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DISCRETE SEARCH ALLOCATION GAME WITH ENERGY CONSTRAINTS

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Abstract This paper deals with a search game. In a search space, a target wants to avoid a searcher by selecting his path. The searcher has superior mobility and makes effort to detect the target by distributing divisible searching effort anywhere he wants. The target might move diffusively from the starting points but his motion is restricted by some constraints on his maximum speed and energy consumption. The searcher also has limits on the total amount of searching effort. A payoff of the game is assumed to be the detection probability of the target, which is represented by an exponential function of the cumulative searching effort weighted by the probability distribution of the target. Regardless of the payoff function, we name the game the search allocation game with energy constraints. We formulate it as a two-person zero-sum game and propose a linear programming method to solve it. Our formulation and method have the flexibility to be applied to other search models by a small modification.

1. Introduction

This paper deals with a search game. In a search space, a target wants to avoid a searcher by selecting his path. The searcher has superior mobility and makes effort to detect the target by distributing divisible searching effort anywhere he wants. The target might move diffusively from the starting points but his motion is restricted by some constraints within his maximum speed and energy consumption. The searcher also has limits on the total amount of his searching effort. This type of search problem has many applications such as the search-and-rescue operation and military ones in the ocean. As an example, we can take so-called datum search for a smuggler ship by custom airplanes, which is motivated by the information of position of the smuggler/target. We call the position the datum point. The datum information, which is composed of the datum point, the time when the datum point is exposed and other data, is reported to the custom and it triggers the datum search by the custom. They are going to detect the target by dispatching some airplanes at hand and the target acts as an evader in the search.

Koopman[13] compiled the results of the naval Operations Research by the U.S. Navy in the Second World War and put them into a book. He studied the datum search where the target took a diffusive motion after randomly selecting his course from the datum point. Meinardi[14] modeled the datum search as a search game. He considered the discrete model, in which the search took place on a discrete space at discrete time points. In order to solve the game, he investigated the target transition so as to make the probability distribution of the target as uniform as possible on the space any time. That is why his method is difficult to be applied to other datum search problems. The direct application of the datum search model could be military operations such as the anti-submarine warfare(ASW). Danskin[2] dealt with a search game of ASW, where a submarine selected a course and speed once at the beginning of the search, while an ASW helicopter chose a sequence of points to dip a sonar. He found an equilibrium point. The optimal target's strategy was the uniform distribution of speed on a speed space and the optimal searcher's one was the uniformly scattering the covering area of his sonar on the space. Baston and Bostock[1] and Garnaev[4] discussed games to determine the best points of hiding a submarine and throwing down depth charges by an anti-submarine helicopter on a one-dimensional cell space. Washburn's work[17] was about a multi-stage game for target's and searcher's discrete motions, where both players had no restriction on their motions and the payoff was the total traveling distance until the coincidence of positions of the target and the searcher. Nakai[15] dealt with an interesting model in the sense that a safety zone was set up for the target. His model was also a multi-stage game with the payoff of the detection probability of the target. Kikuta[11, 12] studied a game with the payoff of traveling cost. Eagle and Washburn[3] worked on a single-stage game, where the payoff was defined as the total of the value determined by players' positions. They assumed constraints on the cells which both players were able to move to. In these studies, the searcher's strategy was to choose his cells. But it could be regarded as the distribution of searching effort on search space, especially in the case that the searcher can move quite faster than the target. Let us name such a game the search allocation game regardless of the payoff function. For the search allocation game, a basic problem is to determine a point of hiding a stationary target and a distribution of searching effort[5]. Nakai[16], and Iida, Hohzaki and Sato[10] did research in such stationary target games with the payoff of the detection probability or some reward. For moving target games, there were Iida's and Hohzaki's works[9, 6, 7]. Hohzaki and Iida[8] proposed a numerical method to solve more generalized games, where it is just required that the payoff is concave for the searcher's strategy and linear for the target's strategy.

In the games of the datum search for a moving target outlined above, authors did not set so many constraints on the capability of target motion. Even when they did, those were just on the maximum speed or the selection of a target path among the limited number of preplanned ones. Under the former constraint, the problem preserves a kind of uniformity for optimal solution and then the solution is easy to be estimated. Under the latter constraint, the solver does not need to worry about the computational complexity of the problem, which often becomes burden when the large size of problems are solved. In this paper, an additional constraint on target energy is assumed, which makes the problem more practical, e.g. problems with limited capacities of batteries and the energy consumption of players by waves created on the sea or in the air. It carries off uniformity from optimal solutions. Washburn and Hohzaki[18] considered a datum search game with energy constraint in a continuous search space but failed to derive optimal solution because the continuous space is more difficult to deal with than the discrete one for the optimization problem. In this paper, we deal with a search allocation game with energy constraint on a discrete search space. We can regard the problem as a generalized discrete version of Washburn and Hohzaki's study[18] and also as the specialization and the extension of Hohzaki and Iida's study[6].

In the next section, we describe assumptions of the problem and formulate it. In Section 3, we prove that there exists an equilibrium point for the problem. We propose two formulations where two kinds of representations are adopted for the target strategy in Section 3 and 4. The method in Section 4 will be useful for solving large size of problems. In Section 5, we show that our basic formulation is easily applied to some other models after a small modification and illustrate several examples.

2. Description and Formulation of Discrete Search Allocation Game

We consider the following two-person zero-sum game, which a searcher and a target participate in as two players.

(1) A search space consists of two spaces: discrete cell space $\boldsymbol{K} = \{1, \cdots, n\}$ and time space $\boldsymbol{T} = \{1, \cdots, m\}$.

(2) In order to evade a searcher, a target chooses a path among feasible paths, which satisfy the constraints stated later, in advance and goes along it throughout the search game. There are possibly n^m paths in all in the search space. We denote a set of the paths by Ω. Number $\omega = 1, \cdots, n^m$ is assigned to each path as its identification. Path $\omega \in \Omega$ goes through cell $\omega(t)$ at time point $t = 1, \cdots, m$. The target has some constraints on his movement. He is supposed to start from one of starting cells $S_0 \subseteq \boldsymbol{K}$ at time $t = 1$. The target can move to cells $N(i,t)$ from cell i at time t. The set $N(i,t)$ defines the neighborhood of cell i at time t. He consumes energy $\mu(i,j)$ by moving from cell i to cell j. He has an initial energy e_0 at the beginning but its exhaustion means his staying where he is ever since. We call a path satisfying these constraints a feasible path. A pure strategy of the target is to choose a feasible path ω from Ω.

(3) In order to detect the target, the searcher distributes divisible searching effort in the search space. However he can not begin searching earlier than time τ. Now we denote the searching time period by $\widehat{\boldsymbol{T}} := \{\tau, \cdots, m\} \subseteq \boldsymbol{T}$. The total of searching effort is limited up to $\Phi(t)$ for each time $t \in \widehat{\boldsymbol{T}}$. Letting $\varphi(i,t)$ denote the amount of searching effort to be distributed in cell i at time t, a pure strategy of the searcher is to decide a plan of $\{\varphi(i,t), i \in \boldsymbol{K}, t \in \boldsymbol{T}\}$.

