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A REVISION OF MINTY'S ALGORITHM FOR FINDING A MAXIMUM WEIGHT STABLE SET OF A CLAW-FREE GRAPH

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Abstract The maximum weight/cardinality stable set problem is to find a maximum weight/cardinality stable set of a given graph. It is well known that these problems for general graphs belong to the class of NP-hard. However, for several classes of graphs, e.g., for perfect graphs and claw-free graphs and so on, these problems can be solved in polynomial time. For instance, Minty (1980), Sbihi (1980) and Lovász and Plummer (1986) have proposed polynomial time algorithm finding a maximum cardinality stable set of a claw-free graph. Moreover, it has been believed that Minty's algorithm is the unique polynomial time algorithms finding a maximum weight stable set of a claw-free graph up to date. Here we show that Minty's algorithm for the weighted version fails for some special cases, and give modifications to overcome it.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex-set V and edge-set E. A subset S of V is called a *stable set* (or an independent set or a vertex packing) if any two elements of S are nonadjacent. A subset M of E is called a *matching* if no two elements of M are incident to the same vertex. Given a weight function $w : V \to \mathbf{R}$, a *maximum weight stable set* is a stable set S maximizing the sum of weights of all of its elements, $w(S) = \sum_{v \in S} w(v)$. Similarly, given a weight function $\hat{w} : E \to \mathbf{R}$, a *maximum weight matching* is a matching M maximizing $\hat{w}(M) = \sum_{e \in M} \hat{w}(e)$. We will deal with the problems of finding a maximum weight stable set/matching, the so-called *maximum weight stable set/matching problem.* Particularly, if $w(v) = 1$ for all $v \in V$ ($\hat{w}(e) = 1$ for all $e \in E$), they are called the *maximum cardinality stable set/matching problems.* The maximum weight matching problem can be easily transformed to a maximum weight stable set problem by using line graphs. The line graph $\ell(G)$ of G is a graph whose vertex set is E and in which two distinct vertices e and f are adjacent if and only if e and f have a common endpoint in G. Since the matchings of G correspond to the stable sets of $\ell(G)$, it is easily seen that the maximum weight matching problem is a special case of the maximum weight stable set problem. However, it is well known that there is a big gap between these two problems. The maximum weight stable set problem is NP-hard, even if $w(v) = 1$ for $v \in V$ (see [8]). On the other hand, many polynomial time algorithms for the maximum weight/cardinality matching problem have been proposed, e.g., [11, 9, 5] for bipartite graphs and [2, 3, 15, 6] for general graphs. Moreover, these polynomial time algorithms have been extended to those solving more general problems, for instance, the maximum weight/cardinality stable set problem for claw-free graphs [18, 16, 14], the linear matroid parity problem [12, 13, 7, 17] and the linear

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delta-matroid parity problem [10].

This paper deals with the maximum weight stable set problem for claw-free graphs. The complete bipartite graph $K_{1,3}$ is called a *claw*. A graph is said to be *claw-free* if it does not contain an induced subgraph isomorphic to a claw (see [4] for a survey on claw-free graphs). The claw is one of the forbidden subgraphs of line graphs [1]. That is, line graphs are claw-free, and the maximum weight stable set problem for claw-free graphs is a generalization of the maximum weight matching problem. Up to date, three polynomial time algorithms, one by Minty [16], one by Sbihi [18] and the third by Lovász and Plummer [14] are known for the maximum cardinality stable set problem for claw-free graphs. Furthermore, it has been believed that Minty's algorithm is the only algorithm which can be extended to the weighted problem. This algorithm is based on a scheme by constructing "semi-optimal" stable sets for increasing cardinalities until it finds a maximum one. Here we say that a stable set S is semi-optimal if there is no stable set T such that $|T| \leq |S|$ and $w(T) > w(S)$. In order to find a next semi-optimal stable set from a current one, the algorithm transforms the problem to the maximum weight matching problem by constructing a graph called the Edmonds' graph. However, the construction does not always work well for the weighted problem (it is correct for the cardinality version). We will describe an example in which the construction fails. And in this paper we shall show how to revise the original definition of Edmonds' graphs by Minty [16] so that an optimal solution is obtained for the weighted case as well.

For combinatorial optimization problems which can be solved in polynomial time, polyhedral descriptions of the feasible regions of those problems are generally known (e.g., matching polytopes of general graphs [2, 3]). Although polyhedral descriptions of stable set polytopes of claw-free graphs are unknown so far, the maximum weight stable set problem on them can be solved in polynomial time. From this point of view, Minty's result seems to be important. Our contribution is to give that reassurance.

Section 2 briefly explains Minty's algorithm, gives an example in which it fails and analyzes why such an error occurs. Section 3 proposes our revision.

