1 \le j \le N.$$

Assumption 5 *There exist at least one j such that*

$$\sum_{i=1}^{N} s_{ij} < 1 \quad or \quad \sum_{i=1}^{N} f_{ij} < 1.$$

Assumption 1 provides the same liability structure supposed by Merton[14]. From Assumptions 4 and 5, it follows that there must be at least one firm which has stockholders or bondholders outside of the N firms. Stated differently, there can exist a firm that all stockholders and all debt holders are among the N firms. From Assumptions 1 through 3, we can define *cross-holdings of stock* and *cross-holdings of debt.*

Definition 1 (Cross-Holdings of Stock) *If there exists a pair of firms $(i,j), i \ne j$, such that $s_{ij} > 0, s_{ji} > 0$, then firms i and j are called to be cross-holdings of stock.*

Definition 2 (Cross-Holdings of Debt) *If there exists a pair of firms $(i,j), i \ne j$, such that $f_{ij} > 0, f_{ji} > 0$, then firms i and j are called to be cross-holdings of debt.*

Cross-holdings of stock are called *mochiai* (mo-chee-eye) in Japanese. Many companies own various kinds of stocks, and usually cross-hold each other's stock intentionally. Hence there exist complicated multiple cross-holdings of stock among Japanese companies. Cross-holdings of debt exist occasionally in Japan. An industrial company that receives a loan from a bank will occasionally deposit part of the loan in the bank. When it makes this deposit, called a counter balance, cross-holdings of debt are said to exist between the industrial company and the bank.

3. Corporate Debt Payoff Functions with Cross-Holdings

In this section, we derive a payoff function for corporate debt and stock at the common maturity date T. First, we derive a payoff function when there exist cross-holdings between two firms that have a capital structure satisfying Assumptions 1 through 5. We then extend it to the $N \ge 3$ firm case.

Assume that bankruptcy can only occur at bond maturity T. Assume further that in this case, bondholders receive the remaining assets $A_i(T)$ and stockholders receive nothing. Then, from the Assumption 1, the corporate securities payoff functions can be expressed as

$$F_i(T) = \min[A_i(T), B_i], \tag{2}$$

$$S_i(T) = \max[A_i(T) - B_i, 0], \quad i = 1, \cdots, N, \tag{3}$$

where $A_i(T)$ follows equation (1). Here, we suppose that the exogeneous variables are *business* asset values.[1] Then we should note that the payoffs $F_i(T)$, $S_i(T)$ are also functions of $F_i(T)$, $S_i(T)$ because asset value $A_i(T)$ is a function of $F_i(T)$ and $S_i(T)$. Thus payoff functions are not clear when cross-holdings exist. We will solve this problem in the present section.

3.1. Two firms case

Assume there are two firms ($N = 2$) which satisfy Assumptions 1 through 5. Firm 1 owns stocks and bonds of firm 2, and firm 2 also owns stocks and bonds of firm 1. In this case, equation (1), (2) and (3) can be rewritten as

$$\begin{cases} S_1(T) &= \max[A_1(T) - B_1, 0], \\ S_2(T) &= \max[A_2(T) - B_2, 0], \\ F_1(T) &= \min[A_1(T), B_1], \\ F_2(T) &= \min[A_2(T), B_2], \end{cases}$$

where

$$\begin{aligned} A_1(T) &= \sum_{k=1}^{M} v_{1k} V_k(T) + s_{12} S_2(T) + f_{12} F_2(T), \\ A_2(T) &= \sum_{k=1}^{M} v_{2k} V_k(T) + s_{21} S_1(T) + f_{21} F_1(T). \end{aligned}$$

This is a system of equations which have max and min where $F_1(T)$, $F_2(T)$, $S_1(T)$, $S_2(T)$ are unknown; $V_k(T)(k = 1, \cdots, M)$, B_1, B_2 are known. Next, noting that $V_k(T) = V_k$, $S_i(T) = S_i$, $F_i(T) = F_i$, we solve these equations under the following assumption that is needed in order to derive closed form expression.

Assumption 6

$$s_{12}s_{21} < 1, \quad f_{12}f_{21} < 1, \quad s_{12}f_{21} < 1, \quad s_{21}f_{12} < 1.$$

Proposition 1 *If firm 1 and firm 2 satisfy Assumptions* 1 *through* 6, *then their debt and equity payoff functions are given as*

$$F_1(T) = \begin{cases} B_1, & \boldsymbol{W} \in A_{ss} \\ B_1, & \boldsymbol{W} \in A_{sd} \\ \dfrac{W_1 + s_{12}(W_2 - B_2) + f_{12}B_2}{1 - f_{21}s_{12}}, & \boldsymbol{W} \in A_{ds} \\ \dfrac{W_1 + f_{12}W_2}{1 - f_{12}f_{21}}, & \boldsymbol{W} \in A_{dd} \end{cases} \qquad S_1(T) = \begin{cases} \dfrac{W_1 - B_1' + s_{12}(W_2 - B_2')}{1 - s_{12}s_{21}}, & \boldsymbol{W} \in A_{ss} \\ \dfrac{W_1 - B_1' + f_{12}(W_2 - B_2')}{1 - s_{21}f_{12}}, & \boldsymbol{W} \in A_{sd} \\ 0, & \boldsymbol{W} \in A_{ds} \\ 0, & \boldsymbol{W} \in A_{dd} \end{cases}$$

$$F_2(T) = \begin{cases} B_2, & \boldsymbol{W} \in A_{ss} \\ \dfrac{s_{21}(W_1 - B_1) + W_2 + f_{21}B_1}{1 - f_{12}s_{21}}, & \boldsymbol{W} \in A_{sd} \\ B_2, & \boldsymbol{W} \in A_{ds} \\ \dfrac{f_{21}W_1 + W_2}{1 - f_{21}f_{12}}, & \boldsymbol{W} \in A_{dd} \end{cases} \qquad S_2(T) = \begin{cases} \dfrac{s_{21}(W_1 - B_1') + W_2 - B_2'}{1 - s_{21}s_{12}}, & \boldsymbol{W} \in A_{ss} \\ 0, & \boldsymbol{W} \in A_{sd} \\ \dfrac{f_{21}(W_1 - B_1') + W_2 - B_2'}{1 - s_{12}f_{21}}, & \boldsymbol{W} \in A_{ds} \\ 0, & \boldsymbol{W} \in A_{dd} \end{cases}$$

where

$$B_1' = B_1 - f_{12}B_2, \quad B_2' = B_2 - f_{21}B_1,$$

and where

$$\boldsymbol{W} = (W_1, W_2), \quad W_1 = \sum_{k=1}^{M} v_{1k} V_k, \quad W_2 = \sum_{k=1}^{M} v_{2k} V_k,$$

[1] Merton[14] supposes that the firm asset value is the exogeneous variable.

and where

$$\begin{aligned}
A_{ss} &\equiv \{\boldsymbol{W} : \ W_1 + s_{12}(W_2 - B_2') > B_1', \ \ W_2 + s_{21}(W_1 - B_1') > B_2'\}, \\
A_{sd} &\equiv \{\boldsymbol{W} : \ W_1 + f_{12}(W_2 - B_2') > B_1', \ \ W_2 + s_{21}(W_1 - B_1') < B_2'\}, \\
A_{ds} &\equiv \{\boldsymbol{W} : \ W_1 + s_{12}(W_2 - B_2') < B_1', \ \ W_2 + f_{21}(W_1 - B_1') > B_2'\}, \\
A_{dd} &\equiv \{\boldsymbol{W} : \ W_1 + f_{12}(W_2 - B_2') < B_1', \ \ W_2 + f_{21}(W_1 - B_1') < B_2'\}.
\end{aligned}$$

Proof: See Appendix.

This proposition shows that the securities of firms with cross-holdings can be valued as basket type options by considering the *business* assets as underlying assets. Also, these payoff functions are dependent on the state of the two firms, which can fall into one of four areas:

A_{ss}) Both firms are solvent;
A_{ds}) Firm 1 is in default, and firm 2 is solvent;
A_{sd}) Firm 1 is solvent, and firm 2 is in default;
A_{dd}) Both firms are in default.

Figure 1 shows these four areas. A number of important insights about the risk structure of cross-holdings emerge from Figure 1. For example, the slope of the four lines depends on ratios s_{12}, s_{21}, f_{12}, and f_{21} respectively. Using Proposition 1, the risk structure of cross-holdings is clarified in a later section.