(4) The searcher and the target decide their own strategies in advance of the search game. The payoff is assumed to be the detection probability of the target and be given by an exponential function of the total amount of the searching effort cumulated along the target path. The effectiveness of unit searching effort on the payoff depends on cells, which is indicated by parameter α_i for cell i. The searcher acts to maximize the payoff and the target does to minimize it.

The problem is to solve a single-stage two-person zero-sum game with a maximizer/searcher and a minimizer/target. The nonnegativeness of searching effort is represented by $\varphi(i,t) \geq 0$ and the amount of available effort $\sum_{i \in \boldsymbol{K}} \varphi(i,t)$ is limited by $\Phi(t)$ each time t. Then we can represent the feasible region of φ by

$$\Psi = \{\varphi(i,t) \geq 0, i \in \boldsymbol{K}, t \in \boldsymbol{T} \mid \sum_{i \in \boldsymbol{K}} \varphi(i,t) \leq \Phi(t), t \in \boldsymbol{T}\} .$$

For the target strategy, his mobility constraint is given by $\omega(t+1) \in N(\omega(t),t)$ for each path ω. The expression $\sum_{t=1}^{m-1} \mu(\omega(t), \omega(t+1)) \leq e_0$ is for his energy constraint. Now we obtain a set of feasible paths $\widehat{\Omega} := \Omega \bigcap P$, where P is a set of path ω satisfying the following constraints.

$$\omega(1) \in S_0, \tag{1}$$

$$\omega(t+1) \in N(\omega(t),t), \; t = 1, \cdots, m-1, \tag{2}$$

$$\sum_{t=1}^{m-1} \mu(\omega(t), \omega(t+1)) \leq e_0 . \tag{3}$$

For a search strategy φ and a path strategy ω, a payoff function is defined by

$$R(\varphi, \omega) = 1 - \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t)\right) . \tag{4}$$

We already know that the so-called convex game has an equilibrium point within the range of pure strategies. Noting that the payoff function is concave for φ and its feasible region Ψ is convex and bounded, we don't have to consider any mixed strategy for φ. We denote a mixed strategy of path selection by $\pi = \{\pi(\omega), \omega \in \hat{\Omega}\}$, where $\pi(\omega)$ is the probability of selecting path ω. Now we can state the following theorem about the solution of the game.

Theorem 1 *The discrete search allocation game has an equilibrium point for the pure strategy φ of the searcher and a mixed strategy π of the target.*

Since we can express the expected payoff by $R(\varphi,\pi) = \sum_{\omega\in\hat{\Omega}} \pi(\omega) R(\varphi,\omega)$ for two strategies φ and π, the value of the game is given by the optimal value of the following problem.

$$(P_1^p) \quad \max_{\varphi} \min_{\pi} R(\varphi,\pi)$$

$$s.t. \quad \varphi(i,t) \geq 0, \ i \in \boldsymbol{K}, \ t \in \hat{\boldsymbol{T}}, \tag{5}$$

$$\sum_{i\in\boldsymbol{K}} \varphi(i,t) \leq \Phi(t), \ t \in \hat{\boldsymbol{T}}, \tag{6}$$

$$\pi(\omega) \geq 0, \ \omega \in \hat{\Omega}, \tag{7}$$

$$\sum_{\omega\in\hat{\Omega}} \pi(\omega) = 1. \tag{8}$$

From now on, we denote the feasible region of π by Π, which satisfies conditions (7), (8).

3. Optimal Solution

In this section, we propose a method to give an optimal solution. We already know that there exists an equilibrium point for the game and it satisfies $\min_\pi \max_\varphi R(\varphi,\pi) = \max_\varphi \min_\pi R(\varphi,\pi)$. We can transform the problem (P_1^p) as follows.

$$\max_{\varphi\in\Psi} \min_{\pi\in\Pi} R(\varphi,\pi) = \max_{\varphi\in\Psi} \min_{\omega\in\hat{\Omega}} R(\varphi,\omega) = \max \left\{\gamma | R(\varphi,\omega) \geq \gamma, \omega \in \hat{\Omega}, \varphi \in \Psi\right\}. \tag{9}$$

Since $R(\varphi,\omega) = 1-\exp\left(-\sum_{t\in\hat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t)\right) \geq \gamma$ is equivalent to $\sum_{t\in\hat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \geq \log 1/(1-\gamma)$, writing $\eta = \log 1/(1-\gamma)$, the transformation (9) points a linear programming method to solve the game.

$$(P_2^p) \quad \max_{\{\eta,\varphi(i,t)\}} \eta$$

$$s.t. \sum_{t\in\hat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) \geq \eta, \ \omega \in \hat{\Omega}, \ \sum_{i\in\boldsymbol{K}} \varphi(i,t) \leq \Phi(t), \ t \in \hat{\boldsymbol{T}}, \ \varphi(i,t) \geq 0, \ i \in \boldsymbol{K}, \ t \in \hat{\boldsymbol{T}}.$$

Using an optimal solution η^* of this problem, we can obtain the value of the game by $1 - \exp(-\eta^*)$. An optimal distribution of searching effort φ^* is given by another formulation, too. For simplicity, we redefine the payoff function of the game as $R(\varphi,\omega) = \sum_{t\in\hat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t)$ from now on. Now we can transform the expectation of the redefined payoff as follows.

$$\begin{aligned}\sum_{\omega\in\hat{\Omega}} \pi(\omega) R(\varphi,\omega) &= \sum_{\omega\in\hat{\Omega}} \pi(\omega) \sum_{t\in\hat{\boldsymbol{T}}} \alpha_{\omega(t)}\varphi(\omega(t),t) = \sum_{t\in\hat{\boldsymbol{T}}} \sum_{i\in\boldsymbol{K}} \sum_{\omega\in\hat{\Omega}} \pi(\omega)\delta_{i\omega(t)}\alpha_i\varphi(i,t) \\ &= \sum_{t\in\hat{\boldsymbol{T}}} \sum_{i\in\boldsymbol{K}} \left(\sum_{\omega\in\hat{\Omega}_{it}} \pi(\omega)\right) \alpha_i\varphi(i,t),\end{aligned}$$

where δ_{ij} is Kronecker's delta and $\hat{\Omega}_{it}$ is a set of paths going through cell i at time t, namely, $\hat{\Omega}_{it} := \{\omega \in \hat{\Omega} | \omega(t) = i\}$. Now we have

$$\max_{\varphi\in\Psi} R(\varphi,\pi) = \max_{\varphi\in\Psi} \sum_{t\in\hat{\boldsymbol{T}}} \sum_{i\in\boldsymbol{K}} \left(\sum_{\omega\in\hat{\Omega}_{it}} \pi(\omega)\right) \alpha_i\varphi(i,t) = \sum_{t\in\hat{\boldsymbol{T}}} \Phi(t) \max_{i\in\boldsymbol{K}} \left(\alpha_i \sum_{\omega\in\hat{\Omega}_{it}} \pi(\omega)\right)$$

and another linear programming problem D_2^p to give an optimal selection of target paths π^*.

$$(D_2^p)\quad \min_{\{\nu(t),\pi(\omega)\}} \sum_{t\in\hat{\boldsymbol{T}}} \Phi(t)\nu(t)$$

$$s.t.\ \ \alpha_i \sum_{\omega\in\hat{\Omega}_{it}} \pi(\omega) \le \nu(t),\ i\in\boldsymbol{K},\ t\in\hat{\boldsymbol{T}},\quad \sum_{\omega\in\hat{\Omega}} \pi(\omega)=1,\quad \pi(\omega)\ge 0,\ \omega\in\hat{\Omega}\ .$$

Two problems (P_2^p) and (D_2^p) are dual each other.