2. Minty's Algorithm

The claw-freeness is the property that the set of all neighbors of any vertex have no stable set of size greater than or equal to three. This guarantees that the symmetric difference of any two stable sets, referred to as "the black vertices" and the "purple vertices" respectively, consists of a family of disjoint paths and cycles in which the black and the purple vertices appear alternately. Minty's algorithm is based on this property. Here, we first define basic notations and explain Minty's idea.

We denote the difference and the symmetric difference of two sets by using symbols '$-$' and '$\triangle$'. Fix a claw-free graph $G = (V, E)$, a weight function $w : V \rightarrow \mathbf{R}$ and a stable set S of G. We call the elements of S *black* and other vertices *white*. A white vertex is adjacent to at most two black vertices since G is claw-free. A white vertex is said to be *bounded* if it is adjacent to two black vertices, *free* if it is adjacent to exactly one black vertex and otherwise *super free*. A simple path (or cycle) is called an *alternating path* (or an *alternating cycle*) of S, if white and black vertices appear alternately, and no two of its white vertices are adjacent to each other. We call an alternating path *white* (or *black*) if both of its endpoints are white (black), and otherwise *white-black*. We define the *weight* of a path P (or cycle C) by the sum of weights of its white vertices minus the sum of weights of its black vertices and denote it by $\delta(P)$ (or $\delta(C)$). An alternating path P is called an *augmenting path* if $\delta(P) > 0$, and endpoints of P are not bounded (i.e. free, super free, or black). Note that $S \triangle P$ is a stable set and $w(S \triangle P) = w(S) + \delta(P)$.

The scheme of Minty's algorithm is described below.

1. $S \leftarrow \emptyset$;
2. **if** there exists no white augmenting path of S **then output** S; **stop**
   else find a white augmenting path P^* having the maximum weight;
3. $S \leftarrow S \triangle P^*$; **go to** Step 2.

The correctness of the scheme follows from the next fact:

Fact 2.1 ([16, Lemma 11]) *Let S be a semi-optimal but not maximum weight stable set. Then, for any maximum weight white augmenting path P^* of S, $S \triangle P^*$ is a semi-optimal stable set with cardinality $|S| + 1$.*

The essential part, Step 2, is divided as below:

2-1. Generate all white alternating paths of length 0 or 2;
2-2. **for** each pair of non-adjacent free vertices a and b **do**
2-3. Let x_a and x_b be the black vertices adjacent to a and b respectively;
2-4. **if** $x_a \neq x_b$ **then**
2-5. find a maximum weight white alternating path between a and b;
2-6. **end for**;
2-7. **if** all the white alternating paths generated above have nonpositive weight **then output** S; **stop**;
2-8. Choose a maximum weight path P^* among all the generated white alternating paths.

The above scheme is correct, but the execution of Line 2-5 in Minty's algorithm contains an error. In [16], Line 2-5 is transformed to an instance of the maximum weight matching problem by constructing a graph, called the "Edmonds' graph."

We will explain the construction of the Edmonds' graph briefly. We first ignore all the super free vertices, free vertices except a and b, and all the white vertices that are adjacent to a or b, since they never appear in any white alternating path between a and b. The reduced graph is called an *RBS* (a reduced basic structure), including the weight function w and the semi-optimal stable set S. In the rest of this paper, we will deal with the RBS instead of G.

A nonempty set of all bounded vertices, which are adjacent to the same two black vertices x and y, is called a *wing*. Vertices x_a and x_b are called *regular I*. Other black vertices are classified as follows: a black vertex is called *regular II* if it is adjacent to three or more wings, *irregular* if it is adjacent to exactly two wings, and otherwise *useless*. Let $(v_0, v_1, v_2, \ldots, v_{2\ell-1}, v_{2\ell})$ be a black alternating path. If $v_2, v_4, \ldots, v_{2\ell-2}$ are all irregular, then the subpath $(v_1, v_2, \ldots, v_{2\ell-1})$ is called an *IWAP* (irregular white alternating path) between v_0 and $v_{2\ell}$.

We partition the neighbor set $N(x_a)$ of the vertex x_a into two sets $N^1(x_a) = \{a\}$ and $N^2(x_a) = N(x_a) - \{a\}$, and define $N^1(x_b)$ and $N^2(x_b)$ similarly. Obviously, the unique element a of $N^1(x_a)$ is nonadjacent to any element of $N^2(x_a)$. This property can be extended to regular II vertices as below.

Fact 2.2 ([16, Theorem 1]) *For any regular II vertex v, $N(v)$ is uniquely (except for exchanging) partitioned into $N^1(v)$ and $N^2(v)$ so that for any x and y in distinct wings*

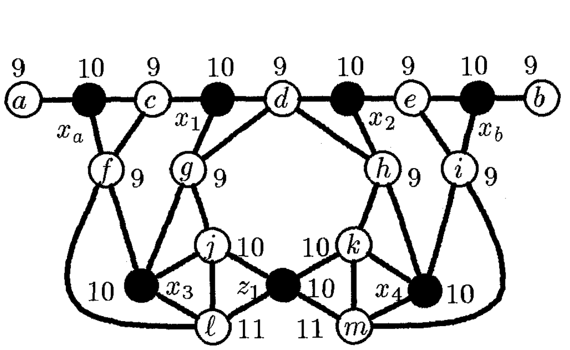

Figure 1: RBS.

