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ON A MULTICOMMODITY FLOW NETWORK RELIABILITY MODEL AND ITS APPLICATION TO A CONTAINER-LOADING TRANSPORTATION PROBLEM

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Abstract This paper is mainly to extend the MMCF (multi-commodity minimum cost flow) problem from deterministic flow networks to stochastic cases. Under the transportation budget constraint, this paper proposes an approach to calculate the probability that the required amount of multi-commodity can be transmitted successfully through a stochastic flow network. Such a probability is called transportation reliability. An algorithm is proposed first to find out all lower boundary points for the requirement in terms of minimal paths. The transportation reliability can then be calculated in terms of such lower boundary points. A numerical example, in which two types of commodities are shipped in container under two container-loading policies, is presented to illustrate the application of such a reliability model.

1. Introduction and Problem Description

The minimum cost flow (MCF) problem [1] to determine the least cost shipment or transportation of commodities of the same type through a flow network to satisfy the given demands at destinations from available supplies at sources is one most fundamental of all network flow problems. This problem is studied under the assumption that such a flow network is deterministic (i.e., the capacity of each arc being an integer-valued constant). The MCF problem whenever generalized to commodities of multiple different types is called a multi-commodity minimum cost flow (MMCF) problem [5, 6, 9, 10, 12, 17, 18] in which each type commodity can be transmitted from several sources to several destinations. Such a MMCF problem can be applied to solve the optimal energy policy, optimal resource allocation, etc. [2]. However, in the real situation, the flow network is stochastic (i.e., the capacity of each arc is stochastic) due to that the arc may be in failure, maintenance, consumed or pre-occupied by another agency, etc. Hence, the need to generalize MCF and MMCF problems to stochastic cases arises.

For simplicity of our purpose, the flow network concerned here is assumed to have the unique source s and destination t. Lin *et al.*[14] and Xue [19] had proposed a reliability model in single-commodity case which calculates the probability that a desired amount d^1 of the same type commodity can be transported from s to t without budget constraint. Lin [13] improved the above reliability model to include the budget requirement. In this article, we will further extend it to the multi-commodity reliability model. Our presentation will first restrict to two-commodity case. Similar studies can be easily extended to the general multi-commodity case. The approach is briefly described as follows. Given the system demand (d^1, d^2) at sink t and the budget K, an algorithm is proposed to generate all lower boundary points for $(d^1, d^2; K)$ in terms of minimal paths (MPs). An MP is a path whose proper subsets cannot be paths. The transportation reliability can then be calculated in terms of such lower boundary points for $(d^1, d^2; K)$. We will apply this model to an example to evaluate the transportation reliability that desired amounts of *1st* and *2nd* types commodity are shipped in containers from s to t

1.1. Nomenclature and notation

G	$G = (N, A)$ is a stochastic flow network with the unique source s and sink t where N and $A = \{a_i \mid 1 \leq i \leq n\}$ denote the sets of nodes and arcs, respectively.
$n; m$	number of arcs in G; number of minimal paths of G from s to t
C_i	(integer) the maximum capacity (or resource) of a_i $(i = 1, 2, \cdots, n)$
x_i	(integer) the current capacity of a_i $(i = 1, 2, \cdots, n)$
mp_j	minimal path j $(j = 1, 2, \cdots, m)$
(F^1, F^2)	system flow vector where $F^1 = (f_1^1, f_2^1, \cdots, f_m^1)$ and $F^2 = (f_1^2, f_2^2, \cdots, f_m^2)$ with f_j^1 and f_j^2 denoting the current flow (integer value) of *1st* and *2nd* types commodity on mp_j, respectively.
ω_i^1	(real number) the amount of capacity of a_i consumed per *1st* type commodity [2]. For instance, if a container loads 10 commodities exactly, then a commodity consumes 1/10 amount of the container. If the capacity in counted in terms of the number of containers, then $\omega_i^1 = 1/10$.
ω_i^2	(real number) the amount of capacity of a_i consumed per *2nd* type commodity.
c_i^1	the transportation cost of each *1st* type commodity through a_i
c_i^2	the transportation cost of each *2nd* type commodity through a_i
(d^1, d^2)	the system demand to mean that both d^1 amount of *1st* type commodity and d^2 amount of *2nd* type commodity are required at t.
K	transportation budget
$R_{d^1,d^2;K}$	transportation reliability
$\lfloor x \rfloor$	the greatest integer such that $\lfloor x \rfloor \leq x$
$\lceil x \rceil$	the smallest integer such that $\lceil x \rceil \geq x$
$Y \leq X$	$(y_1, y_2, \cdots, y_n) \leq (x_1, x_2, \cdots, x_n)$ if and only if $y_i \leq x_i$ for each $i = 1, 2, \cdots, n$
$Y < X$	$(y_1, y_2, \cdots, y_n) < (x_1, x_2, \cdots, x_n)$ if and only if $Y \leq X$ and $y_i < x_i$ for at least one i

1.2. Assumptions

1. Each node is perfectly reliable (i.e., the reliability of each node is 1).
2. Two types commodity can be transported from s to t.
3. The capacity of arc a_i takes values in $\{0, 1, 2, \cdots, C_i\}$ with a given probability distribution.
4. The capacities of different arcs are statistically independent.
5. Flow of each type commodity must satisfy the so-called flow conservation [7].