The proposed method looks simple but the applicability of the theory depends on the size of the feasible paths $\hat{\Omega}$ satisfying conditions (1)-(3). We are confronted with the difficulty of the combinatorial explosion by the reason that $\hat{\Omega}$ possibly has n^m elements for n cells and m time points. From the practical point of view, it is crucial how much extent we can reduce the total number of feasible paths by the constraints on the target path to.

4. A Method Based on Transition Probability

To avoid the combinatorial explosion, we consider the transition probability of the target instead of directly dealing with target paths, noting that any mixed strategy of target's selecting paths can be represented by its transition probability. Eagle and Washburn[3] took the similar idea for two moving players. For simplicity, it is assumed that the energy consuming function $\mu(\cdot)$ is an integer-valued function and initial energy e_0 of the target is a positive integer. We denote energy states of the target by $\boldsymbol{E}=\{0,\cdots,e_0\}$. Variable $q(i,t,e)$ indicates the probability that the target has energy e in cell i at time t and $v(i,j,t,e)$ the probability that the target in state (i,t,e) moves to cell j at the next time $t+1$. In this section, we take variables $\{q(i,t,e)\}$ and $\{v(i,j,t,e)\}$ as the target strategy while $\{\varphi(i,t)\}$ still represents the searcher's strategy.

As stated above in the formulation (P_2^p), we let $R(\varphi,\omega)=\sum_{t\in\hat{\boldsymbol{T}}}\alpha_{\omega(t)}\varphi(\omega(t),t)$ be a new payoff function. Variable $h(t)$ represents the maximum expected payoff gained by the search after time t. Since reachable cells from state (i,t,e) are denoted by $N(i,t,e)=\{j\in N(i,t)|\mu(i,j)\le e\}$, the cells from which the target comes into state (i,t,e) at time t are given by $N^*(i,t,e)=\{j\in\boldsymbol{K}|i\in N(j,t-1,e+\mu(j,i))\}$. It is self-evident that $N(i,t,e)$ are empty sets for $e>e_0$. The following recurrence relation is valid for $h(t)$.

$$\begin{aligned} h(t) &= \max_{\varphi\in\Psi}\left\{\sum_{i\in\boldsymbol{K}}\alpha_i\varphi(i,t)\sum_{e\in\boldsymbol{E}}q(i,t,e)+h(t+1)\right\} \\ &= \Phi(t)\max_{i\in\boldsymbol{K}}\left(\alpha_i\sum_{e\in\boldsymbol{E}}q(i,t,e)\right)+h(t+1). \end{aligned} \tag{10}$$

In the brace, there are two terms. The first is the expected payoff at time t and the second is the maximal payoff expected after t. Taking account of constraint $\sum_{i\in\boldsymbol{K}}\varphi(i,t)\le\Phi(t)$, the maximum of the first term is given by concentrating all searching effort $\Phi(t)$ in a cell i of $\arg\max_i\ \alpha_i\sum_e q(i,t,e)$. Now we have Equation (10) and then $h(t)\ge\Phi(t)\alpha_i\sum_e q(i,t,e)+h(t+1)$ holds for any i. The target intends to minimize the searcher's reward $h(\tau)$ after time τ. In the result, we can obtain optimal solutions q^*, v^* for the target's strategy by solving the following linear programming problem.

$$(P_1^e)\quad \min\ h(\tau)$$

$$s.t.\quad h(t)\ge\Phi(t)\alpha_i\sum_{e\in\boldsymbol{E}}q(i,t,e)+h(t+1),\ i\in\boldsymbol{K},\ t=\tau,\cdots,m-1, \tag{11}$$

$$h(m) \geq \Phi(m)\alpha_i \sum_{e \in \boldsymbol{E}} q(i,m,e), \ i \in \boldsymbol{K}, \tag{12}$$

$$q(i,t,e) = \sum_{j \in N(i,t,e)} v(i,j,t,e), \ i \in \boldsymbol{K}, \ t = 1, \cdots, m-1, \ e \in \boldsymbol{E}, \tag{13}$$

$$q(i,t,e) = \sum_{j \in N^*(i,t,e)} v(j,i,t-1,e+\mu(j,i)), \ i \in \boldsymbol{K}, \ t = 2, \cdots, m, \ e \in \boldsymbol{E}, \tag{14}$$

$$\sum_{i \in S_0} q(i,1,e_0) = 1, \tag{15}$$

$$\sum_{i \in \boldsymbol{K}} \sum_{e \in \boldsymbol{E}} q(i,t,e) = 1, \ t \in \boldsymbol{T}, \tag{16}$$

$$v(i,j,t,e) \geq 0, \ i,j \in \boldsymbol{K}, \ t = 1, \cdots, m-1, \ e \in \boldsymbol{E}. \tag{17}$$

Conditions (11), (12) come from recursive Equation (10) and conditions (13), (14) from the conservation law of the transition probability, and furthermore Eqs. (15), (16) from the facts that the total of the distribution probability of the target should be one at every time. It ought to be noted that the memory size is reduced to the order $O(n^2 \cdot m \cdot e_0)$ comparing with the order $O(n^m)$ for the previous path-based method.

The next step is to obtain an optimal strategy of the searcher. Let us construct the dual problem of (P_1^e). We assign dual variables $\eta(i,t)$ to condition (11) and do $y(i,t,e), z(i,t,e), \lambda$ and $\nu(t)$ to (13), (14), (15) and (16), respectively.

