Journal of the Operations Research
Society of Japan
Vol. 44, No. 4, December 2001

STACKELBERG HUB LOCATION PROBLEM

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(Received February 23, 2001; Revised July 2, 2001)

Abstract In this paper, we consider a new competitive hub location model called Stackelberg hub location problem where a big firm competes with several medium firms to maximize its own profit. We assume that the medium firms' service sets are mutually disjoint and there is no competition among them. The big firm first locates a new hub on a plane as a leader on the condition that the other firms locate their new hubs after that. We formulate the leader's problem as a bilevel programming problem with followers' problem as lower level problems, and solve it using SQP method. Computational results show the significance of the proposed competitive hub location model.

1. Introduction

Hubs play an important role in designing a transportation network as well as an airline network. Because of this fact, many researchers have focused on hub location problems in the past decade. They have proposed several types of hub location models along with solution methods for those problems. We may classify hub location models into several classes according to three different criteria. In terms of the type of space in which hubs are located, they are classified into two classes. One is the discrete location model in which a hub can be located only at one of a finite number of candidate points, while the other is the continuous location model in which a hub can be located arbitrarily in a region on a plane. In terms of the assignment rule between hub and non-hub nodes, they are classified into single allocation model and multiple allocation model. In the former model, each demand node can connect to a single hub only, while the latter model allows each demand node to connect to more than one hub. In terms of the number of intermediate hubs, they are classified into 1-stop model and 2-stop model. They restrict the number of intermediate hubs in a trip to be no greater than one and two, respectively.

O'Kelly [15] first formulated a 2-stop discrete hub location problem with single allocation rule as a quadratic integer programming problem. This type of hub location problems were mainly studied in the 80's. In the 90's, multiple allocation models have become the main subject of research instead of single allocation models [2, 3, 9, 10, 21].

In 2-stop models, the number of variables and constraints drastically grows with the size of models and hence we can expect to solve problems of practical size only approximately. Sasaki et al. [18, 19] considered a 1-stop hub location model in which every trip uses at most one hub other than origin and destination. Since the number of variables of 1-stop models is relatively small compared with 2-stop models, we can expect to obtain their exact optimal solutions. Recently, developing new formulations of 2-stop models has received much attention. Specifically, Skorin-Kapov et al. [22] propose new formulations of the 2-stop model and show that their models may yield tight linear programming relaxations. O'Kelly et al. [17] further reduce the size of the problem and show that the linear programming relaxation often results in integer solutions. Ernst et al. [5, 6, 7] formulate the 2-stop model as a multi-commodity flow problem to reduce the number of variables and constraints, and propose a practical heuristic method as well as an exact method that utilize lower bounds obtained by the shortest path calculations.

Continuous hub location problems have not been studied as extensively as discrete models. O'Kelly [14] first formulated a continuous model in which hubs could be located anywhere on a plane. O'Kelly [16] applied a clustering approach to a continuous model so as to minimize the sum of squared distances. Aykin [1] studied the same type of model both on a plane and on a sphere. These models assume that demands are distributed discretely. On the other hand, Suzuki et al. [23] formulated a continuous model in which demands are evenly spread in a given area and proposed a solution method using Voronoi diagram. Note that the complexity of discrete models grows drastically as the number of hub candidates increases. Continuous models do not suffer from such a difficulty, which is an advantage of those models.

As mentioned above, a variety of hub location models have been studied in the last decade. However, studies on competitive hub location problems are scarce. In the real situation, several firms usually exist in a market and compete with each other to capture market share. We can easily imagine that hub locations are affected by competing with rival firms. Recently, Marianov et al. [11] formulated a competitive hub location problem on a network, which seems to be the first hub location model that takes into account competition. In their model, the sum of captured flows is maximized under some passengers allocation rules. Sasaki et al. [20] considered a continuous hub location model in which two firms of similar size locate their own new hubs in an arbitrary order, and formulated a leader's problem as a bilevel programming problem. The numerical experiments reported in [20] show that the leader firm may suffer heavy losses if it neglects to consider the competitor's strategies.

In this paper, we consider a 1-stop continuous hub location problem involving a number of firms as a natural extension of the model introduced in [20]. More precisely, we focus on a hub location problem with the following situation. Suppose that there are a big firm and several medium firms who intend to take part in a market. We assume that the medium firms' service sets are mutually disjoint and there is no competition among them. After the big firm locates its own hub, the other firms also locate their own hubs, simultaneously. We formulate the model as a bilevel programming problem and solve it using SQP method. We call this model Stackelberg hub location model because its situation is similar to Stackelberg game [13]. We also compare the solution of this model with that of the non-competitive model and discuss the effect of taking into account the competition among the rival firms.

