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AN OPTIMAL DISTRIBUTION OF SEARCHING EFFORT RELAXING THE ASSUMPTION OF LOCAL EFFECTIVENESS

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Abstract In this paper, taking account of the detectability of a sensor for a target being in the neighborhood of a searching point, an optimal distribution of searching effort maximizing detection probability for a stationary target is investigated. Defining a function of extended effect of searching effort, the problem is formulated as a variational problem and theorems of the optimal distribution of searching effort are derived. By this model, the assumption of local effectiveness for searching effort in Koopman's model is removed and reality of the model is improved. A procedure for calculating the optimal (or suboptimal) solution is presented, and applying it, an optimal search plan for the datum search situation is analyzed in detail. Meaning of the conditions of the optimal solution and its properties are elucidated. Generalizations of the model and problems to be investigated in future are also discussed.

1. Introduction

During World War II, Dr. Koopman, B.O. and his colleagues in the Operations Research Group of U.S.Navy devoted intensive effort to developing efficient search operations in antisubmarine warfares. They established many fundamental concepts of search theory and Koopman compiled the studies in "*Search and Screening* [6]." He also presented basic theorems for the optimal distribution of searching effort which maximizes detection probability of a stationary target [7]. After the war, problems of optimal distribution of searching effort for various search situations have been investigated by many authors enthusiastically, such as optimal searches for a moving target, survivor searches for a mortal target, two-sided searches taking account of the forestalling detection by the target and optimal searches with criterion other than detection probability of the target, etc. These theoretical development of the studies on the optimal search plan were compiled and reviewed in the text books [3,5,8], and theories have been extended to more general resource allocation problems [2]. In those studies on the optimal distribution of searching effort, it should be noted that a crucial assumption called "local effectiveness of searching effort" has been imposed consistently. In this assumption, the searching effort allocated to a point is assumed to be effective on detecting the target being at the searching point only and does not have any effect on the target in the neighborhood of the searching point, However, this assumption is not valid in real world applications, since a sensor used in search operations usually has its effective range and the target being in the range is detectable. Therefore the assumption of local effectiveness of searching effort may be said that it spoils the reality of search model, especially for search problems in continuous search space. It is very strange that the assumption of local effectiveness of searching effort has been kept rigidly in the studies on the optimal distribution of searching effort until now.

In this paper, taking account of the detectability of a sensor for a target in the neighborhood of searching point, we formulate a search model relaxing the assumption of local effectiveness of searching effort, and obtain an optimal (or suboptimal) solution. In the next section, assumptions of our model are described in detail and system parameters are defined precisely. Our problem is formulated as a variational problem and theorems for the optimal solution are given in Section 3. In Section 4, we analyze the optimal distribution of searching effort in datum search situation in detail, and finally in Section 5, elucidations of the theorems obtained in this paper, generalization of our model and problems to be investigated in future are discussed.

2. Assumptions of Model

Assumptions of the model dealt with in this paper are described as follows.

(1). A stationary target hides in a continuous n-dimensional Euclidean space R^n and its position is denoted by $X \in R^n$. A searcher wants to locate the target. Although the target position is uncertain, a probability distribution of the target, p.d.f. $p(X)$, is assumed to be known to the searcher.

(2). Let Φ be a limit of searching effort available to the searcher such as search time, total search man-hour or effective search area in the target space. Φ is assumed to be continuously divisible when it is allocated in the target space. Let $\phi(X)$ be searching effort density allocated to a point $X \in R^n$ in the target space. $\phi = \{\phi(X)\}$ is called a search plan or an effort distribution. If we need to discriminate the target position from the searching point, the target position is denoted by X_T with subscript T.

(3). Effectiveness of searching effort density $\phi(X)$ at $X \in R^n$ on detecting the target being at X_T is assumed to be given by $\phi(X)g(X-X_T)$. Here, the function $g(X-X_T)$ specifies diminishing ratio of effectiveness of unit searching effort density by a vector $X-X_T$ from the target position X_T to the searching point X. Denoting the vector by z and the distance $|z| = |X-X_T|$, $g(z)$ is defined by

$$g(z) = \begin{cases} \delta_z & \text{if } |z| < \varepsilon, \\ g_r(z) & \text{if } |z| \geqq \varepsilon, \end{cases} \tag{1}$$

where δ_z is Dirac's delta function and $0 < \varepsilon \ll 1$. Hereafter, the function $g(z)$ is called the extended effect function of searching effort, and $g_r(z)$ is assumed to be integrable on R^n. The scale factor γ of the extended effect function g is given by

$$\gamma = \int_{R^n} g(z)\,dz = 1 + \lim_{\varepsilon \to 0} \int_{R^n,\ |z|>\varepsilon} g_r(z)\,dz. \tag{2}$$

(4). The conditional detection probability of the target (called as detection function) at X_T by the searching effort $\phi(X)\Delta X$ allocated in $[X, X+\Delta X]$ is assumed to be given by the exponential detection function (the random search formula) as

$$f(\phi(X)\,|\,X_T) = 1 - \exp(-\alpha(X_T)\,\phi(X)\,g(X-X_T)\Delta X). \tag{3}$$

(5). We assume that the detections of the target by searches at different points X_i's are independent each other,

(6). Measure of effectiveness of a search plan ϕ is assumed to be the detection probability $P(\phi)$ of the target and a search plan ϕ which maximizes $P(\phi)$ is called an optimal search plan and denoted by $\phi^* = \{\phi^*(X), X \in R^n\}$.

Under the assumptions described above, we formulate the problem and give an optimal search plan in the next section.