$$(D_1^e) \quad \max \ \lambda + \sum_{t=1}^{m} \nu(t)$$

$s.t.$

$$\lambda + y(i,1,e_0) + \nu(1) = 0, \ i \in S_0, \tag{18}$$

$$y(i,1,e) + \nu(1) = 0, \ i \in \boldsymbol{K}, \ e \in \boldsymbol{E} - \{e_0\}, \tag{19}$$

$$y(i,1,e) + \nu(1) = 0, \ i \in \boldsymbol{K} - S_0, \ e \in \boldsymbol{E}, \tag{20}$$

$$z(i,t,e) + y(i,t,e) + \nu(t) = 0, \ i \in \boldsymbol{K}, \ t = 2, \cdots, \tau - 1, \ e \in \boldsymbol{E}, \tag{21}$$

$$z(i,t,e) + y(i,t,e) + \nu(t) - \alpha_i \Phi(t)\eta(i,t) = 0, \ i \in \boldsymbol{K}, \ t = \tau, \cdots, m-1, \ e \in \boldsymbol{E}, \tag{22}$$

$$z(i,m,e) + \nu(m) - \alpha_i \Phi(m)\eta(i,m) = 0, \ i \in \boldsymbol{K}, \ e \in \boldsymbol{E}, \tag{23}$$

$$-y(i,t,e) - z(j,t+1,e-\mu(i,j)) \leq 0, i \in \boldsymbol{K}, t = 1, \cdots, m-1, j \in N(i,t,e), e \in \boldsymbol{E}, \tag{24}$$

$$\sum_{i \in \boldsymbol{K}} \eta(i,\tau) = 1, \tag{25}$$

$$-\sum_{i \in \boldsymbol{K}} \eta(i,t) + \sum_{i \in \boldsymbol{K}} \eta(i,t+1) = 0, \ t = \tau, \cdots, m-1, \tag{26}$$

$$\eta(i,t) \geq 0, \ i \in \boldsymbol{K}, \ t = \tau, \cdots, m. \tag{27}$$

There is redundancy between variables y and z, as seen from the transformation $y(i,t,e) = -z(i,t,e) - \nu(t)$ of equation (21). After carefully replacing $y(i,t,e)$ with $z(i,t,e)$ by equations (21), (22), that is,

$$y(i,t,e) \ = \ -z(i,t,e) - \nu(t), \ i \in \boldsymbol{K}, \ t = 2, \cdots, \tau - 1, \ e \in \boldsymbol{E},$$

$$y(i,t,e) \ = \ -z(i,t,e) - \nu(t) + \alpha_i \Phi(t)\eta(i,t) = 0, \ i \in \boldsymbol{K}, \ t = \tau, \cdots, m-1, \ e \in \boldsymbol{E}$$

and putting together equations (25) and (26), we have another formulation. There we insert a new variable $z(i,1,e) := -y(i,1,e) - \nu(1)$ which has not been defined in problem (D_1^e).

$$(D_2^e) \quad \max \ \lambda + \sum_{t=1}^{m} \nu(t)$$

$s.t.$

$$z(i,1,e_0) = \lambda, \ i \in S_0,$$

$$z(i,1,e)=0,\ i\in\boldsymbol{K},\ e\in\boldsymbol{E}-\{e_0\},$$
$$z(i,1,e)=0,\ i\in\boldsymbol{K}-S_0,\ e\in\boldsymbol{E},$$
$$z(i,t,e)+\nu(t)\le z(j,t+1,e-\mu(i,j)), i\in\boldsymbol{K}, j\in N(i,t,e), t=1,\cdots,\tau-1, e\in\boldsymbol{E},$$
$$z(i,t,e)+\nu(t)\le \alpha_i\Phi(t)\eta(i,t)+z(j,t+1,e-\mu(i,j)),$$
$$i\in\boldsymbol{K}, j\in N(i,t,e), t=\tau,\cdots,m-1, e\in\boldsymbol{E},$$
$$z(i,m,e)+\nu(m)=\alpha_i\Phi(m)\eta(i,m),\ i\in\boldsymbol{K},\ e\in\boldsymbol{E},$$
$$\sum_{i\in\boldsymbol{K}}\eta(i,t)=1,\ t=\tau,\cdots,m,$$
$$\eta(i,t)\ge 0,\ i\in\boldsymbol{K},\ t=\tau,\cdots,m.$$

Using $w(i,t,e):=z(i,t,e)+\sum_{k=t}^{m}\nu(k)$ instead of $z(i,t,e)$, we modify the problem further.

$$(D_3^e)\quad \max\ \lambda+\sum_{t=1}^{m}\nu(t)$$
$$s.t.\quad w(i,1,e_0)=\lambda+\sum_{t=1}^{m}\nu(t),\ i\in S_0, \tag{28}$$
$$w(i,1,e)=\sum_{t=1}^{m}\nu(t),\ i\in\boldsymbol{K},\ e\in\boldsymbol{E}-\{e_0\}, \tag{29}$$
$$w(i,1,e)=\sum_{t=1}^{m}\nu(t),\ i\in\boldsymbol{K}-S_0,\ e\in\boldsymbol{E}, \tag{30}$$
$$w(i,t,e)\le w(j,t+1,e-\mu(i,j)), i\in\boldsymbol{K}, j\in N(i,t,e), t=1,\cdots,\tau-1, e\in\boldsymbol{E}, \tag{31}$$
$$w(i,t,e)\le \alpha_i\Phi(t)\eta(i,t)+w(j,t+1,e-\mu(i,j)),$$
$$i\in\boldsymbol{K}, j\in N(i,t,e), t=\tau,\cdots,m-1, e\in\boldsymbol{E}, \tag{32}$$
$$w(i,m,e)=\alpha_i\Phi(m)\eta(i,m),\ i\in\boldsymbol{K},\ e\in\boldsymbol{E}, \tag{33}$$
$$\sum_{i\in\boldsymbol{K}}\eta(i,t)=1,\ t=\tau,\cdots,m, \tag{34}$$
$$\eta(i,t)\ge 0,\ i\in\boldsymbol{K},\ t=\tau,\cdots,m. \tag{35}$$

We can prove that the following formulation gives the same optimal value as problem (D_3^e).

$$(D_4^e)\quad \max\ \xi$$
$$s.t.\quad w(i,1,e_0)=\xi,\ i\in S_0, \tag{36}$$
$$w(i,1,e)=0,\ i\in\boldsymbol{K},\ e\in\boldsymbol{E}-\{e_0\}, \tag{37}$$
$$w(i,1,e)=0,\ i\in\boldsymbol{K}-S_0,\ e\in\boldsymbol{E}, \tag{38}$$
$$w(i,t,e)\le w(j,t+1,e-\mu(i,j)), i\in\boldsymbol{K}, j\in N(i,t,e), t=1,\cdots,\tau-1, e\in\boldsymbol{E}, \tag{39}$$
$$w(i,t,e)\le \alpha_i\Phi(t)\eta(i,t)+w(j,t+1,e-\mu(i,j)),$$
$$i\in\boldsymbol{K}, j\in N(i,t,e), t=\tau,\cdots,m-1, e\in\boldsymbol{E}, \tag{40}$$
$$w(i,m,e)=\alpha_i\Phi(m)\eta(i,m),\ i\in\boldsymbol{K},\ e\in\boldsymbol{E}, \tag{41}$$
$$\sum_{i\in\boldsymbol{K}}\eta(i,t)=1,\ t=\tau,\cdots,m, \tag{42}$$
$$\eta(i,t)\ge 0,\ i\in\boldsymbol{K},\ t=\tau,\cdots,m. \tag{43}$$

