Journal of the Operations Research
Society of Japan
Vol. 44, No. 4, December 2001

MARKOV MODULATED FLUID QUEUES WITH BATCH FLUID ARRIVALS

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(Received November 15, 2000; Revised May 15, 2001)

Abstract In the conventional Markov modulated fluid queue, the buffer content process has a continuous sample path. This paper concerns a Markov modulated fluid queue whose buffer content process may have jumps. This model extends not only the conventional fluid queue but also the $MAP/G/1$ model. We give a procedure to get a Laplace-Stieltjes transform of the stationary joint distribution of the buffer content and background state. Some numerical examples are presented as well.

1. Introduction

The conventional fluid queue is an input-output system of fluid with a buffer. Suppose that input and output rates change according to a continuous-time Markov chain with a finite state space. This model is referred to as a *Markov modulated fluid queue*, and the Markov chain is called a *background process.* There are many applications for Markov modulated fluid queues. For instance, they are successfully applied to modern ATM systems, where the fluid flow is interpreted as a packet data stream which is processed by ATM switches.

Fluid queues have been studied in the much literature for more than twenty years. For example, see Anick, Mitra and Sondhi [1], Gaver and Lehoczky [5], Mitra [7], Elwalid and Stern [4], Rogers [9] and Asmussen [3]. In particular, Rogers [9] and Asmussen [3] consider the first passage time of the buffer content to obtain the empty probability and the stationary joint distribution of the buffer content and the background state. They show that the stationary joint distribution has the matrix-exponential form, which is determined by a certain matrix equation. In Rogers [9], the matrix equation is determined by Wiener-Hopf factorization. Asmussen [3] directly derives the matrix equation, then solves it by iteration. Furthermore, he considers the case that the fluid flow is subject to Brownian motion.

In this paper, we are interested in the situation that a fluid model has an extra input in addition to the conventional fluid flow. For example, such an input describes very high rate arrivals in short periods, e.g., big file transfers from a high speed source in a telecommunication network. If they infrequently occur, it is natural to process them together with the conventional fluid to gain better performance per cost. It may be also natural to have separate buffers for different types of sources. However it would further complicates an analysis. As the first step, we here concentrate on the total buffer contents, which can be also considered for the separate buffer model. Thus, we extend the Markov modulated fluid queue in such a way that its input flow may have upward jumps which is determined by the background Markov chain. We assume that amounts of jumps are independent and identically distributed under given state transitions. We refer to such jumps as *batch fluid arrivals.*

Since our model has a piecewise linear sample path and the background process is Markovian, the model includes the workload process of the $MAP/G/1$ queue, in which services may depend on its arrival process. Here, the workload and the arrival process can be interpreted as the buffer content and the background process, respectively, in which there are no input flow except for jumps and the output flow rates are always 1. Hasegawa and Takine [12] and Asmussen [2] use iteration approaches with respect to the first passage time to the idle state to get the LST (Laplace-Stieltjes transform) of the stationary joint distribution of the workload and the background state. Similar approaches can be found for the fluid queue in Asmussen [3] and for the $GI/PH/1$ queue in Sengputa [11].

The purpose of this paper is to get the LST of the stationary joint distribution for the buffer content and the background state. Using the rate conservation law (e.g., see [8]), we see that the problem to get the LST reduces to the problem to obtain the empty buffer probabilities jointly with the background states. We show that the empty probabilities are given by the stationary vector of a rate matrix determined by certain matrix equations. We then introduce a matrix iteration to solve the matrix equations, and prove that it converges to the rate matrix.

This paper is organized by six sections. In Section 2, we introduce the Markov modulated fluid queue with the batch fluid arrivals, and get the LST which includes the empty buffer probabilities as unknown terms. We then consider a random time change for simplifying arguments in Section 3, 4 and 5. In Section 3, we consider the first passage time to the empty state. In Section 4, we introduce a matrix iteration, and we show the main result. In Section 5, we compute the empty probabilities. In Section 6, we give an algorithm to compute moments of the buffer content. Numerical examples are also given. In Appendix, we derive the LST using the rate conservation law.

2. Background Markov Chain and Buffer Process

Let us describe the fluid system by a stochastic process. We first introduce a background Markov chain. To define a transition rate matrix of the Markov chain, we introduce some auxiliary notation. Let S be a state space of the Markov chain. We assume that S is a finite set, and denote the number of elements in S by $|S|$. Let C be a $|S| \times |S|$-matrix whose diagonal and off-diagonal elements are negative and nonnegative, respectively. Let $D(x)$ be a $|S| \times |S|$-matrix whose (i,j)th element $[D(x)]_{ij}$ is nonnegative and non-decreasing function of $x > 0$ for all $i, j \in S$. Define $|S| \times |S|$-matrix D as

$$D = \int_{0+}^{\infty} D(dx),$$

where the integration is performed in component-wise. We assume that

$$(C + D)\boldsymbol{e} = \mathbf{0}, \tag{2.1}$$

where $\boldsymbol{e}$ is the $|S|$-dimensional column vector whose all components are 1. We define $\{M(t)\}$ to be a Markov chain with transition rate matrix $C+D$. We assume that $C+D$ is irreducible. Since S is finite, there exists a stationary distribution for $C + D$, which is denoted by $|S|$-dimensional row vector $\boldsymbol{\pi}$ i.e

$$\boldsymbol{\pi}(C + D) = \mathbf{0}, \quad \boldsymbol{\pi}\boldsymbol{e} = 1.$$

A transition of the background process due to C is called C-type, while the one due to D ($D(x)$) is called D-type ($D(x)$-type, respectively).

To describe the buffer content mathematically, we next define function $v : S \to v(S)(\subset \mathcal{R} - \{0\})$. v does not take value 0, but the other structure of v is not assumed. Function v specifies increasing or decreasing rates of the buffer content when it is not empty. That is, if $M(t) = i \in S$ and $v(i) = -2$, then the buffer content decrease with rate 2 at time t. Let $Y(t)$ be the accumulated fluids and batch fluids received up to time t. That is,

$$Y(t) = \int_0^t v(M(u))du + \int_0^t B_u(M(u-), M(u))N(du), \tag{2.2}$$

where $B_u(i,j)$ is the amount of the batch fluids when background process $M(t)$ jumps from i to j at time u, and N is a counting process of D-type transitions which include transitions from any state to itself. Note that $B_u(i,j)$ is subject to the distribution $[D(x)]_{ij}/[D]_{ij}$. Thus, a jump of $\{Y(t)\}$ only occur by D-type transition of the background process. $D(x)$-type transition is a D-type with a jump amount of $\{Y(t)\}$ which is less than or equal x. $Y(t)$ may be negative. Then, buffer content process $X(t)$ starting with $X(0)$ is defined by

$$X(t) = Y(t) + \max\{X(0), -\inf_{0 \le u \le t} Y(u)\}.$$

We define $|S| \times |S|$ matrices:

$$V = \mathrm{diag}(v(i); i \in S), \quad V_{abs} = \mathrm{diag}(|v(i)|; i \in S),$$

where diag$(\cdot)$ denotes a diagonal matrix.

Define sets S^+ and S^- as

$$S^+ = \{i \in S | v(i) > 0\}, \quad S^- = \{i \in S | v(i) < 0\}.$$

$|S| \times |S|$-matrix A and $|S|$-dimensional row vector $\boldsymbol{x}$ are partitioned into S^+ and S^- sections in the following way.

$$A = \begin{pmatrix} A^{++} & A^{+-} \\ A^{-+} & A^{--} \end{pmatrix}, \quad \boldsymbol{x} = (\boldsymbol{x}^+, \boldsymbol{x}^-).$$

For convenience, we also define $|S^+| \times |S|$-matrix $A^{+\bullet}$ and $|S| \times |S^+|$-matrix $A^{\bullet +}$ as

$$A^{+\bullet} = \begin{pmatrix} A^{++} & A^{+-} \end{pmatrix}, \quad A^{\bullet +} = \begin{pmatrix} A^{++} \\ A^{-+} \end{pmatrix},$$

respectively. $A^{-\bullet}$ and $A^{\bullet -}$ are similarly defined as $|S^-| \times |S|$- and $|S| \times |S^-|$-matrices, respectively. For the $MAP/G/1$ queue, we put $S^+ = \emptyset$, $S = S^-$, $A = A^{--}$, $\boldsymbol{x} = \boldsymbol{x}^-$, $V = -I$, while, for the conventional fluid queue, we put $D(x) = 0$ and $D = 0$.

For an increasing and bounded function f, its LST (Laplace-Stieltjes transform) is denoted by $f^*(\theta)$, and its LT (Laplace transform) is denoted by $f^{**}(\theta)$. That is,

$$f^*(\theta) = \int_0^\infty e^{-\theta t} f(dt), \quad f^{**}(\theta) = \int_0^\infty e^{-\theta t} f(t)dt.$$

Note that $f^*(\theta) = \theta f^{**}(\theta) - f(0-)$.

Since we are concerned with the stationary distribution of $\{(M(t), X(t))\}$, we can assume without loss of generality that $\{Y(t)\}$ has stationary increments. By the well-known result due to Loynes [6], $X(t)$ has a stationary distribution, i.e. $X(t)$ is stable, if

$$E(Y(1)) = \boldsymbol{\pi} V \boldsymbol{e} + \boldsymbol{\pi} \int_0^\infty x D(dx) \boldsymbol{e} < 0. \tag{2.3}$$

We assume that $E(Y(1)) < 0$ throughout this paper, and we denote $(X(t), M(t))$ in the steady state by (X, M).

We calculate the LST of stationary joint distribution $P(X \leq x, M = i)$. For $i \in S$ and $t \geq 0$, define $|S|$-dimensional vector $\boldsymbol{F}(x)$ as

$$[\boldsymbol{F}(x)]_i = \lim_{t\to\infty} P(X(t) \leq x, M(t) = i) = P(X \leq x, M = i).$$

Proposition 2.1 For $\theta > 0$,

$$-\theta \boldsymbol{F}^*(\theta)V + \theta \boldsymbol{F}(0)V = -\boldsymbol{F}^*(\theta)(C + D^*(\theta)). \tag{2.4}$$

Hence, $\boldsymbol{F}^*(\theta)$ is given by

$$\boldsymbol{F}^*(\theta) = \theta \boldsymbol{F}(0)V\left(\theta V - (C + D^*(\theta))\right)^{-1}, \quad (\theta > 0).$$

This proposition is proved in Appendix. Note that $[\boldsymbol{F}(0)]_i = 0$ for $i \in S^+$ and $[\boldsymbol{F}(0)]_i = P(X = 0, M = i)$ is unknown yet for $i \in S^-$. This will be considered in Section 5.