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3. Formulation of the Problem and Theorems for Optimal Search Plan

3.1. Formulation of the problem

Let $P(\phi|X_T)$ be the conditional detection probability of the target being at X_T by a search plan ϕ. Here, we consider a discrete search space $\{X_i, i=1, \cdots, N\}$ which is made by dividing the continuous space X with sufficiently small mesh ΔX, where X_i is the central point of ith mesh. Then, by the assumption (4), the conditional detection probability $f(\phi(X_i)|X_T)$ of the target at X_T by the searching effort allocated to X_i is given by

$$f(\phi(X_i)|X_T) = 1 - \exp(-\alpha(X_T)\phi(X_i)g(X_i-X_T)\Delta X).$$

By the assumption (5) of the independent detection of the target, $P(\phi|X_T)$ is calculated as follows.

$$P(\phi|X_T) = \lim_{\Delta X\to 0,\ (N\to\infty)} \{1-\prod_{i=1}^{N}(1-f(\phi(X_i)|X_T))\}.$$

On the other hand, we can easily prove the next relation for an integrable function $F(X)$.

$$\lim_{\Delta X\to 0,\ (N\to\infty)} \exp(-\sum_{i=1}^{N} F(X_i)\Delta X) = \exp(-\lim_{\Delta X\to 0,\ (N\to\infty)} \sum_{i=1}^{N} F(X_i)\Delta X) = \exp(-\int_{R^n} F(X)dX).$$

Applying the above relation to $P(\phi|X_T)$, we have

$$P(\phi|X_T) = 1 - \exp(-\alpha(X_T)\int_{R^n}\phi(X)g(X-X_T)dX).$$

Therefore, the detection probability of the target by a plan ϕ is presented by

$$P(\phi) = \int_{R^n} p(X_T)\{1-\exp(-\alpha(X_T)\int_{R^n}\phi(X)g(X-X_T)dX)\}dX_T. \quad (4)$$

Since the searching effort $\phi(X)$ in Eq. (4) is non-negative and the total searching effort is limited to Φ, the next conditions are imposed.

$$\int_{R^n}\phi(X)\ dX \leqq \Phi, \quad \phi(X) \geqq 0. \quad (5)$$

Therefore, our problem is formulated as a variational problem in which a function ϕ maximizes the functional $P(\phi)$ defined by Eq. (4) subject to Eq. (5). Hereafter, this problem is referred as Problem (P1).

Problem (P1) : $\max_\phi P(\phi)$ given by Eq. (4),

subject to $\int_{R^n}\phi(X)dX \leqq \Phi$ and $\phi(X) \geqq 0$.

3.2. Optimal search plan

In Eq. (4), we define $\psi(X_T)$ for $X_T \in R^n$ by

$$\psi(X_T) \equiv \int_{R^n}\phi(X)g(X-X_T)dX. \quad (6)$$

Eq. (6) is a sort of transformation which converts ϕ to ψ by a kernel g and $\psi(X_T)$ means effective searching effort at the target position X_T by a search plan ϕ. Substituting Eq. (6) into Eq. (4), we have the next equation instead of Eq. (4).

$$P(\psi) = \int_{R^n} p(X_T)\{1-\exp(-\alpha(X_T)\psi(X_T))\}dX_T. \quad (7)$$

On the other hand, integrating $\psi(X_T)$ over $X_T \in R^n$, we have

$$\int_{R^n}\psi(X_T)dX_T = \int_{R^n}\int_{R^n}\phi(X)g(X-X_T)dXdX_T \leqq \gamma\Phi, \quad (8)$$

where γ is the scale factor of the extended effect function g defined by Eq. (2). Therefore, the problem (P1) is rewritten to the next problem (P2). In the following, we ommit the subscript T in X_T since any confusion wouldn't be expected.

Problem (P2) : $\max_\psi P(\psi)$ given by Eq. (7),

subject to $\int_{R^n}\psi(X)dX \leqq \gamma\Phi$ and $\psi(X) \geqq 0$.

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Problem (P2) is the well-known classical Koopman problem in which the detection probability $P(\psi)$ of the target is maximized by continuous searching effort ψ subject to the total searching effort $\gamma\Phi$ and the exponential detection function: $f(\psi(X)|X) = 1-\exp(-\alpha(X)\psi(X))$ with local effectiveness. It should be noted that the available total searching effort increases by γ times of Φ in Problem (P2) compared with Problem (P1), where γ is larger than 1 as shown in Eq. (2). Therefore, $\psi(X)$ given by Eq. (6) is interpreted as a transformation by which the searching effort ϕ is converted to equivalent effective effort ψ having the local effectiveness and expanded volume $\gamma\Phi$. Thus, Problem (P1) becomes Problem (P2). On the other hand, necessary and sufficient conditions of Koopman's problem have been established and the optimal distribution of searching effort has been presented. The theorem gives the optimal solution for not only the case with the exponential detection function [7] but also more general regular detection function such that the conditional detection probability $f(\psi(X)|X)$ given the target position X is a continuous, strictly increasing, differentiable and concave function of $\psi(X)$ [1]. We shall quote it by the next lemma [3, 4, 8] without proof.

Lemma: If the detection function f is regular, necessary and sufficient conditions for optimal distribution of effective searching effort $\psi^*(X)$ of Problem (P2) are

$$\text{iff } \psi^*(X) > 0, \quad p(X)\left.\frac{df(\psi(X)|X)}{d\psi(X)}\right|_{\psi(X)=\psi^*(X)} = \lambda, \tag{9}$$

$$\text{iff } \psi^*(X) = 0, \quad p(X)\left.\frac{df(\psi(X)|X)}{d\psi(X)}\right|_{\psi(X)=0} \leqq \lambda,$$

$$\int_{R^n} \psi(X)dX = \gamma\Phi \quad \text{and} \quad \psi(X) \geqq 0, \tag{10}$$

where λ is a positive constant determined by the total searching effort $\gamma\Phi$. □

It should be noted that although the first condition of Eq. (5) holds for inequality (≦) in feasible search plans, but in our problem, an optimal plan ϕ^* always uses the available searching effort Φ exhaustively since any value of reserved effort in a search plan is not evaluated.

