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OPTIMUM REQUIREMENT HAMILTON CYCLE PROBLEM WITH A MONGE-LIKE PROPERTY

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Abstract Given a simple graph G with a vertex set V and a set of "requirements" $\{r_{vw} \mid v, w \in V\}$, we consider a problem to find a Hamilton cycle minimizing an objective function similar to that in the optimum requirement spanning tree (ORST) problem studied by Hu. And we show that a particular Hamilton cycle C^* which is explicitly definable is a solution to the problem when G is complete and $\{r_{vw}\}$ satisfies inverse Supnick property closely related to Monge property. It is of great interest that C^* is also a solution to the symmetric traveling salesman problem with (not inverse) Supnick property. The result in this paper can be applied to the construction of ring networks with high reliability in case where the inverse Supnick property is naturally assumed.

1. Introduction

The *optimum requirement spanning tree problem* (ORST problem) studied by Hu [5] is often discussed as a problem of finding a communication network of tree type with the minimum average cost. However, from the viewpoint of reliability, k-connected spanning subgraph ($k \geq 2$) should be considered than spanning trees. Hence, we take up in this paper 2-connected graphs with the minimum size, i.e. Hamilton cycles, and consider a problem of finding one which minimizes an objective function similar to that in the ORST problem.

Before detailed discussion, let us define some basic notations and review the ORST problem. Suppose that a simple graph $G = (V, E)$ is given, where $V = \{0, 1, \ldots, n-1\}$ is the set of vertices and E is the set of edges. Throughout this paper, we assume that G is a complete graph. An edge connecting two vertices v and w is denoted by an unordered pair (v, w). Let $\binom{V}{2}$ be the set of all pairs of distinct vertices in V. We define the length of a path as the number of edges forming the path. For a subgraph G' of G, let $d(v, w; G')$ be the distance (the length of the shortest path(s)) between two vertices v and w of G'. Assume that a nonnegative value r_{vw} (called *requirement*, representing the frequency of communication between v and w) is given to each pair $\{v, w\} \in \binom{V}{2}$, where $r_{vw} = r_{wv}$ holds. Hu [5] defined an ORST as a spanning tree T of G which minimizes

$$f(T) = \sum_{\{v,w\} \in \binom{V}{2}} d(v, w; T) r_{vw},$$

and showed that an ORST is obtained by the Gomory-Hu algorithm [4] when the degrees of vertices are *not* restricted. On the other hand, Anazawa [1] considered a problem of finding a spanning tree T of G to

$$\begin{array}{ll} \text{minimize} & f_g(T) = \sum_{\{v,w\}\in\binom{V}{2}} g(d(v,w;T))r_{vw}, \\ \text{subject to} & \deg(v;T) \leq l_v \ (v \in V), \\ & l_0 \geq l_1 \geq \cdots \geq l_{n-1} \geq 1, \ \sum_{v=0}^{n-1} l_v \geq 2(n-1), \end{array}$$

where $g(x)$ is an arbitrary real-valued function of real variable x such that it is monotone nondecreasing on $[0, n-1]$, $\deg(v;T)$ denotes the degree of v in T, and l_v is a positive integer given to each $v \in V$. And he showed that if $\{r_{vw}\}$ satisfies

$$r_{vw} \geq r_{vw'} \quad \text{for all} \quad 0 \leq v \leq n-1, \ 0 \leq w < w' \leq n-1 \quad (w \neq v, w' \neq v) \tag{1}$$

and

$$r_{vw} + r_{v'w'} \geq r_{vw'} + r_{v'w} \quad \text{for all} \quad 0 \leq v < v' \leq n-1, \ 0 \leq w < w' \leq n-1 \quad (v \neq w, v \neq w', v' \neq w, v' \neq w'), \tag{2}$$