In Section 3, 4 and 5, we assume that function v only takes either 1 or -1. This is sufficient for our analysis since we can always reduce the general case to this simpler case by a random time change. In fact, let

$$\tilde{C} = (V_{abs})^{-1} C, \quad \tilde{D}(x) = (V_{abs})^{-1} D(x), \quad \tilde{v}(i) = \begin{cases} 1 & (v(i) > 0) \\ -1 & (v(i) < 0) \end{cases},$$

and consider the joint process $\{(\tilde{X}(t), \tilde{M}(t))\}$ defined by these matrices and the rate function. Let $\tilde{\boldsymbol{\pi}}$ be the stationary distribution of $\tilde{C} + \tilde{D}$. Since a sojourn time in state $i \in S$ is unchanged in distribution by this modification, the buffer content process is stochastically unchanged as well. Since $\tilde{\boldsymbol{\pi}}(V_{abs})^{-1}(C + D) = \tilde{\boldsymbol{\pi}}(\tilde{C} + \tilde{D}) = \mathbf{0}$, $\tilde{\boldsymbol{\pi}}$ is proportional to $\boldsymbol{\pi} V_{abs}$. In a similar way, we get

$$\begin{aligned} [\boldsymbol{F}(0)]_i &= P(X = 0, M = i) = \frac{P(\tilde{X} = 0, \tilde{M} = i)/|v(i)|}{\sum_{j\in S} P(\tilde{M} = j)/|v(j)|} \\ &= \begin{cases} \dfrac{1}{\tilde{\boldsymbol{\pi}}\,(V_{abs})^{-1}\boldsymbol{e}} \left[\tilde{\boldsymbol{F}}^-(0)\left(V_{abs}^-\right)^{-1}\right]_i & (i \in S^-) \\ 0 & (i \in S^+) \end{cases}, \end{aligned} \tag{2.5}$$

where $[\tilde{\boldsymbol{F}}(0)]_i = P(\tilde{X}(0) = 0, \tilde{M}(0) = i)$. Thus, our problem is reduced to compute $\tilde{\boldsymbol{F}}^-(0)$.

3. The First Passage Time to The Empty State

As we discussed in Section 2, we could assume that $v(S) = \{-1, 1\}$. This assumption will be used in Sections 3, 4 and 5. As we shall see, the first passage time to the empty state is a key to get the empty probabilities jointly with the background states. We consider the LST of the first passage time to the empty state in this section.

Let τ be the first passage time to the empty state measured from time 0, i.e.,

$$\tau = \inf\{u > 0 | X(u) = 0\}.$$

Note that, if $X(0) = 0$ and $M(0) \in S^-$, then $\tau = 0$. Define $|S| \times |S^-|$-matrix $H(t|x)$ as

$$[H(t|x)]_{ij} = P(\tau \leq t, M(\tau) = j | X(0) = x, M(0) = i), \quad (j \in S^-, i \in S).$$

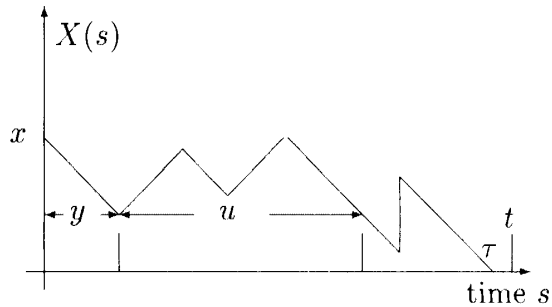

Figure 1: No jump at time y for H^{--}

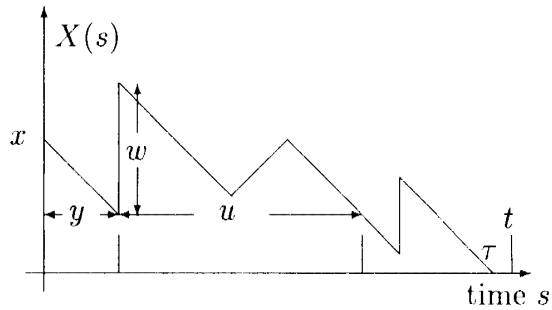

Figure 2: A jump at time y for H^{--}

Clearly, we have $H^{+-}(0|0) = 0^{+-}$ and $H^{--}(0|0) = I^{--}$. For convenience, we introduce the following matrix functions.

$$\begin{aligned}
\Phi^{--}(u) &= C^{-+}H^{+-}(u|0) + \int_0^\infty D^{-\bullet}(dw)H^{\bullet -}(u|w) - \int_0^u D^{--}(dw)H^{--}(w|w), \\
\Phi^{+-}(u) &= C^{+-}1(u=0) + \int_0^\infty D^{+\bullet}(dw)H^{\bullet -}(u|w) - \int_0^u D^{+-}(dw)H^{--}(w|w).
\end{aligned}$$

Taking the LST of these formulas, we have

$$\Phi^{--*}(\theta) = C^{-+}H^{+-*}(\theta|0) + \int_0^\infty D^{-\bullet}(dw)H^{\bullet -*}(\theta|w), \tag{3.1}$$

$$\Phi^{+-*}(\theta) = C^{+-} + \int_0^\infty D^{+\bullet}(dw)H^{\bullet -*}(\theta|w). \tag{3.2}$$

Lemma 3.1 The following equations hold for $\theta > 0$ and $x \geq 0$.

$$H^{--*}(\theta|x) = e^{-(\theta I^{--}-C^{--})x}\left\{I^{--} + \int_0^x e^{(\theta I^{--}-C^{--})y}\Phi^{--*}(\theta)H^{--*}(\theta|y)dy\right\}, \tag{3.3}$$

$$H^{+-*}(\theta|x) = e^{(\theta I^{++}-C^{++})x}\int_x^\infty e^{-(\theta I^{++}-C^{++})y}\Phi^{+-*}(\theta)H^{--*}(\theta|y)dy. \tag{3.4}$$

Proof. To derive (3.3), we first observe the following fact. Given $X(0) = x > 0$, there is no possibility to attain the empty state in time interval $[0, x)$. Given $X(0) = 0$, if $M(0) \in S^-$ then $\tau = 0 \leq t$. Note that the (i, j)th element of matrix $e^{C^{--}x}$ is a conditional joint probability that, in time interval $(0, x]$, the buffer content process has no jumps and the background process stays in S^- with $M(x) = j$, given $M(0) = i$. If either the buffer content process has a jump or the background process enters S^+ in time interval $(0, x]$, we decompose time interval $[0, \tau]$ into three parts. See Figures 1 and 2. Let y be the first time when the

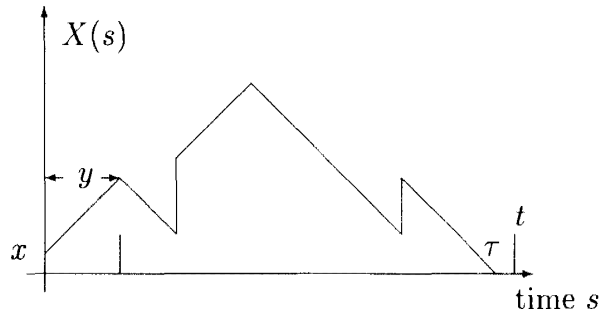

Figure 3: No jump at time y for H^{+-}

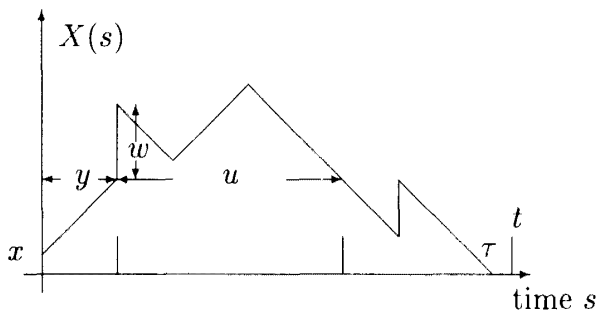

Figure 4: A jump at time y for H^{+-}

buffer content increases due to either a jump or a state change of the background process into S^+. Let u be the first return time for the buffer content being to the level at time $y-$ measured from time y. Thus, time interval $(0, \tau]$ is partitioned into three sections, the first section is $[0, y)$, the second section is $[y, y+u)$, and the remaining section is $[y+u, \tau]$. These observations yield

$$\begin{aligned}
H^{--}(t|x) &= 1(x=0)I^{--} + 1(0 < x \le t)\{e^{C^{--}x} \\
&\quad + \int_0^x e^{C^{--}y} dy C^{-+} \int_0^{t-x} H^{+-}(du|0) H^{--}(t-y-u|x-y) \\
&\quad + \int_0^x e^{C^{--}y} dy \int_0^{t-x} D^{-\bullet}(dw) \int_w^{t-x} H^{\bullet -}(du|w) H^{--}(t-y-u|x-y)\} \\
&\quad \text{(Changing variable } y \text{ to } x-y) \\
&= 1(x=0)I^{--} + 1(0 < x \le t)\{e^{C^{--}x} \\
&\quad + \int_0^x e^{C^{--}(x-y)} dy C^{-+} \int_0^{t-x} H^{+-}(du|0) H^{--}(t-(x-y)-u|y) \\
&\quad + \int_0^x e^{C^{--}(x-y)} dy \int_0^{t-x} D^{-\bullet}(dw) \int_w^{t-x} H^{\bullet -}(du|w) H^{--}(t-(x-y)-u|y)\} \\
&= 1(x=0)I^{--} + 1(0 < x \le t)\left\{e^{C^{--}x}\right. \\
&\quad \left. + \int_0^x e^{C^{--}(x-y)} dy \int_0^{t-x} \Phi^{--}(du) H^{--}(t-(x-y)-u|y)\right\},
\end{aligned} \tag{3.5}$$