This paper is organized as follows. In Section 2, we explain a background of the Stackelberg hub location model. In Section 3, we formulate the model as a bilevel programming problem. In Section 4, we show computational results using real airlines' data. In Section 5, we give concluding remarks and mention some future work.

2. Stackelberg Hub Location Model

As mentioned in the previous section, much effort has recently been made to formulate various types of hub location problems. One assumption common to those hub location problems is that the firm is always a monopolist in a market. In other words, it is implicitly assumed that a firm can capture all of the demands in the market regardless of its hub locations. Under this premise, passengers are forced to use the service provided by the firm, even if it is inconvenient, because of the absence of alternatives. As a result, hub locations would tend to be determined by the firm's convenience with no regard to passengers' convenience or preferences. In practice, however, passengers are supposed to choose which service to use according to their own preferences. In order to reflect such situations, it is important to consider a hub location problem in a competitive environment.

By taking a competitive factor into consideration, a new problem comes up to us. That is, we need to specify how to reflect passengers' preferences in the model. This can be done by using an assignment rule or an assignment function, which determines the level of captured passengers in terms of the disutilities of available services. An example of such a rule is the all-or-nothing assignment rule. Under this rule, the firm who provides the most convenient service captures all passengers and others capture nothing. Marianov et al. [11] use the all-or-nothing assignment rule in which the level of capture is predetermined based on cost differences among competitors. In this paper, we assume that the level of captured passengers is determined by a logit function [12]. Since services are, in general, not equally attractive to passengers, all of the passengers who travel between an OD pair (origin and destination pair) do not necessarily select the same service. The logit function is often used to determine the assignment of passengers to available services so as to reflect their various preferences. Specifically, we assume that there are k services available for an OD pair and let $u_i, i = 1, \cdots, k$, be the disutility of the i-th service. Then the level of capture for the i-th service is determined by

$$L_i(u) = \frac{\exp[-\alpha u_i]}{\sum_{j=1}^{k} \exp[-\alpha u_j]}, \quad i = 1, \cdots, k, \tag{1}$$

where $\alpha > 0$ is a parameter. The assignment given by (1) may represent various types of passenger preferences by adjusting the value of parameter α appropriately. In particular, it approaches the all-or-nothing assignment as α becomes large. In our model, we use the ratio of the actual travel distance to the direct distance between an OD pair as a service disutility.

In general, there are various types of firms in a real market. On one hand, some big firms may provide their services in the whole market; on the other hand, some medium size firms may provide their services in a confined area. In order to reflect such a circumstance in a real market, we consider the situation in which one big firm and several medium firms exist in a market. Suppose that they are planning to locate their own new hubs so as to maximize their own profit that consist of the total airfare revenues. A big firm usually has a dominating power in a market and medium firms often subordinate to the big firm's decision or strategy. So we assume that the big firm is the leader and the other firms are the followers. Namely, after the leader locates its hub, the followers locate their hubs simultaneously. In addition, we assume that the followers' service sets are subsets of the leader's service set and the followers' service sets are mutually disjoint, i.e., there is no competition among the followers. Here, the service set of a firm is the set of OD pairs for which the firm provides its service. Usually medium firms are not strong enough to compete with the big firm. Therefore they may tend to concentrate their service effort in a relatively small area so that they can capture market share more efficiently. Taking into account such a situation, we assume that the followers' service sets are mutually disjoint. Moreover, we assume the following conditions: (i) The trip demands among all OD pairs are assumed to be known and symmetric. (ii) Each hub can be located anywhere on the plane (continuous location model) and there is no capacity limit on the passengers who use it. (iii) Hubs are only for the use of a facility for transfer and they have no trip demand of their own. Under these assumptions, each firm determines the location of its new hub. The leader firm knows that the follower firms are going to locate their new hubs after knowing the leader's decision. So the leader firm has to locate its new hub, given that the follower firms make optimal decisions.

Another type of competitive location problem was proposed by Hakimi [8]. He considered the problem where the leader locates p facilities and the follower locates q facilities on a weighted network to maximize the captured market share. He assumed that the consumers patronize the closest facilities and, in case of equal distance, the leader captures the consumer. All-or-nothing assignment rule was employed in the model. The leader's and the follower's problems were called the centoroid problem and the medianoid problem, respectively. We can regard Stackelberg hub location model as an extension of Hakimi's model where the number of followers is not necessarily one. Here we employ a continuous location model since it is generally more tractable than a discrete model. In the next section, we formulate Stackelberg hub location problem as a bilevel programming problem.

3. Formulation

Let Firm A denote the leader firm and Firms $\mathrm{B}_1, \cdots, \mathrm{B}_m$ denote m follower firms. The decision variables of the firms are as follows:

x: the location of Firm A's new hub, $x \in \Re^2$,

y_l: the location of Firm B_l's new hub, $y_l \in \Re^2, l = 1, \cdots, m$.