then a particular spanning tree T^* of G which is explicitly definable is a solution to the problem. Roughly speaking, the tree T^* is constructed by the following "greedy algorithm": First, to vertex 0, connect the remaining vertices by ascending order of vertex number as many as possible; secondly, to vertex 1, connect the remaining vertices by the same order as many as possible; and continue to connect the remaining vertices in the same manner until all n vertices are connected. For rigid definition of T^*, see [1]. Condition (2) is called *inverse Supnick* property, since we can obtain so-called *Supnick* property by reversing the inequality sign of the first inequality in (2) (cf. [3]). Supnick and inverse Supnick properties are known to make the symmetric traveling salesman problem explicitly solvable, which is discovered by Supnick [7] (see also [3]). Also, they are closely related to *Monge* property which is known to make some NP-hard problems polynomially solvable (see e.g. [6] and [3]). Although condition (2) seems a little tight, there is a case where the condition reflects a practical situation, which is shown in the last section (see also [1]).

Here, we define the problem to be considered in this paper. For a Hamilton cycle C in G and two vertices v and w on C, there exist two paths between v and w, say P_1 and P_2. Suppose that the length of P_1 is shorter than or equal to that of P_2. Then the length of P_1 equals $d(v, w; C)$ and that of P_2 equals $n - d(v, w; C)$. Let p_{vw} $(0 \leq p_{vw} \leq 1)$ be the relative frequency of using P_1 and $\bar{p}_{vw}$ that of using P_2 where $p_{vw} + \bar{p}_{vw} = 1$ holds. Then the *average distance* between v and w on C is defined by

$$d_{\text{AVG}}(v, w; C) = p_{vw} d(v, w; C) + \bar{p}_{vw}(n - d(v, w; C)).$$

We assume in this paper that, for any $\{v, w\} \in \binom{V}{2}$, p_{vw} and $\bar{p}_{vw}$ are expressed by $p(d(v, w; C))$ and $p(n - d(v, w; C))$ respectively, where $p(d)$ is a monotone nonincreasing function of d defined on $[0, n]$. Then it follows from $p(d) + p(n - d) = 1$ that $p(d) \geq \frac{1}{2}$ holds for $d \in [0, \frac{n}{2}]$. For example,

$$p(d) = \frac{n-d}{n} \ (0 \leq d \leq n) \quad \text{and} \quad p(d) = \begin{cases} 1 & (0 \leq d < \frac{n}{2}) \\ \frac{1}{2} & (d = \frac{n}{2}) \\ 0 & (\frac{n}{2} < d \leq n) \end{cases}$$

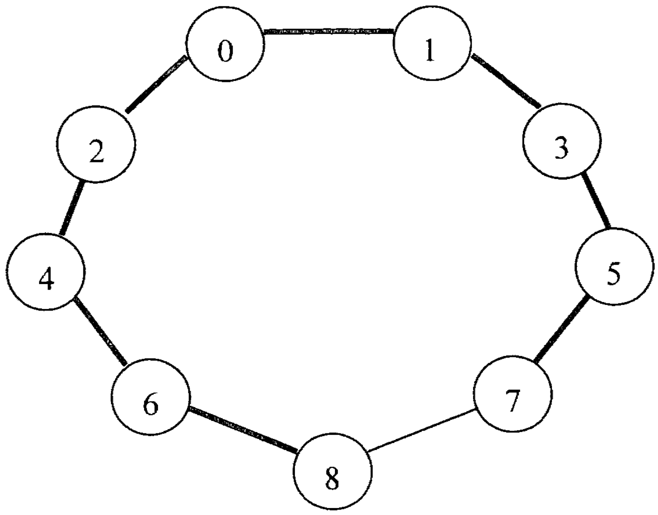

Figure 1: C^* for $n = 9$. In this figure, the subgraph drawn by bold lines coincides with T^* for $n = 9$ and $l_v = 2$ $(v \in V)$.

satisfy the condition stated above. The problem we want to solve is to find a Hamilton cycle C in G which minimizes a function

$$f_{\text{AVG}}(C) = \sum_{\{v,w\} \in \binom{V}{2}} d_{\text{AVG}}(v, w; C) r_{vw},$$

and we call a Hamilton cycle minimizing this function an *optimum requirement Hamilton cycle* (ORHC).