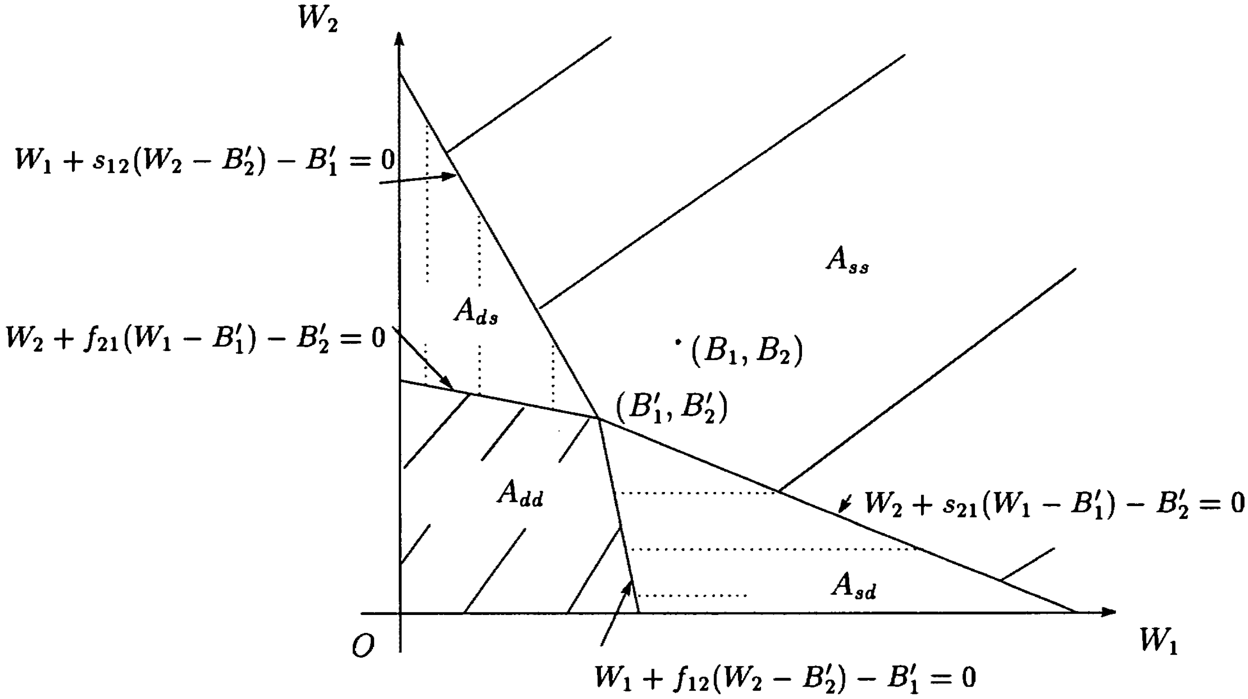


Figure 1: Default Areas (Proposition 1)

(A_{ss}) Both firms are solvent; (A_{ds}) Firm 1 is in default, firm 2 is solvent; (A_{sd}) Firm 1 is solvent, firm 2 is in default; (A_{dd}) Both firms are in default.

3.2. *N* firms case

It is difficult to extend Proposition 1 to an N firms case because with N firms, we need 2^N areas to clarify the payoff functions. We next show that payoff functions in the case of N firms are given by a simulation with the following notation.

1. Let $\boldsymbol{S}(t) = (S_1(t), \cdots, S_N(t))^t$, denote equity values of N firms at time t, and let $\boldsymbol{F}(t) = (F_1(t), \cdots, F_N(t))^t$ denote debt values of N firms at time t. Then equity and debt values of N firms can be defined as

$$\boldsymbol{Z}(t) = (\boldsymbol{S}(t), \boldsymbol{F}(t))^t, \tag{4}$$

where X^t is the transposed matrix of X.

2. Let $\boldsymbol{V}(t) = (V_1(t), \cdots, V_M(t))^t$ denote business asset values at time t. Then the total asset values of the N firms are defined as

$$\boldsymbol{A}(t) = (A_1(t), \cdots, A_N(t))^t = \boldsymbol{v}\boldsymbol{V}(t) + \boldsymbol{z}\boldsymbol{Z}(t).$$

where $\boldsymbol{v}$ denotes the ratio of investment value to business assets of N firms,

$$\boldsymbol{v} = \begin{pmatrix} v_{11} & \cdots & v_{1M} \\ \vdots & & \vdots \\ v_{N1} & \cdots & v_{NM} \end{pmatrix},$$

and where $\boldsymbol{z}$ denotes the ratio of investment value to bonds and stocks of N firms,

$$\boldsymbol{z} = \begin{pmatrix} s_{11} & \cdots & s_{1N} & f_{11} & \cdots & f_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ s_{N1} & \cdots & s_{NN} & f_{N1} & \cdots & f_{NN} \end{pmatrix}.$$

3. Let $\boldsymbol{B} = (B_1, \cdots, B_N)^t$ denote the face value of N firms.

Using these notations, equation (3) can be rewritten as

$$\boldsymbol{Z}(T) = g(\boldsymbol{Z}(T)), \tag{5}$$

where

$$g(\boldsymbol{Z}(t)) \equiv \begin{pmatrix} \max[\boldsymbol{v}\boldsymbol{V}(t) + \boldsymbol{z}\boldsymbol{Z}(t) - \boldsymbol{B}, \boldsymbol{0}] \\ \min[\boldsymbol{v}\boldsymbol{V}(t) + \boldsymbol{z}\boldsymbol{Z}(t), \boldsymbol{B}] \end{pmatrix}. \tag{6}$$

It is clear that equation (5) is a fixed point problem for $\boldsymbol{Z}(T)$. Now, we will show that it can be solved by the contraction principle.

Proposition 2 *Let $\boldsymbol{R}^k_{[0,\infty)} = \{(x_1, \ldots, x_k)^t : x_1, \ldots, x_k \in [0,\infty)\}$ denote k-dimensional non-negative real number space. Suppose there exist N firms which satisfy Assumptions 1 through 5. Then using function g which is defined by equation* (5), (6), *payoff functions $\boldsymbol{Z}(T)$ can be given as*

$$\boldsymbol{Z}(T) = \lim_{n\to\infty} g^n(\boldsymbol{Z}_0(T)), \text{ for all } \boldsymbol{Z}_0(T) \in \boldsymbol{R}^{2N}_{[0,\infty)}$$

where

$$\begin{aligned} \boldsymbol{Z}_1(T) &= g(\boldsymbol{Z}_0(T)), \\ \boldsymbol{Z}_2(T) &= g(\boldsymbol{Z}_1(t)) = g^2(\boldsymbol{Z}_0(T)), \\ &\vdots \\ \boldsymbol{Z}_n(T) &= g(\boldsymbol{Z}_{n-1}(T)) = \cdots = g^n(\boldsymbol{Z}_0(T)). \end{aligned}$$

Proof: See Appendix.

This proposition shows that we can obtain payoff values by simulation when cross-holdings exist among firms. Moreover, if N firms have at least one investor outside the N firms (Assumption 5), the payoff value can converge (See Proof). In the case of Japanese *mochiai*, the convergence is certain because banks, who play a central role in the *mochiai* structure, have many outside investors in the form of depositors.

4. Modigliani-Miller Theorem with Cross-Holdings

Merton[14], [15] prove that the Modigliani-Miller theorem holds even in the presence of bankruptcy. In this section, we extend Merton's result when there exists cross-holdings of stock.

We make similar assumptions as in Merton[14], [15] by using different technical terms. We assume that financial markets are complete, frictionless, and trading continuously. Under this assumption, Harrison and Kreps[8] have shown that there exists a unique risk neutral measure Q under which the continuously discounted price of any security is a Q-martingale. We assume further that firms keep business *identical* even if they establish cross-holdings of stock. We define *identical business* as follows.

Definition 3 (identical business) *Let $V_1(t), V_2(t)$ denote the value of business 1 and business 2 at time t. Let $T, 0 \leq t \leq T$ be a pre-specified time. Then, business 1 is identical with business 2, if and only if*

1. *$V_1(T) = V_2(T)$,*
2. *Business 1 yields the same amount of payout per unit time as business 2.*

The next result can be derived from the fact that in Proposition 1, we could specify the payoff functions of securities issued by cross-holding firms.

Proposition 3 *Suppose that two firms establish cross-shareholdings by issuing new stocks and immediately buying each other's stock. Then, the value of each firm's business assets is invariant to the establishment of cross-holdings of stock.*

Remark 4.1 In this cross-shareholding, both firms expand their firm size with no cash. However, the cross-holding does not change the value of either firm's *business* assets. Hence, in a market that allows cross-shareholding, there exist many stocks that are not collateralized by real business assets.

Proof:

First, we consider two pairs of firms as follows. Consider two plain firms that have only single business assets whose values are denoted by V_1^*, V_2^* respectively. Let them satisfy Assumptions 1 through 5 with $s_{12} = s_{21} = f_{12} = f_{21} = 0$ and $N = 2$. Moreover, we denote each firm's continuous dividend per unit time by d_1^*, d_2^* and each firm's continuous coupon payment per unit time by c_1^*, c_2^*. Let C_1^*, C_2^* denote net continuous cash out per unit time by each business asset. Then the following equations must be satisfied.

$$C_1^* = c_1^* + d_1^*, \quad C_2^* = c_2^* + d_2^*. \tag{7}$$

Consider two firms that establish cross-shareholdings. Let them satisfy Assumptions 1 through 6 with $f_{12} = f_{21} = 0$ and $N = 2$. Let the firm 1 have a single business asset V_1 and the firm 2 have a single business asset V_2. We denote each firm's continuous dividend per unit time by d_1, d_2 and each firm's continuous coupon payment per unit time by c_1, c_2. Here, from the assumption that business V_1 and V_2 are identical with V_1^* and V_2^* respectively, we obtain the following equation:

$$C_1^* + s_{12}d_2 = c_1 + d_1, \quad C_2^* + s_{21}d_1 = c_2 + d_2. \tag{8}$$

From the equations (7) and (8), for given coupons c_1, c_2, the dividend of each cross-holding firm must be given by

$$d_1 = \frac{(c_1^* - c_1 + d_1^*) + s_{12}(c_2^* - c_2 + d_2^*)}{1 - s_{12}s_{21}}, \quad d_2 = \frac{s_{21}(c_1^* - c_1 + d_1^*) + (c_2^* - c_2 + d_2^*)}{1 - s_{12}s_{21}}. \tag{9}$$

Second, we show that there exist self-financing trading strategies that finance the cross-holding firm's debts and stocks.