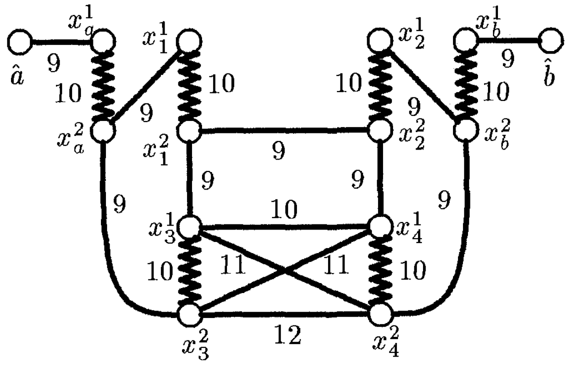

Figure 2: Edmonds' graph.

adjacent to v, x is not adjacent to y if and only if one of x and y is in $N^1(v)$ and the other is in $N^2(v)$.

This and the following fact are keys in constructing the Edmonds' graph.

Fact 2.3 ([16, see Lemma 4]) *A simple path $(v_1, v_2, \ldots, v_\ell)$ is an alternating path if and only if white and black vertices appear alternately, each white vertex v_{i-1} is not adjacent to v_{i+1}, neither v_1 nor v_ℓ is adjacent to two black vertices of the path, and v_1 and v_ℓ are not adjacent to each other if they are white. A simple cycle $(v_1, v_2, \ldots, v_{2\ell}, v_{2\ell+1} = v_1)$ is an alternating cycle if and only if white and black vertices appear alternately and each white vertex v_{i-1} is not adjacent to v_{i+1}.*

The *Edmonds' graph* is defined as below. Let $x_1, \ldots, x_r$ be all the regular (I or II) vertices of the RBS. The Edmonds' graph has $2r+2$ vertices labeled by x_i^1, x_i^2 $(i = 1, \ldots, r)$ and $\hat{a}, \hat{b}$. Each edge is painted black or white and assigned a weight $\hat{w}(\cdot)$ in the following manner. For each regular vertex x_i, join x_i^1 and x_i^2 by a black edge with weight $\hat{w}((x_i^1, x_i^2)) = w(x_i)$. Join $\hat{a}$ and x_a^1 by a white edge with $\hat{w}((\hat{a}, x_a^1)) = w(a)$, and join $\hat{b}$ and x_b^1 by a white edge with $\hat{w}((\hat{b}, x_b^1)) = w(b)$. These two white edges correspond to white vertices a and b. For each pair of regular vertices x_i and x_j and for $p, q \in \{1, 2\}$, if there exists an IWAP between x_i and x_j whose endpoints are in $N^p(x_i)$ and $N^q(x_j)$, join x_i^p and x_j^q by a white edge whose weight is the maximum weight among such IWAPs. Such an edge can be found in polynomial time, since it can be transformed to the longest path problem of an acyclic directed graph from Fact 2.3.

We briefly explain the construction of the Edmonds' graph by using an example in Figure 1. The RBS has four regular II vertices $\{x_1, x_2, x_3, x_4\}$. For example, let us consider two black vertices x_3 and x_4. By Fact 2.2, $N(x_3)$ and $N(x_4)$ are partitioned into $N^1(x_3) = \{g, j\}$ and $N^2(x_3) = \{f, \ell\}$, and $N^1(x_4) = \{h, k\}$ and $N^2(x_4) = \{i, m\}$, respectively. Between x_3 and x_4, there are four IWAPs (j, z_1, k), (j, z_1, m), (ℓ, z_1, k) and (ℓ, z_1, m) which join vertices belonging to distinct pairs of partitions of $N(x_3)$ and $N(x_4)$. Thus, the Edmonds' graph of the RBS has four white edges (x_3^1, x_4^1), (x_3^1, x_4^2), (x_3^2, x_4^1) and (x_3^2, x_4^2) whose weights are 10, 11, 11 and 12, respectively. Figure 2 in which the waved lines represent the black edges is the Edmonds' graph of the RBS in Figure 1.

We call a simple path/cycle in the Edmonds' graph an *alternating path/cycle* if white and black edges appear alternately, and also define white alternating path and so on, in a manner similar to the vertex case. We define the *weight* of a path $\hat{P}$ (or cycle $\hat{C}$) in the Edmonds' graph by the sum of weights of its white edges minus the sum of weights of its black edges and denote it by $\hat{\delta}(\hat{P})$ (or $\hat{\delta}(\hat{C})$).