where $1(\cdot)$ denotes the indicator function of statement '$\cdot$'. For $x > 0$, multiplying $e^{-\theta t}$ to both sides of (3.5) and integrating from 0 to ∞, we have

$$H^{--**}(\theta|x) = \int_x^\infty e^{-\theta t} e^{C^{--}x} dt$$

$$
\begin{aligned}
&+\int_x^\infty e^{-\theta t}dt\int_0^x e^{C^{--}(x-y)}dy\int_0^{t-x}\Phi^{--}(du|0)H^{--}(t-(x-y)-u)|y)\\
&=\frac{1}{\theta}e^{-(\theta I^{--}-C^{--})x}\\
&\quad+\int_{y=0}^x\int_{t=x}^\infty\int_{u=0}^{t-x}e^{-(\theta I^{--}-C^{--})(x-y)}e^{-\theta u}\Phi^{--}(du|0)\\
&\qquad\qquad\times e^{-\theta(t-(x-y)-u)}H^{--}(t-(x-y)-u|y)dydt\\
&=\frac{1}{\theta}e^{-(\theta I^{--}-C^{--})x}\\
&\quad+\int_{y=0}^x\int_{u=0}^\infty\int_{t=x+u}^\infty e^{-(\theta I^{--}-C^{--})(x-y)}e^{-\theta u}\Phi^{--}(du|0)\\
&\qquad\qquad\times e^{-\theta(t-(x-y)-u))}H^{--}(t-(x-y)-u|y)dydt\\
&=\frac{1}{\theta}e^{-(\theta I^{--}-C^{--})x}+\int_{y=0}^x e^{-(\theta I^{--}-C^{--})(x-y)}\Phi^{--*}(\theta|0)H^{--**}(\theta|y)dy\\
&=\frac{1}{\theta}e^{-(\theta I^{--}-C^{--})x}\left\{I^{--}+\int_{y=0}^x e^{(\theta I^{--}-C^{--})y}\Phi^{--*}(\theta|0)H^{--*}(\theta|y)dy\right\}.
\end{aligned}
$$

Multiplying θ to both sides of it, applying $H^{--*}(\theta|x)=\theta H^{--**}(\theta|x)$ to it, we obtain (3.3) for $x>0$. This is also valid for $x=0$ since

$$H^{--**}(\theta|0)=\frac{1}{\theta}I^{--}.$$

For H^{+-}, we decompose time interval $[0,\tau]$ into two or three parts. Let y be the first time when either the buffer content has a jump or the background state change into S^-. In the latter case, the time interval is divided into intervals $[0,y)$ and $[y,\tau]$(see Figure 3). In the former case, it is divided into intervals $[0,y)$, $[y,y+u)$ and $[y+u,\tau]$, where u is the first return time to level $X(y-)$ measured from time y (see Figure 4). From these observations, we have

$$
\begin{aligned}
H^{+-}(t|x)&=1[t\geq x\geq 0]\left\{\int_0^{(t-x)/2}e^{C^{++}y}dyC^{+-}\int_0^{t-x-2y}H^{--}(du|0)H^{--}(t-y|x+y)\right.\\
&\left.+\int_0^{(t-x)/2}e^{C^{++}y}dy\int_0^{t-x-2y}D^{+\bullet}(dw)\int_w^{t-x-2y}H^{\bullet-}(du|w)H^{--}(t-y-u|x+y)\right\}\\
&\quad\text{(Changing variable } y \text{ to } y-x)\\
&=1[t\geq x\geq 0]\left\{\int_x^{(t+x)/2}e^{C^{++}(y-x)}dyC^{+-}\int_0^{t+x-2y}H^{--}(du|0)H^{--}(t-(y-x)|y)\right.\\
&+\int_x^{(t+x)/2}e^{C^{++}(y-x)}dy\int_0^{t+x-2y}D^{+\bullet}(dw)\\
&\qquad\qquad\left.\times\int_w^{t+x-2y}H^{\bullet-}(du|w)H^{--}(t-(y-x)-u|y)\right\}\\
&=1[t\geq x\geq 0]\left\{\int_x^{(t+x)/2}e^{C^{++}(y-x)}dy\int_0^{t+x-2y}\Phi^{+-}(du)H^{--}(t-(y-x)-u|y)\right\}.
\end{aligned}
$$

Then its LT is calculated as

$$
\begin{aligned}
&H^{+-**}(\theta|x)\\
&\quad=\int_x^\infty e^{-\theta t}dt\int_x^{(t+x)/2}e^{C^{++}(y-x)}dy\int_0^{t+x-2y}\Phi^{+-}(du)H^{--}(t-(y-x)-u|y)
\end{aligned}
$$

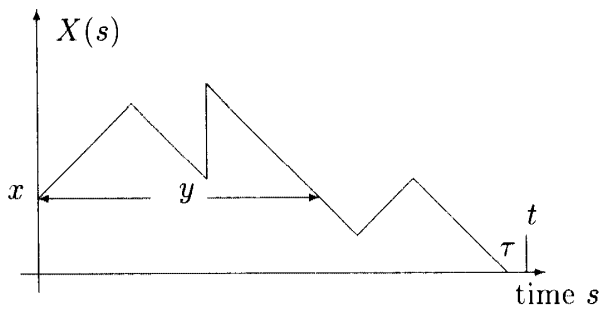

Figure 5: The another observation for H^{+-}

$$
\begin{aligned}
&= \int_{t=0}^{\infty}\int_{y=x}^{(t+x)/2} e^{-(\theta I^{++}-C^{++})(y-x)} \int_{u=0}^{t+x-2y} e^{-\theta u}\Phi^{+-}(du) \\
&\qquad\qquad \times e^{-\theta(t-(y-x)-u)} H^{--}(t-(y-x)-u|y)dydt \\
&= \int_{y=x}^{\infty}\int_{t=2y-x}^{\infty} e^{-(\theta I^{++}-C^{++})(y-x)} \int_{u=0}^{t+x-2y} e^{-\theta u}\Phi^{+-}(du) \\
&\qquad\qquad \times e^{-\theta(t-(y-x)-u)} H^{--}(t-(y-x)-u|y)dydt \\
&= \int_{y=x}^{\infty}\int_{u=0}^{\infty}\int_{t=u+2y-x}^{\infty} e^{-(\theta I^{++}-C^{++})(y-x)} e^{-\theta u}\Phi^{+-}(du) \\
&\qquad\qquad \times e^{-\theta(t-(y-x)-u)} H^{--}(t-(y-x)-u|y)dydt \\
&= \int_{x}^{\infty} e^{-(\theta I^{++}-C^{++})(y-x)}\Phi^{+-*}(\theta)H^{--**}(\theta|y)dy.
\end{aligned}
$$

Multiplying θ to both sides of it, we obtain (3.4). □

From Lemma 3.1, we shall derive a matrix exponential form for the LST of H. To this end, define

$$Q^{--}(\theta) = C^{--} - \theta I^{--} + \Phi^{--*}(\theta).$$

Lemma 3.2

$$
\begin{aligned}
H^{--*}(\theta|x) &= e^{Q^{--}(\theta)x}, && (3.6)\\
H^{+-*}(\theta|x) &= H^{+-*}(\theta|0)e^{Q^{--}(\theta)x}. && (3.7)
\end{aligned}
$$

Proof. Differentiating both sides of (3.3) with respect to x yields

$$
\begin{aligned}
\frac{\partial}{\partial x}H^{--*}(\theta|x) &= -(\theta I^{--} - C^{--})H^{--*}(\theta|x) + \Phi^{--*}(\theta)H^{--*}(\theta|x) \\
&= [C^{--} - \theta I^{--} + \Phi^{--*}(\theta)]H^{--*}(\theta|x) \\
&= Q^{--}(\theta)H^{--*}(\theta|x).
\end{aligned}
$$

a solution of this differential equation is given by (3.6). For (3.7), we need another observation different from Lemma 3.1. We decompose time interval $[0,\tau]$ into two parts. Let y be the first return time to level x measured from time 0. The first part is interval $[0,y)$ and the second part is interval $[y,\tau]$ (see Figure 5). From these, we have

$$H^{+-}(t|x) = \int_0^{t-x} H^{+-}(dy|0)H^{--}(t-y|x). \qquad (3.8)$$

Multiplying $e^{-\theta t}$ to both sides of it and integrating it from 0 to ∞ with respect to t, we get

$$H^{+-**}(\theta|x) = H^{+-*}(\theta|0)H^{--**}(\theta|x).$$

From this equation and (3.6), we obtain (3.7). □

Matrices $Q^{--}(\theta)$ and $H^{+-*}(\theta|0)$ are unknown yet. We need these matrices at $\theta = 0+$ to calculate the empty probabilities $\boldsymbol{F}(0)$ later. We so define matrices Q^{--} and R^{+-} by

$$Q^{--} = \lim_{\theta\to 0+} Q^{--}(\theta), \quad R^{+-} = \lim_{\theta\to 0+} H^{+-*}(\theta|0).$$

To compute these matrices, we first note the following facts.

Lemma 3.3 For $\theta > 0$, the following equations hold.

$$\begin{aligned} Q^{--}(\theta) &= C^{--} - \theta I^{--} + C^{-+}H^{+-*}(\theta|0) \\ &\quad + \int_0^\infty D^{-+}(dw)H^{+-*}(\theta|0)e^{Q^{--}(\theta)w} + \int_0^\infty D^{--}(dw)e^{Q^{--}(\theta)w}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} H^{+-*}(\theta|0)Q^{--}(\theta) &= (\theta I^{++} - C^{++})H^{+-*}(\theta|0) - C^{+-} \\ &\quad - \int_0^\infty D^{++}(dw)H^{+-*}(\theta|0)e^{Q^{--}(\theta)w} - \int_0^\infty D^{+-}(dw)e^{Q^{--}(\theta)w}. \end{aligned} \tag{3.10}$$

Proof. From (3.1) and Lemma 3.2, we have

$$\begin{aligned} &\Phi^{--*}(\theta) \\ &= C^{-+}H^{+-*}(\theta|0) + \int_0^\infty D^{-+}(dw)H^{+-*}(\theta|0)e^{Q^{--}(\theta)w} + \int_0^\infty D^{--}(dw)e^{Q^{--}(\theta)w}. \end{aligned}$$

Substituting this equation into the definition of $Q^{--}(\theta)$, we obtain (3.9).