From Proposition 1, we specify the payoff functions of the cross-holding firm's debts and stocks. Hereafter, let these functions be rewritten as follows:

$$F_1(T) = F_1(V_1(T), V_2(T), T), \quad S_1(T) = S_1(V_1(T), V_2(T), T),$$
$$F_2(T) = F_2(V_1(T), V_2(T), T), \quad S_2(T) = S_2(V_1(T), V_2(T), T).$$

Because of the assumption of complete markets, we can obtain self-financing trading strategies with terminal values $F_1(V_1^*(T), V_2^*(T), T)$ and $F_2(V_1^*(T), V_2^*(T), T)$ using the plain firm's business assets. Now let $c_1 < C_1^*, c_2 < C_2^*$. Then the strategies can consume c_1, c_2 per unit time respectively.

For the same reason, we can obtain self-financing trading strategies with terminal values $S_1(V_1^*(T), V_2^*(T), T)$ and $S_2(V_1^*(T), V_2^*(T), T)$. Here, it can be shown that the self-financing trading strategies consume dividends d_1, d_2, which are the same dividends as the stocks issued by the cross-holding firms. This is because from equation (7), the residual cash of business assets V_1^* and V_2^* is shown to be

$$C_1^* - c_1 = c_1^* - c_1 + d_1^*, \quad C_2^* - c_2 = c_2^* - c_2 + d_2^*.$$

Moreover, the exposures of the strategies are shown to be

$$\frac{V_1^* + s_{12}V_2^*}{1 - s_{12}s_{21}}, \quad \frac{s_{21}V_1^* + V_2^*}{1 - s_{12}s_{21}}$$

by the definition of payoff functions in Proposition 1. Hence, the strategies can consume cash that is equal to equation (9). Thus, by following the four self-financing strategies, a firm can receive interim payments exactly equal to those on the debt and stock of the cross-holding firms.

Third, we can show that the four self-financing strategies and cross-holding firms' debt and stock are the same value respectively at time t as follows:

$$F_1(V_1(t), V_2(t), t) = F_1(V_1^*(t), V_2^*(t), t), \quad S_1(V_1(t), V_2(t), t) = S_1(V_1^*(t), V_2^*(t), t),$$
$$F_2(V_1(t), V_2(t), t) = F_2(V_1^*(t), V_2^*(t), t), \quad S_2(V_1(t), V_2(t), t) = S_2(V_1^*(t), V_2^*(t), t). \tag{10}$$

This occurs due to the existence of risk neutral measure Q, and to the fact that $V_1(T) = V_1^*(T), V_2(T) = V_2^*(T)$ must be satisfied from the assumption of identical investment policy.

Finally, from the definition of each payoff function and the existence of risk neutral measure Q, we obtain the following equation

$$S_1(V_1^*(t), V_2^*(t), t) + D_1(V_1^*(t), V_2^*(t), t) - s_{12}S_2(V_1^*(t), V_2^*(t), t) = V_1^*(t). \tag{11}$$

Moreover, from the definition of cross-holding firm, we can obtain

$$V_1(t) = S_1(V_1(t), V_2(t), t) + D_1(V_1(t), V_2(t), t) - s_{12}S_2(V_1(t), V_2(t), t). \tag{12}$$

Thus from equations (11), (10) and (12), we have $V_1(t) = V_1^*(t)$. From this it follows that Proposition 3 holds, because we can show $V_2(t) = V_2^*(t)$ in the same way.

□

Remark 4.2 The sum of dividends and coupons paid to the original stock holders and debt holders is not changed by the establishment of cross-shareholding.

This is explained as follows. The holding ratio of the original stockholder who owns firm 1's stock is changed from 1 to $1-s_{21}$. Thus with cross-shareholding, the change in dividend to the original stockholder of firm 1 is equal to $(1-s_{21})d_1-d_1^*$. In the same way, the change in dividend to the original stockholder of firm 2 is equal to $(1-s_{12})d_2-d_2^*$. Here, we can use equations (7) and (8) from the assumption of identical business. Hence, we obtain the following equation

$$\{(1-s_{21})d_1-d_1^*\}+\{(1-s_{12})d_2-d_2^*\}+\{c_1-c_1^*\}+\{c_2-c_2^*\}=0.$$

This means that the net change in cash paid out to the original stockholders and debt holders is equal to zero.

Example 4.1 Suppose cross-shareholdings are formed with $s_{12}=s_{21}, s_{12}\neq 0$. Suppose further that $c_1=c_1^*, c_2=c_2^*$, which is the general case for cross-holdings because a coupon payment is promised even if there exist cross-holdings. Then the following equation is derived from equation (9).

$$(1-s_{21})d_1^*-d_1<0 \text{ if and only if } d_1^*>d_2^*.$$

Hence, holding an equal amount of stock in cross-holdings is disadvantageous for the stockholder who originally owned stock paying the higher dividend.

5. Effect of Cross-Holdings

The payoff function presented in this article has many implications for the risk structure of companies with cross-holdings. In this section, we show some of the implications through several examples. It should be noted that the following implications are not dependent on the stochastic process of underlying *business* asset value.

In example A, we indicate by $S_i^A(t)$ and $F_i^A(t)$ the equity value and debt value issued by firm i. In examples B and C, we indicate securities value in the same way.

5.1. Effect of stock cross-holdings

Below we examine the effect of firm cross-holdings of stock.

Suppose once again we have the plain firms and cross-holding firms in the proof of Proposition 3. Each firm's capital structures is shown in the Example A and Example B which follow. Suppose further that the plain firms change their capital structures and become cross-holding firms by issuing new stock and immediately buying the other's stock. Moreover, we suppose the same assumption as in section 3: (1) complete market, (2) the plain firm and cross-holding firm have identical businesses. Hence, from Proposition 3, the business asset values of both firms are not affected by the establishment of stock cross-holdings.

Example A (plain firms)

firm 1

$V_1(t)$	$F_1^A(t)$
	$S_1^A(t)$

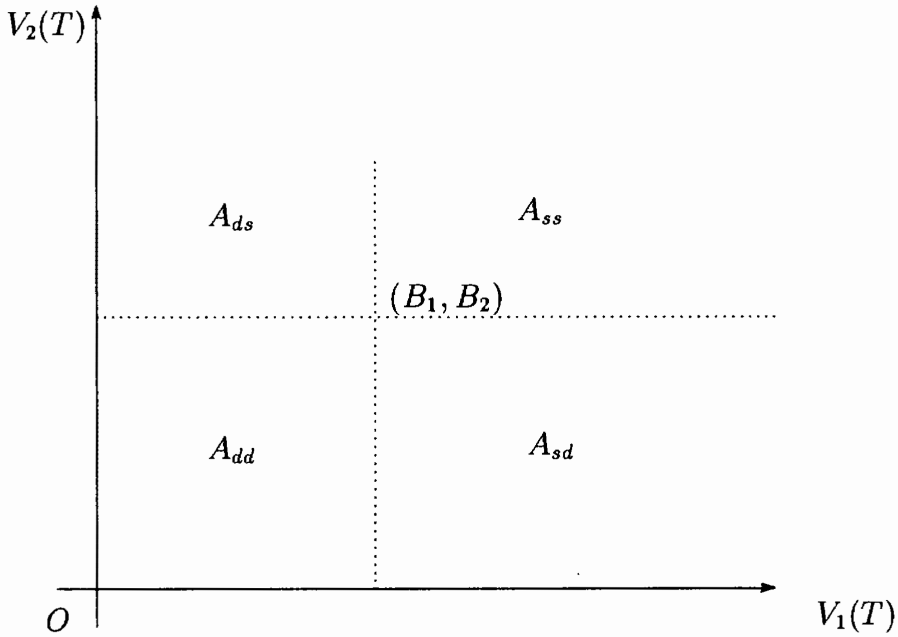
firm 2

$V_2(t)$	$F_2^A(t)$
	$S_2^A(t)$

The debt of both firms, as investigated by Merton, have the following payoff functions:

$$\begin{aligned} F_1^A(T)&=\min[V_1(T),B_1], \quad S_1^A(T)=\max[V_1(T)-B_1,0],\\ F_2^A(T)&=\min[V_2(T),B_2], \quad S_2^A(T)=\max[V_2(T)-B_2,0]. \end{aligned} \tag{13}$$

Figure 2 shows the areas that determine whether each firm is solvent or in default.



(A_{ss}) Both firms are solvent; (A_{ds}) Firm 1 is in default, firm 2 is solvent; (A_{sd}) Firm 1 is solvent, firm 2 is in default; (A_{dd}) Both firms are in default.