Since (D_4^e) is given by adding one more condition $\sum_{t=1}^{m}\nu(t)=0$ to (D_3^e), the optimal value of (D_3^e) is equal to or larger than (D_4^e). At the same time, we can say the value of (D_4^e) is equal to or larger than (D_3^e) by the following reason. The variable $w(i,m,e)$ is nonnegative from Eq. (33). Since the optimization of the problem is to maximize $w(i,1,e_0), i\in S_0$, variables $w(i,t,e), t=m-1,\cdots,1$ vary as large as possible while satisfying inequalities (31) and

(32). In the result, the values of expressions (29) and (30) become nonnegative. Given that an optimal solution $w^*(i,t,e)$, $\nu^*(t)$, λ^*, $\eta^*(i,t)$ for (D_3^e), we can obtain the same objective value by changing $w(i,1,e)=0$ for either of $i \in \boldsymbol{K}-S_0$ or $e \in \boldsymbol{E}-\{e_0\}$, $\nu(t)=0$ for $t \in \boldsymbol{T}$ and $\lambda=\lambda^*+\sum_t \nu^*(t)$ but remaining other variables unchanged. Therefore we conclude that a feasible solution of equations (36)-(43) gives the optimal value of (D_3^e).

We are ready to prove that an optimal strategy $\varphi(i,t)$ of the searcher is given by the optimal solution $\Phi(t)\eta(i,t)$ of problem (D_4^e). Let $w(i,t,e)$ be the minimum payoff given that the target starts from state (i,t,e). We have $w(i,t,e)=\min_{j\in N(i,t,e)}\{\alpha_i\varphi(i,t)+w(j,t+1,e-\mu(i,j))\}$ for $t=\tau,\cdots,m$ because the target's existence in cell i yields the payoff $\alpha_i\varphi(i,t)$ and $w(j,t+1,e-\mu(i,j))$ if he moves to cell j after then. By definition, $w(i,t,e)$ is given by minimizing the yields. On the contrary, there is no payoff occurred before starting the search and then we have $w(i,t,e)=\min_{j\in N(i,t,e)} w(j,t+1,e-\mu(i,j))$ for $t=1,\cdots,\tau-1$. Since the target is in cells S_0 and has energy e_0 at starting time, $w(i,1,e)=0$ holds in either case of $i \notin S_0$ or $e \neq e_0$. The searcher wants to increase the minimum payoff $w(i,1,e_0)$ as large as possible. This explanation gives us the following formulation.

(D_5^e) max ξ

$$\begin{aligned}
s.t.\ & w(i,1,e_0) \geq \xi,\ i \in S_0, && (44)\\
& w(i,1,e)=0,\ i \in \boldsymbol{K},\ e \in \boldsymbol{E}-\{e_0\}, && (45)\\
& w(i,1,e)=0,\ i \in \boldsymbol{K}-S_0,\ e \in \boldsymbol{E}, && (46)\\
& w(i,t,e) \leq w(j,t+1,e-\mu(i,j)), i \in \boldsymbol{K}, j \in N(i,t,e), t=1,\cdots,\tau-1, e \in \boldsymbol{E}, && (47)\\
& w(i,t,e) \leq \alpha_i\varphi(i,t)+w(j,t+1,e-\mu(i,j)), && \\
& \qquad i \in \boldsymbol{K}, j \in N(i,t,e), t=\tau,\cdots,m-1, e \in \boldsymbol{E}, && (48)\\
& w(i,m,e)=\alpha_i\varphi(i,m),\ i \in \boldsymbol{K},\ e \in \boldsymbol{E}, && (49)\\
& \sum_{i\in \boldsymbol{K}} \varphi(i,t)=\Phi(t),\ t=\tau,\cdots,m, && (50)\\
& \varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t=\tau,\cdots,m. && (51)
\end{aligned}$$

In the case of $\xi=w(i,1,e_0)<w(j,1,e_0)$ for optimal solution, the replacing $w(j,1,e_0)=\xi$ never affects the optimality of ξ. That is the reason why we can replace inequality (44) by $w(i,1,e_0)=\xi$. We can easily recognize relation $\varphi(i,t)=\Phi(t)\eta(i,t)$ by comparing (D_4^e) with (D_5^e). Summing up the discussion so far, we obtain the following theorem.

Theorem 2 *For the discrete search allocation game, problems (P_1^e) and (D_5^e) give us the optimal strategies of the target's movement and the searcher's distribution of the searching effort, respectively.*

5. Extended Models and Numerical Examples

The formulation of (P_1^e) has a kind of flexibility such that we can apply it to other types of models by just a small modification. Here we clarify the properties of optimal solutions of our game by some examples. At the same time, we deal with the extension of the formulation. Let us keep in mind that some round-error cannot be avoided in output figures of the following examples.

(1) Energy-free model

To make clear the property of the optimal solution, it is a good way to compare the solution with one for energy-free model. Here the target does not have any energy constraint like assumptions of $\mu(\cdot)$ and e_0, but still have the mobility constraint $N(i,t)$. On the analogy of the formulation (P_1^e), we easily derive the following problem, which is made by dropping energy index e.

(P^n) min $h(\tau)$

$$s.t.\ \ h(t) \geq \alpha_i \Phi(t) q(i,t) + h(t+1),\ i \in \boldsymbol{K},\ t = \tau, \cdots, m-1, \tag{52}$$

$$h(m) \geq \alpha_i \Phi(m) q(i,m),\ i \in \boldsymbol{K}, \tag{53}$$

$$q(i,t) = \sum_{j \in N(i,t)} v(i,j,t),\ i \in \boldsymbol{K},\ t = 1, \cdots, m-1, \tag{54}$$

$$q(i,t) = \sum_{j \in N^*(i,t)} v(j,i,t-1),\ i \in \boldsymbol{K},\ t = 2, \cdots, m, \tag{55}$$

$$\sum_{i \in S_0} q(i,1) = 1, \tag{56}$$

$$v(i,j,t) \geq 0,\ i,j \in \boldsymbol{K},\ t = 1, \cdots, m-1. \tag{57}$$

Let us consider an example of a search game played in cell space $\boldsymbol{K} = \{1, \cdots, 10\}$. Two cells with consecutive numbers are assumed to be adjacent each other. A time space is made of ten points, $m = 10$. We set other parameters as $\tau = 3, \Phi(t) = 1$ for $t \in \widehat{\boldsymbol{T}}$ and $\alpha_i = 1$ for $i \in \boldsymbol{K}$. The target starts from the first cell $S_0 = \{1\}$ and can always move to a cell within 3-neighbored cells, which defines $N(i,t)$. An optimal probability distribution of the target and an optimal distribution of searching effort are given in Table 1-(a) and 1-(b). It is evident that the feature of the optimal solution lies in uniform distribution. That is, the target should flatten his probability distribution in his reachable region and the searcher should distribute his effort uniformly there. This means that the target must not generate any point with dense probability, where the searcher would surely concentrate searching effort, and the searcher must not make the target anticipate easily where to avoid or where to pass through from the density of searching effort. This characteristic of optimal solution is always true.