On the other hand, differentiating both sides of (3.4) and (3.7), we have

$$\begin{aligned} \frac{\partial}{\partial x}H^{+-*}(\theta|x) &= (\theta I^{++} - C^{++})H^{+-*}(\theta|x) - \Phi^{+-*}(\theta)H^{--*}(\theta|x) \\ &= \{(\theta I^{++} - C^{++})H^{+-*}(\theta|0) - \Phi^{+-*}(\theta)\}e^{Q^{--}(\theta)x}, \end{aligned} \tag{3.11}$$

$$\frac{\partial}{\partial x}H^{+-*}(\theta|x) = H^{+-*}(\theta|0)Q^{--}(\theta)e^{Q^{--}(\theta)x}, \tag{3.12}$$

respectively. (3.2) and Lemma 3.2 imply that,

$$\Phi^{+-*}(\theta) = C^{+-} + \int_0^\infty D^{++}(dw)H^{+-*}(\theta|0)e^{Q^{--}(\theta)w} + \int_0^\infty D^{+-}(dw)e^{Q^{--}(\theta)w}.$$

From this equation, (3.11) and (3.12), we obtain (3.10). □

Taking the limit as $\theta \to 0+$ for (3.9) and (3.10), we have for any η,

$$\begin{aligned} Q^{--} &= C^{-+}R^{+-} + C^{--} \\ &\quad + \int_0^\infty D^{-+}(dw)R^{+-}e^{Q^{--}w} + \int_0^\infty D^{--}(dw)e^{Q^{--}w}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} R^{+-}(\eta I^{--} - Q^{--}) &= (\eta I^{++} + C^{++})R^{+-} + C^{+-} \\ &\quad + \int_0^\infty D^{++}(dw)R^{+-}e^{Q^{--}w} + \int_0^\infty D^{+-}(dw)e^{Q^{--}w}. \end{aligned} \tag{3.14}$$

Remark 3.1 1. Equations (3.13) and (3.14) can be written as:

$$\begin{pmatrix} I^{++} & 0^{+-} \\ 0^{-+} & -I^{--} \end{pmatrix} C \begin{pmatrix} R^{+-} \\ I^{--} \end{pmatrix} + \begin{pmatrix} I^{++} & 0^{+-} \\ 0^{-+} & -I^{--} \end{pmatrix} \int_{0+}^{\infty} D(dw) \begin{pmatrix} R^{+-} \\ I^{-} \end{pmatrix} e^{Q^{--}w}$$
$$= -\begin{pmatrix} R^{+-} \\ I^{--} \end{pmatrix} Q^{--}. \tag{3.15}$$

Recall that we have assumed the case of $v(S) = \{-1, 1\}$. We now rewrite the above equation by the original notation in Section 2. Let,

$$\tilde{V} = \tilde{V}^{-1} = \begin{pmatrix} I^{++} & 0^{+-} \\ 0^{-+} & -I^{--} \end{pmatrix},$$

and change C and $D(x)$ to $\tilde{C}$ and $\tilde{D}(x)$ in (3.15). Since $\tilde{C} = V_{abs}C$, $\tilde{D}(x) = V_{abs}D(x)$ and $V = V_{abs}\tilde{V}$, we have the following expression.

$$V^{-1}C \begin{pmatrix} R^{+-} \\ I^{--} \end{pmatrix} + V^{-1} \int_{0+}^{\infty} D(dw) \begin{pmatrix} R^{+-} \\ I^{--} \end{pmatrix} e^{Q^{--}w} = -\begin{pmatrix} R^{+-} \\ I^{--} \end{pmatrix} Q^{--}.$$

This generalizes a part of the Wiener-Hopf factorization of a Markov chain in Rogers [9], which is the case that $D = 0$, i.e,

$$V^{-1}C \begin{pmatrix} R^{+-} \\ I^{--} \end{pmatrix} = -\begin{pmatrix} R^{+-} \\ I^{--} \end{pmatrix} Q^{--}.$$

2. When $D = 0$, (3.13) and (3.14) are similar to the corresponding equations in Asmussen [3]. However, they are not the same. The reason is that the first passage time of $\{Y(t)\}$ to level 0 starting with $Y(0) = 0$ is considered for $v(M(0)) < 0$ in Asmussen [3], while $v(M(0)) > 0$ in this paper, respectively.
3. In the case of the $MAP/G/1$ queue, since $v(i) = -1$ for all $i \in S$, we can omit (3.14) and terms concerning S^+. From (3.13), we have

$$Q^{--} = C^{--} + \int_{0}^{\infty} D^{--}(dw) e^{Q^{--}w}.$$

This is the equation obtained in Hasegawa and Takine [12]. A similar equation is also obtained in Asmussen [2].

Definition 3.1 When a matrix has negative diagonal elements and nonnegative non-diagonal elements, the matrix is called an *ML-matrix* (see Seneta [10]). If ML-matrix A satisfies $A\boldsymbol{e} \leq \mathbf{0}$, then matrix A is called a *subrate matrix.* In particular, when $A\boldsymbol{e} = \mathbf{0}$, A is called a *rate matrix.*

Lemma 3.4 Q^{--} is a rate matrix, i.e.

$$Q^{--}\boldsymbol{e}^{-} = \mathbf{0}^{-}, \tag{3.16}$$

where $\boldsymbol{e}^{-}$ is the $|S^{-}|$-dimensional column vector all of whose elements are 1, and $\mathbf{0}^{-}$ is the $|S^{-}|$-dimensional column zero vector. Furthermore, R^{+-} is a nonnegative matrix.

Proof. By the definition of R^{+-}, R^{+-} is obviously a nonnegative matrix. Since $Y(t)$ goes to $-\infty$ as t tends to $+\infty$ by the stability condition (2.3), τ is finite with probability one. Thus, H is the proper joint distribution of τ and $M(\tau)$. Hence,

$$\lim_{\theta\to 0+} H^*(\theta|w)\boldsymbol{e}^- = \int_0^\infty H(dt|w)\boldsymbol{e}^- = \boldsymbol{e}.$$

Thus, from (2.1) and (3.13), we have

$$\begin{aligned}\lim_{\theta\to 0+} Q^{--}(\theta)\boldsymbol{e}^- &= C^{--}\boldsymbol{e}^- + C^{-+}\boldsymbol{e}^+ + \int_0^\infty D^{-\bullet}(dw) \lim_{\theta\to 0+} H^{\bullet -*}(\theta|w)\boldsymbol{e}^- \\ &= C^{-\bullet}\boldsymbol{e} + \int_0^\infty D^{-\bullet}(dw)\boldsymbol{e} = \boldsymbol{0}^-.\end{aligned}$$

Since C^{--} is a subrate matrix and other matrices are nonnegative in the right side of (3.13), Q^{--} also is the ML-matrix. These imply that Q^{--} is a rate matrix. □

4. Iteration Algorithm and Its Verification

To compute matrices Q^{--} and R^{+-}, we use iteration. We choose an η such that $\eta > \max\{|C_{ii}|; i \in S\}$. Define matrices Q_n^{--} and R_n^{+-} for $n \geq 0$ by

$$\begin{aligned}&R_0^{+-} = 0^{+-}, \quad Q_0^{--} = C^{--}, \\ &Q_{n+1}^{--} = C^{--} + C^{-+}R_n^{+-} \\ &\qquad + \int_0^\infty D^{-+}(dw)R_n^{+-}e^{Q_n^{--}w} + \int_0^\infty D^{--}(dw)e^{Q_n^{--}w}, \qquad (4.1)\\ &R_{n+1}^{+-} = \Big\{C^{+-} + (\eta I^{++} + C^{++})R_n^{+-} + \int_0^\infty D^{++}(dw)R_n^{+-}e^{Q_{n+1}^{--}w} \\ &\qquad + \int_0^\infty D^{+-}(dw)e^{Q_{n+1}^{--}w}\Big\}(\eta I^{--} - Q_{n+1}^{--})^{-1}. \qquad (4.2)\end{aligned}$$

Note that (i) $\eta I^{++} + C^{++}$ is nonnegative, (ii) it can be shown by induction that Q_n^{--} and R_n^{+-} are subrate matrix and substochastic matrix, respectively, and (iii) it is well known from Perron-Frobenius theorem (for example, see Chapter 2 of Seneta [10]) that $\eta I^{--} - Q_n^{--}$ is invertible and the inverse matrix is nonnegative. Properties (i), (ii), and (iii) are used later.

The next theorem is a key for our computation, which verifies that the iterations (4.1) and (4.2) indeed converge to the right values.