2. Model Formulation

2.1. Reliability model building

The two-commodity flow model for G is described in terms of two vectors: the system capacity vector $X = (x_1, x_2, \cdots, x_n)$ and the system flow vector (F^1, F^2). Such an (F^1, F^2) is feasible under X if

$$\left\lceil \sum_{a_i \in mp_j} (\omega_i^1 \cdot f_j^1 + \omega_i^2 \cdot f_j^2) \right\rceil \leq x_i \quad \text{for each } i = 1, 2, \cdots, n. \tag{1}$$

The value $\sum_{a_i \in mp_j} (\omega_i^1 \cdot f_j^1 + \omega_i^2 \cdot f_j^2) = \omega_i^1 \cdot \sum_{a_i \in mp_j} f_j^1 + \omega_i^2 \cdot \sum_{a_i \in mp_j} f_j^2$ is the total amount of capacity of a_i consumed by (F^1, F^2). For convenience, let Ψ_X denote the set of such (F^1, F^2)s which are feasible under X, and Ψ_C the set of system flow vectors where $C = (C_1, C_2, \cdots, C_n)$. That is, $(F^1, F^2) \in \Psi_C$ if and only if (F^1, F^2) satisfies constraint (2),

$$\left\lceil \sum_{a_i \in mp_j} (\omega_i^1 \cdot f_j^1 + \omega_i^2 \cdot f_j^2) \right\rceil \le C_i \quad \text{for each } i = 1, 2, \cdots, n. \tag{2}$$

The network G satisfies the given demand (d^1, d^2) at t under the budget K if and only if there exists an (F^1, F^2) such that

$$\sum_{j=1}^{m} f_j^1 = d^1 \quad \text{and} \quad \sum_{j=1}^{m} f_j^2 = d^2 \tag{3}$$

and

$$\sum_{i=1}^{n} \left\{ c_i^1 \cdot \sum_{a_i \in mp_j} f_j^1 + c_i^2 \cdot \sum_{a_i \in mp_j} f_j^2 \right\} \le K. \tag{4}$$

The value $\sum_{i=1}^{n} \left\{ c_i^1 \cdot \sum_{a_i \in mp_j} f_j^1 + c_i^2 \cdot \sum_{a_i \in mp_j} f_j^2 \right\}$ is thus the total transportation cost under (F^1, F^2). Let $\Omega_{d^1,d^2;K} = \{X \mid$ there exists an $(F^1, F^2) \in \Psi_X$ which satisfies (3) and (4)$\}$. The transportation reliability $R_{d^1,d^2;K}$ is thus

$$R_{d^1,d^2;K} = \Pr\{\Omega_{d^1,d^2;K}\} = \sum_{X \in \Omega_{d^1,d^2;K}} \Pr\{X\},$$

where $\Pr\{X\} = \mathrm{P}x_1 \times \mathrm{P}x_2 \times \cdots \times \mathrm{P}x_n$ by Assumption 4. (Note that $\mathrm{P}x_i$ is the probability that the capacity of a_i is x_i.) Each minimal vector X in $\Omega_{d^1,d^2;K}$ is called a lower boundary point for $(d^1, d^2; K)$, i.e., X is a lower boundary point for $(d^1, d^2; K)$ if and only if i) $X \in \Omega_{d^1,d^2;K}$ and ii) $Y \notin \Omega_{d^1,d^2;K}$ for any system capacity vector Y such that $Y < X$. However, the necessary and sufficient condition for $Y \notin \Omega_{d^1,d^2;K}$ can be restated as follows.

Lemma 1. For any system capacity vector Y, $Y \notin \Omega_{d^1,d^2;K}$ if and only if for each $(F^1, F^2) \in \Psi_Y$, at least one of the following statements holds;

(i) $\sum_{j=1}^{m} f_j^1 < d^1$,

(ii) $\sum_{j=1}^{m} f_j^2 < d^2$,

(iii) $\sum_{i=1}^{n} \left\{ c_i^1 \cdot \sum_{a_i \in mp_j} f_j^1 + c_i^2 \cdot \sum_{a_i \in mp_j} f_j^2 \right\} > K$.