In addition, the following notations are employed:

N: the set of demand nodes, $|N| = n$,

d_i: the location of demand node $i \in N$, $d_i \in \Re^2$,

M: the set of follower firms, $M = \{1, \cdots, m\}$,

Π: the set of OD pairs, $\Pi \subseteq N \times N$,

Π_A: the set of OD pairs for which Firm A provides its services, $\Pi_A \subseteq \Pi$,

Π_{B_l}: the set of OD pairs for which Firm B_l provides its services, $\Pi_{B_l} \subseteq \Pi_A$, $l = 1, \cdots, m$, $(\Pi_{B_l} \cap \Pi_{B_{l'}} = \emptyset, l \neq l')$,

W_π: the trip demand (the number of passengers) for OD pair $\pi \in \Pi$,

F_π: the airfare for OD pair $\pi \in \Pi$,

$D_\pi(w)$: the travel distance between OD pair π via a hub located at $w \in \Re^2$.

Throughout we assume that the distance is Euclidean, and for each OD pair $\pi = (i, j)$, the travel distance $D_\pi(w)$ is given by

$$D_\pi(w) = \| w - d_i \| + \| w - d_j \| \qquad w \in \Re^2,$$

where $\| \cdot \|$ denotes the Euclidean norm. Each firm provides its service using its own hub. For OD pairs for which more than one services are provided, passengers have to choose one of the available services according to their preferences. As mentioned earlier, we suppose that the passengers of each OD pair are distributed among available services according to the logit function given by (1), which is a function of those services' disutility. The service disutility $\eta_\pi(w)$ between OD pair $\pi = (i,j)$ using a hub located at $w \in \Re^2$ is defined as the ratio of the actual travel distance to the direct distance between the pair (i,j), i.e., $\eta_\pi(w) = D_\pi(w)/\parallel d_i - d_j \parallel$. If a firm provides no service on a particular OD pair, the service disutility between the OD pair is defined to be infinity. Therefore, the disutilities of the firms are given by

$$\eta_\pi^A(w) = \begin{cases} D_\pi(w)/\parallel d_i - d_j \parallel & \text{if } \pi \in \Pi_A, \\ \infty & \text{if } \pi \notin \Pi_A, \end{cases}$$

$$\eta_\pi^{B_l}(w) = \begin{cases} D_\pi(w)/\parallel d_i - d_j \parallel & \text{if } \pi \in \Pi_{B_l} \\ \infty & \text{if } \pi \notin \Pi_{B_l} \end{cases}, \quad \forall l \in M.$$

Let Firm A and Firms $\mathrm{B}_l (l = 1, \cdots, m)$ locate their hubs at $x \in \Re^2$ and $y_l \in \Re^2 (l = 1, \cdots, m)$, respectively. Then the number of passengers of OD pair π who use Firm A's hub is given by

$$\Phi_\pi(x, y_1, \cdots, y_m) = \frac{W_\pi \exp[-\alpha \eta_\pi^A(x)]}{\exp[-\alpha \eta_\pi^A(x)] + \sum_{l \in M} \exp[-\alpha \eta_\pi^{B_l}(y_l)]},$$

with a constant $\alpha > 0$. From the assumption $\Pi_{B_l} \cap \Pi_{B_{l'}} = \emptyset (l \neq l')$, for each $\pi \in \Pi$, there is at most one l such that $\pi \in \Pi_{B_l}$, which we denote $l(\pi)$. Thus, using $l(\pi)$, we can simplify the representation of $\Phi_\pi(x, y_1, \cdots, y_m)$ as

$$\Phi_\pi(x, y_1, \cdots, y_m) = \frac{W_\pi \exp[-\alpha \eta_\pi^A(x)]}{\exp[-\alpha \eta_\pi^A(x)] + \exp[-\alpha \eta_\pi^{B_{l(\pi)}}(y_{l(\pi)})]}.$$

In a similar manner, the number of passengers of OD pair π who use Firm B_l's hub is given by

$$\Psi_\pi^l(x, y_l) = \begin{cases} \dfrac{W_\pi \exp[-\alpha \eta_\pi^{B_l}(y_l)]}{\exp[-\alpha \eta_\pi^A(x)] + \exp[-\alpha \eta_\pi^{B_l}(y_l)]} & \text{if } l = l(\pi), \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the total revenues of Firm A and Firms $\mathrm{B}_l (l = 1, \cdots, m)$ are given by

$$f(x, y_1, \cdots, y_m) = \sum_{\pi \in \Pi_A} F_\pi \Phi_\pi(x, y_1, \cdots, y_m)$$

and

$$g_l(x, y_l) = \sum_{\pi \in \Pi_{B_l}} F_\pi \Psi_\pi^l(x, y_l), \qquad \forall l \in M.$$

respectively.