In the above lemma, if the detection function is the exponential function $f(\psi(X)|X) = 1-\exp(-\alpha(X)\psi(X))$ as defined in Eq. (3), $\psi^*(X)$ is obtained more explicitly by the next corollary.

Corollary: For the case of the exponential detection function given by Eq. (3), we have

$$\psi^*(X) = \frac{1}{\alpha(X)}[\log\alpha(X)p(X) - \log\lambda]^+, \tag{11}$$

$$\log\lambda = \frac{\int_A (\log\alpha(X)p(X)/\alpha(X))dX - \gamma\Phi}{\int_A (1/\alpha(X))dX},$$

where A is defined by $A = \{X|\psi^*(X)>0\}$, and $[B]^+$ is defined by $[B]^+ = B$ if $B > 0$ and $[B]^+ = 0$ if $B \leqq 0$. □

(Proof) Eq. (11) is easily derived by substituting $f(\psi(X)|X) = 1-\exp(-\alpha(X)\psi(X))$ into Eq. (9) and determining λ by Eq. (10). (Q. E. D.)

Since the optimal $\psi^*(X)$ of Problem (P2) is given by Lemma or Corollary stated above, we can calculate the optimal ϕ^* of Problem (P1) from Eq. (6). Thus the next theorem is presented.

Theorem: The optimal distribution of searching effort ϕ^* of Problem (P1) is the solution of the next integral equation,

$$\phi(X) + \int_{R^n,\, Y\neq X} \phi(Y)g_r(Y-X)dY = \psi^*(X), \tag{12}$$

where $\psi^*(X)$ is the optimal distribution of effective searching effort given by Lemma or Corollary. We call the solution ϕ obtained from Eq. (12) a tentative

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solution $\hat{\phi}$. If the tentative solution $\hat{\phi}$ satisfies the following conditions

$$\int_{R^n} \hat{\phi}(X)\ dX = \Phi, \qquad \hat{\phi}(X) \geqq 0, \tag{13}$$

then $\hat{\phi}$ is optimal : $\phi^* = \hat{\phi}$. □

(Proof) Eq. (12) is given by Eq. (6) and $\hat{\phi}$ is the inverse-transformation derived from it. Since $\hat{\phi}$ is feasible if it satisfies Eq. (13), the above theorem is obvious. (Q. E. D.)

The optimal plan is given by the above theorem, however, the tentative solution $\hat{\phi}$ does not always satisfy Eq. (13), since usually a function ϕ does not satisfy simultaneously two independent integral equations (12) and (13). Then, we must find a feasible solution $\bar{\phi}$ which satisfies Eq. (13) and approximates to Eq. (12). Here, we can consider several approaches that approximately satisfy Eq. (12) as follows.

Approximation 1. If $\hat{\phi}(X) \geqq 0$ for all X and $\int_{R^n} \hat{\phi}(X)dX \neq \Phi$, change appropriate parameters of $\hat{\phi}$ so as to give the total searching effort Φ, or calculate an adjusting factor $\alpha = \Phi / \int_{R^n} \hat{\phi}(X)dX$ and set $\bar{\phi}(X) = \alpha\hat{\phi}(X)$.

Approximation 2. If $\hat{\phi}$ is negative in some interval of X, reset $\hat{\phi}(X) = 0$ in the interval and apply Approximation 1.

Approximation 3. Choose a trial function $\bar{\phi}$ which approximates to $\hat{\phi}$. It is desirable that $\bar{\phi}$ is non-negative and closely resembles $\hat{\phi}$ by selecting adequate value of parameters contained in ϕ. Determine the value of parameters of the trial function $\bar{\phi}$ so as to satisfy Eq. (13) and to agree with moments of both sides of Eq. (12) by ϕ (the left-hand side) and ψ^* (the right-hand side).

Approximation 4. By calculating numerically the tentative solution $\hat{\phi}$ of Eq. (12), determine the value of parameters of $\bar{\phi}$ by Eq. (13) and conditions which give least squares deviation between $\hat{\phi}$ and $\bar{\phi}$.

These procedures stated above are explained in detail by examles in Section 4.

Here, we summarize an algorithm to calculate the optimal (or suboptimal) searching effort distribution ϕ^*.

Step 1. Calculate the optimal solution $\{\psi^*(X)\}$ of Problem (P2) by Eqs. (9) and (10) or Eq. (11).

Step 2. Substitute ψ^* into the right-hand side of Eq. (12) and solve it, then we obtain a tentative solution $\hat{\phi}$. If $\hat{\phi}$ satisfies Eq. (13), set $\phi^* = \hat{\phi}$ and calculate $P(\phi^*)$ by Eq. (4), and terminate calculation. Otherwise, go to the next step.

Step 3. Select a trial function $\bar{\phi}$ which contains several unknown parameters, and go to Step 4.

Step 4. Determine the unknown parameters of $\bar{\phi}$ by either method : Approximation 1 ～ 4 stated before. Here, the constraint of Eq. (13) is adopted always as one of conditions determining the parameters of $\bar{\phi}$. Go to Step 5.

Step 5. Calculate $P(\psi^*)$ and $P(\bar{\phi})$ by Eqs. (7) and (4), respectively. If error $(P(\psi^*)-P(\bar{\phi}))$ or relative error $(P(\psi^*)-P(\bar{\phi}))/P(\psi^*)$ is smaller than a prescribed accuracy ε, set $\phi^* = \bar{\phi}$ and terminate calculation. Otherwise, select another trial function $\bar{\phi}$ similar to $\hat{\phi}$ more closely and go back to Step 4.

It should be noted that the algorithm described above does not depend on the dimension of the target space X, the target distribution $p(X)$ and the function

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of the extended effect of searching effort $g(z)$.