It is easy to find that $R_{d^1,d^2;K} = \Pr(\bigcup_X \{Y \mid Y \ge X\})$ over all lower boundary points Xs for $(d^1, d^2; K)$ and so $R_{d^1,d^2;K}$ can be reduced to be evaluated in term of all lower boundary points for $(d^1, d^2; K)$. Hence, we may calculate it by applying the inclusion-exclusion method [8, 11, 15, 16], state-space decomposition [13, 14] or disjoint subsets [19]. The problem remains is how to generate all such points for $(d^1, d^2; K)$.

2.2. Generate all lower boundary points for $(d^1, d^2; K)$

Let $\Phi_{d^1,d^2;K} = \{(F^1, F^2) \mid (F^1, F^2)$ satisfies constraints (2)-(4)$\}$, i.e., $\Phi_{d^1,d^2;K}$ is the set of system flow vectors satisfying the system demand (d^1, d^2) and whose transportation cost does not exceed the budget K.

We can generate $\Phi_{d^1,d^2;K}$ by applying the implicit enumeration method (e.g., branch-and-bound or backtracking), which is always denoted by a search tree composed nodes and arcs. The arcs from level j to level $j+1$ nodes are labeled with possible values of f_j. A search tree for $\sum_{j=1}^{m} f_j^1 = d^1$ is illustrated in Figure 1 [13], where constraints (2) and (4) are used to be bounding constraints.

Given each $(F_1, F_2) \in \Phi_{d^1,d^2;K}$, generate the vector $Z_{F^1,F^2} = (z_1, z_2, \cdots, z_n)$ where

$$z_i = \left\lceil \sum_{a_i \in mp_j} (\omega_i^1 \cdot f_j^1 + \omega_i^2 \cdot f_j^2) \right\rceil \quad \text{for each } i = 1, 2, \cdots, n. \tag{5}$$

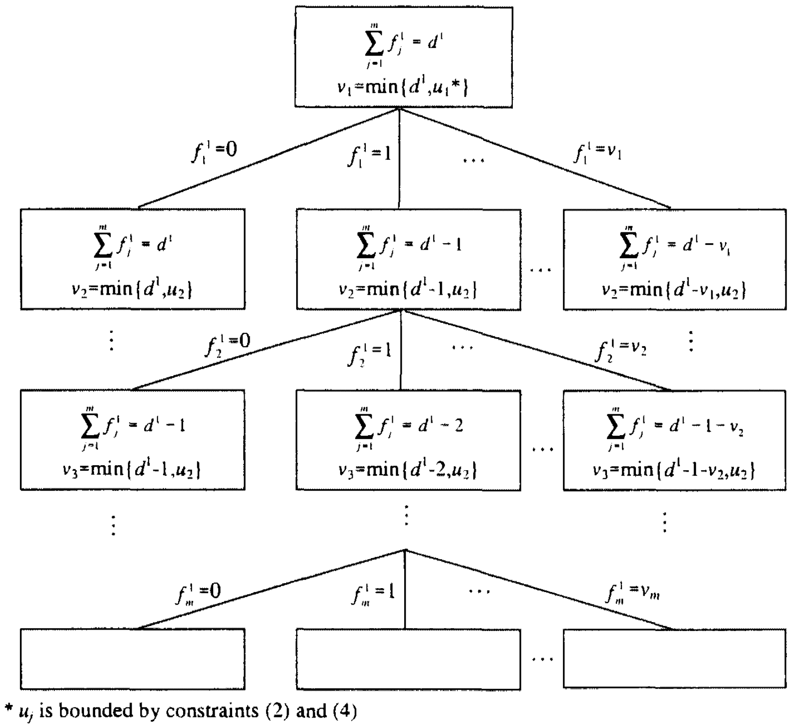

Figure 1: A search tree for implicit enumeration method

In fact, such obtained Z_{F^1,F^2} is a system capacity vector as $z_i \in \{0, 1, 2, \cdots, C_i\}$ $\forall i$ (according to Equation (2)) and $(F^1, F^2) \in \Psi_{Z_{F^1,F^2}}$ from Equation (5). A necessary condition for a lower boundary point for $(d^1, d^2; K)$ is shown in the following lemma.

Lemma 2. If X is a lower boundary point for $(d^1, d^2; K)$. Then $X = Z_{F^1,F^2}$ for each $(F^1, F^2) \in \Psi_X$ satisfying constraints (2)-(4).

Proof: From Equation (1), $Z_{F^1,F^2} \leq X$ $\forall (F_1, F_2) \in \Psi_X \cap \Phi_{d^1,d^2;K}$. Suppose $Z_{F^1,F^2} < X$. Then $Z_{F^1,F^2} \notin \Omega_{d^1,d^2;K}$ as X is minimal in $\Omega_{d^1,d^2;K}$. This is a contradiction. Hence, $X = Z_{F^1,F^2}$. Q.E.D.