Now we proceed to formulating the problem. First we consider Firm B_l's problem. Given that Firm A locates a hub at $x \in \Re^2$, Firm B_l will locate a hub so as to maximize its total revenue. So Firm B_l's problem, which is called SHLP-B_l, is written as follows:

[SHLP-B$_l$]

$$\begin{array}{ll} \text{maximize}_{y_l} & g(x, y_l) \\ \text{subject to} & y_l \in Y_l \subseteq \Re^2 \ , \end{array}$$

where Y_l denotes the feasible region for locating Firm B$_l$'s hub. Assuming Firm B$_l$ always finds an optimal location $y \in Y_l$ by solving SHLP-B$_l$ for any hub location $x \in X$ of Firm A, Firm A solves its own problem subject to the condition that y_l is the optimal solution of SHLP-B$_l$. More precisely, $y_l \in \arg\max\{g_l(x, y_l)|y_l \in Y_l \subseteq \Re^2\}$ should be a constraint in Firm A's problem. Hence, Firm A's problem is stated as the following bilevel programming problem:

$$\begin{array}{ll} \text{maximize} & f(x, y_1, \cdots, y_m) \\ \text{subject to} & x \in X \subseteq \Re^2 \\ & y_l \in \arg\max\{g_l(x, y_l)|y_l \in Y_l \subseteq \Re^2\} \ , \ l = 1, \cdots, m, \end{array}$$

where X denotes the feasible region for locating Firm A's hub. Hereafter we assume that SHLP-B$_l$ has a unique optimal solution for any given $x \in \Re^2$, which is denoted by $\xi_l(x)$. That is,

$$\xi_l(x) = \text{argmax}_{y_l \in Y_l} g_l(x, y_l).$$

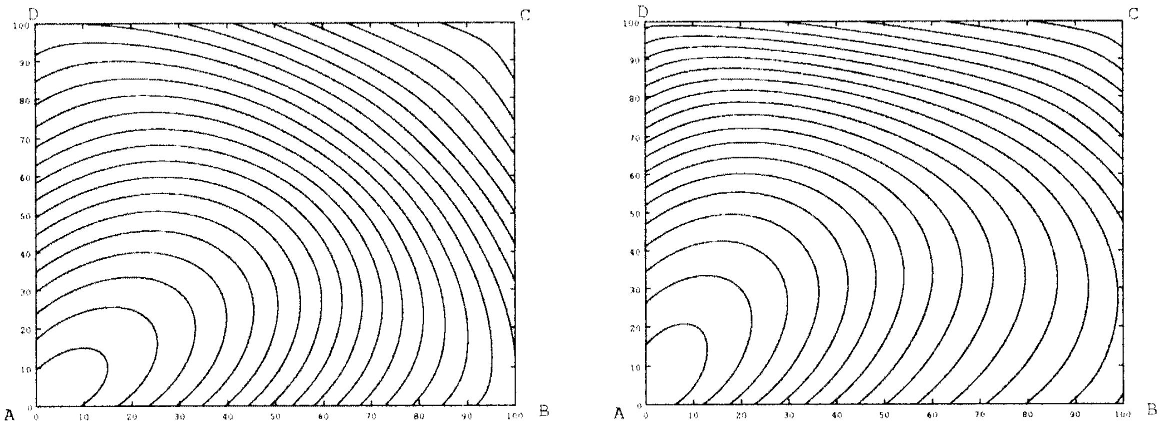
Consequently, by defining the function $\Theta(x)$ as $\Theta(x) = f(x, \xi_1(x), \cdots, \xi_m(x))$, Firm A's problem can be reformulated simply as follows:

[Stackelberg hub location problem: SHLP]

$$\begin{array}{ll} \text{maximize} & \Theta(x) \\ \text{subject to} & x \in X \subseteq \Re^2. \end{array}$$