Now we add energy constraints of $e_0 = 9$ and $\mu(i,j) = (i-j)^2$ to problem (P^n) and generate the original problem (P_1^e). Optimal solution is shown in Table 2. Although the uniformity strategy still looks effective in the interior of the reachable area of the target, we can also see randomness on the boundary of the area. The expansion of the reachable area is limited by the energy constraint, of course. The target which intends to run fast in the early time does not have mobility enough to go far later because he exhausts his energy. The energy constraint has such effect that trades off the expansion of reachable area for the uniformity of his probability distribution.

Table 1-(a). Optimal probability distribution of target for energy-free model

Cells										
10	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
9	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
8	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
7	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
6	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
5	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
4	0	0.25	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
3	0	0.25	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
2	0	0.25	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
1	1	0.25	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
Time	1	2	3	4	5	6	7	8	9	10

Table 1-(b). Optimal distribution of searching effort for energy-free model

Cells										
10	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
9	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
8	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
7	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
6	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
5	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
4	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
3	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
2	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
1	0	0	1/7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
Time	1	2	3	4	5	6	7	8	9	10

Table 2-(a). Optimal probability distribution of target

Cells										
10	0	0	0	0	0	0	0	0	0	0.041
9	0	0	0	0	0	0	0	0	0.055	0.065
8	0	0	0	0	0	0	0.068	0.125	0.118	0.111
7	0	0	0	0	0	0.143	0.133	0.125	0.118	0.112
6	0	0	0	0.085	0.167	0.143	0.133	0.125	0.118	0.112
5	0	0	0.2	0.183	0.167	0.143	0.133	0.125	0.118	0.112
4	0	0.002	0.2	0.183	0.167	0.143	0.133	0.125	0.118	0.112
3	0	0.341	0.2	0.183	0.167	0.143	0.133	0.125	0.118	0.112
2	0	0.389	0.2	0.183	0.167	0.143	0.133	0.125	0.118	0.112
1	1	0.268	0.2	0.183	0.167	0.143	0.133	0.125	0.118	0.112
Time	1	2	3	4	5	6	7	8	9	10

Table 2-(b). Optimal distribution of searching effort

Cells										
10	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0.244	0.136	0.125
7	0	0	0	0	0	0.109	0.148	0.108	0.123	0.125
6	0	0	0	0	0.167	0.221	0.236	0.108	0.123	0.125
5	0	0	0.2	0.2	0.167	0.134	0.123	0.108	0.123	0.125
4	0	0	0.2	0.2	0.167	0.134	0.123	0.108	0.123	0.125
3	0	0	0.2	0.2	0.167	0.134	0.123	0.108	0.123	0.125
2	0	0	0.2	0.2	0.167	0.134	0.123	0.108	0.123	0.125
1	0	0	0.2	0.2	0.167	0.134	0.123	0.108	0.123	0.125
Time	1	2	3	4	5	6	7	8	9	10

Table 2-(b) shows that the searcher distributes his searching effort uniformly on almost all area except the boundary. He allocates more effort on the boundary cell 8 at times 8

and 9 and aims to cover the slow-starter target who saves his energy in the early time and goes farther later on. On the other hand, he concentrates his effort a little more on interior cell 6 at times 6 and 7, which covers the target distribution of the quick starter staying in the interior area. In this case, the value of the game is 1.1803, which corresponds to 0.693 in terms of the detection probability of the target. While the value is 0.8429 for the previous energy-free case of Table 1. The energy constraint brings the restriction on the target motion so that the searching effort is distributed over smaller reachable region of the target more efficiently.

(2) Shelter for target

We can consider another energy-free model, in which optimal solution does not have the uniformity. Now let us think of a datum search game on a one-dimensional search space, where a target begins to evade from cell 1, $S_0 = \{1\}$, and can move to neighbor cells. Time space is made of $m = 10$ time points, $\boldsymbol{T} = \{1, \cdots, 10\}$. As a part of a cell space $\boldsymbol{K} = \{1, \cdots, 5\}$, the target has shelter cells $\widetilde{\boldsymbol{K}} \subset \boldsymbol{K}$, where he is never detected or captured. Let $\widetilde{\boldsymbol{K}}$ be a single cell $\{5\}$. In this case, we just need to replace conditions (52) and (53) with two conditions,

$$h(t) \geq \alpha_i \Phi(t) q(i,t) + h(t+1), \ i \in \boldsymbol{K}/\widetilde{\boldsymbol{K}}, \ t = \tau, \cdots, m-1,$$

$$h(m) \geq \alpha_i \Phi(m) q(i,m), \ i \bullet \boldsymbol{K}/\widetilde{\boldsymbol{K}} \ .$$

The search starts from time $\tau = 1$ and the amount of searching effort $\Phi(t) = 5$ is available for the searcher each time. Other parameters are set as $\alpha_i = 1, \mu(i,j) = (i-j)^2$. An optimal solution is given in Table 3, where the optimal searching effort given in Table 3-(b) are normalized such that figures indicate the ratios of distributed searching effort to available effort $\Phi(t)$ and they sum up to one. From now on, we will use such normalized figures for optimal distribution of search effort. In this case, the value of the game is 13.5, which approximately corresponds to detection probability 1.0.

Table 3-(a). Optimal probability distribution of target

Cells \ Time	1	2	3	4	5	6	7	8	9	10
5	0	0	0	0	0.250	0.438	0.578	0.684	0.762	0.822
4	0	0	0	0.25	0.188	0.141	0.106	0.079	0.059	0.045
3	0	0	0.333	0.25	0.188	0.141	0.106	0.079	0.059	0.045
2	0	0.5	0.333	0.25	0.188	0.141	0.106	0.079	0.059	0.045
1	1	0.5	0.333	0.25	0.188	0.141	0.106	0.079	0.059	0.045

Table 3-(b). Optimal distribution of searching effort

Cells \ Time	1	2	3	4	5	6	7	8	9	10
5	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0.8665	0.8220	0.7627	0.6835	0.5781	0.4375	0.25
3	0	0	0.3333	0.0445	0.0593	0.0791	0.1055	0.1406	0.1875	0.25
2	0	0.5	0.3333	0.0445	0.0593	0.0791	0.1055	0.1406	0.1875	0.25
1	1	0.5	0.3333	0.0445	0.0593	0.0791	0.1055	0.1406	0.1875	0.25

The target can reach a shelter in cell 5 at the earliest time $t = 5$ but he does not always do that. He takes the uniform distribution strategy and gradually enters the shelter. The searcher ambushes the coming target in cell 4, an entrance way for the shelter, after time $t = 4$. The concentration of searching effort into cell 4 decreases as the probability of the target is anticipated to be getting lower time by time.

The next case is the same shelter-model but has a slight different parameter of available effort $\Phi(t) = \{1,2,3,4,10,10,5,5,5,5\}$. The searching effort totals 50 as the previous case but more effort are assigned for the latter half of time. An optimal solution is shown in Table 4 and an optimal value of the game is 8.5317, which yields detection probability 1.0. As the available searching effort are less at early time $t = 1,2,3,4$, the target chooses a strategy of accelerating to get into the shelter more than the strategy of uniform probability distribution. The value of the game is less than the previous case because the searcher can not use so much effort before the target evacuates in the shelter.