The key property connecting the RBS and the Edmonds' graph is the next fact.

Fact 2.4 ([16, Theorem 2]) *There exists a (in general many-to-one) mapping of the set of white alternating paths between a and b in the RBS onto the set of alternating paths between $\hat{a}$ and $\hat{b}$ in the Edmonds' graph. Furthermore, maximum weight augmenting paths in the RBS and the Edmonds' graph have the same weight.*

Minty's algorithm checks whether or not the current RBS has a heavier stable set of size one greater than the semi-optimal stable set S as below. Let n be the size of S, and B'_{n+1} be a maximum weight matching in the Edmonds' graph. For preciseness, we borrow statements from [16, pp. 302]:

> Form a set of vertices W'_{n+1} in the RBS consisting of (a) the (black regular) vertices corresponding to the black branches of B'_{n+1}, (b) the black (irregular) vertices appearing in IWAPs corresponding to white branches not appearing in B'_{n+1}, (c) the white vertices which appear in IWAPs corresponding to white branches appearing in B'_{n+1}, and (d) those free white vertices in the RBS which correspond to white branches appearing in B'_{n+1}.
>
> It is routine but somewhat tedious to verify that W'_{n+1} is an independent set of vertices...

However, this construction fails. For example, let us consider the case described in Figures 1 and 2. The Edmonds' graph in Figure 2 has a maximum weight matching

$$B'_8 = \{(\hat{a}, x_a^1), (x_a^2, x_1^1), (x_1^2, x_2^2), (x_b^2, x_2^1), (\hat{b}, x_b^1), (x_3^1, x_4^1), (x_3^2, x_4^2)\}.$$

By using the above construction, we obtain $\{a, b, c, d, e, j, \ell, k, m\}$ as W'_8, which is not a stable set.

The reason of occurrence of the above failure is as follows: even though Fact 2.4 holds, there is no such relation between the sets of alternating cycles. More precisely, some alternating cycle of length 4 in the Edmonds' graph (e.g., $(x_3^1, x_3^2, x_4^2, x_4^1, x_3^1)$ in Figure 2) does not correspond to an alternating cycle in the RBS.

From Fact 2.4, we may overcome the above error by directly finding a maximum weight alternating path between $\hat{a}$ and $\hat{b}$ in the Edmonds' graph in polynomial time. This approach, however, seems to be difficult. Given a graph $\hat{G}$, a matching M, two specified unmatched vertices s and t and a weight function on edges, the problem finding a maximum weight alternating path between s and t is NP-hard, because the longest simple path problem between two specified vertices on weighted directed graphs which is NP-hard [8], can be easily transformed to the problem.

The problem, however, can be solved in polynomial time in some special cases. For instance, let us consider the case where M is semi-optimal, where we define semi-optimal matchings in the same way as in the case of stable sets. Without loss of generality, we assume that all vertices other than s and t are matched. If M has the maximum weight then there is no alternating path of positive weight; otherwise, for any maximum weight matching M^* of $\hat{G}$, $M \triangle M^*$ contains a maximum weight alternating path between s and t. Hence, this case can be solved in polynomial time by using polynomial time algorithms for the maximum weight matching problem. We note that the matching consisting of black edges in the Edmonds' graph is semi-optimal in the cardinality case, that is, Minty's algorithm works well. Unfortunately, in the weighted case, this does not always hold (see Figure 2). The fact seems the other weak point of Minty's algorithm.

In the next section, we fix the error of Minty's algorithm by modifying the Edmonds' graph whose black edges form a semi-optimal matching.

3. Revised Edmonds' Graphs

Before constructing our revised Edmonds' graph, we analyze the case where the matching consisting of black edges in the Edmonds' graph is not semi-optimal. In general, any graph with non semi-optimal matching contains one of the following subgraphs:

(1) an augmenting cycle,
(2) a white-black augmenting path whose one endpoint is unmatched,
(3) a black alternating path P and a white alternating path Q such that the endpoints of Q are unmatched, P and Q are vertex-disjoint and $\hat{\delta}(P) + \hat{\delta}(Q) > 0$ (we call these an *augmenting path pair*),
(4) a black augmenting path.

However, the Edmonds' graph contains none of them except one special case.

Lemma 3.1 *The Edmonds' graph contains none of augmenting cycles of length more than or equal to* 6*, white-black augmenting paths, augmenting path pairs or black augmenting paths.*

Proof. Assume that there is an augmenting cycle C of length more than or equal to 6 in the Edmonds' graph. Each white edge of C corresponds an IWAP in the RBS. Since the length of C is greater than or equal to 6, these IWAPs do not join the same pair of regular vertices (this does not hold for cycles of length 4). By Facts 2.2 and 2.3 and the construction of the Edmonds' graph, C corresponds to an alternating cycle C' in the RBS. Moreover $\delta(C') = \hat{\delta}(C)$. Hence C' is an augmenting cycle in the RBS, contradicting the assumption that the given stable set is semi-optimal.