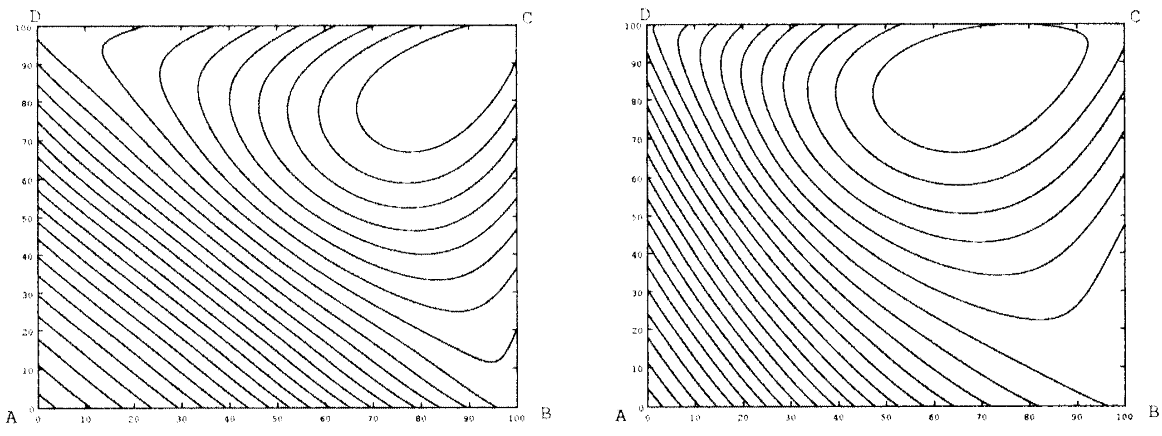
When $X = \Re^2$, this problem is an unconstrained optimization problem. If Y_l is a proper subset of $\Re^2$, the optimal solution of SHLP-B$_l$, i.e., $\xi_l(x)$, is not necessarily smooth with respect to x. As a result, $\Theta(x)$ may not be smooth with respect to x. However, if SHLP-B$_l$ is an unconstrained problem or $\xi_l(x)$ is an interior point of Y_l, then we may expect that $\xi_l(x)$ is smooth with respect to x. In this case, $\Theta(x)$ also becomes smooth with respect to x, and we can apply some gradient-based methods to solve SHLP. Follower's optimal hub location is usually inside the convex hull of all demand nodes regardless of leader's hub location. Therefore, we may assume that $\xi_l(x)$ is an interior point of Y_l when Y_l is a region containing the convex hull of all demand nodes.

In general, there is no guarantee that SHLP-B$_l$ has a unique optimal solution for any given $x \in \Re^2$. However, our computational experience suggests that there is a great likelihood for the assumption to hold. We illustrate it with a simple example containing 4 demand nodes: A(0,0), B(100,0), C(100,100) and D(0,100). We assume that the demands for OD pairs are all equal. Suppose that OD pairs (A,B), (A,C) and (A,D) comprise Firm B$_1$'s service set, while OD pairs (B,C), (C,D) and (D,A) comprise Firm B$_2$'s service set. Figure 1(a) and Figure 1(b) show contours of Firm B$_1$'s cost function when Firm A locates its hub on (57.3,57.8) and (20,80), respectively. Similarly, Figure 2(a) and Figure 2(b) show contours of Firm B$_2$'s cost function when Firm A locates its hub on (57.3,57.8) and (20,80), respectively. Figure 3 displays contours of Firm A's cost function and the white circle represents the optimal solution, whose coordinate is (57.3,57.8). These figures indicate that the follower firms' problems often have a unique optimal solution, although the shape of their cost functions changes depending on the Firm A's hub location.



(a) Firm A's hub is located at (57.3,57.8) (b) Firm A's hub is located at (20,80)

Figure 1: Contours of Firm B_1's cost function



(a) Firm A's hub is located at (57.3,57.8) (b) Firm A's hub is located at (20,80)

Figure 2: Contours of Firm B_1's cost function

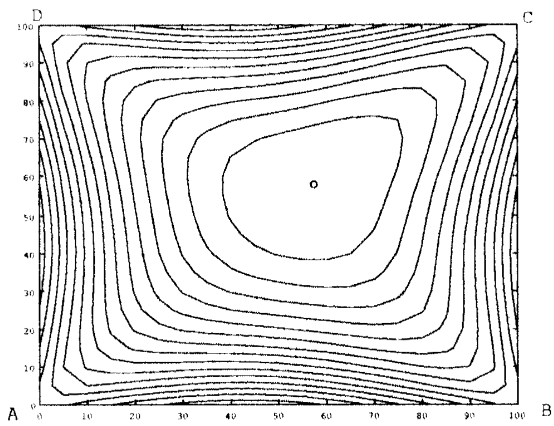


Figure 3: Contours of Firm A's cost function for a simple example

Table 1: Test Data No.1 ~ No.7

No.	Firm B_1's services	market scale
1	Top 150 OD pairs in terms of demand	82.06%
2	Top 150 OD pairs in terms of revenue	85.00%
3	Randomly selected 150 OD pairs	48.87%
4	All OD pairs originated at the top 6 cities in terms of demand (129 pairs)	73.67%
5	All OD pairs originated at the top 2 cities in terms of demand (47 pairs)	45.84%
6	All OD pairs originated at the city with the greatest demand (24 pairs)	33.77%
7	All OD pairs originated at 8 cities located in Middle West (164 pairs)	47.57%