Our algorithm stated above to obtain the suboptimal distribution of serach-ing effort is considered as one of the direct method of calculus of variations (such as Ritz's method). Hence, the accuracy of the approximation is depend on the trial function $\bar{\Phi}(X)$, and the calculation to obtain the suboptimal distri-bution with required (prescribed) accuracy must be conducted by trial-and-error manner. However, Approximation 3 (the moment maching method) and Approximation 4 (the least squares deviation method between $\hat{\Phi}$ and $\bar{\Phi}$) are expected to give good approximations usualy if the trial function $\bar{\Phi}$ is selected appropriately. Approx-imation 2 may also give a good approximation. Because the approximation $\bar{\Phi}(X) = 0$ in the interval $\hat{\Phi} < 0$ does not give large effect to $P(\phi)$ since the region $\hat{\Phi}(X) < 0$ is the marginal region of A $(= \{X | \psi^*(X) > 0\})$ and $p(X) f'(0|X)$ is small from the lemma: Eq. (9), therefore, the marginal effectiveness of the searching effort at X is small compared with the region $\hat{\Phi}(X) > 0$. On the other hand, Approximation 1 sometimes may have large error since the adjustment of total effort $\bar{\Phi}$ to Φ is conducted in all X with $\hat{\Phi}(X) > 0$. In future, we must investi-gate more detail properties of the procedure and accuracy of the approximations. In the next section, to show our algorithm concretely, we analyze an optimal distribution of searching effort for the datum search situation by applying the procedure stated above.

4. An Optimal Searching Effort Distribution for a Datum Search

One of frequently experienced search situations in practice is so-called datum search operation in which an uncomfirmed information of target position X_0 is reported and the contact with the target is lost before confirming the information. Therefore, in order to comfirm it, the region around the datum point X_0 must be searched again in detail. In this case, since error of the reported point X_0 is inevitable, the target distribution $p(X)$ is estimated by a normal distribution $N(X_0, \sigma^2)$, where σ^2 is the variance of datum error. Here, since the problem formulation and the theorems given in the previous section are not influenced by the dimension of the target space, we assume a target space R^1 in this section for simplicity of calculations and set the origin of the target space X on the reported position X_0. Then, a vector X is denoted by a scalar x and p.d.f. $p(x)$ of the target distribution is given by the next formula.

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right). \tag{14}$$

On the other hand, the exponential detection function given by Eq. (3) is assumed and the continuous searching effort Φ is available to the searcher. Therefore, Problem (P2) is defined as

$$\max_{\psi} P(\psi) = 1 - \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2} - \alpha(x)\psi(x)\right) dx,$$

$$\text{subject to } \int_{-\infty}^{\infty} \psi(x)\,dx \leqq \gamma\Phi \quad \text{and} \quad \psi(x) \geqq 0.$$

Hereafter, $\alpha(x) \equiv \alpha$ is assumed. The optimal effective searching effort ψ^* of the above problem (P2) is derived from Corollary stated before as

$$\psi^*(x) = \frac{1}{2\alpha\sigma^2}(r_0^2 - x^2) \qquad |x| \leqq r_0, \tag{15}$$

$$= 0 \qquad |x| > r_0,$$

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where $\quad r_0 = (\frac{3}{2}\alpha\gamma\sigma^2\Phi)^{1/3}$.

The detection probability $P(\psi^*)$ by the optimal ψ^* is calculated as follows.

$$P(\psi^*) = 2\int_0^{r_0}\frac{1}{\sqrt{2\pi}\,\sigma}\exp(-\frac{x^2}{2\sigma^2})\,dx - \sqrt{\frac{2}{\pi}}\,\frac{r_0}{\sigma}\exp(-\frac{r_0^2}{2\sigma^2}). \qquad (16)$$

As for the function of extended effect of searching effort $g(z) = \delta_z + g_r(z)$, we examine two cases.

Definite range type $g_r(z)$:

$$g_r(z) = \begin{cases} 1 & |z| \leqq a, \\ 0 & |z| > a. \end{cases} \qquad \gamma = 2a+1. \qquad (17)$$

Gaussian type $g_r(z)$:

$$g_r(z) = \exp(-\frac{|z|^2}{2\beta^2}). \qquad \gamma = \sqrt{2\pi}\,\beta+1. \qquad (18)$$

In the former case given by Eq. (17), the searching effort allocated to x is effective on detecting the target being in the effective range a from x: $|z| = |x-x_T| \leqq a$, and has no effect out of a from x. On the other hand, in the later case given by Eq. (18), the effectiveness of searching effort decreases monotonically by increasing distance $|z|$.

4. 1. The case of definite range type $g(z)$

Substituting Eqs. (15) and (17) into Eq. (12), we have

$$\hat{\phi}(x) + \int_{x-a,\ x\neq y}^{x+a}\hat{\phi}(y)\,dy = \frac{1}{2\alpha\sigma^2}(r_0^2-x^2) \qquad |x| \leqq r_0, \qquad (19)$$
$$= 0 \qquad |x| > r_0.$$
$$r_0 = (\frac{3}{2}\alpha(2a+1)\sigma^2\Phi)^{1/3}.$$

The searching effort is allocated in the interval $-r_0 \leqq x \leqq r_0$. Here, we assume that the effective range a of the searching effort is smaller than r_0 : $a < r_0$ (otherwise, the optimal solution ; $\phi^*(x) = \Phi\delta_x$ is obvious). A tentative solution $\hat{\phi}$ is obtained from Eq. (19) as follows.

$$\hat{\phi}(x) = \frac{r_0^2+2a^3/(6a+3)-x^2}{2(2a+1)\alpha\sigma^2} \qquad |x| \leqq r_0, \qquad (20)$$
$$= 0 \qquad |x| > r_0.$$