Next let us consider the case where an augmenting path pair (P, Q) exists. In the same way as above, P and Q correspond to alternating paths P' and Q' in the RBS, respectively. Since P and Q are vertex-disjoint, P' and Q' are also vertex-disjoint. From the construction of the Edmonds' graph, the white vertices of P' (or Q') are nonadjacent to each other. Assume to the contrary that a white vertex v of Q' is adjacent to a white vertex u of P'. Since P' is a black alternating path, u is adjacent to exactly two black vertices x and y. By claw-freeness, v must be adjacent to either x or y. Thus, either x or y must be contained in Q'. However, this contradicts that P' and Q' are vertex-disjoint. Hence the white vertices of P' and Q' are nonadjacent to each other. The symmetric difference of the given stable set S and $(P' \cup Q')$ is a stable set of the RBS, because two endpoints of Q' are a and b and because a white vertex of P' and Q' is adjacent to none of $S - (P' \cup Q')$. Furthermore, $\delta(P') + \delta(Q') = \delta(P) + \delta(Q)$ holds. This contradicts that S is semi-optimal.

We can similarly show the remaining cases. ∎

By the above lemma, we modify the Edmonds' graph by eliminating all augmenting cycles of length 4, while preserve all alternating paths between $\hat{a}$ and $\hat{b}$.

Since the Edmonds' graph has no alternating cycle containing (x_a^1, x_a^2) or (x_b^1, x_b^2), let us fix distinct regular II vertices x_i and x_j. Suppose that P_{pq} denotes the maximum weight IWAP corresponding to the edge (x_i^p, x_j^q) in the Edmonds' graph if the edge exists, for $p, q \in \{1, 2\}$. We consider the following cases where an augmenting cycle of length 4 exists in the Edmonds' graph:

Case A: there exist both P_{11} and P_{22}, and $\delta(P_{11}) + \delta(P_{22}) > w(x_i) + w(x_j)$, or

Case B: there exist both P_{12} and P_{21}, and $\delta(P_{12}) + \delta(P_{21}) > w(x_i) + w(x_j)$.

We first discuss an easy situation. A wing W is said to be *reachable by irregular vertices* to a regular vertex x if there exist an integer $\ell \geq 1$, distinct irregular vertices $z_1, \ldots, z_{\ell-1}$ and distinct wings $W_1(= W), W_2, \ldots, W_\ell$ such that W_1 is adjacent to z_1 and W_k is adjacent to z_{k-1} and z_k for $k = 2, \ldots, \ell$, where $z_\ell = x$. Let $W(x_i, x_j)$ denote the union of all the wings that are reachable by irregular vertices to both x_i and x_j.

Lemma 3.2 *If $N^1(x_j) \subseteq W(x_i, x_j)$, then any white alternating path between a and b in the RBS passes through neither P_{12} nor P_{22}. That is, we can delete the edges (x_i^1, x_j^2) and (x_i^2, x_j^2) from the Edmonds' graph. Similarly if $N^2(x_j) \subseteq W(x_i, x_j)$, we can delete (x_i^1, x_j^1) and (x_i^2, x_j^1). If $N^1(x_i) \subseteq W(x_i, x_j)$, we can delete (x_i^2, x_j^1) and (x_i^2, x_j^2). If $N^2(x_i) \subseteq W(x_i, x_j)$, we can delete (x_i^1, x_j^1) and (x_i^1, x_j^2).*

Proof. Assume to the contrary that a white alternating path from a to b passes x_i, P_{12} (or P_{22}) and x_j. Before x_j, it passes a vertex in $N^2(x_j)$. Hence it must pass a vertex in $N^1(x_j)$ after x_j. However, since $N^1(x_j) \subseteq W(x_i, x_j)$, it must pass x_i again. ∎

Under the condition of Lemma 3.2, we can delete an augmenting cycle of length 4.

In the sequel, we suppose that none of $N^1(x_i)$, $N^2(x_i)$, $N^1(x_j)$ and $N^2(x_j)$ belongs to $W(x_i, x_j)$. Let us consider Case A. (It is symmetric to consider Case B.) We next introduce key lemmas in our revision and will give proofs of those in the last part of this section.