4. Computational Results

In this section, we report some computational results for the proposed model SHLP and examine the differences between optimal solutions of SHLP and the non-competitive model. Computer programs were coded in MatlabR12 with optimization toolbox [4]. We used function fmincon included in the optimization toolbox, which finds a constrained minimum of a nonlinear multivariable function. It uses SQP (Sequential Quadratic Programming) method that updates the Hessian at each iteration according to the BFGS formula. Each inner iteration which solves SHLP-B_l uses five different initial points. All programs were run on a Sun Ultra10 computer operated under SunOS Release 5.6 with 256 Mb memory. We prepared the demand data based on the well-known U.S. 25 cities data evaluated in 1970 by CAB (Civil Aeronautics Board). For airfare data, we used the data supplied by http://www.airfare.com. We assume that the leader Firm A provides its services on all OD pairs among 25 cities, i.e., 300 pairs. We solved SHLP with two follower firms, i.e., l=2. The feasible region is the rectangle that ranges from 20° N to 50° N and from 70° W to 130° W. The execution time of SHLP largely depends on the choice of an initial point, and it ranged from 450 CPU seconds to 3200 CPU seconds in our examples.

A firm is able to capture all passengers regardless of its hub location if there is no competitor. In this case, it is natural to suppose that the firm may locate its hub on the unweighted median of the demand nodes. Therefore, we compare the results for SHLP with those for the problem in which Firm A locates its hub on the unweighted median.

Follower firms may provide their services in a particular area or for a limited number of cities. Taking into account these circumstances, we prepared seven service set data (No.1 ~ No.7) as shown in Table 1. Top 6 cities with large demand are New York, Chicago, Los Angeles, Boston, Washington D.C. and Miami, in descending order. Middle West 8 cities are Seattle, San Francisco, Los Angeles, Phoenix, Denver, Dallas, Kansas City and Minneapolis. Market scale means the ratio of the total revenue yielded in each service set to those in the whole market. Firm B_1's service set is displayed in the table and Firm B_2's service set is its complement. The results for these test examples are displayed in Table 2. For each test example, the first row shows the results for SHLP and the second row shows the results obtained by locating Firm A's hub on the unweighted median of the nodes. Firm A's market share is given by the ratio of its revenue to the total revenue of all three firms. The numbers in the parentheses show the decrease in the market share of Firm A. Follower's market share is estimated in two ways. The column labeled "B_1 area" shows Firm B_1's share in its service set. Namely, it is the ratio of Firm B_1's revenue to the total revenue of Firm A and Firm B_1 in Firm B_1's service set. On the other hand, the column labeled "whole area" shows Firm B_1's share in the whole market. Firm B_2's share is estimated in a similar manner. Figures 4, 5 and 6 show the results for test examples No.5, No.6 and No.7, respectively. In these figures, white circles represent 25 cities and other white marks indicate the hub locations obtained from SHLP, while black marks show the results when Firm A locates its own hub on the unweighted median of the nodes. Moreover, a square shows Firm A's hub location, a triangle shows Firm B_1's hub location, and a diamond shows Firm B_2's hub location. Figure 7 also shows the results for example No.7 together with contours of the cost function of Firm A. The white circle indicates the optimal location.

These results indicate that, in all tests, Firm A increases its market share by taking advantage of the leadership in the market. We may also observe that hub locations are affected by competition substantially. However, we cannot extract clear relationship between Firm A's share and the difference of two followers' market scale. Although the difference of two followers' market scale is relatively small in No.3, No.5 and No.7, Firm A's share decrease ranges from below 9.68% to 15.40%. In case of No.7, Firm B_1's market share is almost 70% in B_1 area in spite of the fact that Firm A takes into account the competition with the followers. It is interesting to observe that a follower often captures a great deal of demand by restricting its services to a limited area. For example, in No.6, Firm B_1 provides services on only those OD pairs which contain New York as shown in Figure 4, where black and white triangles overlap at New York. Then it turns out that Firm B_1's optimal hub location is at New York regardless of Firm A's hub location and indeed provides non-stop services for all OD pairs containing New York. Firm A provides 1-stop services on those OD pairs. Figure 4 shows that Firm A provides services originated from New York with relatively short travel distance except for some services whose destinations are near New York or in the South. This is the reason why more than 35% market share is captured by Firm A, because the disutility of services depends only on the actual travel distance without taking into account transfer. If non-stop services are allowed in the model, Firm B_1 may capture more market share. In fact, medium firms often provide non-stop services in a particular area at a bargain price. It is an interesting future work to study a model that takes into account non-stop services.