We can easily confirm that $\hat{\phi}(x) \geqq 0$ and $\{\hat{\phi}(x)\}$ does not satisfy the constraint of total searching effort Φ of Eq. (13). Therefore, we apply Approximation 1 mentioned before, and the trial function $\bar{\phi}(x)$ is set by modifying r_0 in Eq. (20) to r_1 so as to satisfy the next equation of the total searching effort Φ, and $\bar{\phi}(x) = 0$ in $|x| \geqq r_1-a$ since searching effort allocated at $|x| = r_1-a$ is equally effective in $r_1-a < |x| \leqq r_1$ by the assumption of definit range $g_r(z)$ given by Eq. (17).

$$\int_{-\infty}^{\infty}\bar{\phi}(x)\,dx = 2\int_0^{r_1-a}\frac{r_1^2+2a^3/(6a+3)-x^2}{2(2a+1)\alpha\sigma^2}\,dx = \Phi.$$

From the above, we have

$$2(2a+1)r_1^3 - (4a+3)a^2r_1 + a^3 - 3(2a+1)^2\alpha\sigma^2\Phi = 0.$$

Since the above cubic equation has unique positive solution r_1 if $a < r_0$, the feasible $\bar{\phi}$ is obtained by setting $r_0 = r_1 - a$ in Eq. (20) as follows.

$$\bar{\phi}(x) = \frac{r_1^2+2a^3/(6a+3)-x^2}{2(2a+1)\alpha\sigma^2} \qquad |x| \leqq r_1 - a, \qquad (21)$$
$$= 0 \qquad |x| > r_1 - a.$$

Calculating $P(\psi^*)$ and $P(\bar{\phi})$ by Eqs. (16) and (4), respectively, Step 5 stated

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before is examined. For example, Table 1 is shown for cases with parameters; α=1, σ=20, Φ=25, and a = 0, 1, ⋯, 5. As seen in this table, the absolute error or the relative error is sufficiently small, and therefore, we can adopt $\bar{\phi}$ given by Eq. (21) as a suboptimal solution: $\phi^* = \bar{\phi}$. Figure 1-A shows the optimal solution ϕ^* (a = 1) and the optimal solution of Koopman's model (a = 0) to see the difference of the optimal solutions of our model and Koopman's model. It should be noted that ϕ^* of our model considerably differs from Koopman's solution; the former is distributed broadly with a gentle peak at x = 0 in contrast with the steep peak of the latter. However, this does not mean that ϕ^* of our model makes little of searching in the neighborhood of the datum point. Since the effective searching effort ψ is accumulated in the central part (the neighborhood of datum point) when the searching effort has the extended effect, the searcher should not concentrate his effort at the central part too much, and he must allocate it broadly. This fact is shown in Figure 1-B by the distribution of $\psi(\bar{\phi})$ which is calculated by substituting $\bar{\phi}$ into the left-hand side of Eq. (19). This figure shows clearly concentration of $\psi(\bar{\phi})$ in the neighborhood of the datum point. In this figure, the optimal ψ^* given by Eq. (15) is also shown by a broken line to see similarity of $\psi(\bar{\phi})$ and ψ^*. As seen in this figure, both curves are quite similar. If we neglect the extended effects of searching effort and adopt the search plan given

Table 1. $P(\psi^*)$, $P(\phi)$ and relative error

System parameters: α=1, σ=20, Φ=25,

a	γ	r_0	r_1	$P(\psi^*)$	$P(\bar{\phi})$	absolute error $P(\psi^*)-P(\bar{\phi})$	relative error $\frac{P(\psi^*)-P(\bar{\phi})}{P(\psi^*)}$
0	1	24.662	24.662	0.32246	0.32246	0	0
1	3	35.569	35.580	0.63281	0.63281	0	6.321 E-6
2	5	42.172	42.206	0.78285	0.78284	0.00001	2.044 E-5
3	7	47.177	47.245	0.86515	0.86511	0.00004	4.046 E-5
4	9	51.299	51.409	0.91340	0.91335	0.00005	5.912 E-5
5	11	54.848	55.006	0.94297	0.94290	0.00007	7.523 E-5

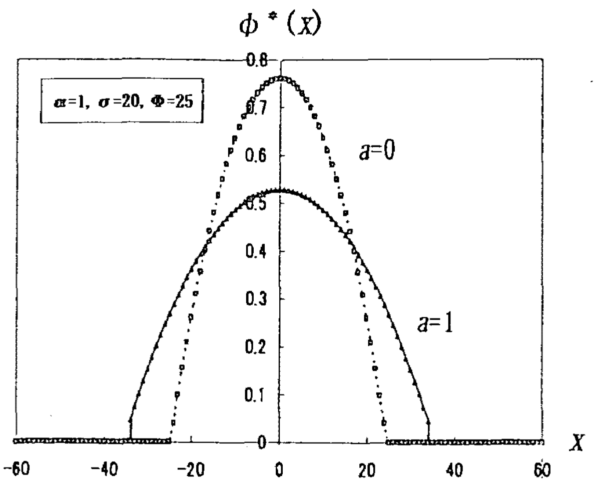

Figure 1-A ϕ^*

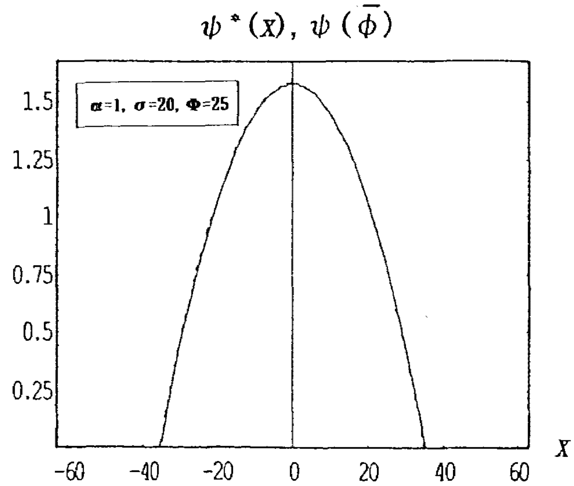

Figure 1-B ψ^* and $\psi(\phi)$

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by Koopman's solution, concentration of searching effort near the datum point exceeds extremly and it is not optimal clearly.