For convenience, let $\rho = \{Z_{F^1,F^2} \mid (F^1, F^2) \in \Phi_{d^1,d^2;K}\}$. Lemma 2 implies that ρ contains all lower boundary points for $(d^1, d^2; K)$. The following lemma further proves that

$\rho_{\min} = \{X \mid X$ is minimal w.r.t. $\leq$ in $\rho\}$ is the set of all lower boundary points for $(d^1, d^2; K)$.

Lemma 3. $\{X \mid X$ is a lower boundary point for $(d^1, d^2; K)\} = \rho_{\min}$.

Proof: Firstly, suppose X is a lower boundary point for $(d^1, d^2; K)$ (note that $X \in \rho$ by Lemma 2) but $X \notin \rho_{\min}$, i.e., there exist a $Y \in \rho$ such that $Y < X$. Then $Y \in \Omega_{d^1,d^2;K}$. This contradicts to that X is a lower boundary point for $(d^1, d^2; K)$. Hence $X \in \rho_{\min}$.

Conversely, suppose $X \in \rho_{\min}$ (note that $X \in \Omega_{d^1,d^2;K}$) but it is not a lower boundary point for $(d^1, d^2; K)$. Then there exists a lower boundary point Y for $(d^1, d^2; K)$ such that $Y < X$. By Lemma 2, $Y \in \rho$. This contradicts to that $X \in \rho_{\min}$. Hence, X is a lower boundary point for $(d^1, d^2; K)$. Q.E.D.

3. Algorithm

Given the system demand (d^1, d^2) and the budget K, an algorithm to generate all lower boundary points for $(d^1, d^2; K)$ in terms of MPs is shown in the following steps;

Step 1. Obtain all feasible solutions (F^1, F^2) with $F^1 = (f_1^1, f_2^1, \cdots, f_m^1)$ and $F^2 = (f_1^2, f_2^2, \cdots, f_m^2)$ of the following constraints by applying the implicit enumeration:

$$\left\lceil \sum_{a_i \in mp_j} (\omega_i^1 \cdot f_j^1 + \omega_i^2 \cdot f_j^2) \right\rceil \leq C_i \quad \text{for each } i = 1, 2, \cdots, n, \tag{6}$$

$$\sum_{j=1}^{m} f_j^1 = d^1 \quad \text{and} \quad \sum_{j=1}^{m} f_j^2 = d^2, \tag{7}$$

$$\sum_{i=1}^{n} \left\{ c_i^1 \cdot \sum_{a_i \in mp_j} f_j^1 + c_i^2 \cdot \sum_{a_i \in mp_j} f_j^2 \right\} \leq K. \tag{8}$$

Step 2. Transform all feasible solutions (F^1, F^2) into $X = (x_1, x_2, \cdots, x_n)$ according to

$$x_i = \left\lceil \sum_{a_i \in mp_j} (\omega_i^1 \cdot f_j^1 + \omega_i^2 \cdot f_j^2) \right\rceil \quad \text{for each} \quad i = 1, 2, \cdots, n. \tag{9}$$

Step 3. Suppose the result of Step 2 is: $X_1, X_2, \cdots, X_k$. Remove those non-minimal ones in $\{X_1, X_2, \cdots, X_k\}$ to obtain all lower boundary points for $(d^1, d^2; K)$ as follows.

3.1) $I = \emptyset$
3.2) For $i = 1$ to k with $i \notin I$
3.3) For $j = i + 1$ to k with $j \notin I$
3.4) If $X_j < X_i$, X_i is not a lower boundary points for $(d^1, d^2; K)$. $I = I \cup \{i\}$ and goto Step 3.7)
elseif $X_j \geq X_i$, X_j is not a lower boundary points for $(d^1, d^2; K)$. $I = I \cup \{j\}$
3.5) $j = j + 1$
3.6) X_i is a lower boundary point for $(d^1, d^2; K)$
3.7) $i = i + 1$
3.8) END.

4. Application to A Container-Loading Transportation Problem

The supplier likes to have its d^1 amount of *1st* type commodity and d^2 amount of *2nd* type commodity transmitted in containers of a same type from s to t responsible by one specific transportation company. Such commodities will be packaged and then loaded into containers. The supplier's budget for this transportation is K. Each route from s in general might need to pass through other intermediate nodes (transfer stations) to finally arrive at t. The capacity of each arc is stochastic due to that either containers or traffic tools (e.g., cargo airplane, cargo ship, etc.) through each arc may be in maintenance, reserved by other suppliers or in other conditions. Hence, the supplier needs to know the reliability that desired amounts of commodities of two different types can be transported successfully from s to t under the budget K. The transportation reliability will depend on the chosen policy from two different container-loading policies as follows;

Policy I: The container is rented as a whole (i.e., the supplier is not allowed to share each container with another supplier) and commodities of different types should be loaded into different containers. The transportation cost per container through arc a_i is c_i.