Lemma 3.3 *Paths P_{11} and P_{22} have the same set of irregular vertices (which may be empty).*

Lemma 3.3 says that P_{11} and P_{22} can be represented as follows:

$$P_{11} = (y_1^1, z_1, y_2^1, z_2, \ldots, y_{\ell-1}^1, z_{\ell-1}, y_\ell^1),\ P_{22} = (y_1^2, z_1, y_2^2, z_2, \ldots, y_{\ell-1}^2, z_{\ell-1}, y_\ell^2). \tag{3.1}$$

Here $z_1, \ldots, z_{\ell-1}$ are irregular vertices, both y_k^1 and y_k^2 are in a common wing W_k for $k = 1, \ldots, \ell$, $y_1^1 \in N^1(x_i)$, $y_1^2 \in N^2(x_i)$, $y_\ell^1 \in N^1(x_j)$ and $y_\ell^2 \in N^2(x_j)$.

We first consider the case where $\ell = 1$.

Lemma 3.4 *If $\ell = 1$, any white alternating path from a to b passes through neither P_{11} nor P_{22}. Hence we can delete the edges (x_i^1, x_j^1) and (x_i^2, x_j^2) from the Edmonds' graph. Moreover, Case B does not occur for the same pair x_i and x_j.*

Next we consider the case where $\ell \geq 2$.

Lemma 3.5 *If $\ell \geq 2$, the followings hold.*

(1) There exists k such that $2 \leq k \leq \ell - 1$ and $y_k^1 = y_k^2$, or there exists k such that $1 \leq k \leq \ell - 1$, y_k^1 is not adjacent to y_{k+1}^2 and y_k^2 is not adjacent to y_{k+1}^1.

(2) For such k, let

$$P_{11i} = (y_1^1, z_1, y_2^1, \ldots, z_{k-1}, y_k^1), \qquad P_{11j} = (y_{k+1}^1, z_{k+1}, \ldots, y_{\ell-1}^1, z_{\ell-1}, y_\ell^1),$$
$$P_{22i} = (y_1^2, z_1, y_2^2, \ldots, z_{k-1}, y_k^2) \quad \text{and} \quad P_{22j} = (y_{k+1}^2, z_{k+1}, \ldots, y_{\ell-1}^2, z_{\ell-1}, y_\ell^2),$$

and let $P'_{12} = (P_{11i}, z_k, P_{22j})$ and $P'_{21} = (P_{22i}, z_k, P_{11j})$. Then $\delta(P'_{12}) + \delta(P'_{21}) = \delta(P_{11}) + \delta(P_{22})$, P'_{12} is an IWAP between $N^1(x_i)$ and $N^2(x_j)$, and P'_{21} is an IWAP between $N^2(x_i)$ and $N^1(x_j)$.

(3) $\delta(P_{11}) + \delta(P_{22}) = \delta(P_{12}) + \delta(P_{21})$.

(4) $\delta(P'_{12}) = \delta(P_{12})$ and $\delta(P'_{21}) = \delta(P_{21})$.

Summing up the above discussion, we propose a revision of the Edmonds' graph. The above discussion deals with three cases, Lemmas 3.2, 3.4 and 3.5. In the first two cases, elimination of augmenting cycles can be easily done by deleting edges. Lemmas 3.2 and 3.4

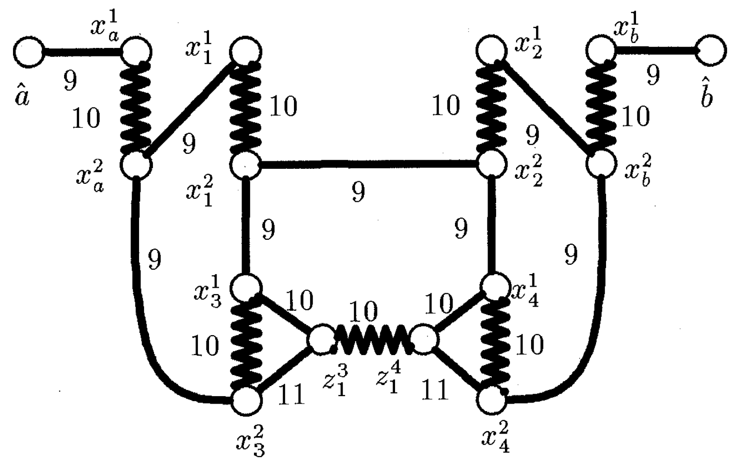

Figure 3: The revised Edmonds' graph of the RBS in Figure 1.

guarantee that the alternating paths between $\hat{a}$ and $\hat{b}$ are preserved and no new alternating path is generated. Let us consider the last case. We revise the Edmonds' graph as below (see Figure 3):

- delete the four edges (x_i^1, x_j^1), (x_i^2, x_j^2), (x_i^1, x_j^2) and (x_i^2, x_j^1) (Lemma 3.5 guarantees the existence of these four edges),
- add two new vertices z_k^i and z_k^j, join z_k^i and z_k^j by a black edge and assign its weight $\hat{w}((z_k^i, z_k^j))$ to be $w(z_k)$, where k satisfies the conditions of Lemma 3.5,
- add the four white edges (x_i^1, z_k^i), (x_i^2, z_k^i), (x_j^1, z_k^j) and (x_j^2, z_k^j), and assign their weights to be $\hat{w}((x_i^1, z_k^i)) = \delta(P_{11i})$, $\hat{w}((x_i^2, z_k^i)) = \delta(P_{22i})$, $\hat{w}((x_j^1, z_k^j)) = \delta(P_{11j})$ and $\hat{w}((x_j^2, z_k^j)) = \delta(P_{22j})$.