4.2. The case of Gaussian type $g(z)$

Substituting Eqs. (15) and (18) into Eq. (12), we have

$$\hat{\phi}(x) + \int_{-\infty,\ y\neq x}^{\infty} \hat{\phi}(y)\exp\left(-\frac{(y-x)^2}{2\beta^2}\right)dy = \frac{1}{2\alpha\sigma^2}(r_0^2-x^2) \qquad |x| \leqq r_0, \quad (22)$$
$$= 0 \qquad |x| > r_0.$$
$$r_0 = \left(\frac{3}{2}\alpha(\sqrt{2\pi}\beta+1)\sigma^2\Phi\right)^{1/3}.$$

We can prove easily that the tentative solution $\hat{\phi}$ obtained from Eq. (22) does not satisfy the non-negative restriction of $\hat{\phi}(x)$. To prove it by the reductive absurdity, we assume that there exists a non-negative function $\hat{\phi}$ satisfying Eq. (22). Then, the left-hand side of Eq. (22) is positive for any x and it contradicts the second condition of Eq. (22) ; it must be zero for $|x| > r_0$. Therefore, non-negative solution $\hat{\phi}$ can not exist. Here, we calculate $\phi(x)$ numerically by dividing the interval x: $[-3\sigma, 3\sigma]$ with equi-interval $h = r_0/M$, and converting the integral equation (22) to a set of simultaneous linear equations of $\hat{\phi}(\cdot)$:

$$\hat{\phi}(x_j) + \sum_{i=0}^{2bM} \hat{\phi}(y_i)\exp\left(-\frac{(y_i-x_j)^2}{2\beta^2}\right)h = \frac{r_0^2-x_j^2}{2\alpha\sigma^2} \qquad |x_j| \leqq r_0, \quad (23)$$
$$\hat{\phi}(x_j) + \sum_{i=0}^{2bM} \hat{\phi}(y_i)\exp\left(-\frac{(y_i-x_j)^2}{2\beta^2}\right)h = 0 \qquad |x_j| > r_0,$$

where $y_i = -3\sigma+ih$, $x_j = -3\sigma+jh$, $(0 \leqq i, j \leqq 2bM$, i and j are integer), $h = r_0/M$, $b = [3\sigma/r_0]+1$. A solution $\hat{\phi}$ of Eq. (23) is obtained by the Gauss-Jordan method numerically and shown in Figure 2-A for a case with parameters : α=1, σ=20, Φ=25, (in these values are the same as Figure 1-A) and β=1 (γ=3.507). As seen in this figure, $\hat{\phi}$ vibrates at the neighborhood of r_0 and becomes negative. Therefore, applying Approximation 3 stated before, we approximate to $\hat{\phi}$ by the next positive trial function $\bar{\phi}$ with two unknown parameters c_1 and c_2 which are determined by the constraint of Φ and the moment matching condition stated below.

$$\bar{\phi}(x) = c_1 \exp(-c_2x^2), \quad c_1, c_2 > 0.$$

Since $\bar{\phi}(x)$ must satisfy the constraint of the total searching effort Φ, we have the next equation.

$$2\int_0^{\infty} \bar{\phi}(x)\, dx = 2\int_0^{\infty} c_1\exp(-c_2x^2)\,dx = c_1\sqrt{\frac{\pi}{c_2}} = \Phi. \qquad (24)$$

Furthermore, in order to approximate the left-hand side of Eq. (22) to ψ^* by Approximation 3, set absolute moment of $\{\psi^*(x)\}$ equally to that of the left-hand side of Eq. (22).

$$\int_0^{\infty} c_1x\exp(-c_2x^2)\,dx + \int_0^{\infty} c_1x\int_{-\infty}^{\infty}\exp\left(-c_2y^2-\frac{(y-x)^2}{2\beta^2}\right)dydx = \int_0^{r_0}\frac{x}{2\alpha\sigma^2}(r_0^2-x^2)\,dx.$$

Calculating the above, we have

$$\frac{c_1}{c_2}\left(1+\beta\sqrt{2\pi(2\beta^2c_2+1)}\right) = \frac{r_0^4}{4\alpha\sigma^2}. \qquad (25)$$

c_1 and c_2 are obtained by solving the simultaneous equations (24) and (25) numerically by the Newton-Raphson method. For the case shown by $\hat{\phi}$ in Figure 2-A, we obtain c_1 = 0.5670 and c_2 = 1.616×10^{-3} for β=1 and ϕ is shown in this figure. For this case, $P(\psi^*)$ = 0.6805, and $P(\bar{\phi})$ = 0.6666 are obtained from Eqs. (16) and (4), respectively, and we have $P(\psi^*)-P(\bar{\phi})$ = 0.0139 and $(P(\psi^*)-P(\bar{\phi}))/P(\psi^*)$ = 0.0204. Since these errors are very small, we can adopt $\bar{\phi}$ as a suboptimal

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solution $\phi^* = \bar{\phi}$. Figure 2-B shows ψ^* and $\psi(\bar{\phi})$ given by the left-hand sides of Eq. (22) to see the similalty of both curves. In Table 2, a sensitivity analysis of β for the case with parameters : $\alpha=1$, $\sigma=20$, $\Phi=25$ (in these values are the same as Table 1) is shown. It may be said that errors shown in Table 2 are small and ϕ^* has sufficient accuracy for practical use. All the matters discussed on the case of definite range type g in the previous section are also valid in this case of Gaussian type g.