Policy II: The container is divided into q unit-spaces. The container can be rented either as a whole in cost c_i through a_i or in unit-space in cost b_i through a_i where $q \cdot b_i > c_i$. Suppose the supplier needs $(xq + q')$ unit-spaces to transport the total amount of commodities where x is the number of whole containers and $q' < q$. Then the supplier will rent x containers plus q' extra unit-spaces with cost $(xc_i + q'b_i)$ through a_i. Also, commodities of different types can be loaded into a same container but different space-units.

4.1. The proposed reliability model under Policy I

Under Policy I, x_i, C_i, f_j^1 and f_j^2 are all counted in terms of number of containers. The system demand (d^1, d^2) is thus transformed into (D^1, D^2) where D^1 (resp. D^2) is the number of containers needed to load d^1 (resp. d^2) commodities of *1st* (resp. *2nd*) type. Let c_i be the transport cost of each container through a_i. Hence, the constraints (6)–(8) are modified correspondingly into (10)–(12) (i.e., (F^1, F^2) satisfies $(D^1, D^2; K)$ if it satisfies constraints (10)–(12)) as follows:

$$\sum_{a_i \in mp_j} (f_j^1 + f_j^2) \le C_i, \tag{10}$$

$$\sum_{j=1}^{m} f_j^1 = D^1 \quad \text{and} \quad \sum_{j=1}^{m} f_j^2 = D^2, \tag{11}$$

$$\sum_{i=1}^{n} \left\{ c_i \cdot \sum_{a_i \in mp_j} (f_j^1 + f_j^2) \right\} \le K. \tag{12}$$

The equation (9) is also modified to Equation (13) as follows:

$$x_i = \sum_{a_i \in mp_j} (f_j^1 + f_j^2) \quad \text{for each } i = 1, 2, \cdots, n. \tag{13}$$

Hence, the proposed reliability model with $\omega_i^1 = \omega_i^2 = 1$ and $c_i^1 = c_i^2 = c_i \; \forall i$ can be applied to Policy I case.

4.2. The proposed reliability model under Policy II

C_i and x_i are both counted in terms of number of unit-spaces but in $F = (F^1, F^2)$, f_j^1 and f_j^2 are both counted in terms of amount of commodity for all j. One unit of *1st* (resp. *2nd*) type commodity consumes α_1 (resp. α_2) unit-spaces for all arcs. That is, $\omega_i^1 = \alpha_1$ and

$\omega_i^2 = \alpha_2\ \forall i$. Hence, (F^1, F^2) satisfies $(d^1, d^2; K)$ if it satisfies the following constraints:

$$\left\lceil \sum_{a_i \in mp_j} (\alpha_1 \cdot f_j^1 + \alpha_2 \cdot f_j^2) \right\rceil \le C_i, \tag{14}$$

$$\sum_{j=1}^{m} f_j^1 = d^1 \quad \text{and} \quad \sum_{j=1}^{m} f_j^2 = d^2, \tag{15}$$

$$\sum_{i=1}^{n} \left\{ c_i \cdot \left\lfloor \left\lceil \sum_{a_i \in mp_j} (\alpha_1 \cdot f_j^1 + \alpha_2 \cdot f_j^2) \right\rceil \Big/ q \right\rfloor + b_i \cdot \left(\left\lceil \sum_{a_i \in mp_j} (\alpha_1 \cdot f_j^1 + \alpha_2 \cdot f_j^2) \right\rceil - \left\lfloor \left\lceil \sum_{a_i \in mp_j} (\alpha_1 \cdot f_j^1 + \alpha_2 \cdot f_j^2) \right\rceil \Big/ q \right\rfloor \cdot q \right) \right\} \le K, \tag{16}$$

where $x = \left\lfloor \left\lceil \sum_{a_i \in mp_j} (\alpha_1 \cdot f_j^1 + \alpha_2 \cdot f_j^2) \right\rceil \Big/ q \right\rfloor$ is the number of whole containers rented by the supplier and $q' = \left\lceil \sum_{a_i \in mp_j} (\alpha_1 \cdot f_j^1 + \alpha_2 \cdot f_j^2) \right\rceil - \left\lfloor \left\lceil \sum_{a_i \in mp_j} (\alpha_1 \cdot f_j^1 + \alpha_2 \cdot f_j^2) \right\rceil \Big/ q \right\rfloor \cdot q$ which is less than q is the number of unit-spaces rented for the surplus by the supplier. Then transform all feasible (F^1, F^2) into $X = (x_1, x_2, \cdots, x_n)$ according to

$$x_i = \left\lceil \sum_{a_i \in mp_j} (\alpha_1 \cdot f_j^1 + \alpha_2 \cdot f_j^2) \right\rceil \quad \text{for each } i = 1, 2, \cdots, n. \tag{17}$$

Hence, the proposed reliability model can be applied to Policy II case after modifying constraints (6)–(9) to (14)–(17), respectively.