The main result of this paper is the following

Main Theorem *Let*

$$e_i^* = \begin{cases} (0,1) & \text{for } i = 1 \\ (i-2, i) & \text{for } i = 2, 3, \ldots, n-1 \\ (n-2, n-1) & \text{for } i = n \end{cases},$$

and $C^* = (V, E_{C^*})$ *where* $E_{C^*} = \{e_1^*, e_2^*, \ldots, e_n^*\}$. *If* $\{r_{vw}\}$ *satisfies condition (2), then* C^* *is an ORHC.*

Remark Here, $\{r_{vw}\}$ does not need to satisfy (1). Also, C^* is obtained by the "greedy algorithm" stated above. In fact, construct T^* by the "greedy algorithm" with $l_v = 2$ $(v \in V)$, and add an edge $(n-2, n-1)$ to T^*. Then we obtain the Hamilton cycle C^* (see Figure 1). Further, it is of great interest that C^* is expressed by a cycle permutation $< 0, 2, 4, \ldots, n-1, \ldots, 5, 3, 1 >$ which is an explicit solution to the symmetric traveling salesman problem with the Supnick property (see e.g. [7], [2] and [3]).

In this paper, after giving mathematical preliminaries in Section 2, we will show a property of connected subgraphs of C^* in Section 3. The proof of Main Theorem will be given in Section 4. In the last section, we will show an application of the ORHC problem to the construction of ring networks with high reliability, and a case where the inverse Supnick property is naturally assumed.

2. Preliminaries

Lemma 1 *For any Hamilton cycle C in G and any vertices v, w, v' and w' on C, if $d(v,w;C) < d(v',w';C)$, then $d_{\mathrm{AVG}}(v,w;C) \le d_{\mathrm{AVG}}(v',w';C)$ holds.*

Proof. From the assumptions of p_{vw} and $\bar{p}_{vw}$, we can express $d_{\mathrm{AVG}}(v,w;C)$ by $d_{\mathrm{AVG}}(d(v,w;C))$ for any $\{v,w\} \in \binom{V}{2}$, where

$$d_{\mathrm{AVG}}(d) = p(d) \cdot d + (1 - p(d))(n - d) \quad \left(d \in \left[0, \frac{n}{2}\right]\right).$$

Then we find from the assumption of $p(\cdot)$ that

$$d_{\mathrm{AVG}}(d+1) - d_{\mathrm{AVG}}(d) = (n - 2d)(p(d) - p(d+1)) + 2\left(p(d+1) - \frac{1}{2}\right) \ge 0$$

holds for any $d \in [0, \frac{n}{2} - 1]$. □

For a Hamilton cycle $C = (V, E_C)$ in $G = (V, E)$ $(E_C \subset E)$ and a path $P = (u_1, \ldots, u_k)$ $(k = 2$ or $3)$ of C, let

$$m = \begin{cases} \lfloor n/2 \rfloor & \text{if } k = 2 \\ \lfloor (n-1)/2 \rfloor & \text{if } k = 3 \end{cases},$$

where $\lfloor x \rfloor$ is the maximum integer not exceeding x, and let $P(u_1) = (V(u_1), E(u_1))$ and $P(u_k) = (V(u_k), E(u_k))$, where $V(u_1) \cup V(u_k) \subset V$, $V(u_1) \cap V(u_k) = \emptyset$,

$$V(u_1) = \{s_1(= u_1), s_2, \ldots, s_m\}, \quad E(u_1) = \{(s_i, s_{i+1}) \in E_C | i = 1, 2, \ldots, m-1\},$$

$$V(u_k) = \{t_1(= u_k), t_2, \ldots, t_m\}, \quad E(u_k) = \{(t_i, t_{i+1}) \in E_C | i = 1, 2, \ldots, m-1\}$$

are satisfied. For the path $P = (u_1, \ldots, u_k)$, we define an isomorphism $\sigma_P : V(u_1) \to V(u_k)$ by $\sigma_P(s_i) = t_i$ $(i = 1, 2, \ldots, m)$. Also, we consider the following transformation of C which may reduce the f_{AVG} value: Let $V_{\mathrm{GT}} = \{v \in V(u_1) | v > \sigma_P(v)\}$, and exchange v and $\sigma_P(v)$ for all $v \in V_{\mathrm{GT}}$. We call such a transformation *biasing* with respect to σ_P. Further, let C' be a Hamilton cycle obtained from C by biasing with respect to σ_P (see Figure 2).