Figure 2: Default Areas (Example A, plain firms)

Example B (cross-shareholding firms)

firm 1

$V_1(t)$	$F_1^B(t)$
$+s_{12}S_2^B(t)$	$S_1^B(t)$

firm 2

$V_2(t)$	$F_2^B(t)$
$+s_{21}S_1^B(t)$	$S_2^B(t)$

In this example, cross-shareholding exists between firm 1 and firm 2, and each firm owns a business asset. Hence, under the following conditions

$$\begin{aligned} v_{11} &= 1, \quad f_{12} = 0, \quad s_{12} > 0, \\ v_{21} &= 1, \quad f_{21} = 0, \quad s_{21} > 0, \\ v_{1k} &= v_{2k} = 0, (k = 2, \cdots, M), \end{aligned}$$

payoff functions are given by Proposition 1 as follows:

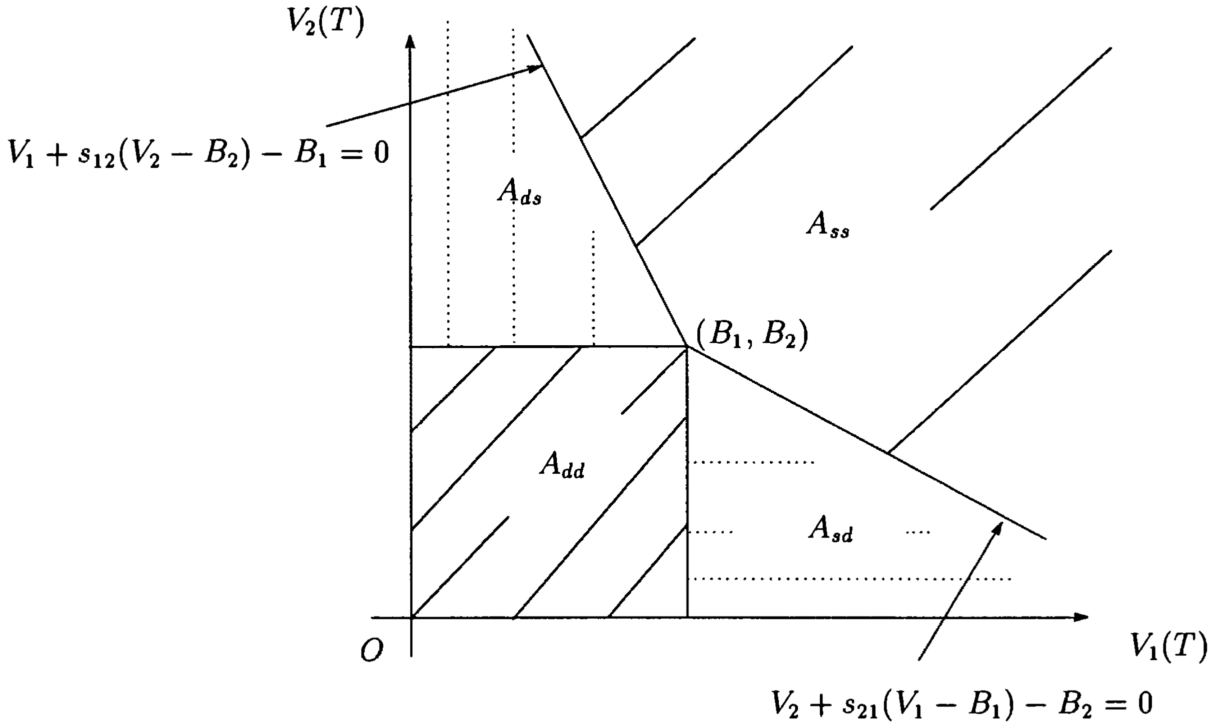
$$F_1^B(T) = \begin{cases} B_1, & (V_1, V_2) \in A_{ss} \\ B_1, & (V_1, V_2) \in A_{sd} \\ V_1 + s_{12}(V_2 - B_2), & (V_1, V_2) \in A_{ds} \\ V_1, & (V_1, V_2) \in A_{dd}, \end{cases} \quad S_1^B(T) = \begin{cases} \dfrac{V_1 - B_1 + s_{12}(V_2 - B_2)}{1 - s_{12}s_{21}}, & (V_1, V_2) \in A_{ss} \\ V_1 - B_1, & (V_1, V_2) \in A_{sd} \\ 0, & (V_1, V_2) \in A_{ds} \\ 0, & (V_1, V_2) \in A_{dd}, \end{cases}$$

$$F_2^B(T) = \begin{cases} B_2, & (V_1, V_2) \in A_{ss} \\ V_2 + s_{21}(V_1 - B_1), & (V_1, V_2) \in A_{sd} \\ B_2, & (V_1, V_2) \in A_{ds} \\ V_2, & (V_1, V_2) \in A_{dd}, \end{cases} \quad S_2^B(T) = \begin{cases} \dfrac{s_{21}(V_1 - B_1) + V_2 - B_2}{1 - s_{21}s_{12}}, & (V_1, V_2) \in A_{ss} \\ 0, & (V_1, V_2) \in A_{sd} \\ V_2 - B_2, & (V_1, V_2) \in A_{ds} \\ 0, & (V_1, V_2) \in A_{dd}, \end{cases} \tag{14}$$

where

$$\begin{array}{lll} A_{ss} \equiv \{(V_1, V_2): & V_1 + s_{12}(V_2 - B_2) > B_1 & V_2 + s_{21}(V_1 - B_1) > B_2\}, \\ A_{sd} \equiv \{(V_1, V_2): & V_1 > B_1, & V_2 + s_{21}(V_1 - B_1) < B_2\}, \\ A_{ds} \equiv \{(V_1, V_2): & V_1 + s_{12}(V_2 - B_2) < B_1, & V_2 > B_2\}, \\ A_{dd} \equiv \{(V_1, V_2): & V_1 < B_1, & V_2 < B_2\}. \end{array}$$

Figure 3 shows the areas that determine whether each firm is solvent or in default.



(A_{ss}) Both firms are solvent; (A_{ds}) Firm 1 is in default, firm 2 is solvent; (A_{sd}) Firm 1 is solvent, firm 2 is in default; (A_{dd}) Both firms are in default.

Figure 3: Default Areas (Example B, cross-shareholding firms)

By comparing Figure 2 and Figure 3, we can confirm that the area in which both firms are solvent(A_{ss}) is widened by cross-shareholding. Intutively this follows that both firms can reduce the credit risks by cross-holdings of stock. We will examine this by comparing payoff values of Example A with Example B. Hereafter, we define changes of original security holders' payoff value as follows:

$$\Delta F_1 = F_1^B(T) - F_1^A(T), \quad \Delta S_1 = (1 - s_{21})S_1^B(T) - S_1^A(T), \tag{15}$$

$$\Delta F_2 = F_2^B(T) - F_2^A(T), \quad \Delta S_2 = (1 - s_{12})S_2^B(T) - S_2^A(T). \tag{16}$$

From these definitions, we can show following equation by comparing equation (13) with equation (14):

$$\Delta F_1 \begin{cases} = 0, & (V_1, V_2) \in A_0 \\ > 0, & (V_1, V_2) \in A_1 \\ = 0, & (V_1, V_2) \in A_1' \\ > 0, & (V_1, V_2) \in A_2 \\ = 0, & (V_1, V_2) \in A_2' \\ = 0, & (V_1, V_2) \in A_3 \\ = 0, & (V_1, V_2) \in A_3' \end{cases} \quad \Delta S_1 \begin{cases} = 0, & (V_1, V_2) \in A_0 \\ = 0, & (V_1, V_2) \in A_1 \\ < 0, & (V_1, V_2) \in A_1' \\ > 0, & (V_1, V_2) \in A_2 \\ < 0, & (V_1, V_2) \in A_2' \\ > 0, & (V_1, V_2) \in A_3 \\ < 0, & (V_1, V_2) \in A_3' \end{cases} \tag{17}$$

$$\Delta F_2 \begin{cases} = 0, & (V_1, V_2) \in A_0 \\ = 0, & (V_1, V_2) \in A_1 \\ > 0, & (V_1, V_2) \in A_1' \\ = 0, & (V_1, V_2) \in A_2 \\ > 0, & (V_1, V_2) \in A_2' \\ = 0, & (V_1, V_2) \in A_3 \\ = 0, & (V_1, V_2) \in A_3' \end{cases} \qquad \Delta S_2 \begin{cases} = 0, & (V_1, V_2) \in A_0 \\ < 0, & (V_1, V_2) \in A_1 \\ = 0, & (V_1, V_2) \in A_1' \\ < 0, & (V_1, V_2) \in A_2 \\ > 0, & (V_1, V_2) \in A_2' \\ < 0, & (V_1, V_2) \in A_3 \\ > 0, & (V_1, V_2) \in A_3' \end{cases} \tag{18}$$

where

$$\begin{aligned} A_0 &= \{(V_1, V_2): \ V_1 \geq 0, V_1 < B_1, \ V_2 \geq 0, V_2 \leq B_2\}, \\ A_1 &= \{(V_1, V_2): \ V_1 \geq 0, V_2 \geq B_2, \ V_1 + s_{12}(V_2 - B_2) - B_1 \leq 0\}, \\ A_1' &= \{(V_1, V_2): \ V_2 \geq 0, V_1 \geq B_1, \ V_2 + s_{21}(V_1 - B_1) - B_2 < 0\}, \\ A_2 &= \{(V_1, V_2): \ V_1 \geq 0, V_1 \leq B_1, \ V_1 + s_{12}(V_2 - B_2) - B_1 > 0\}, \\ A_2' &= \{(V_1, V_2): \ V_2 \geq 0, V_2 < B_2, \ V_2 + s_{21}(V_1 - B_1) - B_2 \geq 0\}, \\ A_3 &= \{(V_1, V_2): \ V_1 > B_1, \ -s_{21}(1 - s_{12})(V_1 - B_1) + s_{12}(1 - s_{21})(V_2 - B_2) \geq 0\}, \\ A_3' &= \{(V_1, V_2): \ V_2 \geq B_2, \ -s_{21}(1 - s_{12})(V_1 - B_1) + s_{12}(1 - s_{21})(V_2 - B_2) < 0\}. \end{aligned} \tag{19}$$

In addition to this, from the restriction of firm total value we can show following equation:

$$\Delta F_1 + \Delta S_1 + \Delta F_2 + \Delta S_2 = 0 \tag{20}$$

for all $(V_1, V_2) \in \{[0, \infty), [0, \infty)\}$. Figure 4 shows the areas that are defined by A_0 through A_3'.