4.3. A numerical example

According to the law of Taiwan, no direct route from s (Taichung, Taiwan) to t (Shanghai, Mainland China) is permitted. Hence, each route from s should pass either Hongkong or Tokyo before arriving t. The routes through each arc are all by ship. Each container has 3 unit-spaces (i.e., $q = 3$) and one unit of commodity means 60 commodities of a same type. The supplier wants to have its 6 units of 15-inches monitors and 3 units of 17-inches monitors (i.e., $d^1 = 6$ and $d^2 = 3$) to be transported from s to t under the budget $K = 70 \times 100$ US dollars. The sizes of each container, 15-inches monitor and 17-inches monitor are $591 \times 230 \times 220$ (length $\times$ breadth $\times$ high) cm^3, $48 \times 45 \times 52$ cm^3 and $56 \times 56 \times 57$ cm^3, respectively. Hence, each container can load 3 units (i.e., 180) of 15-inches monitors or 2 units (i.e., 120) of 17-inches monitors completely. Thus, one unit of *1st* (resp. *2nd*) type commodity consumes 1 (resp. 1.5) unit-space, i.e., $\alpha_1 = 1$ (resp. $\alpha_2 = 1.5$).

There are four MPs from s to t in Figure 2, $mp_1 = \{a_1, a_2\}$, $mp_2 = \{a_1, a_3, a_6\}$, $mp_3 = \{a_5, a_4, a_2\}$, $mp_4 = \{a_5, a_6\}$. In advance, the transportation cost through each arc and the arc data under Policy I and II are given in Tables 1-3, respectively.

4.3.1. Transportation reliability evaluation under Policy I

Under Policy I, the supplier needs 2 whole containers for each type commodity, i.e., $D^1 = D^2 = 2$. The transportation reliability is denoted by $R_{2,2,70}$ and thus can be derived as follows:

Step 1. Find all feasible solutions (F^1, F^2) of the following constraints

$$f_1^1 + f_2^1 + f_1^2 + f_2^2 \le 3, \quad f_1^1 + f_3^1 + f_1^2 + f_3^2 \le 2, \quad f_2^1 + f_2^2 \le 2,$$

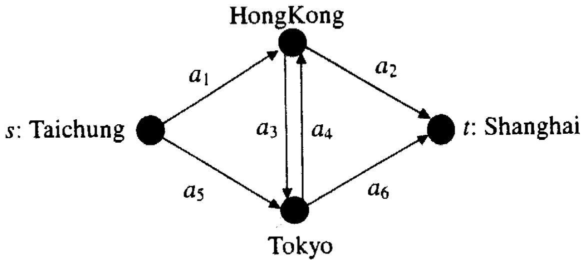

Figure 2: A transportation network from Taichung to Shanghai

Table 1: The transportation costs through arcs (unit: 100 US dollars)

Arc	c_i The transportation cost per container (Policy I & II)	b_i The transportation cost per unit-space of the container (Policy II)
a_1	4	2
a_2	11	5
a_3	7	3
a_4	7	3
a_5	9	4
a_6	9	4

Table 2: Arc data under Policy I

Arc	x_i (number of whole container)	Probability	Arc	x_i (number of whole container)	Probability
a_1	0	.05	a_4	0	.15
	1	.10		1	.15
	2	.15		2	.70
	3	.70	a_5	0	.05
a_2	0	.10		1	.10
	1	.10		2	.15
	2	.80		3	.70
a_3	0	.10	a_6	0	.10
	1	.15		1	.10
	2	.75		2	.80

$$f_3^1 + f_3^2 \leq 2, \quad f_3^1 + f_4^1 + f_3^2 + f_4^2 \leq 3, \quad f_2^1 + f_4^1 + f_2^2 + f_4^2 \leq 2, \tag{18}$$

$$f_1^1 + f_2^1 + f_3^1 + f_4^1 = 2 \quad \text{and} \quad f_1^2 + f_2^2 + f_3^2 + f_4^2 = 2, \tag{19}$$

$$\begin{aligned} &4(f_1^1 + f_2^1 + f_1^2 + f_2^2) + 11(f_1^1 + f_3^1 + f_1^2 + f_3^2) + 7(f_2^1 + f_2^2) + \\ &\quad 7(f_3^1 + f_3^2) + 9(f_3^1 + f_4^1 + f_3^2 + f_4^2) + 9(f_2^1 + f_4^1 + f_2^2 + f_4^2) \leq 70. \end{aligned} \tag{20}$$

We obtain 7 feasible (F^1, F^2)s: (2,0,0,0,0,1,0,1), (1,1,0,0,1,0,0,1), (1,0,0,1,1,1,0,0), (0,1,0,1,2, 0,0,0), (2,0,0,0,0,0,0,2), (1,0,0,1,1,0,0,1), (0,0,0,2,2,0,0,0).