Table 4-(a). Optimal probability distribution of target

Cells										
5	0	0	0	0	0.333	0.500	0.625	0.719	0.789	0.842
4	0	0	0	0.333	0.167	0.125	0.094	0.070	0.053	0.040
3	0	0	0.333	0.241	0.167	0.125	0.094	0.070	0.053	0.040
2	0	0.5	0.333	0.237	0.167	0.125	0.094	0.070	0.053	0.040
1	1	0.5	0.333	0.189	0.167	0.125	0.094	0.070	0.053	0.040
Time	1	2	3	4	5	6	7	8	9	10

Table 4-(b). Optimal distribution of searching effort

Cells										
5	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0.630	0.506	0.684	0.578	0.438	0.25
3	0	0	0.844	0	0.123	0.165	0.106	0.141	0.188	0.25
2	0	0.5	0.078	0	0.123	0.165	0.106	0.141	0.188	0.25
1	1	0.5	0.078	0	0.123	0.165	0.106	0.141	0.188	0.25
Time	1	2	3	4	5	6	7	8	9	10

We take one more similar case, in which less searching effort are assigned in the early half part of time such as $\Phi(t) = \{1,1,1,1,1,10,10,10,10,5\}$. The results are shown in Table 5 and the value of the game is 3.0 (0.95 in terms of the detection probability).

Table 5-(a). Optimal probability distribution of target

Cells										
5	0	0	0	0	0.5	1	1	1	1	1
4	0	0	0	0.5	0.5	0	0	0	0	0
3	0	0	0.5	0.5	0	0	0	0	0	0
2	0	0.5	0.5	0	0	0	0	0	0	0
1	1	0.5	0	0	0	0	0	0	0	0
Time	1	2	3	4	5	6	7	8	9	10

Table 5-(b). Optimal distribution of searching effort

Cells										
5	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0.642	1	0.380	0.344	0.325	0.301	0.273
3	0	0	0.665	0.358	0	0.221	0.214	0.213	0.213	0.225
2	0	0.692	0.335	0	0	0.217	0.232	0.240	0.250	0.257
1	1	0.308	0	0	0	0.182	0.210	0.222	0.236	0.246
Time	1	2	3	4	5	6	7	8	9	10

As shown in the table, the target biases his probability distribution only in two cells or less and accelerates to enter the shelter more remarkably than Table 4-(a) before a lot of searching effort become available for the searcher at time 6 or later. Corresponding to it, searching effort are distributed on two cells in the early time. The value of the game decreases further.

(3) Supply of energy on the way

Let us consider such a case that the target is supplied with energy e_f at some bases on the way of his evasion. The supply bases are located in cells K_f and the target wants to know the best time t_f to stop by there. Fixing t_f, we modify problem (P_1^e) as follows. Energy states can be represented by $\boldsymbol{E} = \{0, \cdots, e_0 + e_f\}$. Let $Q(i, e)$ be the probability of the target who reaches a supply base in cell i and is filled up to energy e. At time t_f, his energy e changes to $e+e_f$ and then we have $Q(i, e+e_f) = q(i, e)$, out of which the transition probability $v(i, j, t_f, e)$ flows. The adequate formulation for the model is the following.

(P^f) min $h(\tau)$

$$\begin{aligned}
s.t.\ & h(t) \geq \Phi(t)\alpha_i \sum_{e\in\boldsymbol{E}} q(i,t,e) + h(t+1),\ i \in \boldsymbol{K},\ t = \tau, \cdots, m-1, \\
& h(m) \geq \Phi(m)\alpha_i \sum_{e\in\boldsymbol{E}} q(i,m,e),\ i \in \boldsymbol{K}, \\
& q(i,t,e) = \sum_{j\in N(i,t,e)} v(i,j,t,e),\ i \in \boldsymbol{K},\ t = 1, \cdots, t_f - 1, t_f + 1, \cdots, m-1,\ e \in \boldsymbol{E}, \\
& q(i,t,e) = \sum_{j\in N^*(i,t,e)} v(j,i,t-1,e+\mu(j,i)),\ i \in \boldsymbol{K},\ t = 2, \cdots, m,\ e \in \boldsymbol{E}, \\
& Q(i, e+e_f) = q(i, t_f, e),\ i \in K_f,\ e = 0, \cdots, e_0, \\
& Q(i,e) = \sum_{j\in N(i,t,e)} v(i,j,t_f,e),\ i \in K_f, e \in \boldsymbol{E}, \\
& \sum_{i\in S_0} q(i,1,e_0) = 1, \\
& \sum_{i\in K_f} \sum_{e\in\boldsymbol{E}} q(i,t_f,e) = 1, \\
& v(i,j,t,e) \geq 0,\ i,j \in \boldsymbol{K},\ t = 1, \cdots, m-1,\ e \in \boldsymbol{E}.
\end{aligned}$$

In a search space with twelve cells, $n = 12$, and twelve time points, $m = 12$, the target starts to run away from cell 1 with energy $e_0 = 8$ and can move within 3-neighbored cells anytime. Other parameters are set up as follows: $\tau = 2, \alpha_i = 1, \Phi(t) = 1, \mu(i,j) = (i-j)^2$. Two bases are in cells $K_f = \{1, 6\}$ and they supply the target with energy $e_f = 4$. In five cases of $t_f = 4, 5, 6, 7, 8$, we have obtained the game values 2.868, 2.069, 2.059, 2.135, 2.210. From the result, the target should visit the bases at time $t_f = 6$. Table 6-(a), (b) and (c) illustrate optimal probability distributions of target for $t_f = 4, 6$ and 8, respectively. In the case of early visiting time $t_f = 4$, the target does not have time enough to go to the base on cell 6 and he needs to be back to the base 1 after expanding his reachable area. In the case of $t_f = 8$, stopping by the base in cell 6 seems to put bounds to the expansion of reachable area around times 6, 7, 8. On the contrary, the setting of $t_f = 6$ seems not to disturb the natural expansion of reachable area, which is in favor of the target. Anyway, the probability of visiting two bases at time t_f is fifty-fifty for both cases of $t_f = 6, 8$.

(4) On other models

In the models stated so far, all we have to do is just a minor modification of the original formulation (P_1^e). The reason is that we change only assumptions concerning with the target in those models. Considering transformation (10), the modifications are possible only based on the no-change of assumptions on searching effort. We can say that our formulation (D_5^e) is still flexible for the feasibility conditions of searching effort. For example, when we assume a total amount constraint of searching effort in the whole search space, say $\sum_t \sum_i \varphi(i,t) \leq M$, it would be done by adding the constraint to (D_5^e). Our formulations (P_1^e) and (D_5^e) have such flexibility for the change of assumptions on players' strategy.