Lemma 2 *If $\{r_{vw}\}$ satisfies condition (2), then $f_{\mathrm{AVG}}(C') \le f_{\mathrm{AVG}}(C)$ holds.*

Proof. Note that if $k = 2$ and n is odd, or $k = 3$ and n is even, then C has a vertex c adjacent to both s_m and t_m. Let $\bar{V}_{\mathrm{GT}} = V(u_1) \backslash V_{\mathrm{GT}}$ and $D_{vw} = \{d_{\mathrm{AVG}}(v,w;C') - d_{\mathrm{AVG}}(v,w;C)\} r_{vw}$. First, we consider the case where $k = 2$ and n is even. Then $f_{\mathrm{AVG}}(C') - f_{\mathrm{AVG}}(C)$ is expressed by

$$\begin{aligned} & \sum_{v,w \in V_{\mathrm{GT}}: v<w} D_{vw} + \sum_{v \in V_{\mathrm{GT}}, w \in \sigma_P(V_{\mathrm{GT}})} D_{vw} + \sum_{v,w \in \sigma_P(V_{\mathrm{GT}}): v<w} D_{vw} \\ + & \sum_{v,w \in \bar{V}_{\mathrm{GT}}: v<w} D_{vw} + \sum_{v \in \bar{V}_{\mathrm{GT}}, w \in \sigma_P(\bar{V}_{\mathrm{GT}})} D_{vw} + \sum_{v,w \in \sigma_P(\bar{V}_{\mathrm{GT}}): v<w} D_{vw} \qquad (3) \\ + & \sum_{v \in V_{\mathrm{GT}}, w \in \bar{V}_{\mathrm{GT}}} D_{vw} + \sum_{v \in V_{\mathrm{GT}}, w \in \sigma_P(\bar{V}_{\mathrm{GT}})} D_{vw} + \sum_{v \in \sigma_P(V_{\mathrm{GT}}), w \in \bar{V}_{\mathrm{GT}}} D_{vw} + \sum_{v \in \sigma_P(V_{\mathrm{GT}}), w \in \sigma_P(\bar{V}_{\mathrm{GT}})} D_{vw}, \end{aligned}$$

where $\sigma_P(S) = \{\sigma_P(v) \in V(u_k) | v \in S \subset V(u_1)\}$. However, noting that the first six summations are all equal to zero, we have

$$f_{\mathrm{AVG}}(C') - f_{\mathrm{AVG}}(C) = \sum_{v \in V_{\mathrm{GT}}, w \in \bar{V}_{\mathrm{GT}}} (D_{vw} + D_{v\sigma_P(w)} + D_{\sigma_P(v)w} + D_{\sigma_P(v)\sigma_P(w)}).$$

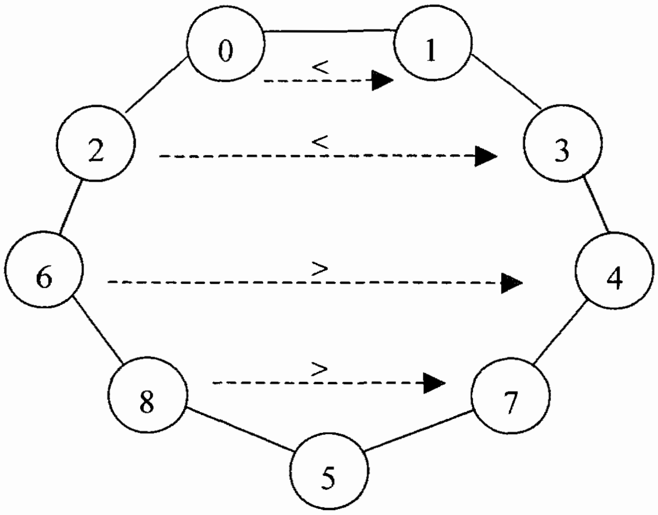

Figure 2: In the above cycle, if we set $P = (0,1)$ where $u_1 = 0$ and $u_k = 1$ $(k = 2)$, then σ_P is denoted by a set of dotted arrows. Then, for each pair of vertices $\{6,4\}$ and $\{8,7\}$, two vertices of the pair are exchanged by biasing with respect to σ_P.