All alternating paths passing through black edges (x_i^1, x_i^2) and (x_j^1, x_j^2) can be preserved by our revision, because (x_i^p, x_j^q) in the original Edmonds' graph $(p, q \in \{1, 2\})$ is interpreted by the path $\{x_i^p, z_k^i, z_k^j, x_j^q\}$ in the revised Edmonds' graph. Furthermore, Lemma 3.5 guarantees that weights of these four edges are equal to those of such four paths, respectively.

Lemma 3.6 *For every pair of regular II vertices x_i and x_j, if Case A or Case B occurs, we apply the above revision. Then, the black edges in the revised Edmonds' graph are a semi-optimal matching.*

Proof. Obviously, all the augmenting cycles of length 4 are eliminated and no new augmenting cycle of length 4 is generated. In the same way as in the proof of Lemma 3.1, we can prove that the revised Edmonds' graph contain no augmenting cycle of length more than or equal to 6, no white-black augmenting path, no augmenting path pair or no black augmenting path. □

Theorem 3.7 *A maximum weight stable set of a claw-free graph can be found in polynomial time in the numbers of vertices and edges.*

Proof. The number of alternating cycles of length 4 is polynomially bounded. For each pair of regular II vertices x_i and x_j, $W(x_i, x_j)$ can be found in polynomial time. Hence the revised Edmonds' graph can be constructed in polynomial time. □

We now prove Lemmas 3.3, 3.4 and 3.5.

Proof of Lemma 3.3 If P_{11} and P_{22} have no irregular vertices, the assertion clearly holds. Suppose that P_{11} or P_{22} contains an irregular vertex. Assume to the contrary that the assertion does not hold. Since P_{11} and P_{22} have no regular vertex, the sets of irregular vertices of these IWAPs have no intersection. Thus, all white vertices of P_{11} and P_{22} belong to distinct wings. Let us consider the cycle $C = (x_i, P_{11}, x_j, P_{22}, x_i)$. By Facts 2.2 and 2.3 and the assumption $\delta(C) = \delta(P_{11}) + \delta(P_{22}) - w(x_i) - w(x_j) > 0$, C is an augmenting cycle in the RBS. However this contradicts the fact that the given stable set is semi-optimal. □

Lemma 3.8 *For P_{11} and P_{22} in (3.1), there exists k such that $2 \le k \le \ell - 1$ and $y_k^1 = y_k^2$, or there exists k such that $1 \le k \le \ell$ and y_k^1 is adjacent to y_k^2.*

Proof. Suppose that $y_k^1 \ne y_k^2$ for all k $(2 \le k \le \ell - 1)$. Note that $y_1^1 \ne y_1^2$ and $y_\ell^1 \ne y_\ell^2$ since $N^1(x_i) \cap N^2(x_i) = \emptyset = N^1(x_j) \cap N^2(x_j)$. Let $z_0 = x_i$ and $z_\ell = x_j$. Since $0 < \delta(P_{11}) + \delta(P_{22}) - w(x_i) - w(x_j) = \sum_{k=1}^{\ell}(w(y_k^1) - w(z_{k-1}) + w(y_k^2) - w(z_k))$, we can choose k such that $w(y_k^1) - w(z_{k-1}) + w(y_k^2) - w(z_k) > 0$. If y_k^1 and y_k^2 are not adjacent, $C = \{y_k^1, z_{k-1}, y_k^2, z_k\}$ is an augmenting cycle in the RBS, contradicting the assumption that the given stable set is semi-optimal. Hence y_k^1 and y_k^2 must be adjacent. □

Lemma 3.9 *Suppose that $\ell = 1$. Then, x_i is adjacent to precisely three wings: W_1, a wing $W_i^1 \subseteq N^1(x_i)$ and a wing $W_i^2 \subseteq N^2(x_i)$, and x_j is also adjacent to precisely three wings: W_1, $W_j^1 \subseteq N^1(x_j)$ and $W_j^2 \subseteq N^2(x_j)$. Moreover, W_i^1 and W_j^1 are adjacent to a common black vertex v_1, and W_i^2 and W_j^2 are adjacent to a common black vertex v_2.*