Table 6-(a). Optimal probability distribution of target for $t_f = 4$

Cells \ Time	1	2	3	4	5	6	7	8	9	10	11	12
12	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0.002
9	0	0	0	0	0	0	0	0	0	0.111	0.111	0.111
8	0	0	0	0	0	0	0	0	0.125	0.111	0.111	0.111
7	0	0	0	0	0	0	0.098	0.143	0.125	0.111	0.111	0.111
6	0	0	0	0	0	0	0.150	0.143	0.125	0.111	0.111	0.111
5	0	0	0	0	0	0.2	0.150	0.143	0.125	0.111	0.111	0.111
4	0	0	0	0	0.25	0.2	0.150	0.143	0.125	0.111	0.111	0.111
3	0	0.333	0.333	0	0.25	0.2	0.150	0.143	0.125	0.111	0.111	0.111
2	0	0.333	0.333	0	0.25	0.2	0.150	0.143	0.125	0.111	0.111	0.111
1	1	0.333	0.333	1	0.25	0.2	0.150	0.143	0.125	0.111	0.111	0.111

Table 6-(b). Optimal probability distribution of target for $t_f = 6$

Cells \ Time	1	2	3	4	5	6	7	8	9	10	11	12
12	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0.091	0.091	0.091
10	0	0	0	0	0	0	0	0	0.1	0.091	0.091	0.091
9	0	0	0	0	0	0	0	0.111	0.1	0.091	0.091	0.091
8	0	0	0	0	0	0	0.125	0.111	0.1	0.091	0.091	0.091
7	0	0	0	0	0	0	0.125	0.111	0.1	0.091	0.091	0.091
6	0	0	0	0	0.167	0.5	0.125	0.111	0.1	0.091	0.091	0.091
5	0	0	0	0.2	0.167	0	0.125	0.111	0.1	0.091	0.091	0.091
4	0	0	0.25	0.2	0.167	0	0.125	0.111	0.1	0.091	0.091	0.091
3	0	0.333	0.25	0.2	0.167	0	0.125	0.111	0.1	0.091	0.091	0.091
2	0	0.333	0.25	0.2	0.167	0	0.125	0.111	0.1	0.091	0.091	0.091
1	1	0.333	0.25	0.2	0.167	0.5	0.125	0.111	0.1	0.091	0.091	0.091

Table 6-(c). Optimal probability distribution of target for $t_f = 8$

Cells												
12	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0.091
10	0	0	0	0	0	0	0	0	0	0	0.1	0.091
9	0	0	0	0	0	0	0	0	0	0.111	0.1	0.091
8	0	0	0	0	0	0	0	0	0.125	0.111	0.1	0.091
7	0	0	0	0	0	0	0	0	0.125	0.111	0.1	0.091
6	0	0	0	0	0.167	0.167	0.167	0.5	0.125	0.111	0.1	0.091
5	0	0	0	0.2	0.167	0.167	0.167	0	0.125	0.111	0.1	0.091
4	0	0	0.25	0.2	0.167	0.167	0.167	0	0.125	0.111	0.1	0.091
3	0	0.333	0.25	0.2	0.167	0.167	0.167	0	0.125	0.111	0.1	0.091
2	0	0.333	0.25	0.2	0.167	0.167	0.167	0	0.125	0.111	0.1	0.091
1	1	0.333	0.25	0.2	0.167	0.167	0.167	0.5	0.125	0.111	0.1	0.091
Time	1	2	3	4	5	6	7	8	9	10	11	12

6. Conclusions

This paper deals with a discrete search allocation game which is a two-person zero-sum game with two players of a searcher and a target. The target moves around the search space to evade the searcher but the moving inevitably consumes some energy. The searcher distributes a limited amount of searching effort in the space to detect the target. The payoff is the detection probability of the target. The target motion is modeled in two ways. One enumerates each route for the target path. In another model, the target motion are represented by the flow of transition probability. By both models, the game is formulated into a linear programming problem and easy to solve. The former model gives us direct and simple image about the target motion, by which we prove the existence of equilibrium point for the game. However it needs the enumeration of each target path so that it requires a large size of memories. The latter model is formulated by the dynamic programming so that we can understand the search process from the point of view of the payoff and additional constraints are comparatively easy to be embedded in the formulation. In this paper, the problem is dealt with in a discrete search space. Energy states are regarded as integers too, but it becomes burden on the extension of the problem and the numerical method for optimal solution. That is to be settled in the near future.

References

[1] V.J. Baston and F.A. Bostock: A one-dimensional helicopter-submarine game. *Naval Research Logistics*, **36**(1989) 479–490.

[2] J.M. Danskin: A helicopter versus submarine search game. *Operations Research*, **16** (1968) 509-517.

[3] J.N. Eagle and A.R. Washburn: Cumulative search-evasion games. *Naval Research Logistics*, **38**(1991) 495–510.

[4] A.Y. Garnaev: A remark on a helicopter-submarine game. *Naval Research Logistics*, **40**(1993) 745-753.

[5] A.Y. Garnaev: *Search Games and Other Applications of Game Theory* (Springer-Verlag, Tokyo, 2000).

[6] R. Hohzaki and K. Iida: A search game with reward criterion. *Journal of the Operations Research Society of Japan*, **41**(1998) 294–320.

[7] R. Hohzaki and K. Iida: A search game when a search path is given. *European Journal of Operational Research*, **124**(2000) 114–124.

[8] R. Hohzaki and K. Iida: A solution for a two-person zero-sum game with a concave payoff function. In W. Takahashi and T. Tanaka(eds.): *Nonlinear Analysis and Convex Analysis* (World Science Publishing Co., London, 1999), 157–166.

[9] K. Iida, R. Hohzaki and S. Furui: A search game for a mobile target with the conditionally deterministic motion defined by paths. *Journal of the Operations Research Society of Japan*, **39**(1996) 501–511.

[10] K. Iida, R. Hohzaki and K. Sato: Hide-and-search game with the risk criterion. *Journal of the Operations Research Society of Japan*, **37**(1994) 287–296.

[11] K. Kikuta: A hide and seek game with traveling cost. *Journal of the Operations Research Society of Japan*, **33**(1990) 168–187.

[12] K. Kikuta: A search game with traveling cost. *Journal of the Operations Research Society of Japan*, **34**(1991) 365–382.

[13] B.O. Koopman: *Search and Screening* (Pergamon, 1980), 221-227.

[14] J.J. Meinardi: A sequentially compounded search game. In A. Mensh(ed.): *Theory of Games: Techniques and Applications* (The English Universities Press, London, 1964), 285–299.

[15] T. Nakai: A sequential evasion-search game with a goal. *Journal of the Operations Research Society of Japan*, **29**(1986) 113–122.

[16] T. Nakai: Search models wiht continuous effort under various criteria. *Journal of the Operations Research Society of Japan*, **31**(1988) 335–351.

[17] A.R. Washburn: Search-evasion game in a fixed region. *Operations Research*, **28**(1980) 1290-1298.

[18] A.R. Washburn and R. Hohzaki: The diesel submarine flaming datum problem. *Military Operations Research*, **6** (2001) 19–30.

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