Theorem 4.1 For each $\eta > \max\{|C_{ii}|; i \in S\}$, Q_n^{--} and R_n^{+-} are increasing for n and R_n^{+-} is nonnegative, and

$$\lim_{n\to\infty} Q_n^{--} = Q^{--}, \quad \lim_{n\to\infty} R_n^{+-} = R^{+-}. \qquad (4.3)$$

Proof. We show the monotonicity of Q_n^{--} and R_n^{+-} by induction. Since matrices D and $e^{C^{--}}$ are nonnegative, we have, from (4.1),

$$Q_1^{--} = C^{--} + \int_0^\infty D^{--}(dw)e^{C^{--}w} \geq C^{--} = Q_0^{--}.$$

Obviously Q_1^{--} is an ML-matrix. So, $e^{Q_1^{--}}$ and $(\eta I^{--} - Q_1^{--})^{-1}$ are nonnegative. We then have, from (4.2),

$$R_1^{+-} = \left(C^{+-} + \int_0^\infty D^{+-}(dw)e^{Q_1^{--}w}\right)(\eta I^{--} - Q_1^{--})^{-1} \geq 0^{--}.$$

We now assume the monotonicity of Q_k^{--} and R_k^{+-} for $k = 0, 1, ..., n$. Then, we get

$$\begin{aligned}
&Q_{n+1}^{--} - Q_n^{--} \\
&= C^{-+}(R_n^{+-} - R_{n-1}^{+-}) + \int_0^\infty D^{++}(dw)(R_n^{+-}e^{Q_n^{--}w} - R_{n-1}^{+-}e^{Q_{n-1}^{--}w}) \\
&\qquad + \int_0^\infty D^{--}(dw)(e^{Q_n^{--}w} - e^{Q_{n-1}^{--}w}) \\
&\geq \int_0^\infty D^{++}(dw)(R_n^{+-}e^{Q_n^{--}w} - R_{n-1}^{+-}e^{Q_{n-1}^{--}w}) \\
&\geq \int_0^\infty D^{++}(dw)R_{n-1}^{+-}(e^{Q_n^{--}w} - e^{Q_{n-1}^{--}w}) \geq 0^{--}.
\end{aligned}$$

Since $Q_{n+1}^{--} \geq Q_n^{--}$, we obtain

$$(\eta I^{--} - Q_n^{--})^{-1} \leq (\eta I^{--} - Q_{n+1}^{--})^{-1}.$$

On the other hand, $(\eta I^{++} + C^{++})$ is a nonnegative matrix. It follows from these facts that

$$\begin{aligned}
R_{n+1}^{+-} &= \Big(C^{+-} + (\eta I^{++} + C^{++})R_n^{+-} + \int_0^\infty D^{++}(dw)R_n^{+-}e^{Q_{n+1}^{--}w} \\
&\quad + \int_0^\infty D^{+-}(dw)e^{Q_{n+1}^{--}w}\Big)(\eta I^{--} - Q_{n+1}^{--})^{-1} \\
&\geq \Big(C^{+-} + (\eta I^{++} + C^{++})R_{n-1}^{+-} + \int_0^\infty D^{++}(dw)R_{n-1}^{+-}e^{Q_n^{--}w} \\
&\quad + \int_0^\infty D^{+-}(dw)e^{Q_n^{--}w}\Big)(\eta I^{--} - Q_n^{--})^{-1} = R_n^{+-}.
\end{aligned}$$

Thus, we obtain the monotonicity of Q_n^{--} and R_n^{+-}. By the monotonicity of R_n^{+-} and definition $R_0^{+-} = 0^{+-}$, we also have the nonnegativity of R_n^{+-}.

We next show that Q_n^{--} and R_n^{+-} are bounded by Q^{--} and R^{+-}, respectively. Since $Q_0^{--} = C^{--}$ and $Q^{--} = C^{--} +$ (nonnegative terms), we have $Q_0^{--} \leq Q^{--}$. The facts that $R_0^{+-} = 0^{+-}$ and R^{+-} is nonnegative matrix yield $R_0^{+-} \leq R^{+-}$. Hence, we have

$$\begin{aligned}
Q_1^{--} &= C^{--} + C^{-+}R_0^{+-} + \int_0^\infty D^{-+}(dw)R_0^{+-}e^{Q_0^{--}w} + \int_0^\infty D^{--}(dw)e^{Q_0^{--}w} \\
&\leq C^{--} + C^{-+}R^{+-} + \int_0^\infty D^{-+}(dw)R^{+-}e^{Q^{--}w} + \int_0^\infty D^{--}(dw)e^{Q^{--}w} = Q^{--}.
\end{aligned}$$

In a similar way, we have $R_1^{+-} \leq R^{+-}$. If $Q_n^{--} \leq Q^{--}$ and $R_n^{+-} \leq R^{+-}$, then replacing subscripts 0 and 1 by n and $n+1$ in above equations, respectively, we also have $Q_{n+1}^{--} \leq Q^{--}$ and $R_{n+1}^{+-} \leq R^{+-}$. Hence, the inequalities hold true for all n. Thus we have

$$\lim_{n\to\infty} R_n^{+-} \leq R^{+-}, \quad \lim_{n\to\infty} Q_n^{--} \leq Q^{--}. \tag{4.4}$$

To complete the proof, we show that

$$\lim_{n\to\infty} R_n^{+-} \geq R^{+-}, \quad \lim_{n\to\infty} Q_n^{--} \geq Q^{--}. \tag{4.5}$$

Clearly, (4.4) and (4.5) conclude (4.3). To prove (4.5), we use the uniformization of a Markov chain. Let $\{\hat{M}(t)\}$ be a uniformized Markov chain obtained from $\{M(t)\}$ with respect to uniform rate η, let and $\hat{N}$ be a counting process that counts all transition instants of $\{\hat{M}(t)\}$. Define $|S| \times |S^-|$-matrix $\hat{H}_n(t|x)$ by

$$[\hat{H}_n(t|x)]_{ij} = P(\tau \leq t, M(\tau) = j, \hat{N}(\tau) \leq n | M(0) = i, X(0) = x).$$

Obviously, we have

$$\hat{H}_n(t|x) \le H(t|x), \qquad \lim_{n\to\infty} \hat{H}_n(t|x) = H(t|x),$$
$$\hat{H}_n^*(\theta|x) \le H^*(\theta|x), \qquad \lim_{n\to\infty} \hat{H}_n^*(\theta|x) = H^*(\theta|x).$$

Define $|S^+| \times |S^-|$-matrix $\hat{R}_n^{+-}$ and $|S^-| \times |S^-|$-matrix $\hat{R}_n^{--}(x)$ by

$$\hat{R}_n^{+-} = \lim_{\theta\to 0+} \hat{H}_n^{+-*}(\theta|0), \quad \hat{R}_n^{--}(x) = \lim_{\theta\to 0+} \hat{H}_n^{--*}(\theta|x),$$

respectively. We then have

$$\lim_{n\to\infty} \hat{R}_n^{+-} = R^{+-}, \quad \lim_{n\to\infty} \hat{R}_n^{--}(x) = e^{Q^{--}x}.$$

We shall prove that

$$\hat{R}_n^{+-} \le R_n^{+-}, \quad \hat{R}_n^{--}(x) \le e^{Q_n^{--}x}, \tag{4.6}$$

for all $n \ge 0$ and all $x \ge 0$. If (4.6) holds, then

$$R^{+-} \le R_\infty^{+-}, \quad e^{Q^{--}x} \le e^{Q_\infty^{--}x}.$$

These inequalities imply

$$\begin{aligned} Q_\infty^{--} - Q^{--} &= C^{-+}(R_\infty^{+-} - R^{+-}) + \int_0^\infty D^{--}(dw)\left(e^{Q_\infty^{--}w} - e^{Q^{--}w}\right) \\ &\quad + \int_0^\infty D^{-+}(dw)\left(R_\infty^{+-}e^{Q_\infty^{--}w} - R^{+-}e^{Q^{--}w}\right) \ge 0^{--}. \end{aligned}$$

We have (4.5). In what follows, we prove (4.6) by induction on n. Since

$$\hat{R}_0^{+-} = 0^{+-}, \quad R_0^{+-} = 0^{+-}, \quad \hat{R}_0^{--}(x) = \exp\{\mathrm{diag}(C_{ii}; i \in S^-)x\}, \quad e^{Q_0^{--}x} = e^{C^{--}x},$$

from the definitions for $n = 0$, both inequalities of (4.6) hold for $n = 0$. We assume that these inequalities hold up to n. We first consider $R_{n+1}^{--}(x)$. Recall that we partition off τ into three sections for $H^{--}(t|x)$ in Section 3. Similarly to $H^{--}(t|x)$, we partition off τ into three sections for $\hat{H}_{n+1}^{--}(t|x)$. Note that the event for $\hat{H}_{n+1}^{--}(t|x)$ has at most $n+1$ transitions of the background process. Suppose that $M(t)$ stays in S^- during time interval $(0, x]$ without jumps for the buffer content process. In this case, there are at most $n+1$ transitions of the background process. Hence, the corresponding transition probability matrix is not greater than $e^{C^{--}x}$. If this is not the case, there are n_1 and n_2 satisfying $0 \le n_1 \le n$ and $1 \le n_2 \le n$ such that the background process has n_1 transitions in S^- on the first section $[0, y)$, has a transition at time y, has n_2 transitions on the second section $(y, y+u)$, and has at most $n - n_1 - n_2$ transitions on the third section $(y+u, \tau]$. From these observations, we have

$$\hat{H}_{n+1}^{--*}(\theta|x) \le e^{-(\theta I^{--} - C^{--})x}\left\{I^{--} + \int_{y=0}^x e^{(\theta I^{--} - C^{--})y}\hat{\Phi}_n^{--*}(\theta)\hat{H}_n^{--*}(\theta|y)dy\right\}, \tag{4.7}$$

where $\hat{\Phi}_n^{--}(u)$ is defined by

$$\hat{\Phi}_n^{--}(u) = C^{-+}\hat{H}_n^{+-}(u) + \int_0^u D^{-\bullet}(dw)\hat{H}_n^{\bullet -}(u|w) - \int_0^u D^{--}(dw)\hat{H}_n^{--}(w|w).$$

Since we also have

$$\hat{H}_n^{+-}(u|w) \le \int_0^u \hat{H}_n^{+-}(dy|0)\hat{H}_n^{--}(u-y|w),$$

we get

$$\begin{aligned}\hat{\Phi}_n^{--*}(\theta) &\le C^{-+}\hat{H}_n^{+-*}(\theta|0) + \int_{w=0}^{\infty} D^{-+}(dw)\hat{H}_n^{+-*}(\theta|0)\hat{H}_n^{--*}(\theta|w) \\ &\quad + \int_{w=0}^{\infty} D^{--}(dw)\hat{H}_n^{--*}(\theta|w).\end{aligned}$$

Taking the limit of the above equation as $\theta \to 0+$, and using (4.1), we have

$$\begin{aligned}\lim_{\theta\to 0+} \hat{\Phi}_n^{--*}(\theta) &\le C^{-+}R_n^{+-} + \int_{w=0}^{\infty} D^{-+}(dw)R_n^{+-}e^{Q_n^{--}w} + \int_{w=0}^{\infty} D^{--}(dw)e^{Q_n^{--}w} \\ &= Q_{n+1}^{--} - C^{--}. \end{aligned} \tag{4.8}$$