Proof. Let us fix a vertex $u \in N^1(x_i) - W_1 (\supseteq N^1(x_i) - W(x_i, x_j) \ne \emptyset)$. Suppose that v_1 is the black vertex adjacent to u other than x_i. Let w be any vertex in $N^1(x_j) - W_1 (\ne \emptyset)$. By Lemma 3.8, y_1^1 is adjacent to y_1^2. Fact 2.2 says that y_1^1 is adjacent to both u and w, but y_1^2 is adjacent to neither u nor w. Thus w must be adjacent to u, since otherwise $\{y_1^1, y_1^2, u, w\}$ induces a claw. Since $w \notin W_1$, w is not adjacent to x_i. Then, by claw-freeness, w must be adjacent to v_1. Now we can conclude that all the vertices in $N^1(x_j) - W_1$ belong to one wing W_j^1 which is adjacent to both x_j and v_1. Similarly, one can prove the other assertions. □

Proof of Lemma 3.4 If a white alternating path from a to b passes through x_i, P_{11} (or P_{22}) and x_j, it must pass through v_2 (or v_1) twice, a contradiction.

Assume to the contrary that Case B occurs. By Lemma 3.9, P_{12} and P_{21} pass through neither v_1 nor v_2, even if v_1 and v_2 are irregular. That is, P_{12} and P_{21} belong to the wing W_1. A parallel discussion to Case A concludes the "twisting" version of Lemma 3.9, but this contradicts the assertion of Lemma 3.9 for Case A. □

Proof of Lemma 3.5

(1): Suppose that $y_k^1 \ne y_k^2$ for all k with $2 \le k \le \ell - 1$. Recall that $y_1^1 \ne y_1^2$ and $y_\ell^1 \ne y_\ell^2$.

Suppose that y_1^1 is adjacent to y_1^2. Fix a vertex $u \in N^1(x_i) - W(x_i, x_j)$. Fact 2.2 says that u is adjacent to y_1^1, but not to y_1^2. Vertices y_1^2 and y_2^2 are not adjacent to each other, since they belong to P_{22}. Vertex u must be nonadjacent to y_2^2, since otherwise, by claw-freeness, u must be adjacent to either z_1 or z_2, a contradiction to $u \notin W(x_i, x_j)$. Thus y_1^1 is not adjacent to y_2^2; otherwise $\{y_1^1, y_1^2, y_2^2, u\}$ induces a claw. Similarly, y_1^2 is not adjacent to y_2^1. Hence $k = 1$ satisfies the latter assertion of (1).

If y_ℓ^1 is adjacent to y_ℓ^2, one can analogously show that $k = \ell - 1$ satisfies the latter assertion.

Finally let us consider the other case. Lemma 3.8 guarantees that there is k such that $2 \le k \le \ell - 1$ and y_k^1 is adjacent to y_k^2, but y_{k-1}^1 is not adjacent to y_{k-1}^2. Obviously, z_{k-1} is adjacent to y_{k-1}^1, y_{k-1}^2, and y_k^1. Vertices y_{k-1}^1 and y_k^1 are nonadjacent since they belong to P_{11}. Thus y_{k-1}^2 is adjacent to y_k^1, since otherwise $\{z_{k-1}, y_{k-1}^1, y_{k-1}^2, y_k^1\}$ induces a claw. Vertices y_{k-1}^2, y_k^2 and y_{k+1}^2 are mutually nonadjacent since they belong to P_{22}. Hence y_k^1 must be nonadjacent to y_{k+1}^2. Similarly, we can prove that y_k^2 is not adjacent to y_{k+1}^1.

(2): Obviously the assertion holds by (1) and Fact 2.3.

(3): By the definition of P_{12} and P_{21}, $\delta(P'_{12}) \leq \delta(P_{12})$ and $\delta(P'_{21}) \leq \delta(P_{21})$. Hence $\delta(P_{11}) + \delta(P_{22}) \leq \delta(P_{12})+\delta(P_{21})$. Assume to the contrary that $\delta(P_{11})+\delta(P_{22}) < \delta(P_{12})+\delta(P_{21})$ holds. By the parallel discussion about P_{12} and P_{21}, and by Lemma 3.4, if P_{12} and P_{21} are of length 0, then Case A does not occur, a contradiction. Thus, lengths of P_{12} and P_{21} are greater than or equal to two (i.e., $\ell \geq 2$). From the assertions (1) and (2) of this lemma for P_{12} and P_{21}, there exist P'_{11} and P'_{22} such that $\delta(P'_{11}) + \delta(P'_{22}) = \delta(P_{12}) + \delta(P_{21}) > \delta(P_{11}) + \delta(P_{22})$. However, this implies $\delta(P'_{11}) > \delta(P_{11})$ or $\delta(P'_{22}) > \delta(P_{22})$, contradicting the maximality of P_{11} or P_{22}.
(4): It follows from $\delta(P'_{12}) \leq \delta(P_{12})$, $\delta(P'_{21}) \leq \delta(P_{21})$ and $\delta(P'_{12}) + \delta(P'_{21}) = \delta(P_{12}) + \delta(P_{21})$. □

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