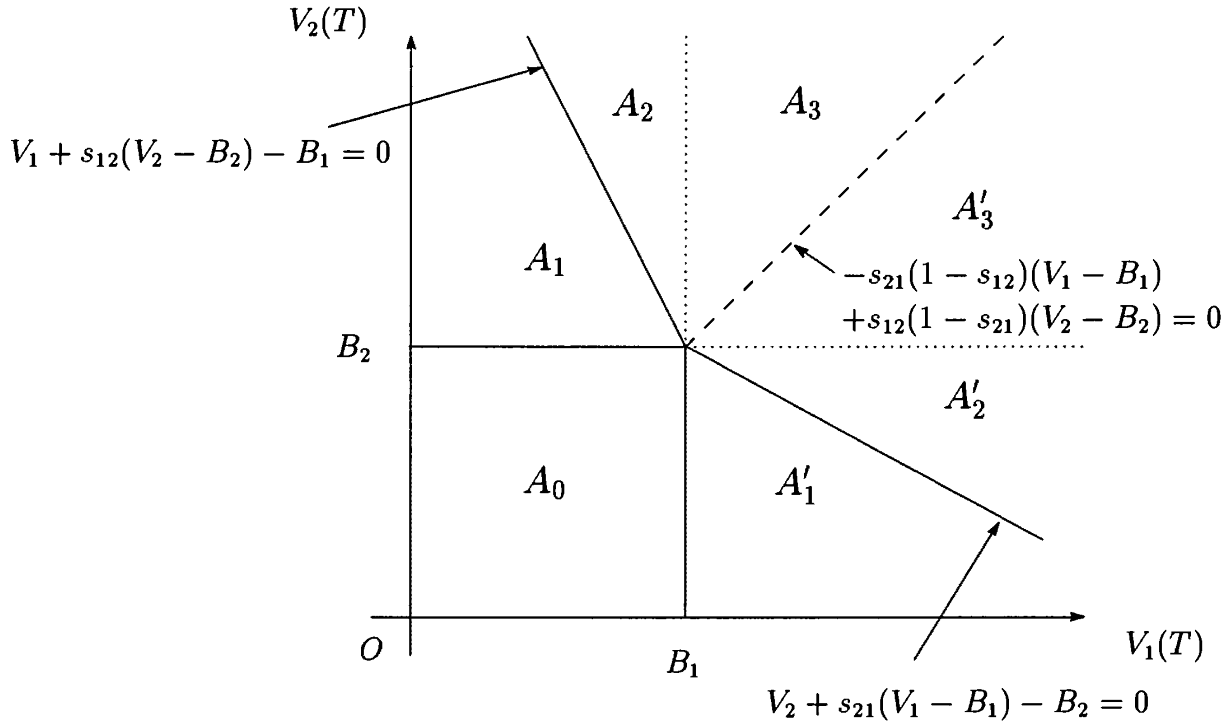


Figure 4: The Areas defined by equation (19)

First, from the equations (17) through (20) we can find that $\Delta F_1 \geq 0, \Delta F_2 \geq 0$ for all $(V_1, V_2) \in \{[0, \infty), [0, \infty)\}$. This means that $F_1^B(t) > F_1^A(t)$ and $F_2^B(t) > F_2^A(t)$ from the fact that we assume the existence of risk neutral measure Q. Remember that the increase in the value at time t means the decrease of credit risk. Hence, cross-shareholding between firm 1 and firm 2 will reduce the credit risks of each firm.

Second, we should remark that the correlation between business assets value can be important measure of credit reducing effect by cross-holdings of stock. Because, when the correlation is lower, there exists higher probability that (V_1, V_2) falls into areas $A_1(A_5)$ or $A_2(A_6)$ which are beneficial to firm 1's (firm 2's) bond holder. We will examine it by numerical example in Section 6.2.

Last, from the equations (17) through (20), we can find that

$$\Delta F_1 = -\Delta S_2 > 0, \quad (V_1, V_2) \in A_1, \tag{21}$$

and

$$\Delta F_1 + \Delta S_1 = -\Delta S_2 > 0, \quad \Delta F_1 > 0, \Delta S_1 > 0, \quad (V_1, V_2) \in A_2. \tag{22}$$

Hence, firm 1's credit risk reduction is caused by cash distribution from the firm 2's stock holder. In these areas (A_1, A_2), firm 2's business is successful while firm 1's business is failed. So, in these areas, there exist cash distribution from successful business firm's stock holder to failed business firm's bond holder. In the same way, we can find the same result with respect to firm 2's credit risk reduction by using the areas A_1', A_2'. Finally, we can find that cross share-holdings decrease both firms' credit risks and the credit risk reduction is caused by the partial loss of the other firm's stock holder. It should be noted that stock holders distribute cash to the other firm's bondholder when their own business is successful while the other firm's business is failed.

Example 5.1 In this paper, we regard stocks as options on business assets. However, as a practical problem in stock investment, investors usually use a mean-variance approach without regarding stocks as options. One simple and practical idea for a mean-variance approach that considers cross-shareholding is as follows.

Let the two plain firms' stocks in Example A be characterized by expected rates of return μ_1, μ_2 and expected variances σ_1^2, σ_2^2. Let both firms be solvent. Then when cross-shareholding is established, the firm's stock (firm 1 in Example B) is characterized by μ_1 and σ_1^2 as follows

$$\frac{1 - s_{21}}{1 - s_{12}s_{21}}(\mu_1 + s_{12}\mu_2), \quad \left(\frac{1 - s_{21}}{1 - s_{12}s_{21}}\right)^2 \left(\sigma_1^2 + s_{12}^2\sigma_2^2 + s_{12}\rho\sigma_1\sigma_2\right)$$

where ρ is the correlation of each firm's business assets. This is because the dependent variable of firm 1's stock is changed from V_1 to $V_1/(1 - s_{12}s_{21}) + s_{12}V_2/(1 - s_{12}s_{21})$ as shown in equation (14), and because the stockholding ratio is reduced from 1 to $1 - s_{21}$ by cross-holding. We must note that this simple approach cannot be applied to a mean-variance approach for debt because it does not consider the firm's default.

5.2. Effect of debt cross-holdings

In the following example, we assume that the firm value is not changed by the establishment of the cross-holding of debt.[2] We will examine the effect of cross-holdings of debt on the firm debt value using the following example:

Example C (cross-holdings of debt)

firm 1

$V_1(t)$	$F_1^C(t)$
$+f_{12}F_2^C(t)$	$S_1^C(t)$

firm 2

$V_2(t)$	$F_2^C(t)$
$+f_{21}F_1^C(t)$	$S_2^C(t)$

[2] This can be easily proved by the analogy of Proposition 3.

In this example, cross-holdings of debt exist between firm 1 and firm 2, and each firm owns a business asset. Hence, payoff functions $F_1(T)$ and $F_2(T)$ are given by Proposition 1 under the following conditions:

$$v_{11}=1,\quad f_{12}>0,\quad s_{12}=0,$$
$$v_{21}=1,\quad f_{21}>0,\quad s_{21}=0,$$
$$v_{1k}=v_{2k}=0,(k=2,\cdots,M).$$

We omit the payoff functions and show only the areas which determine the state of each firm is as follows:

$$\begin{aligned}
A_{ss} &\equiv \{(V_1,V_2): \ V_1>B_1', && V_2>B_2'\},\\
A_{sd} &\equiv \{(V_1,V_2): \ V_1+f_{12}(V_2-B_2')>B_1', && V_2<B_2'\},\\
A_{ds} &\equiv \{(V_1,V_2): \ V_1<B_1', && V_2+f_{21}(V_1-B_1')>B_2'\},\\
A_{dd} &\equiv \{(V_1,V_2): \ V_1+f_{12}(V_2-B_2')<B_1', && V_2+f_{21}(V_1-B_1')<B_2'\},
\end{aligned}$$

where $B_1'=B_1-f_{12}B_2,\ B_2'=B_2-f_{21}B_1$.