For two vertices $v \in V_{\mathrm{GT}}$ and $w \in \bar{V}_{\mathrm{GT}}$, let $\delta_{vw} = d_{\mathrm{AVG}}(v, w; C)$ and $\Delta_{vw} = d_{\mathrm{AVG}}(v, w; C')$. Then we obtain $\delta_{vw} \leq \Delta_{vw}$. In fact,

$$\begin{aligned} d(v,w;C') &= \min\{d(v,w,C) + d(v,\sigma_P(v);C), d(\sigma_P(v),\sigma_P(w);C) + d(w,\sigma_P(w);C)\} \\ &> d(v,w;C) \end{aligned}$$

holds because $d(v,w;C) = d(\sigma_P(v),\sigma_P(w);C)$, and $\delta_{vw} \leq \Delta_{vw}$ comes from Lemma 1. Also, noting that

$$\delta_{vw} = d_{\mathrm{AVG}}(v,w;C) = d_{\mathrm{AVG}}(\sigma_P(v),\sigma_P(w);C) = d_{\mathrm{AVG}}(v,\sigma_P(w);C') = d_{\mathrm{AVG}}(\sigma_P(v),w;C')$$

and

$$\Delta_{vw} = d_{\mathrm{AVG}}(v,w;C') = d_{\mathrm{AVG}}(\sigma_P(v),\sigma_P(w);C') = d_{\mathrm{AVG}}(v,\sigma_P(w);C) = d_{\mathrm{AVG}}(\sigma_P(v),w;C),$$

we obtain

$$f_{\mathrm{AVG}}(C') - f_{\mathrm{AVG}}(C) = -\sum_{v \in V_{\mathrm{GT}}, w \in \bar{V}_{\mathrm{GT}}} \{\Delta_{vw} - \delta_{vw}\} \left(r_{\sigma_P(v)w} + r_{v\sigma_P(w)} - r_{\sigma_P(v)\sigma_P(w)} - r_{vw}\right). \tag{4}$$

Due to the assumption of $\{r_{vw}\}$, the second factor of the summand in (4) is always nonnegative, which means that $f_{\mathrm{AVG}}(C') - f_{\mathrm{AVG}}(C) \leq 0$ holds.

In case where $k = 2$ and n is odd, we have

$$f_{\mathrm{AVG}}(C') - f_{\mathrm{AVG}}(C) = \sum_{v \in V \setminus \{c\}} D_{vc} + (3).$$

However, the first summation is equal to zero. Hence, $f_{\mathrm{AVG}}(C') - f_{\mathrm{AVG}}(C) \leq 0$ is similarly obtained.

When $k = 3$ and n is odd, we have

$$f_{\mathrm{AVG}}(C') - f_{\mathrm{AVG}}(C) = \sum_{v \in V \setminus \{u_2\}} D_{vu_2} + (3).$$

Since the first summation is equal to zero, we find that $f_{\text{AVG}}(C') - f_{\text{AVG}}(C) \leq 0$ holds.

In case where $k = 3$ and n is even, we have

$$f_{\text{AVG}}(C') - f_{\text{AVG}}(C) = D_{u_2c} + \sum_{v\in V\setminus\{u_2,c\},w\in\{u_2,c\}} D_{vw} + (3).$$

Since the first term and the next summation are equal to zero, we find that $f_{\text{AVG}}(C') - f_{\text{AVG}}(C) \leq 0$ holds. □

3. A Property of Connected Subgraphs of C^*

Let $V_\nu = \{0, 1, \ldots, \nu-1\}$ $(1 \leq \nu \leq n)$ and $P^*_\nu = (V_\nu, \{e^*_1, e^*_2, \ldots, e^*_{\nu-1}\})$ where e^*_i $(i \geq 1)$ are defined in Main Theorem. Then P^*_ν is a connected subgraph of C^* for each $\nu \in \{1, 2, \ldots, n\}$. Note that C^* is obtained by adding $e^*_n = (n-2, n-1)$ to P^*_n.