Hence, it follows from (4.7), (4.8) and the monotonicity of Q_n^{--} on n that

$$\begin{aligned}\hat{R}_{n+1}^{--}(x) &\le e^{C^{--}x}\left\{I^{--} + \int_{y=0}^{x} e^{-C^{--}y}\left(Q_{n+1}^{--} - C^{--}\right)e^{Q_n^{--}y}dy\right\} \\ &\le e^{C^{--}x}\left\{I^{--} + \int_{y=0}^{x} e^{-C^{--}y}\left(Q_{n+1}^{--} - C^{--}\right)e^{Q_{n+1}^{--}y}dy\right\} \\ &\le e^{C^{--}x}\left\{I^{--} + \int_{y=0}^{x} e^{-C^{--}y}Q_{n+1}^{--}e^{Q_{n+1}^{--}y}dy - \int_{y=0}^{x} e^{-C^{--}y}C^{--}e^{Q_{n+1}^{--}y}\right\} \\ &\qquad \text{(Integration by parts for the first integral)} \\ &= e^{C^{--}x}\left\{I^{--} + e^{-C^{--}x}e^{Q_{n+1}^{--}x} - I^{--}\right\} = e^{Q_{n+1}^{--}x}.\end{aligned}$$

This proves the second inequality in (4.6). It remains to prove the first inequality in (4.6). For $\hat{H}_{n+1}^{+-}(t|0)$, let y be the first time when the uniformized background process $\{\hat{M}(t)\}$ has a transition. Note that y is exponentially distributed with mean $1/\eta$. Time interval $[0,\tau]$ is partitioned into three sections: $[0,y)$, $[y,y+u)$ and $[y+u,\tau]$, where u is the first return time of the buffer content to level $X(y-)$ measured from time y. Since there is a transition of the uniformized background process at time y, there are at most n transitions of the uniformized background process on each of time intervals $[0,y)$, $[y,y+u)$ and $[y+u,\tau]$. We define $\hat{\Phi}_n^{+-}$ by

$$\hat{\Phi}_n^{+-} = C^{+-} + \int_0^{\infty} D^{+-}(dw)\hat{H}_n^{--*}(0+|w) + \int_0^{\infty} D^{++}(dw)\hat{H}_n^{+-*}(0+|w).$$

It follows from the above observations that,

$$\begin{aligned}\hat{R}_{n+1}^{+-} &\le \int_0^{\infty} \eta e^{-\eta y}\frac{1}{\eta}\hat{\Phi}_n^{+-}\hat{R}_n^{--}(y)dy + \int_0^{\infty} \eta e^{-\eta y}\frac{1}{\eta}(\eta I^{++} + C^{++})\hat{R}_n^{+-}\hat{R}_n^{--}(y)dy. \\ &= (\hat{\Phi}_n^{+-} + (\eta I^{++} + C^{++})\hat{R}_n^{+-})\int_0^{\infty} e^{-\eta y}\hat{R}_n^{--}(y)dy.\end{aligned}$$

From the induction assumption, the monotonicity of Q_n^{--} and the recursion equation (4.2) of R_n^{+-}, we also have

$$\begin{aligned}\hat{\Phi}_n^{+-} &\le C^{+-} + \int_0^{\infty} D^{+-}(dw)e^{Q_n^{--}w} + \int_0^{\infty} D^{++}(dw)R_n^{+-}e^{Q_n^{--}w} \\ &\le C^{+-} + \int_0^{\infty} D^{+-}(dw)e^{Q_{n+1}^{--}w} + \int_0^{\infty} D^{++}(dw)R_n^{+-}e^{Q_{n+1}^{--}w} \\ &= R_{n+1}^{+-}(\eta I^{--} - Q_{n+1}^{--}) - (\eta I^{++} + C^{++})R_n^{+-}.\end{aligned}$$

Hence, we get

$$\begin{aligned}\hat{R}_{n+1}^{+-} &\leq (\hat{\Phi}_n^{+-} + (\eta I^{++} + C^{++})R_n^{+-}) \int_0^\infty e^{-\eta y} e^{Q_n^{--} y} dy \\ &= (\hat{\Phi}_n^{+-} + (\eta I^{++} + C^{++})R_n^{+-})(\eta I^{--} - Q_n^{--})^{-1} \\ &\leq R_{n+1}^{+-}(\eta I^{--} - Q_{n+1}^{--})(\eta I^{--} - Q_{n+1}^{--})^{-1} = R_{n+1}^{+-}.\end{aligned}$$

Thus we get (4.6). This completes the proof. □

Remark 4.1 The iteration (4.2) of R_n^{+-} includes a computation of the inverse at each step n. When $|S^-|$ is very large, this may be inefficient. We can make another recursion of R_n^{+-} by

$$\begin{aligned}R_{n+1}^{+-} &= (-C^{++})^{-1}(C^{+-} + R_n^{+-}Q_{n+1}^{--} + \int_0^\infty D^{++}(dw)R_n^{+-}e^{Q_{n+1}^{--}w} \\ &\quad + \int_0^\infty D^{+-}(dw)e^{Q_{n+1}^{--}w}).\end{aligned}$$

In this algorithm, we only need one inverse operation for C^{++} independent of n. Hence, one might think that this recursion is better than the former recursion for a large $|S^-|$. However, our numerical experience shows that this is not always the case (see Table 6.1 in Section 6). Another problem for this iteration is that it seems hard to verify its convergence to the right values Q^{--} and R^{+-}. In particular, we could not prove that $R_n^{+-} \geq R^{+-}$ for all $n \geq 1$.

5. The Empty Probabilities

To get the empty probabilities, we first consider busy cycles which is the time interval between successive instants when the buffer content attains 0. We then show that the empty probabilities are given by a stationary vector of Q^{--}. Let χ be a busy cycle length. We define $|S^-| \times |S^-|$-matrix $L(t)$ by

$$[L(t)]_{ij} = P(\chi \leq t, M(\chi) = j | M(0) = i), \quad (i, j \in S^-).$$

Conditioning on the first time when the buffer content process leaves the empty state, we have

$$\begin{aligned}L(t) &= \int_0^t e^{C^{--}y} \left\{ C^{-+}H^{+-}(t-y|0) + \int_0^{t-y} D^{-\bullet}(dw)H^{\bullet -}(t-y|w) \right\} dy \\ &= \int_0^t e^{C^{--}(t-y)} \left\{ \Phi^{--}(y) + \int_0^y D^{--}(dw)H^{--}(w|w) \right\} dy.\end{aligned}$$

Hence, we obtain the relationship

$$\begin{aligned}L^{**}(\theta) &= \int_{t=0}^\infty \int_{y=0}^t e^{-\theta t} e^{C^{--}(t-y)} \left\{ \Phi^{--}(y) + \int_0^y D^{--}(dw)H^{--}(w|w) \right\} dy\, dt \\ &= (\theta I^{--} - C^{--})^{-1}\Phi^{--**}(\theta) = \frac{1}{\theta}(\theta I^{--} - C^{--})^{-1}\Phi^{--*}(\theta).\end{aligned}$$

Therefore, the following equality holds.

$$L^*(\theta) = (\theta I^{--} - C^{--})^{-1}\Phi^{--*}(\theta). \tag{5.1}$$

Define $|S| \times |S^-|$-matrix $G(t|x)$ as

$$[G(t|x)]_{ij} = P(X(t) = 0, M(t) = j | X(0) = x, M(0) = i), \quad (x \geq 0, t \geq 0, i \in S, j \in S^-).$$

Since the limiting distribution of $[G(t|x)]_{ij}$ is independent of initial conditions $M(0) = i$ and $X(0) = x$, we have the following form:

$$\lim_{t\to\infty} G(t|x) = \begin{pmatrix} \boldsymbol{F}(0)^- \\ \vdots \\ \boldsymbol{F}(0)^- \end{pmatrix}.$$

$\boldsymbol{F}^-(0)$ is the empty probability vector in the case of $v(S) = \{-1, 1\}$.

Lemma 5.1 For $\theta > 0$ and $x \geq 0$, we have

$$G^{**}(\theta|x)Q^{--}(\theta) = -H^{\bullet -*}(\theta|x). \tag{5.2}$$

Proof. We decompose time t into three parts: the first passage time from level x to the empty state, successive busy cycles and the remaining time up to time t. We then have

$$G(t|x) = \int_{u=0}^{t} H^{\bullet -}(du|x)e^{C^{--}(t-u)} + \sum_{n=1}^{\infty} \int_{u=0}^{t} H^{\bullet -}(du|x) \int_{y=0}^{t-u} L_{(n)}(dy)e^{C^{--}(t-u-y)},$$

where $|S^-| \times |S^-|$-matrix $L_{(n)}(y)$ is the n-fold convolution of L with itself. From (5.1), the LT of $G(t|x)$ is

$$\begin{aligned}
G^{**}(\theta|x) &= \int_{u=0}^{\infty} e^{-\theta u} H^{\bullet -}(du|x) \int_{t=u}^{\infty} e^{-(\theta I^{--} - C^{--})(t-u)} dt \\
&\quad + \sum_{n=1}^{\infty} \int_{u=0}^{\infty} e^{-\theta u} H^{\bullet -}(du|x) \int_{y=0}^{\infty} e^{-\theta y} L_{(n)}(dy) \int_{t=u+y}^{\infty} e^{-(\theta I^{--} - C^{--})(t-u-y)} \\
&= H^{\bullet -*}(\theta|x)(\theta I^{--} - C^{--})^{-1} + \sum_{n=1}^{\infty} H^{\bullet -*}(\theta|x) \left(L^*(\theta)\right)^n (\theta I^{--} - C^{--})^{-1} \\
&= H^{\bullet -*}(\theta|x)(I^{--} - L^*(\theta))^{-1}(\theta I^{--} - C^{--})^{-1} \\
&= H^{\bullet -*}(\theta|x) \left((\theta I^{--} - C^{--})(I^{--} - L^*(\theta))\right)^{-1} \\
&= H^{\bullet -*}(\theta|x)(\theta I^{--} - C^{--} - \Phi^{--*}(\theta))^{-1} \\
&= H^{\bullet -*}(\theta|x)(-Q^{--}(\theta))^{-1}.
\end{aligned}$$

Thus we get (5.2). □

The following theorem gives the representation of $\boldsymbol{F}(0)^-$ in the term of the stationary distribution of Q^{--}, which always exists because S^- is finite.