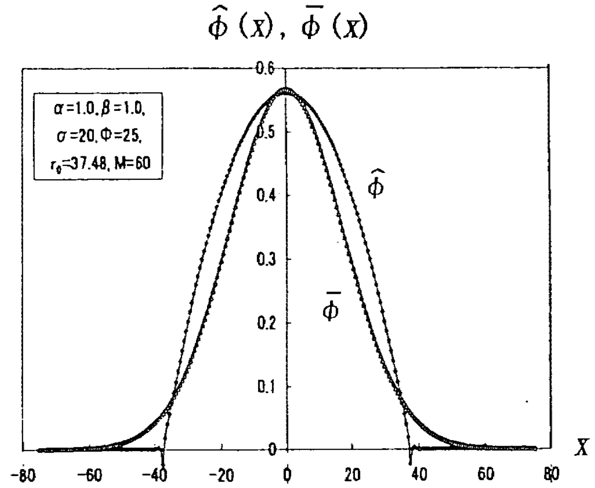

Figure 2-A $\hat{\phi}$ and $\bar{\phi}$

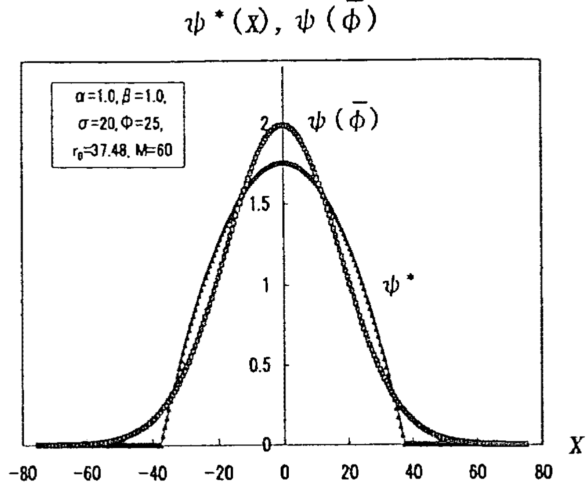

Figure 2-B ψ^* and $\bar{\psi}(\phi)$

Table 2 $P(\psi^*)$, $P(\bar{\phi})$ and relative error

System parameters : $\alpha=1$, $\sigma=20$, $\Phi=25$,

β	γ	c_1	$c_2 \times 10^{-3}$	$P(\psi^*)$	$P(\bar{\phi})$	absolute error $P(\psi^*)-P(\bar{\phi})$	relateive error $\frac{P(\psi^*)-P(\bar{\phi})}{P(\psi^*)}$
0	1.0000	0.86046	3.7216	0.32247	0.31343	0.00904	0.02801
1	3.5066	0.56702	1.6161	0.68048	0.66658	0.01390	0.02043
2	6.0133	0.47496	1.1339	0.83025	0.81771	0.01354	0.01511
3	8.5199	0.42431	0.9049	0.90393	0.89381	0.01012	0.01119
4	11.027	0.39088	0.7680	0.94328	0.93545	0.00783	0.00830
5	13.533	0.36668	0.6758	0.96544	0.95950	0.00594	0.00615

5. Discussions

In this section, we elucidate properties of the optimal solution obtained above, and discuss generalizations of the model and problems remained to be investigated in future.

1. Properties of the optimal solution

Main structure of our model can be stated as follows. The searching effort distribution $\phi(X)$ is transformed to $\psi(X_T)$ by Eq. (6) through a kernel $g(z) = \delta_z + g_r(z)$, in which δ_z shows the local effectiveness of the searching effort $\phi(X)$ and $g_r(z)$ does the extended effect of searching effort by distance $|z|$ ($\neq 0$) between the target position X_T and the searching point X. Hence, $\psi(X_T)$ defined by

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Eq. (6) means effective searching effort of a search plan ϕ at the target position X_T, and $\psi(X_T)$ has the local effectiveness in Problem (P2) as shown in Eq. (7). Here, it should be noted that the transformation ψ eliminates the extended effect from the detection function f, and it diminishes the detectability by unit searching effort. On the other hand, the total effective searching effort of ψ is expanded to $\gamma\Phi$ in volume as shown in Eq. (10), where $\gamma \geqq 1$ and $\gamma-1$ is the increasing ratio of effective searching effort by $g_r(z)$. Thus, Problem (P1) is equivalently converted to Problem (P2) by the transformation ψ. In Problem (P2), since ψ has the local effectiveness, we can apply the results obtained by previous studies on the optimal search plan to derive the optimal ψ^*. However, the optimal solution ψ^* is not complete, because ψ^* is composed of ϕ and we must find ϕ^* which constructs the optimal ψ^*. This means the inverse transformation by Eq. (12) from ψ^* to ϕ^*. However, as mentioned before, ϕ obtained by the inverse transformation of ψ^* by Eq. (12) is not feasible if Eq. (13) is not satisfied, and usually the tentative solution $\hat{\phi}$ of Eq. (12) is not feasible since a function $\hat{\phi}$ cannot satisfy two independent integral equations (12) and (13), simultaneously. Then, ϕ^* obtained by our model is one of suboptimal search plan.

As well known previously, the optimal distribution ψ^* given by Lemma or Corollary has a reasonable property that the marginal detection probability of the target (more generally, the marginal expected reward or the marginal expected utility) $p(X)df(\psi(X))/d\psi(X)$ is balanced to λ for every point to be searched, and if the largest marginal detection probability at X: $p(X)df(\psi(X))/d\psi(X)$ for $\psi(X)=0$ is smaller than λ, searching for the point should be abandoned. It is very interesting that these properties hold for ψ^* and do not valid for ϕ^* if the extended effect of searching effort exists, and it seems to be natural.

2. Generalization of the model

The model proposed in this paper can be generalized to models of various search situations.