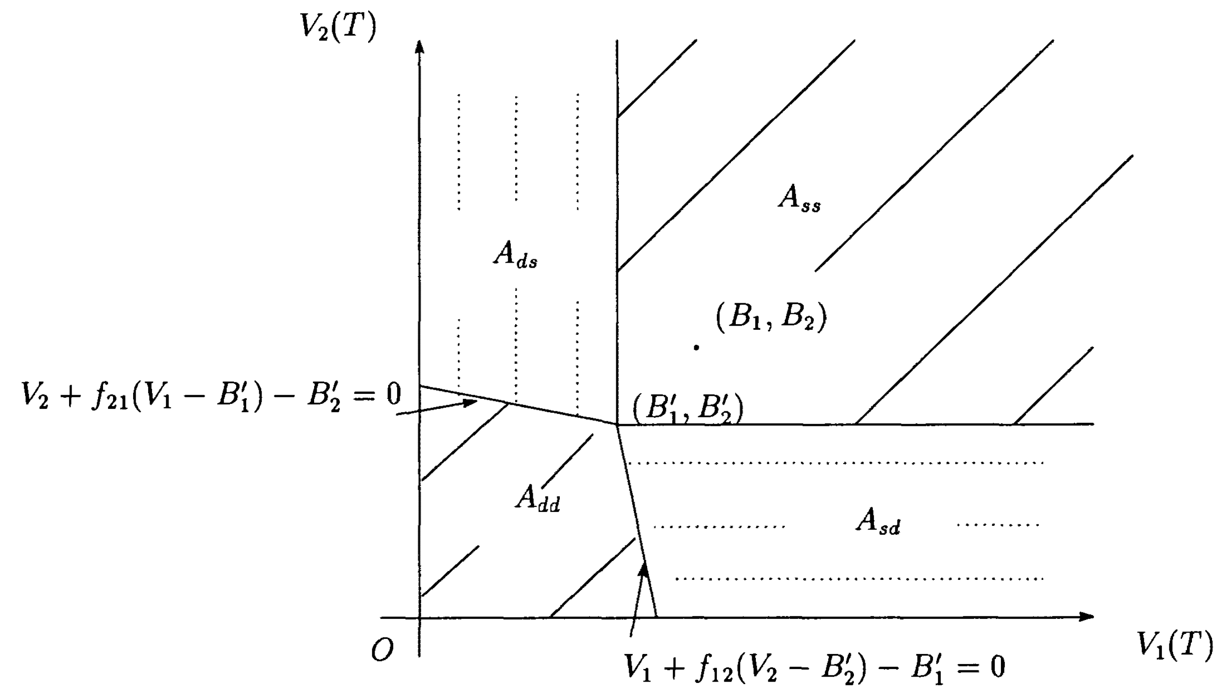


Figure 5: Default Areas (Example C, cross-holdings of debt)

(A_{ss}) Both firms are solvent; (A_{ds}) Firm 1 is in default, firm 2 is solvent; (A_{sd}) Firm 1 is solvent, firm 2 is in default; (A_{dd}) Both firms are in default.

Figure 5 shows these equations. From Figure 2 and 5, we can see that the intersection point of the four lines denoted by (B_1,B_2) moves to (B_1',B_2'). This is not the ordinary offsetting effect of cross-holdings of debt on the face value of each other's debt. We emphasize that both lines

$$V_1+f_{12}(V_2-B_2')=B_1' \quad\text{and}\quad V_2+f_{21}(V_1-B_1')=B_2'$$

intersecting at (B_1',B_2') have negative slope. Hence the area A_{dd} is not rectangular. These facts are important because if we move from (B_1,B_2) to (B_1',B_2') by valuing debt in the conventional method of offsetting face value - thus ignoring cross-holdings of debt – we obtain the payoff functions:

$$\begin{aligned} F_1^B(T) &= \min[V_1(T), B_1'] \\ F_2^B(T) &= \min[V_2(T), B_2']. \end{aligned}$$

From these equations, the area in which both firms are in default is given by

$$A_{dd} = \{(V_1(T), V_2(T)) : 0 < V_1(T) < B_1', 0 < V_2(T) < B_2'\}$$

This is a rectangular area and smaller than the corresponding area A_{dd} in Example C (Figure 5). Therefore, offsetting the face values of debt and ignoring cross-holdings of debt lead to overvaluing each firm's debt.

6. Numerical Examples

In the following numerical examples, we assume that there are no coupon payments and dividends for the purpose of simplicity.

6.1. Valuation method

We assume that the stochastic processes of underlying business assets $V_k(t)$ under risk neutral measure Q are given as

$$\frac{dV_i(t)}{V_k(t)} = rdt + \sigma_k dw_k(t), \quad k = 1, \ldots, M,$$

where r is a constant risk free rate, σ_k are constants, and $w_k(t)$ are standard Wiener Processes. The instantaneous correlation between $dw_i(t)$ and $dw_j(t)$ $(i \neq j)$ is $\rho_{ij}dt$.

The no arbitrage assumption in financial economics implies that the price of any derivative security must equal the discounted value of its payoff function integrated against the appropriate state-price density. Hence the no arbitrage price, where the payoff function is defined by either Proposition 1 or 2, is

$$Z_i(t) = \mathrm{e}^{-r(T-t)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} Z_i(T) \mathrm{d}\Psi(\boldsymbol{V}(T)),$$

where $d\Psi(\cdot)$ is the state-price density.

In our model, it is efficient to use a Monte Carlo simulation to obtain no arbitrage price because business assets are log-normally distributed and the securities are high dimensional multivariate basket type options. If two firms are related through cross-holdings of stock or cross-holdings of debt, then the payoff function is clearly given by Proposition 1, and we can obtain the no-arbitrage price by simple simulation. If $N(> 2)$ firms are related through cross-holdings of stock or debt, then the payoff function is given by Proposition 2. In this case, we can obtain the price as follows:

Step 1 Generate a sample path of $\boldsymbol{V}(T)$. It is denoted by $\boldsymbol{V}^l(T)$;
Step 2 Take $\boldsymbol{Z}_0(T) \in \boldsymbol{R}^{2N}_{[0,\infty)}$, for instance $\boldsymbol{Z}_0(T) = (0, \cdots, 0)^t$. Simulate $\lim_{n=1}^{\infty} g^n(\boldsymbol{Z}_0(T), \boldsymbol{V}^l(T))$. Then obtain the payoff value, $\boldsymbol{Z}^l(T)$(Proposition 2);
Step 3 Calculate the discounted value of $\boldsymbol{Z}^l(T)$. It is denoted by $\boldsymbol{Z}^l(t)$;
Step 4 Repeat steps 1-3 an appropriate number of times, L;
Step 5 The no-arbitrage price is given by $(1/L)\sum_{l=1}^{L} \boldsymbol{Z}^l(t)$.

This is a Monte Carlo simulation in which each path undergoes a simulation to obtain the payoff value.

6.2. Two firms case

As mentioned in Section 5.1, we now show the effect of the correlation in change in business assets value of cross-holding firms. Moreover, in this section we examine the effect of unwinding cross-shareholdings. To examine the effect of establishing and unwinding cross-holdings, we will calculate the default spread that is defined by the equation

$$R - r = \frac{1}{T}\log\frac{B}{F(t)} - r.$$

First, suppose there are plain firms (Example A) and cross-holding firms (Example B). Here we present two cases ($\rho_{12} = \rho = 0.9, -0.9$) for Example B. By using symmetrical balance sheets for the two firms, we need to consider only firm 1's default spread. Figure 6 shows default spreads obtained with a Monte Carlo simulation that includes payoff functions determined by Proposition 1. From Figure 6, we can see that the credit risk barely decreases when ρ is high. Hence, the correlation in two firms' business asset can be a critical condition for reducing credit risk by cross-holdings of stock.

Second, suppose cross-holding firms sell their cross-held stock in the market, and then buy their business assets to expand their business. Then the new asset values, denoted as $V_1^+(t), V_2^+(t)$ respectively, satisfy the equations

$$\begin{aligned} V_1^+(t) &= V_1(t) + s_{12}S_2^B(t), \\ V_2^+(t) &= V_2(t) + s_{21}S_1^B(t). \end{aligned}$$

Hence, the balance sheets are given as follows:

Example D (unwinding of cross-shareholdings)

firm 1

$V_1^+(t)$	$F_1^D(t)$
	$S_1^D(t)$

firm 2

$V_2^+(t)$	$F_2^D(t)$
	$S_2^D(t)$

This is the usual case when cross-shareholdings are unwound. Here, we calculated the default spread when ρ is equal to -0.9. This is shown in Figure 6. From the figure, we can see that the credit risk increases. Moreover, we calculated the default spread when ρ is equal to 0.9. In this case, the default spread decreases when cross-shareholdings are unwound (figure omitted for brevity). Hence, we find that unwinding cross-shareholdings between relatively uncorrelated firms tends to increase their credit risks, while doing so between highly correlated firms tends to reduce their credit risks.

Here, we recall that firm size in Example D is equal to Example B. Moreover, Example D is also a case of a simple capital increase wherein plain firms (Example A) increase their capital by issuing new stock. By comparing the effect of shifting from Example A to Example B on the one hand, and from Example A to Example D on the other, our previous finding can be interpreted as follows: the credit risk reduction by establishing cross-shareholding between relatively uncorrelated firms (Example B, $\rho = -0.9$) is greater than in the case of a simple capital increase (Example D).