Lemma 3 *Suppose that a Hamilton cycle $C = (V, E_C)$ in G contains a subgraph P^*_ν $(1 \leq \nu \leq n)$, that is, $e^*_i \in E_C$ holds for $i = 1, 2, \ldots, \nu-1$. For an arbitrarily-selected path $P = (u_1, \ldots, u_k)$ ($k = 2$ or 3) of C, let $C' = (V, E_{C'})$ be another Hamilton cycle obtained from C by biasing with respect to σ_P. Then C' also contains P^*_ν.*

Proof. Since the case of $\nu = 1$ is trivial, suppose that $\nu \geq 2$ holds. We define $P(u_1) = (V(u_1), E(u_1))$ and $P(u_k) = (V(u_k), E(u_k))$ as those in the previous section. Before showing the proof in detail, let us enumerate the cases to be considered. When $k = 2$ and n is even, $0 \in V(u_1)$ can be assumed without loss of generality. Then either $V_\nu \cap V(u_k) = \emptyset$ or $V_\nu \cap V(u_k) \neq \emptyset$ holds. When $k = 3$ or n is odd, we can assume without loss of generality that either $0 \in V(u_1)$ or $0 \in V \setminus (V(u_1) \cup V(u_k))$ holds. If $0 \in V(u_1)$, then either $V_\nu \cap V(u_k) = \emptyset$ or $V_\nu \cap V(u_k) \neq \emptyset$ holds. If $0 \in V \setminus (V(u_1) \cup V(u_k))$, further assume $1 \in V(u_1)$ without loss of generality. Then we have $V_\nu \cap V(u_k) = \emptyset$ if $\nu = 2$, and $V_\nu \cap V(u_k) \neq \emptyset$ if $\nu > 2$. To sum up, we have only to consider the following three cases:

(i) $V_\nu \cap V(u_k) = \emptyset$,

(ii) $V_\nu \cap V(u_k) \neq \emptyset$ and $0 \in V(u_1)$,

(iii) $V_\nu \cap V(u_k) \neq \emptyset$, $0 \in V \setminus (V(u_1) \cup V(u_k))$ and $1 \in V(u_1)$.

In case (i), if $v \in V_\nu \cap V(u_1)$, then $v < \nu \leq \sigma_P(v)$ holds. Hence, we have $V_\nu \cap V(u_1) \subset \bar{V}_{\text{GT}}$, which implies that C' contains P^*_ν. In case (ii), assume that $s_i > t_i$ holds for some $s_i \in V_\nu$ $(1 \leq i \leq m)$. Then $t_i \in V_\nu$ is obvious. If $s_i = 2j$ holds for some integer j $(j \geq 1)$, then $t_i \leq 2j-1$ holds and $d(s_i, 0; C) = j$ and $d(t_i, 0; C) \leq j$ are satisfied, which means that $0 \notin V(u_1)$ holds (contradiction). If $s_i = 2j+1$ holds for some integer j $(j \geq 0)$, then $t_i \leq 2j$ holds and $d(s_i, 0; C) = j+1$ and $d(t_i, 0; C) \leq j$ are satisfied, which also leads to contradiction. Hence, $s_i < t_i$ holds for all $s_i \in V_\nu$, which implies that $V_\nu \cap V(u_1) \subset \bar{V}_{\text{GT}}$ holds. Also, we find that if $t_l \in V_\nu$ then $s_l \in V_\nu$ holds. In fact, if P^*_ν has s_1 and t_1, then we easily find that at least l vertices $s_1, s_2, \ldots, s_l$ belong to V_ν; if P^*_ν has s_m and t_m, then at least $m-l+1$ vertices $s_m, s_{m-1}, \ldots, s_l$ belong to V_ν. Hence, we have $V_\nu \cap V(u_k) \subset \sigma_P(V_\nu \cap V(u_1))$, which implies that $V_\nu \cap V(u_k) \subset \sigma_P(\bar{V}_{\text{GT}})$ holds. Therefore, C' also contains P^*_ν. In case (iii), we find that

$$s_1 = 1, s_2 = 3, \ldots \text{ and } t_1 = 2, t_2 = 4, \ldots$$

or

$$s_m = 1, s_{m-1} = 3, \ldots \text{ and } t_m = 2, t_{m-1} = 4, \ldots$$

hold, which means that $V_\nu \cap V(u_1) \subset \bar{V}_{\text{GT}}$ and $V_\nu \cap V(u_k) \subset \sigma_P(\bar{V}_{\text{GT}})$ hold. Hence, C' also contains P^*_ν. □