Theorem 5.1 Let $\boldsymbol{\kappa}$ be the stationary distribution of Q^{--}. Then,

$$\boldsymbol{F}(0)^- = -E[Y(1)]\boldsymbol{\kappa}, \tag{5.3}$$

where Y is the stationary accumulated input process with $v(S) = \{-1, 1\}$.

Proof. Using integration by parts, we have

$$\lim_{\theta\to 0+} \theta G^*(\theta|x) = \lim_{\theta\to 0+} \int_0^{\infty} e^{-\theta t} \frac{\partial}{\partial t} G(t|x)dt = \lim_{t\to\infty} G(t|x) = \begin{pmatrix} \boldsymbol{F}(0)^- \\ \vdots \\ \boldsymbol{F}(0)^- \end{pmatrix}.$$

Multiplying θ to (5.2) and taking it in the limit as $\theta \to 0+$, we have

$$\begin{pmatrix} \boldsymbol{F}(0)^- \\ \vdots \\ \boldsymbol{F}(0)^- \end{pmatrix} Q^{--} = \lim_{\theta\to 0+} (\theta G^{**}(\theta|x)) Q^{--}(\theta) = \lim_{\theta\to 0+} \left(-\theta \begin{bmatrix} R^{+-} e^{Q^{--}(\theta)x} \\ e^{Q^{--}(\theta)x} \end{bmatrix} \right) = \mathbf{0}.$$

Thus, we obtain $\boldsymbol{F}(0)^- Q^{--} = \mathbf{0}^-$. Since Q^{--} is irreducible, which follows from the irreducibility of $C + D$, we have

$$\boldsymbol{F}(0)^- = c\boldsymbol{\kappa},$$

for some constant $c > 0$. Multiplying vector $\boldsymbol{e}^-$ to both sides of this equation,

$$c = \boldsymbol{F}(0)^- \boldsymbol{e}^-.$$

To calculate the normalizing constant c, let $U^+(t)$ be the sojourn time in S^+ up to time t, and let $U_1^-(t)$ and $U_0^-(t)$ be the sojourn times in S^- up to time t such that the buffer content is not empty and empty, respectively. Let T_n be the nth jump instant of $\{X(t)\}$. Note that

$$\begin{aligned} &U^+(t) + U_1^-(t) + U_0^-(t) = t, \\ &0 \le \sum_{n=1}^{N(t)} B_{T_n}(M(T_n-), M(T_n)) + U^+(t) - U_1^-(t) \le X(t). \end{aligned}$$

Since $\{X(t)\}$ has the stationary distribution,

$$\lim_{t\to\infty} \frac{X(t)}{t} = 0.$$

Hence, we have

$$\begin{aligned} c &= \lim_{t\to\infty} \frac{U_0^-(t)}{t} \\ &= 1 - \lim_{t\to\infty} \left(\frac{U^+(t)}{t} + \frac{U_1^-(t)}{t}\right) \\ &= 1 - \lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N(t)} B_{T_n}(M(T_n-), M(T_n)) - 2 \lim_{t\to\infty} \frac{U^+(t)}{t}. \end{aligned}$$

Let E_N denote the expectation by the Palm probability with respect to N. Substituting $t = 1$ to (2.2) and taking the expectation of it, we have

$$\begin{aligned} E(Y(1)) &= E(N(1))E_N(B_{T_1}(M(T_1-), M(T_1)) + E(U^+(1)) - E(U_1^-(1)) - E(U_0^-(1)] \\ &= E(N(1))E_N(B_{T_1}(M(T_1-), M(T_1)) + 2E(U^+(1)) - 1. \end{aligned}$$

From the strong law of large numbers,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N(t)} B_{T_n}(M(T_n-), M(T_n)) = E_N(B_{T_1}(M(T_1-), M(T_1)).$$

Hence we have

$$c = 1 - E(N(1))E_N(B_{T_1}(M(T_1-), M(T_1)) - 2E(U^+(1)) = -E(Y(1)).$$

□

Thus, combining Proposition 2.1 and Theorem 5.1, we can determine the LST $\boldsymbol{F}^*(\theta)$ of the stationary joint distribution of X and M. However, to get numerical values from this LST, we need some more work. This will be done in the next section.

6. Moments of The Buffer Content and Numerical Examples

We give an algorithm to compute moments of the buffer content. We also exemplify it for some simple cases. We remove the restriction that $v(S) = \{-1, 1\}$ in this section.

(2.4) can be written as:

$$\boldsymbol{F}^*(\theta)(\theta V - C - D^*(\theta)) = \theta \boldsymbol{F}(0)V. \tag{6.1}$$

Let $\boldsymbol{F}^{*(k)}(\theta)$ and $D^{*(k)}(\theta)$ be the kth derivative of $\boldsymbol{F}^*(\theta)$ and $D^*(\theta)$, respectively. We note that $E(X^k) = (-1)^k \boldsymbol{F}^{*(k)}(0+)\boldsymbol{e}$ for $k \geq 1$ and $\boldsymbol{F}^{*(0)}(0+) = \boldsymbol{\pi}$. Differentiating both sides of (6.1) for n times for $n \geq 1$ and taking $\theta \to 0+$ in it, we have

$$-\boldsymbol{F}^{*(n)}(0)(C + D) = 1(n = 1)\boldsymbol{F}(0)V - \sum_{\ell=1}^{n} \binom{n}{\ell} \boldsymbol{F}^{*(n-\ell)}(0) \left(V 1(\ell = 1) - D^{*(\ell)}(0)\right). \tag{6.2}$$

Define $\boldsymbol{g}^{(n)}$ and $\boldsymbol{x}^{(n)}$ as

$$\boldsymbol{g}^{(n)} = 1(n = 1)\boldsymbol{F}(0)V - \sum_{\ell=1}^{n} \binom{n}{\ell} \boldsymbol{F}^{*(n-\ell)}(0) \left(V 1(\ell = 1) - D^{*(\ell)}(0)\right), \tag{6.3}$$

$$\boldsymbol{x}^{(n)} = \boldsymbol{g}^{(n)}(\boldsymbol{e}\boldsymbol{\pi} - (C + D))^{-1}. \tag{6.4}$$

Since

$$\boldsymbol{\pi}(\boldsymbol{e}\boldsymbol{\pi} - (C + D))^{-1} = \boldsymbol{\pi},$$
$$\boldsymbol{F}^{*(n)}(0)(\boldsymbol{e}\boldsymbol{\pi} - (C + D)) = (\boldsymbol{F}^{*(n)}(0)\boldsymbol{e})\boldsymbol{\pi} + \boldsymbol{g}^{(n)},$$

we obtain

$$\boldsymbol{F}^{*(n)}(0) = (\boldsymbol{F}^{*(n)}(0)\boldsymbol{e})\boldsymbol{\pi} + \boldsymbol{x}^{(n)}. \tag{6.5}$$

Differentiating both sides of (6.1) for $n+1$ times for $n \geq 1$, taking $\theta \to 0+$ to it, multiplying $\boldsymbol{e}$ to the right side of it and applying (6.5) to it, we have

$$(n+1)\boldsymbol{F}^{*(n)}(0)\boldsymbol{e}\boldsymbol{\pi}(V - D^{*(1)}(0))\boldsymbol{e} + (n+1)\boldsymbol{x}^{(n)}(V - D^{*(1)}(0))\boldsymbol{e}$$
$$- \sum_{\ell=2}^{n+1} \binom{n+1}{\ell} \boldsymbol{F}^{*(n+1-\ell)}(0) D^{*(\ell)}(0)\boldsymbol{e} = \boldsymbol{0},$$

and, equivalently,

$$\boldsymbol{F}^{*(n)}(0)\boldsymbol{e} = -\frac{\boldsymbol{x}^{(n)}(V - D^{*(1)}(0))\boldsymbol{e}}{\boldsymbol{\pi}(V - D^{*(1)}(0))\boldsymbol{e}} + \frac{\sum_{\ell=2}^{n+1} \binom{n+1}{\ell} \boldsymbol{F}^{*(n+1-\ell)}(0) D^{*(\ell)}(0)\boldsymbol{e}}{(n+1)\boldsymbol{\pi}(V - D^{*(1)}(0))\boldsymbol{e}}. \tag{6.6}$$

From these calculations, we have the following algorithm:

Step 1. The computation of $\boldsymbol{F}(0)$

- Step 1-1. Compute stationary distribution $\boldsymbol{\pi}$ of $C + D$.
- Step 1-2. Compute $\widetilde{C}$ and $\widetilde{D}(x)$.
- Step 1-3. Compute Q^{--} by the iteration (4.1) and (4.2) for $\widetilde{C}$ and $\widetilde{D}(x)$.
- Step 1-4. Compute stationary distribution $\boldsymbol{\kappa}$ of Q^{--}.
- Step 1-5. Compute $-E(\widetilde{Y}(1))$ by applying $\widetilde{\boldsymbol{\pi}}$, $\widetilde{V}$ and $\widetilde{D}(x)$ in stead of $\boldsymbol{\pi}$, V and $D(x)$ in (2.3).

Table 1: The result of the numerical examples

$v(d)$	$-E[Y(1)]$	$-E[\widetilde{Y}(1)]$	Q^{--}	Computation time Case 1	Case 2
-2	0.065	0.037143	$\begin{pmatrix} -1.036718 & 1.036718 \\ 0.799869 & -0.799869 \end{pmatrix}$	2 sec	40 sec
-6	1.065	0.387273	$\begin{pmatrix} -1.205187 & 1.205187 \\ 0.241365 & -0.241365 \end{pmatrix}$	1 sec	0.5 sec
-10	2.065	0.550667	$\begin{pmatrix} -1.242235 & 1.242235 \\ 0.141360 & -0.141360 \end{pmatrix}$	0.6 sec	0.2 sec

$v(d)$	$\widetilde{\boldsymbol{F}}(0)^-$	$\boldsymbol{F}(0)^-$	$E[X]$	$E[X^2]$
-2	$[0.016168, 0.020955]$	$[0.014147, 0.018336]$	19.253223	766.198930
-6	$[0.064618, 0.322649]$	$[0.088849, 0.147881]$	2.128468	11.782086
-10	$[0.056261, 0.494406]$	$[0.105489, 0.185402]$	1.611229	7.238652

Step 1-6. Compute $\boldsymbol{F}(0)$ by (2.5).