6.3. *N* firms case

In this section, for the purpose of showing the power of Proposition 2, we present a numerical example where four firms establish cross-holdings of stock and of debt, assuming the simple

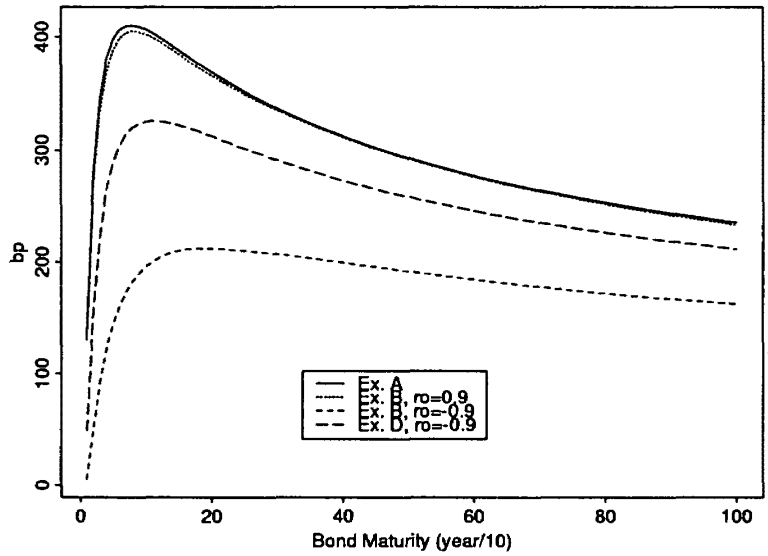


Figure 6: Default Spreads (Examples A, B, and D)

The parameter values used are $V_1(t) = 1.0, V_2(t) = 1.0, B_1 = 0.9, B_2 = 0.9, s_{12} = 0.2, s_{21} = 0.2, \sigma_1 = 0.2, \sigma_2 = 0.2$. The shift from Example A to Example B (ρ(ro) $=$ 0.9, -0.9) represents the establishment of cross-shareholding. The shift from Example B (ρ(ro) $= -0.9$) to Example D represents the unwinding of cross-shareholding. The shift from Example A to Example D represents a simple capital increase.

case as follows:

$$\boldsymbol{v} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{z} = \begin{pmatrix} 0.0 & 0.2 & 0.3 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 \\ 0.2 & 0.0 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 & 0.1 \\ 0.1 & 0.1 & 0.0 & 0.3 & 0.1 & 0.0 & 0.0 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0.8 \\ 0.8 \\ 0.8 \\ 0.8 \end{pmatrix}.$$

As shown in section 6.1, Step 1 and Step 2 are based on Proposition 2. So, we show the convergence of $\boldsymbol{Z}(T)$ with two prescribed steps.

Step 1: For instance, we set $\boldsymbol{V}^l(T)$ as follows:

$$\boldsymbol{V}^l(T) = (2.0, 0.5, 0.6, 0.6)^t.$$

Step 2: Then we obtain the value $\boldsymbol{Z}_n(T)$ as follows:

$$\begin{array}{lllllllll}
\boldsymbol{Z}_0 = (& 0.0, & 0.0, & 0.0, & 0.0, & 0.0, & 0.0, & 0.0, & 0.0 \ )^t \\
\boldsymbol{Z}_1 = (& 1.2, & 0.0, & 0.0, & 0.0, & 0.8, & 0.5, & 0.6, & 0.6 \ )^t \\
\boldsymbol{Z}_2 = (& 1.26, & 0.0, & 0.06, & 0.0, & 0.8, & 0.8, & 0.8, & 0.77 \ )^t \\
\boldsymbol{Z}_3 = (& 1.298, & 0.041, & 0.083, & 0.012, & 0.8, & 0.8, & 0.8, & 0.8 \ )^t \\
\boldsymbol{Z}_4 = (& 1.3143, & 0.0574, & 0.0975, & 0.0222, & 0.8, & 0.8, & 0.8, & 0.8 \ )^t \\
\boldsymbol{Z}_5 = (& 1.32295, & 0.06458, & 0.10383, & 0.02692, & 0.8, & 0.8, & 0.8, & 0.8 \ )^t \\
\quad \vdots & & & & & & & & \\
\boldsymbol{Z}_{19} = (& 1.33014, & 0.0710207, & 0.109434, & 0.0310594, & 0.8, & 0.8, & 0.8, & 0.8 \ )^t \\
\boldsymbol{Z}_{20} = (& 1.33014, & 0.0710207, & 0.109434, & 0.0310595, & 0.8, & 0.8, & 0.8, & 0.8 \ )^t
\end{array}$$

In this example, we can approximate the value of $Z(T)$ as Z_{20}. Remember that when there are four firms with no cross-holdings, we need to calculate eight payoffs functions by max or min operator respectively as seen in equation (13). Thus even when cross-holdings exist, the determination of payoffs by convergence as shown above is not excessively burdensome for valuing securities.

Incidentally, the sample business asset values $V_2(T), V_3(T)$ and $V_4(T)$ are smaller than the bond face values B_2, B_3 and B_4. Hence when the firms do not establish cross-holdings, firms 2, 3 and 4 are in default. Therefore, firms 2, 3 and 4 benefit from the high value of firm 1's business asset $V_1(T)$ by cross-holding.

7. Conclusion

This article develops a new framework for valuing corporate debt by incorporating cross-holdings. We derive payoff functions of corporate securities when cross-holdings exist in corporations. By using the contraction method, the model allows us to value firms within a group characterized by complex multiple cross-holdings of stock.

A number of important insight about cross-holdings emerge from the payoff functions which we derive. We found the following two main results:

(i) The Modigliani-Miller theorem can obtain when there exist cross-holdings of stock. Thus a firm's *business asset* value is invariant to the establishment of cross-holdings. In this analysis, we assume a complete market and firms that maintain identical businesses as in Merton[14], [15].

(ii) By establishing cross-shareholdings stock holders distributes a part of its payoff values to the bond holders of each other's firm, so that both firms can decrease credit risks by cross-shareholdings. We should note that when two highly correlated firms establish cross-shareholdings, credit reductions are not appeared significantly.

Our model allows the following applications: (a) the practical reevaluation of cross-held stocks to determine which are beneficial or disadvantageous for companies with complex multiple cross-holdings, and (b) a value at risk (VaR) approach for business asset exposure and investment exposure involving complex multiple cross-holdings. These applications are made possible by the fact that our model allows for multi dimensional business assets and multi dimensional firms with complex multiple cross-holdings.

Appendix

Proof of Proposition 1:

We will solve the following equations:

$$\begin{cases} S_1(T) &= \max[A_1(T) - B_1, 0], \\ S_2(T) &= \max[A_2(T) - B_2, 0], \\ F_1(T) &= \min[A_1(T), B_1], \\ F_2(T) &= \min[A_2(T), B_2], \end{cases}$$

for $S_1(T)$, $S_2(T)$, $F_1(T)$, $F_2(T)$ where

$$A_1(T) = \sum_{k=1}^{M} v_{1k} V_k(T) + s_{12} S_2(T) + f_{12} F_2(T),$$

$$A_2(T) = \sum_{k=1}^{M} v_{2k}V_k(T) + s_{21}S_1(T) + f_{21}F_1(T).$$

This is a system of equations which have max and min functions. We will solve these equations by assuming four cases:

$$\begin{array}{ll} \text{(i)} & A_1(T) - B_1 > 0, A_2(T) - B_2 > 0, \\ \text{(ii)} & A_1(T) - B_1 \geq 0, A_2(T) - B_2 \leq 0, \\ \text{(iii)} & A_1(T) - B_1 \leq 0, A_2(T) - B_2 \geq 0, \\ \text{(iv)} & A_1(T) - B_1 < 0, A_2(T) - B_2 < 0. \end{array} \quad (23)$$

(i) Let

$$W_1 = \sum_{k=1}^{M} v_{1k}V_k(T), \quad W_2 = \sum_{k=1}^{M} v_{2k}V_k(T).$$

Then, from equation (23), we can obtain a system of equations which do not have max and min functions, as follows:

$$\begin{cases} S_1(T) = W_1 + s_{12}S_2(T) + f_{12}F_2(T) - B_1, \\ S_2(T) = W_2 + s_{21}S_1(T) + f_{21}F_1(T) - B_2, \\ F_1(T) = B_1, \\ F_2(T) = B_2. \end{cases} \quad (24)$$

Let

$$B_1' = B_1 - f_{12}B_2, \quad B_2' = B_2 - f_{21}B_1.$$

Then, we can obtain the solution of equation (24) for $S_1(T)$, $S_2(T)$, $F_1(T)$, $F_2(T)$, as follows:

$$\begin{cases} S_1(T) = \dfrac{W_1 - B_1' + s_{12}(W_2 - B_2')}{1 - s_{12}s_{21}}, \\ S_2(T) = \dfrac{s_{21}(W_1 - B_1') + W_2 - B_2'}{1 - s_{12}s_{21}}, \\ F_1(T) = B_1, \\ F_2(T) = B_2. \end{cases} \quad (25)$$

Further, by using equation (25), equation (23) is rewritten as follows:

$$W_1 + s_{12}(W_2 - B_2') > B_1', W_2 + s_{21}(W_1 - B_1') > B_2'$$

This defines the area A_{ss} in which both firms are solvent.