5. Conclusions

In this paper, we have considered a hub location problem in a competitive environment and formulated Stackelberg hub location problem as a bilevel programming problem. We also made some computational experiments using actual data. We examined how the optimal hub locations of the firms are affected if the major firm takes advantage of its leadership. Although the model adopts the simple assumption that there is no competition among followers, it is still new to incorporate the situation where several firms of different size are competing each other. Further extensions of the model, e.g., considering the competition

Table 2: Market share(%): Competitive model versus non competitive model

No.	Firm A's share	Firm B_1's share		Firm B_2's share	
		whole area	B_1 area	whole area	B_2 area
1	48.25	41.36	50.41	10.38	57.87
	42.76 (▲ 11.39%)	48.13	58.65	9.11	50.77
2	48.35	42.77	50.32	8.88	59.20
	42.61 (▲ 11.87%)	49.61	58.36	7.78	51.89
3	49.75	24.39	49.92	25.86	50.57
	42.95 (▲ 13.66%)	29.83	61.06	27.21	53.22
4	45.83	38.24	51.91	15.93	60.51
	41.94 (▲ 8.48%)	44.86	60.89	13.20	50.12
5	46.62	25.78	56.23	27.60	50.97
	42.11 (▲ 9.68%)	30.12	65.70	27.77	51.28
6	41.21	21.74	64.38	37.05	55.93
	39.78 (▲ 3.48%)	25.56	75.70	34.65	52.32
7	39.92	33.18	69.75	26.90	51.30
	33.77 (▲ 15.40%)	28.14	59.15	38.09	72.65

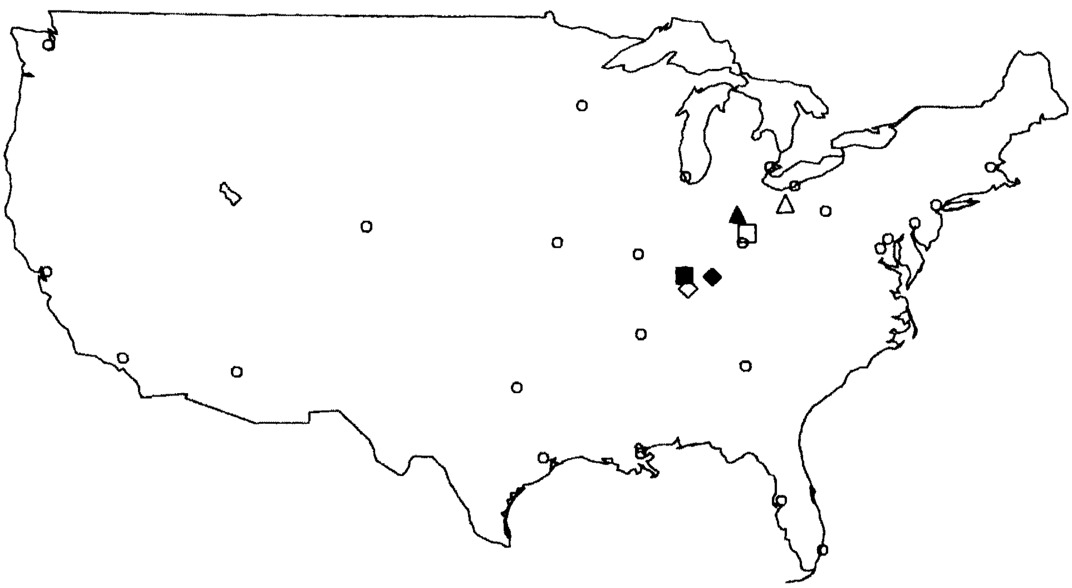


Figure 4: The results for test example No.5

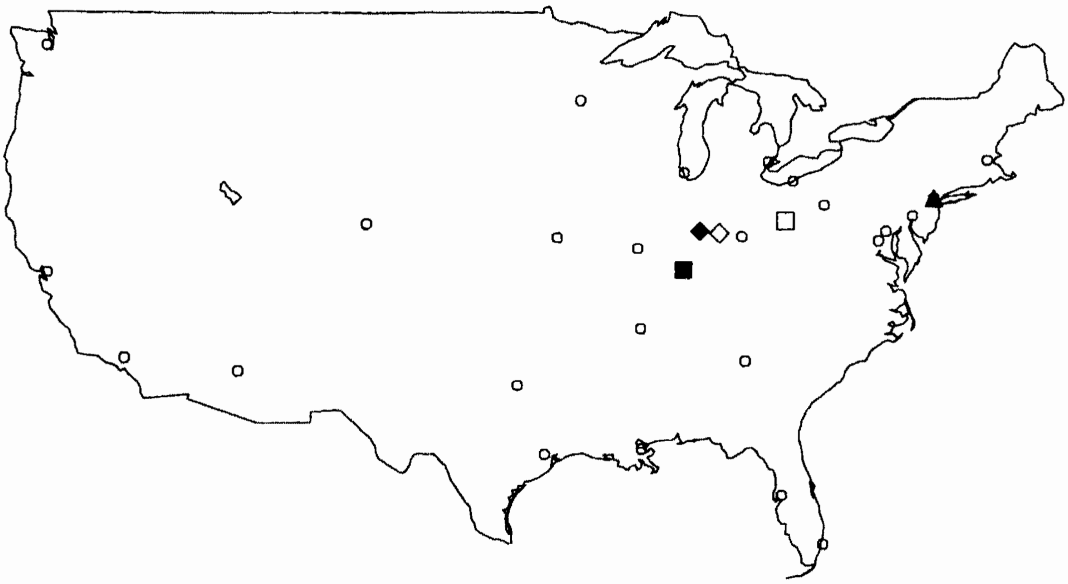


Figure 5: The results for test example No.6 (Black and white triangles overlap.)