(1). Search problem in discrete target space

In this paper, the target space is assumed to be continuous n-dimensional Euclidean spaces. However, the space can be changed to discrete space, say boxes $\{i\}$, without any difficulty. For a problem with discrete target space, $p(i)$ is defined by a probability mass of the target being in box i and $\phi(i)$ is volume of searching effort allocated to box i, and integration over the target space is changed to summation Σ_i. Then, all results obtained in this paper can be applied to the problem of discrete space.

In problems of discrete target space, the searching effort Φ may sometimes become discrete. Since this problem is an integer programming problem, we can not apply the results obtained in this papar to it. However, if this IP problem is solved by a branch-and-bound method, we can use our theorems of continuous searching effort to solve the continuous relaxation problem of discrete effort problem in order to estimate an upper bound value $P(\phi)$ of branches.

(2). Generalization of the detection function

We assume an exponential detection function $f(\phi(X)|X_T)$ given by Eq. (3) in this paper. However, f is generalized easily to a continuous, strictly increasing, differentiable and concave function of $\phi(X)$ (called the regular detection function) as stated in Lemma. The theorem and the algorithm given in Section 3 are all valid irrespective of type of the regular detection function [1].

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(3). Optimal search problems under other measure of effectiveness

In this paper, we deal with basic Koopman's problem taking account of the extended effect of searching effort and employ the detection probability of the target as the measure of effectiveness (abbreviated as M.O.E.) to be maximized. In search theory, optimal search problems with other M.O.E. such as expected searching effort until detection, expected reward (or risk) of search, whereabouts probability, etc. have been investigated by many authors [5,8]. These studies give the conditions for optimal solution under the assumption of local effectiveness of searching effort. We can apply the optimization procedure given in this paper to problems with the extended effect of searching effort and M.O.E. other than the detection probability of the target by applying the transformation ϕ to ψ by Eq.(6). In this case, Lemma given in Section 3.2. must be replaced with the optimal conditions investigated by previous studies. Using them, ψ^* is derived, and then, ϕ^* is inversely transformed by Eq.(12) or approximated by $\bar{\phi}$ as described in Section 3.

(4). Moving target problems

As for the optimal search problems for a moving target, we can say the same stated above. Optimal distribution of searching effort for a moving target have been investigated by many authors thoroughly under the assumption of local effectiveness [3,5,8,9]. These models can be generalized to models with the extended effect of searching effort by using the transformation ψ given by Eq.(6).

(5). Generalized resource allocation problems

Up to here, we deal with problems limiting to the optimal search problems. However, our model presented here can be applied to more general problems of optimal resource allocation. In this case, the search space becomes a discrete space $\{i\}$ defined by set of jobs (or other appropreate target) for which resources are allocated and $p(i)$ is some latent value of job i, and the extended effort function g_{ij} is defined as a function giving the extended effect to job j by unit resource applied to job i. Here the detecion function $f(\phi_j|i)$, Eq.(3), gives some gain earned in job i by ϕ_j units allocated to job j. Thus, the optimal resource allocation model may be formulated and the optimal solution will be obtained by the same approach described in this paper. From a point of view of the non-linear progamming problem, this is an optimization problem of non-separable function in contrast with a separable problem under the assumption of local effectiveness of resources.

3. Problems to be investigated in future

In this paper, we formulate a search problem by considering a function $g(z)$ which gives diminishing ratio of the effectiveness of searching effort by the distance $|z|$ between the searching point and the target. By this definition of $g(z)$, the Koopman model can be generalized without any difficulties. On the other hand, as a fundamental concept of search theory, detection range curve $h(z)$ (instantaneous detection probability) which gives the detection probability of the target by unit search time and a general model of "Inverse n-th power detection law [4]" for continuous search is presented. Furthermore, the detection potential of an encounter are established as micro models of search theory. These concepts and models define the detectability of a target being apart from the searcher by distance $|z|$. If the function $g(z)$ in our model can be formulated by the instantaneous detection probability $h(z)$ mentioned above, it makes clear the relation between the macro model of optimal distribution of searching effort and

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the micro model of target detection, and it may contribute to give a wide view of the theoretical development of search theory.

In this paper, dealing with a generalization of Koopman's model, we do not consider time process of allocating the searching effort. However, search process is usually time schedule and in this case, effect of allocated effort remains and decreases with time elapsed. Hence, the extended effect function of searching effort $g(z)$ becomes a function of both time and distance : $g(t, z)$. Especially, in generalized resource allocation problem, this effect may be important. In this problem, we must deal with a non-stationary search process and it may be one of important problems to be studied in future.

As stated before, we can convert Problem (P1) to Problem (P2) which has the local effectiveness of searching effort by considering the transformation ψ, and it allows us to apply results of previous studies to obtain the optimal ψ^*. However, this approach brings another difficulty to us to solve the integral equation of $\phi(X)$ given by Eq. (12) under the feasible condition : Eq. (13). This integral equation has not any feasible solution frequently as shown in the example of Section 4, and then, we must content ourselves with obtaining a suboptimal solution. If we are not satisfied with the suboptimal solution or want to avoid this difficulties, we must investigate some procedure to derive the optimal solution ϕ^* directly without using the transformation ψ. This is one of the problems to be studied in future.

6. Conclusions

In this paper, we investigate an optimal search problem relaxing the assumption of the local effectiveness of searching effort. Introducing the extended effect function g of searching effort and defining a transformation of the searching effort distribution ϕ to effective searching effort ψ at the target position, we formulate the model and derive the optimal solution. We have shown that the transformation ψ with the kernel g is useful to deal with the problem and have given the procedure to calculate the optimal (or suboptimal) distribution of searching effort. By this model, the reality of the model analyzing the optimal distribution of searching effort may be improved very much. In this paper, as an application of the theorems, the optimal solution for the datum search situation is analyzed in detail. Wide applicability of our model to various optimal search problems studied by many authors previously are discussed and generalization of our model are also suggested.

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