(ii),(iii),(iv) In the same way, we can obtain solutions for $S_1(T)$, $S_2(T)$, $F_1(T)$, $F_2(T)$ and the areas A_{sd}, A_{ds}, A_{dd} respectively. Finally we can obtain Proposition.

□

Proof of Proposition 2:

From the fixed point theorem, if a function f is a contraction map from X to X, then $f^n(x)$ converges on a fixed point for all $x \in X$. Hence we will prove that the function g which is defined by equation (5),(6) is a contraction map.

We denote $\boldsymbol{R}^n_{[0,\infty)} = \{(x_1,\ldots,x_n)^t : x_1,\ldots,x_n \in [0,\infty)\}$ by n-dimensional non-negative real number space. Let $(\boldsymbol{R}^n_{[0,\infty)}, \mathrm{d})$ be a metric space where

$$\mathrm{d}(\boldsymbol{X}^h, \boldsymbol{X}^k) = \sum_{i=1}^{N} |X_i^h - X_i^k|, \quad \boldsymbol{X}^h, \boldsymbol{X}^k \in \boldsymbol{R}^n_{[0,\infty)}.$$

Then, the function g can be a map from $\boldsymbol{R}^{2N}_{[0,\infty)}$ to $\boldsymbol{R}^{2N}_{[0,\infty)}$ where

$$\mathrm{d}(\boldsymbol{Z}^h, \boldsymbol{Z}^k) = \sum_{i=1}^{N} \Big[|S_i^h - S_i^k| + |F_i^h - F_i^k| \Big], \quad \boldsymbol{Z}^h, \boldsymbol{Z}^k \in \boldsymbol{R}^{2N}_{[0,\infty)},$$

and where $\boldsymbol{Z}$ is defined by equation (4).

From the definition of contraction map, g is a contraction map if and only if there exist real number λ such that

$$\mathrm{d}(g(\boldsymbol{Z}^h), g(\boldsymbol{Z}^k)) \le \lambda\, \mathrm{d}(\boldsymbol{Z}^h, \boldsymbol{Z}^k), \quad 0 \le \lambda < 1$$

for all $(\boldsymbol{Z}^h, \boldsymbol{Z}^k)$. This is shown by the following equation:

$$\begin{aligned}
&\mathrm{d}(g(\boldsymbol{Z}^h), g(\boldsymbol{Z}^k)) \\
&= \sum_{i=1}^{N} \Bigg\{ \Bigg| \max\Big[W_i + \sum_{j=1}^{N}(s_{ij}S_j^h + f_{ij}F_j^h) - B_i, 0\Big] - \max\Big[W_i + \sum_{j=1}^{N}(s_{ij}S_j^k + f_{ij}F_j^k) - B_i, 0\Big] \Bigg| \\
&\qquad + \Bigg| \min\Big[W_i + \sum_{j=1}^{N}(s_{ij}S_j^h + f_{ij}F_j^h), B_i\Big] - \min\Big[W_i + \sum_{j=1}^{N}(s_{ij}S_j^k + f_{ij}F_j^k), B_i\Big] \Bigg| \Bigg\} \\
&\le \sum_{i=1}^{N} \Bigg| \sum_{j=1}^{N} s_{ij}(S_j^h - S_j^k) + \sum_{j=1}^{N} f_{ij}(F_j^h - F_j^k) \Bigg| \\
&\le \sum_{j=1}^{N} \Bigg\{ \Big(\sum_{i=1}^{N} s_{ij}\Big) |S_j^h - S_j^k| + \Big(\sum_{i=1}^{N} f_{ij}\Big) |F_j^h - F_j^k| \Bigg\} \\
&= \sum_{j=1}^{N} \Big(s_j |S_j^h - S_j^k| + f_j |F_j^h - F_j^k| \Big).
\end{aligned}$$

The first inequality is shown by the following equation:

$$|\max[x-b,0] - \max[x'-b,0]| + |\min[x,b] - \min[x,b]| \le |x - x'|, \quad \text{for all } x \ge 0, b \ge 0.$$

The second inequality is shown by triangle inequality. In the final equality, we denote $\sum_{i=1}^{N} s_{ij}, \sum_{i=1}^{N} f_{ij}$ by s_j, f_j.

Here, from Assumption 4,

$$0 \le s_j \le 1, \quad 0 \le f_j \le 1, \quad j = 1, \ldots, N.$$

Moreover, from Assumption 5, there exist at least one j such that

$$s_j \ne 1 \quad \text{or} \quad f_j \ne 1.$$

Hence, there exist real number λ such that

$$\begin{aligned}
\mathrm{d}(g(\boldsymbol{Z}^h), g(\boldsymbol{Z}^k)) &\le \lambda \sum_{j=1}^{N} \Big\{ |S_j^h - S_j^k| + |F_j^h - F_j^k| \Big\} \\
&= \lambda\, \mathrm{d}(\boldsymbol{Z}^h, \boldsymbol{Z}^k), \quad 0 \le \lambda < 1
\end{aligned}$$

for all $(\boldsymbol{Z}^h, \boldsymbol{Z}^k)$. So, the function g is a contraction map from $\boldsymbol{R}^{2N}_{[0,\infty)}$ to $\boldsymbol{R}^{2N}_{[0,\infty)}$. □

Acknowledgment

This work was extension of "Valuing Corporate Debt with Cross-Holdings," the author's unpublished master's thesis (in Japanese) at the Graduate School of Systems Management, University of Tsukuba, Tokyo. The author thanks Hisatoshi Suzuki for originally suggesting this topic and for abundant help along the way. The author also thanks Masaaki Kijima for supporting one of the important results and for much useful advise in his seminar held in Kyoto University. In addition, the author thanks James Parker, Kazuhiko Tsuda, Koichi Miyazaki, Hidetaka Kawakita, Hiroshi Yanai, Masanori Fushimi, Naoki Makimoto, Shuuji Tanaka, Tomonari Kurasawa, Yasuhiro Yonezawa, and Yoshihiko Uchida for many helpful discussions. Finally, the author thanks the anonymous referees for their careful reading, many useful suggestions, and some essential comments.

References

[1] H. Bierman, Jr. and N. H. Noyes: Total stock market value with reciprocal ownership: a note on the Japanese situation. *Financial Analyst Journal*, **46**-1 (1990) 77-77.

[2] H. Bierman, Jr. and N. H. Noyes: Price/earnings ratios restructured for Japan. *Financial Analyst Journal*, **47**-2 (1991) 91-91.

[3] F. Black and J. C. Cox: Valuing corporate securities: some effects of bond indenture provisions. *Journal of Finance*, **31** (1976) 351-367.

[4] F. Brioschi, L. Buzzacchi, and G. Massimo: More on stock market value with reciprocal ownership. *Financial Analyst Journal*, **47**-3 (1991) 76-78.

[5] E. Briys and F. de Varenne: Valuing risky fixed debt: an extension. *Journal of Financial and Quantitative Analysis*, **32** (1997) 239-248.

[6] R. Ferguson and N. B. Hitzig: How to get rich quick using GAAP. *Financial Analyst Journal*, **49**-3 (1993) 30-34.

[7] R. Geske: The valuation of corporate liabilites as compound options. *Journal of Financial and Quantitative Analysis*, **12** (1977) 541-552.

[8] M.J. Harrison and D. Kreps: Martingales and arbitrage in multiperiod securities market. *Journal of Economic Theory*, **20** (1979) 381-408.

[9] M. Ikeda: Price/earnings ratios with reciprocal ownership. *Financial Analyst Journal*, **48**-4 (1992) 77-83.

[10] M. Kijima and T. Suzuki: A jump-diffusion model for pricing corporate debt securities in a complex capital structure. *Quantitative Finance*, **1** (2001) 611-620.

[11] M. Kurasawa: Kabusiki mochiai to kigyou kachi. *Finance Kenkyu*, **1** (1984) 43-58, in Japanese.

[12] F. A. Longstaff and E. S. Schwartz: A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance*, **50** (1995) 789-819.

[13] J. McDonald: The mochiai effect: Japanese corporate cross-holdings. *Journal of Portfolio Management*, **16**-1 (1989) 90-94.

[14] R. C. Merton: On the pricing of corporate debt: the risk structure of interest rate. *Journal of Finance*, **29** (1974) 449-470.

[15] R. C. Merton: On the pricing of contingent claims and the modigliani-miller theorem. *Journal of Financial Economics*, **5** (1977) 241-249.

[16] R. Sinha: Company cross-holdings and investment analysis. *Financial Analyst Journal*, **54**-5 (1998) 83-89.

[17] Y. Yonezawa: *Kabusiki Sijou no Keizaigaku*, (Nippon Keizai Shinbun-sha, 1995, in Japanese).

Teruyoshi Suzuki
Financial Research Group
NLI Research Institute
1-1-1 Chiyoda-ku, Yurakucho,
Tokyo 100-0006, JAPAN
E-mail: suzuki@nli-research.co.jp