Step 2. Compute inverse matrix of $(\boldsymbol{e}\boldsymbol{\pi} - (C + D))$.

For $k = 1$ to n:{

Step 3-k. Compute $\boldsymbol{x}^{(k)}$ by (6.3) and (6.4).

Step 4-k. Compute $\boldsymbol{F}^{(k)}(0)\boldsymbol{e}$ by (6.6).

Step 5-k. Compute $\boldsymbol{F}^{(k)}(0)$ by (6.5).

Step 6-k. $E(X^k) = (-1)^k \boldsymbol{F}^{(k)}(0)\boldsymbol{e}$.}.

In the rest of the this section, we consider numerical examples to show how the algorithm works.

Example 6.1 Suppose that jump sizes are deterministic for each possible transitions. Let B be a $|S| \times |S|$-matrix, whose (i, j)th element is the jump size of the buffer process when the background state changes from i to j. Define $D(x)$ by

$$[D(x)]_{ij} = [D]_{ij} 1(x = [B]_{ij}).$$

It is assumed that $(C + D)\boldsymbol{e} = \boldsymbol{0}$. Then the integral terms in the iteration (4.1) and (4.2) for Q^{--} are given by

$$\begin{aligned} \left[\int_0^\infty D^{--}(dw) e^{Q^{--}w}\right]_{ij} &= \sum_{k \in S^-} [D^{--}]_{ik} \left[e^{Q^{--}[B^{--}]_{ik}}\right]_{kj}, \\ \left[\int_0^\infty D^{-+}(dw) R^{+-} e^{Q^{--}w}\right]_{ij} &= \sum_{k \in S^+} \sum_{\ell \in S^-} [D^{-+}]_{ik} [R^{+-}]_{k\ell} [e^{Q^{--}[B^{-+}]_{ik}}]_{\ell j}. \end{aligned}$$

In a similar way, we can get the integral terms of the iteration for R^{+-}. The calculation of $D^{*(k)}(0)$ for $k \geq 1$ is given by

$$\left[D^{*(k)}(0)\right]_{ij} = \int_0^\infty (-x)^k [D(dx)]_{ij} = (-1)^k [B]_{ij}^k [D]_{ij}.$$

For numerical examples, we put the following parameters.

$$S^+ = \{a, b\}, \quad S^- = \{c, d\}, \quad S = S^+ \cup S^-,$$

$$C = \begin{pmatrix} -3 & 0 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$v(a) = 2, \quad v(b) = 1, \quad v(c) = -2, \quad v(d) = -2, -6, \text{ or } -10,$$

where the states are ordered as a, b, c and d. For instance, a $|S| \times |S|$-matrix A has the following form:

$$A = \begin{pmatrix} A_{a,a} & A_{a,b} & A_{a,c} & A_{a,d} \\ A_{b,a} & A_{b,b} & A_{b,c} & A_{b,d} \\ A_{c,a} & A_{c,b} & A_{c,c} & A_{c,d} \\ A_{d,a} & A_{d,b} & A_{d,c} & A_{d,d} \end{pmatrix}.$$

$v(d)$ is varied so as to represent the heavy traffic case ($-E(Y(1))$ close to zero) and the light traffic case ($-E(Y(1))$ far from zero). The results are presented in Table 6.1, where for $\widetilde{\boldsymbol{F}}(0)^-$, $\boldsymbol{F}(0)^-$ and Q^{--}, the background states are ordered as c, d. i.e.,

$$\widetilde{\boldsymbol{F}}(0)^- = (\widetilde{\boldsymbol{F}}(0)_c, \widetilde{\boldsymbol{F}}(0)_d), \quad \boldsymbol{F}(0)^- = (\boldsymbol{F}(0)_c, \boldsymbol{F}(0)_d), \quad Q^{--} = \begin{pmatrix} Q_{cc} & Q_{cd} \\ Q_{dc} & Q_{dd} \end{pmatrix}.$$

For those examples, we also compare computation times of both iterations in Section 4 and Remark 4.1, which are referred to as cases 1 and 2, respectively. Here, numerical results of both iterations are found to be identical as it is expected.

Acknowledgment The author is deeply thankful to Professor Masakiyo Miyazawa at Science University of Tokyo for invaluable discussions, and grateful to anonymous referees for their helpful comments.

References

[1] D. Anick, D. Mitra and M.M. Sondhi: Stochastic theory of a data-handling system with multiple sources. *The Bell System Technical Journal,* **61**-8 (1982) 1871-1894.

[2] S. Asmussen: Ladder heights and the Markov-modulated M/G/1 queue. *Stochastic Processes and Their Applications,* **37** (1991) 313-326.

[3] S. Asmussen: Stationary distributions for fluid flow models with or without brownian noise. *Stochastic Models,* **11**-1 (1995) 21-49.

[4] A.I. Elwailid and T.E. Stern: Analysis of separable Markov-modulated rate models for information-handling systems. *Advances in Applied Probability,* **23** (1991) 105-139.

[5] D.P. Gaver and J.P. Lehoczky: Performance evaluation of voice/data queueing systems. *Applied Probability–Computer Science: the Interface,* **I** (1982) 329-346.

[6] R.M. Loynes: The stability of a queue with non-independent inter-arrival and service times. *Proceedings of Cambridge Philosophical Society,* **58** (1962) 497-520.

[7] D. Mitra: Stochastic theory of a fluid model of producers and consumers coupled by a buffer. *Advances in Applied Probability,* **20** (1988) 646-676.

[8] M. Miyazawa: Rate conservation laws: a survey. *Queueing Systems,* **15** (1994) 1-58.

[9] L.C.G. Rogers: Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Annals of Applied Probability,* **4**-2 (1994) 390-413.

[10] E. Seneta: *Non-negative Matrices and Markov Chains. Second Edition* (Springer-Verlag, New York, Heidelberg, Berlin 1981).

[11] B. Sengupta: Markov processes whose steady state distribution is matrix-exponential with an application to the GI/PH/1 queue. *Advances in Applied Probability,* **21** (1989) 159-180.

[12] T. Takine and T. Hasegawa: The workload in the MAP/G/1 queue with state-dependent services: Its application to a queue with preemptive resume priority. *Stochastic Models,* **10**-1 (1994) 183-204.

Appendix. Proof of Proposition 2.1

To calculate the stationary equation of $\boldsymbol{F}(x)$, we assume that $X(t)$ and $M(t)$ are jointly stationary, and define $Z(t)$ as

$$Z(t) = e^{-\theta X(t)} 1(M(t) = i),$$

for each $i \in S$ and $\theta \geq 0$.

We uniformize $\{M(t)\}$ with rate λ given by

$$\lambda = \max_{i \in S}\{-(C_{ii} + D_{ii})\}.$$

Namely, state transition instants are subject to the Poisson process with rate λ, and at each instant, the state of $M(t)$ changes according to the transition probability matrix $P^C + P^D$, where

$$P^C = \frac{1}{\lambda}(\lambda I + C), \quad P^D(x) = \frac{1}{\lambda}D(x), \quad P^D = \int_0^\infty P^D(dx) = \frac{1}{\lambda}D.$$

For P^D-transition, jumps are associated. Each jump size is subject to distribution $[D(x)]_{ij}/[D]_{ij}$. We denote the Poisson process by Λ.

We apply the rate conservation law (see Miyazawa [8]) to $Z(t)$ with uniformized $M(t)$.

$$E(Z'(0)) = \lambda(E_\Lambda(Z(0-)) - E_\Lambda(Z(0+))), \tag{A.1}$$

where E_Λ is the Palm expectation with respect to point process Λ.

To calculate the left side of rate conservation law, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(t) = -\theta\frac{\mathrm{d}}{\mathrm{d}t}X(t)e^{-\theta X(t)}1(M(t) = i).$$

If $X(t) = 0$ and $i \in S^-$, then $X'(t) = 0$. In the other case, we have $X'(t) = v(i)$. Hence, we have

$$\begin{aligned} Z'(t) &= -\theta v(i)e^{-\theta X(t)}1(M(t) = i)(1(X(t) > 0) + 1(X(t) = 0, i \in S^+)) \\ &= -\theta v(i)e^{-\theta X(t)}1(M(t) = i) + \theta v(i)1(X(t) = 0, i \in S^-, M(t) = i). \end{aligned}$$

This yields

$$E(Z'(0)) = -\theta v(i)[\boldsymbol{F}^*(\theta)]_i + \theta v(i)1(i \in S^-)[\boldsymbol{F}(0)]_i.$$

We calculate the right side of (A.1). Because transition instants of the uniformized process constitute the Poisson process with rate λ, we can apply PASTA to $E_\Lambda(Z(0-))$, i.e.

$$E_\Lambda(Z(0-)) = E(Z(0)).$$

We have

$$\begin{aligned}
\lambda E_\Lambda(Z(0-)) &= \lambda E(Z(0)) = \lambda[\boldsymbol{F}^*(\theta)]_i, \\
\lambda E_\Lambda(Z(0+)) &= \lambda E_N(e^{-\theta X(0+)} 1(M(0+) = i)) \\
&= \lambda \sum_{j \in S} E_N(e^{-\theta X(0+)} 1(M(0+) = i, M(0-) = j)) \\
&= \lambda \sum_{j \in S} [\boldsymbol{F}^*(\theta)]_j P_{ji}^{D*}(\theta) + \lambda \sum_{j \in S} [\boldsymbol{F}^*(\theta)]_j P_{ji}^C \\
&= [\boldsymbol{F}^*(\theta)(\lambda I + C + D^*(\theta))]_i.
\end{aligned}$$

Hence, we get the right side of (A.1),

$$\lambda(E_\Lambda(Z(0-)) - E_\Lambda(Z(0+))) = -[\boldsymbol{F}^*(\theta)(C + D^*(\theta))]_i.$$

Thus, we obtain

$$-\theta \boldsymbol{F}^*(\theta) V + \theta \boldsymbol{F}(0) V = -\boldsymbol{F}^*(\theta)(C + D^*(\theta)).$$

This proves Proposition 2.1.

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