Step 2. Transform each feasible (F^1, F^2) to $X = (x_1, x_2, \cdots, x_6)$ via

$$\begin{aligned} &x_1 = f_1^1 + f_2^1 + f_1^2 + f_2^2, \quad x_2 = f_1^1 + f_3^1 + f_1^2 + f_3^2, \quad x_3 = f_2^1 + f_2^2, \\ &x_4 = f_3^1 + f_3^2, \quad x_5 = f_3^1 + f_4^1 + f_3^2 + f_4^2, \quad x_6 = f_2^1 + f_4^1 + f_2^2 + f_4^2. \end{aligned} \tag{21}$$

Table 3: Arc data under Policy II

Arc	x_i (number of unit space)	Probability	Arc	x_i (number of unit space)	Probability
a_1	0	.01	a_4	0	.03
	1	.02		1	.03
	2	.02		2	.04
	3	.03		3	.05
	4	.03		4	.05
	5	.04		5	.05
	6	.05		6	.75
	7	.05	a_5	0	.01
	8	.05		1	.02
	9	.70		2	.02
a_2	0	.03		3	.03
	1	.03		4	.03
	2	.04		5	.04
	3	.03		6	.05
	4	.03		7	.05
	5	.04		8	.05
	6	.80		9	.70
a_3	0	.03	a_6	0	.03
	1	.03		1	.03
	2	.04		2	.04
	3	.05		3	.03
	4	.05		4	.03
	5	.05		5	.04
	6	.75		6	.80

We obtain two Xs: $X_1 = (3,2,1,0,1,2)$ and $X_2 = (2,2,0,0,2,2)$.

Step 3. Delete those non-minimal ones in $\{X_1, X_2\}$ to obtain all lower boundary points for (2,2;70).

3.1) $I = \emptyset$

3.2) $i = 1$

3.3) $j = 2$

3.4) $X_2 = (2,2,0,0,2,2) \not< X_1 = (3,2,1,0,1,2)$ and $X_2 \ngeq X_1 . I = \emptyset$.

3.6) X_1 is a lower boundary point for (2,2;70).

3.2) $i = 2$

3.6) X_2 is a lower boundary point for (2,2;70).

We find that (3,2,1,0,1,2) and (2,2,0,0,2,2) are the lower boundary points for (2,2;70). To calculate the transportation reliability $R_{2,2,70}$, firstly let $B_1 = \{X \mid X \geq (3,2,1,0,1,2)\}$ and $B_2 = \{X \mid X \geq (2,2,0,0,2,2)\}$. Then by applying the inclusion-exclusion method,

$$\begin{aligned} R_{2,2,70} &= \Pr\{B_1 \cup B_2\} \\ &= \Pr\{B_1\} + \Pr\{B_2\} - \Pr\{B_1 \cap B_2\} \\ &= \Pr\{X \mid X \geq (3,2,1,0,1,2)\} + \Pr\{X \mid X \geq (2,2,0,0,2,2)\} \\ &\quad - \Pr\{X \mid X \geq (3,2,1,0,2,2)\} \\ &= (0.7 \times 0.8 \times 0.9 \times 1 \times 0.95 \times 0.8) + (0.85 \times 0.8 \times 1 \times 1 \times 0.85 \times 0.8) \end{aligned}$$

$$\begin{aligned} & -(0.7 \times 0.8 \times 0.9 \times 1 \times 0.85 \times 0.8) \\ = \ & 0.38304 + 0.4624 - 0.34272 \\ = \ & 0.50272. \end{aligned}$$

Hence, the transportation reliability under Policy I is 0.50272.

4.3.2. Transportation reliability evaluation under Policy II

Similar to the solution procedure under Policy I, the result of each step is briefly described in the third column of Table 4. We can see that the transportation reliability is higher than that under Policy I.

Table 4: Comparison between $R_{d^1,d^2;K}$ under Policies I & II

$d^1 = 6, d^2 = 3$ units of commodities, $K = 70 \times 100$ US dollars	**Policy I** $(D^1, D^2; K) = (2, 2; 70)$	**Policy II** $(d^1, d^2; K) = (6, 3; 70)$
Elements in $\Phi_{d^1,d^2;K}$	7 feasible solutions	63 feasible solutions
Elements in ρ	(3,2,1,0,1,2), (2,2,0,0,2,2)	(9,6,3,0,2,5), (9,5,4,0,2,6),(9,6,4,0,2,5), (8,6,2,0,3,5), (8,6,3,1,3,5), (8,6,3,0,3,5), (8,5,3,0,3,6), (8,5,4,0,3,6), (7,5,2,0,4,6), (7,6,1,0,4,5), (7,6,2,0,4,5), (7,5,3,0,4,6), (6,6,0,0,5,5), (6,5,1,0,5,6), (6,6,1,1,5,5), (6,5,2,0,5,6), (5,6,0,1,6,5), (5,5,0,0,6,6), (4,5,0,1,7,6)
Elements in $\rho_{\min}$	(3,2,1,0,1,2), (2,2,0,0,2,2)	(9,6,3,0,2,5), (9,5,4,0,2,6), (8,6,2,0,3,5), (8,5,3,0,3,6), (7,6,1,0,4,5), (7,5,2,0,4,6), (4,5,0,1,7,6), (6,6,0,0,5,5), (6,5,1,0,5,6), (5,6,0,1,6,5), (5,5,0,0,6,6)
$R_{d^1,d^2;K}$	0.50272	0.694029376