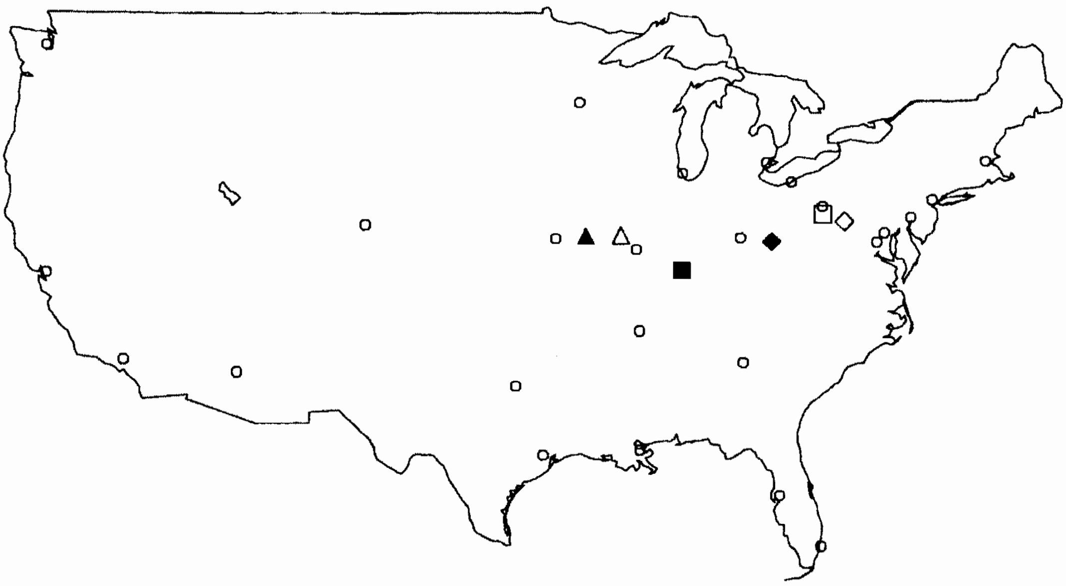


Figure 6: The results for test example No.7

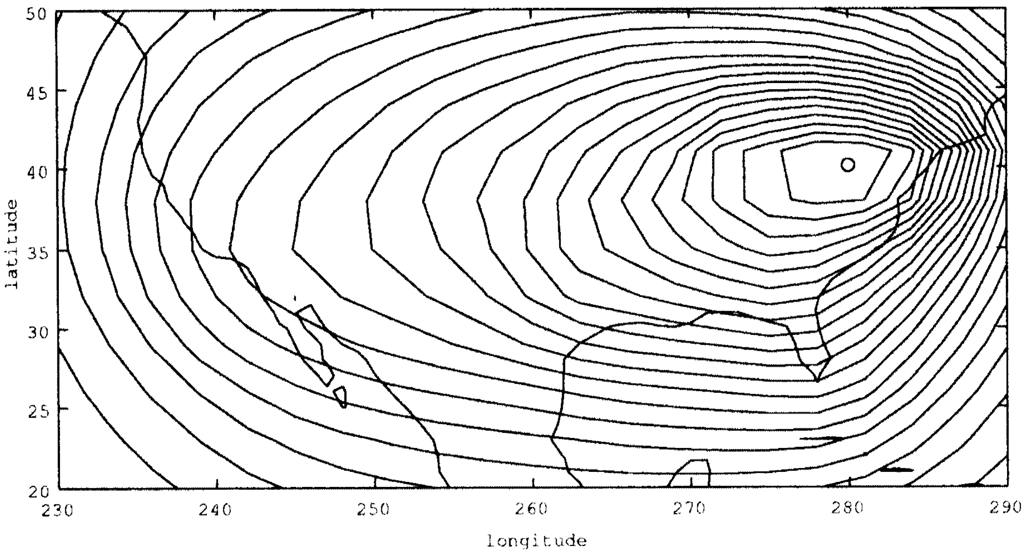


Figure 7: Contours of Firms A's cost function in test example No.7

among followers and incorporating non-stop services, are important subjects of future research. Moreover, it may be worthwhile to consider a hub location problem where medium firms cooperate to compete against the big firm, which is also an interesting future research topic.

Acknowlegement
The work was supported by Nanzan University Pache Subsidy I-A.

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