4. Proof of Main Theorem

Let $C^* = (V, E_{C^*})$ be the Hamilton cycle in G defined in Main Theorem. For a Hamilton cycle $C = (V, E_C)$ in G, let

$$v_C = \begin{cases} \min\{v > 0 | e_v^* \notin E_C\} & \text{if } E_C \neq E_{C^*} \\ n-1 & \text{if } E_C = E_{C^*}. \end{cases}$$

We will show that any ORHC can be transformed into C^* with the f_{AVG} value unchanged.

Let $C = (V, E_C)$ be an ORHC with $v_C < n-1$. Note that C contains a subgraph $P_{v_C}^*$. Also, let v^* be a vertex with $e_{v_C}^* = (v^*, v_C)$, and v^{**} a vertex with $v^{**} > v_C$ and $(v^*, v^{**}) \in E_C$ (such v^{**} obviously exists). We can consider a path $P' = (v^{**}, v^*, \ldots, 0, \ldots, v_C)$ of C, and let v_1 be a vertex on P' adjacent to v_C. Then it is obvious that $v^* < v_1$ holds. Let $P = (u_1, \ldots, u_k)$ ($k = 2$ or 3) be a subpath of P' satisfying

$$d(u_1, v^*; C) = d(u_k, v_1; C) \quad \text{and} \quad P' = (v^{**}, v^*, \ldots, u_1, \ldots, u_k, \ldots, v_1, v_C).$$

Defining σ_P for the path P in the same way with that in Section 2, we find that $\sigma_P(v^*) = v_1$ and $\sigma_P(v^{**}) = v_C$ hold. Also, let C' be a Hamilton cycle in G obtained from C by biasing with respect to σ_P. Then we find from Lemmas 2 and 3 that C' is also an ORHC and contains $P_{v_C}^*$. Also, C' has an edge $e_{v_C}^* = (v^*, v_C)$, which implies that $v_{C'} > v_C$ holds.

By continuing this process, we find that C^* is an ORHC. □

5. Application to the Construction of Ring Networks

We can apply the result in this paper to the construction of ring networks with high reliability. When a certain pair of hosts (vertices) communicate each other by using one of two paths between the hosts, let L be the length of the chosen path. To keep the damage by unauthorized access at the minimum, we should design networks to minimize the expectation of L, denoted by $E(L)$. Let C be a ring network (Hamilton cycle) and r_{vw} the relative frequency of communication between v and w. Suppose that any ring network C is designed to choose a path between v and w so that the average distance between v and w, denoted by $d_{\text{AVG}}(v, w; C)$, is calculated by

$$p(d(v, w; C))d(v, w; C) + (1 - p(d(v, w; C)))(n - d(v, w; C))$$

where $p(d)$ is a monotone nonincreasing function of d on $[0, n]$. Then we have $E(L) = f_{\text{AVG}}(C)$, which means that ring networks with the minimum f_{AVG} value are optimum in the sense stated above.

When $\{r_{vw}\}$ is not completely known, it is natural for us to assume that each r_{vw} is proportional to the product of the number of users at v (say N_v) and that of users at w (say N_w), that is,

$$r_{vw} = cN_vN_w \quad (c \text{ is a positive constant}).$$

If all hosts are labeled so that $N_0 \geq N_1 \geq \cdots \geq N_{n-1}$ holds, then we can easily verify that $\{r_{vw}\}$ satisfies inequality (2) (inverse Supnick property). Therefore, in this case, C^* defined in Main Theorem is the optimum ring network.

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