5. Discussions and Summary

It is known that the problem to search all lower boundary points for $(d^1, d^2; K)$ is NP-hard [4]. The number of feasible solutions of Equation (7) is $\binom{m+d^1-1}{d^1} \cdot \binom{m+d^2-1}{d^2}$. Let $\xi \equiv \binom{m+d^1-1}{d^1} \cdot \binom{m+d^2-1}{d^2}$. Hence, the number of solutions of constraints (6) and (8) is bounded by ξ. Similarly, the number of Xs transformed according to Equation (9) is bounded by ξ. Each solution of Equation (7) needs $O(m)$ time to test whether it satisfies constraint (6) for each arc a_i, $O(m \cdot n)$ time for all arcs and $O(m \cdot n)$ time to test constraint (8). Hence, it takes $O(m \cdot n \cdot \xi)$ time for Step 1 in the worst case. Then each solution needs $O(m \cdot n)$ time to be transformed into X via Equation (9). In the worst case, it takes $O(m \cdot n \cdot \xi)$ time for Step 2. It further takes $O(n \cdot \xi)$ time to test each solution of Step 2 whether it is a lower boundary points for $(d^1, d^2; K)$ and $O(n \cdot \xi^2)$ time for all solutions of Step 2 in the worst case. Hence, the computational time complexity of the algorithm in the worst case is $O(n \cdot \xi^2) = O(m \cdot n \cdot \xi) + O(m \cdot n \cdot \xi) + O(n \cdot \xi^2)$. (Note that m is less than $\binom{m+d^1-1}{d^1}$ in the first two summands). As each X in Step 2 is a n-tuple, it needs at most $O(n \cdot \xi)$ storage space to store all Xs.

For single-commodity case, the proposed algorithm reduces the number of constraints when comparing to the best existing method of [13]. The method [13] includes an extra flow constraint:

$$f_j^1 \leq \min_{a_i \in mp_j} C_i \quad \text{for each } j = 1, 2, \cdots, m. \tag{22}$$

In fact, constraint (6) implies such a constraint. In addition, the proposed algorithm considers the weight factor ω_i^1 for two-commodity case because different commodity competes the same capacity. However, this weight factor is not considered in the single-commodity case.

This article is basically to extend the so-called MMCF problem to a multi-commodity reliability model in the way that the flow networks concerned are from deterministic cases to stochastic cases. We apply this model to solve the transportation reliability that the desired amounts of two types commodity are transported from s to t via a stochastic flow network. The numerical example indicates that different container-loading policies result in different reliability models and also the reliabilities. Policy II is more flexible than Policy I in container-loading, and it can reduce the transportation cost. The cost for vector (F^1, F^2) under Policy I equals that under Policy II. But the cost for (F^1, F^2) under Policy II is larger than that under Policy I if $q' > 0$. For instance, $q' = 1$ under Policy II implies that it will cost an extra c_i under Policy I. Hence, the number of feasible (F^1, F^2) under Policy II is always more than that under Policy I. Similarly, the number of X under Policy II is always more than that under Policy I. That is why the reliability under Policy II is always higher than that under Policy I.

Moreover, the proposed algorithm can be easily extended to r-commodity ($r > 2$) case after modifying constraints (6)–(9) to (23)–(26), respectively.

$$\left\lceil \sum_{a_i \in mp_j} \left(\sum_{k=1}^{r} \omega_i^k \cdot f_j^k\right) \right\rceil \leq C_i \quad \text{for each } i = 1, 2, \cdots, n, \tag{23}$$

$$\sum_{j=1}^{m} f_j^k = d^k, \quad k = 1, 2, \cdots, r, \tag{24}$$

$$\sum_{k=1}^{r} \sum_{i=1}^{n} \left(c_i^k \cdot \sum_{a_i \in mp_j} f_j^k\right) \leq K, \tag{25}$$

$$x_i = \left\lceil \sum_{a_i \in mp_j} \left(\sum_{k=1}^{r} \omega_i^k \cdot f_j^k\right) \right\rceil \quad \text{for each } i = 1, 2, \cdots, n. \tag{26}$$

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